

## A SIMPLE NOISE MODEL WITH MEMORY FOR BIOLOGICAL SYSTEMS

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A noise source model, consisting of a pulse sequence at random times with memory, is presented. By varying the memory we can obtain variable randomness of the stochastic process. The delay time between pulses, i.e. the noise memory, produces different kinds of correlated noise ranging from white noise, without delay, to quasi-periodical process, with delay close to the average period of the pulses. The spectral density is calculated. This type of noise could be useful to describe physical and biological systems where some delay is present. In particular it could be useful in population dynamics. A simple dynamical model for epidemiological infection with this noise source is presented. We find that the time behavior of the illness depends on the noise parameters. Specifically the amplitude and the memory of the noise affect the number of infected people.

*Keywords:* Statistical mechanics; population dynamics; noise induced effects.

### 1. Introduction

Many studies have reported occurrence of fluctuations and noise in biological systems and the role played by the noise, always present in natural systems [1–3]. The stochastic nature of living systems and the related noise-induced effects ranges from bioinformatics [4, 5], population dynamics [6–9], to virus dynamics and epidemics [10, 11]. Noise existing in biological systems is due to environmental fluctu-

ations and it is usually taken into account as a multiplicative white noise [12]. The white noise is usually defined as a continuous stochastic process with zero mean and correlation function  $\langle \xi \xi_t \rangle = C \delta(t)$ , where  $C$  is the noise intensity. However when we reproduce this noise in discrete time with step  $\tau$  for computer simulations, we consider rectangular pulses with width  $\tau$  and some distribution of probabilities for their heights. In this case the randomness is in the amplitude of noise pulses. We call this noise a discrete white noise source. In this paper we will consider both continuous and discrete noise sources. Starting from Markovian white noise we will introduce some memory obtaining a quasi-periodical process, which is strongly non Markovian.

In some population dynamics models, the noise must be positive at all times and often the randomness is due to stochastic natural events, that can be represented as pulses occurring at random times. We consider therefore a white noise source as a series of positive pulses at random times. To obtain zero average, if it is necessary, we can subtract the positive mean value.

In our model the discrete white noise at  $i$ -th step can be expressed as

$$\xi_i = \frac{f}{\tau} \vartheta(\nu\tau - \eta_i), \quad (1)$$

where  $f/\tau$  is the amplitude of pulses, which we set constant for simplicity,  $\eta_i$  is a random value, distributed homogeneously in the interval  $[0, 1]$ ,  $\vartheta(x)$  is the theta function ( $\vartheta(x \geq 0) = 1$ ,  $\vartheta(x < 0) = 0$ ),  $\nu$  is the probability per unit time to have a pulse and  $\nu\tau$  is the probability per one step to have a pulse. To obtain a time series of noise we extract random values  $\eta_i$  at every time step  $\tau$ . The average distance between pulses  $\zeta$ , can be considered as an effective period  $\langle \zeta \rangle = T = 1/\nu$  (with  $T \gg \tau$ ). The noise average is  $\langle \xi \rangle = f/T$ , and the correlation function is different from zero only during the step  $\tau$ . From formal point of view this noise is therefore white.

The continuous white noise is obtained in the limit  $\tau \rightarrow 0$ . This noise is usually expressed as a sequence of  $\delta$ -shape pulses at random times  $t_j$  [13]

$$\xi_j = f \sum_j \delta(t - t_j) \quad (2)$$

where

$$t_j = t_0 + \zeta_1 + \dots + \zeta_j, \quad (3)$$

with  $t_0$  the initial time, and  $\zeta_j$  the random time distances between neighboring pulses. The probability distribution is  $w(\zeta) = \nu \exp(-\nu\zeta)$ , where  $\zeta$  is distributed in the interval  $[0, \infty]$ . Of course the effective period  $T = \langle \zeta \rangle$  and the mean value  $\langle \xi \rangle$  are the same as for discrete noise, and here we consider them as constant parameters. We will use both continuous (2) and discrete (1) descriptions of the noise.

## 2. Noise with Memory

For some biological applications, it should be interesting to consider a noise source with some delay  $\zeta_0$  after each pulse. The system has memory during time  $\zeta_0$  and the next pulse is forbidden during this time. This process is suitable to obtain

noise sources with varying degree of randomness, ranging from white noise, as we defined it above ( $\zeta_0 = 0$ ), to quasi-periodical process ( $\zeta_0 \simeq T$ ). So the probability distribution of random time distances  $\zeta_j$  is

$$w(\zeta) = pe^{p(\zeta_0 - \zeta)}, \tag{4}$$

where  $\zeta$  is distributed in the interval  $[\zeta_0, \infty]$ ,  $p$  is the probability per unit time to have a pulse. If  $\zeta_0 = 0$ , then  $p = \nu$ . The first moment of  $\zeta$  is therefore easily derived as

$$\langle \zeta \rangle = \zeta_0 + \frac{1}{p}. \tag{5}$$

For fixed effective period  $\langle \zeta \rangle$ , the probability density  $p$  increases as memory  $\zeta_0$  increases. This probability is equal to  $1/\tau$ , when we have periodical process in discrete time ( $\zeta_0 \rightarrow \langle \zeta \rangle - \tau$ ). For continuous time, the probability of pulse per unit time  $p \rightarrow \infty$ , when  $\zeta_0 \rightarrow T$ .

For the discrete process the standard deviation of the noise is the weighted sum of this quantity during delay (which is zero) and that after delay  $\tilde{\sigma}^2$ , with the related probabilities. All parameters after delay we will mark by tilde. So we obtain

$$\sigma^2 = \frac{\zeta_0}{T} \times 0 + \frac{T - \zeta_0}{T} \times \tilde{\sigma}^2. \tag{6}$$

For the variance after delay we have

$$\tilde{\sigma}^2 = \langle \tilde{\xi}^2 \rangle - \langle \tilde{\xi} \rangle^2 = \left( \frac{f}{\tau} \right)^2 p\tau - \left( \frac{f}{\tau} p\tau \right)^2, \tag{7}$$

where  $\langle \tilde{\xi} \rangle$  and  $\langle \tilde{\xi}^2 \rangle$  are taken for the time after delay. Substituting (7) into (6) and taking into account (5) we obtain

$$\sigma^2 = \frac{T - \zeta_0}{T} p\tau \left( \frac{f}{\tau} \right)^2 (1 - p\tau) = \frac{(T - \zeta_0 - \tau)}{T\tau(T - \zeta_0)} f^2. \tag{8}$$

So we have an increasing  $\sigma^2$  as the memory of the system decreases. The standard deviation achieves its maximum value  $\sigma^2 = ((T - \tau)/(T^2\tau))f^2 \simeq f^2/\tau T$  for  $\zeta_0 = 0$ . When the memory is maximum  $\zeta_0 = T - \tau$ , the process is not random, so  $\sigma^2 = 0$ .

For the continuous noise process it is useful to calculate the spectral density. The characteristic function for  $\zeta$  is the Fourier transform of the probability density  $w(\zeta)$

$$\Theta(\omega) = pe^{p\zeta_0} \int_{\zeta_0}^{\infty} e^{-p\zeta + i\omega\zeta} d\zeta = \frac{pe^{i\omega\zeta_0}}{p - i\omega}. \tag{9}$$

By using the relation between the spectral density of the process  $\xi$  and the characteristic function of  $\zeta$ , derived in Ref. [13]

$$S[\xi - \langle \xi \rangle, \omega] = \frac{2(1 - |\Theta(\omega)|^2)}{|1 - \Theta(\omega)|^2}, \tag{10}$$

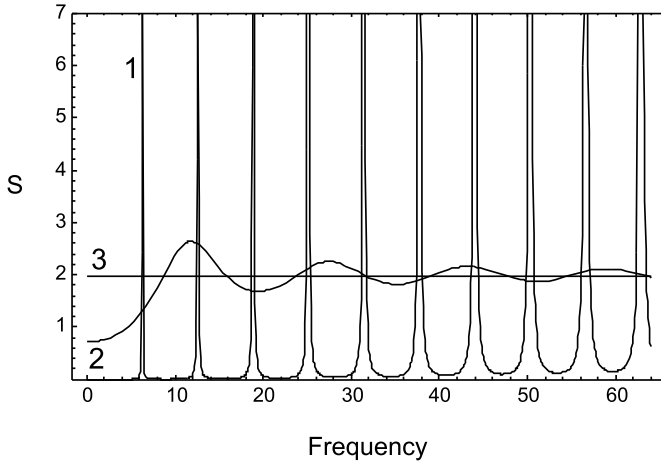


Fig. 1. Power spectrum of the process  $\xi(t)$  for three values of the memory  $\zeta_0$ : curves 1, 2 and 3, correspond respectively to  $\zeta_0 = 0.99, 0.3, 0.01$ .

we get

$$S[\xi - \langle \xi \rangle, \omega] = \frac{2\omega^2(1 - \zeta_0)^2}{2 - 2\cos(\omega\zeta_0) + 2(1 - \zeta_0)\omega \sin(\omega\zeta_0) + \omega^2(1 - \zeta_0)^2}, \quad (11)$$

where we use  $T = \langle \zeta \rangle = 1$ . The power spectrum of the noise  $\xi(t)$  is shown in Fig. 1, for three values of memory parameter, namely  $\zeta_0 = 0.99, 0.3, 0.01$ . We see that at  $\zeta_0 = 0.99$  the spectrum is a series of  $\delta$ -function-like pulses with harmonics at  $\omega_n \simeq 2\pi n$ . For  $\zeta_0 = 0.01$  the spectrum density is almost constant, and corresponds to white noise. For all cases, the spectrum density is quasi-periodical with distance between maxima equal to:  $\omega_n - \omega_{n-1} \simeq 2\pi/\zeta_0$ .

### 3. Some Possible Applications

The pulse noise model can be useful to describe some different physical and biological systems, where random processes with delay are present. It can be useful to describe and to understand the behavior of an epidemiological infection inside a population with a big number of individuals. In particular the number of contacts between ill and healthy people can be modelled as our noise with memory, where the delay describes the incubation period of the illness. So we can compare development of illnesses with different incubation periods.

Another possible application is the neuronal response to an electrochemical pulse. In fact the response time of a neuron, activated by a pulse, can depend on temperature and other parameters. Moreover our pulse noise model allows to regulate the periodicity, and this should play an important role in the neural network field.

The pulse noise source can also represent an alternative way to describe population dynamics of interacting species, previously modelled using stochastic resonance phenomenon [8]. This should be very useful when a periodicity, connected with environmental parameters, is present.

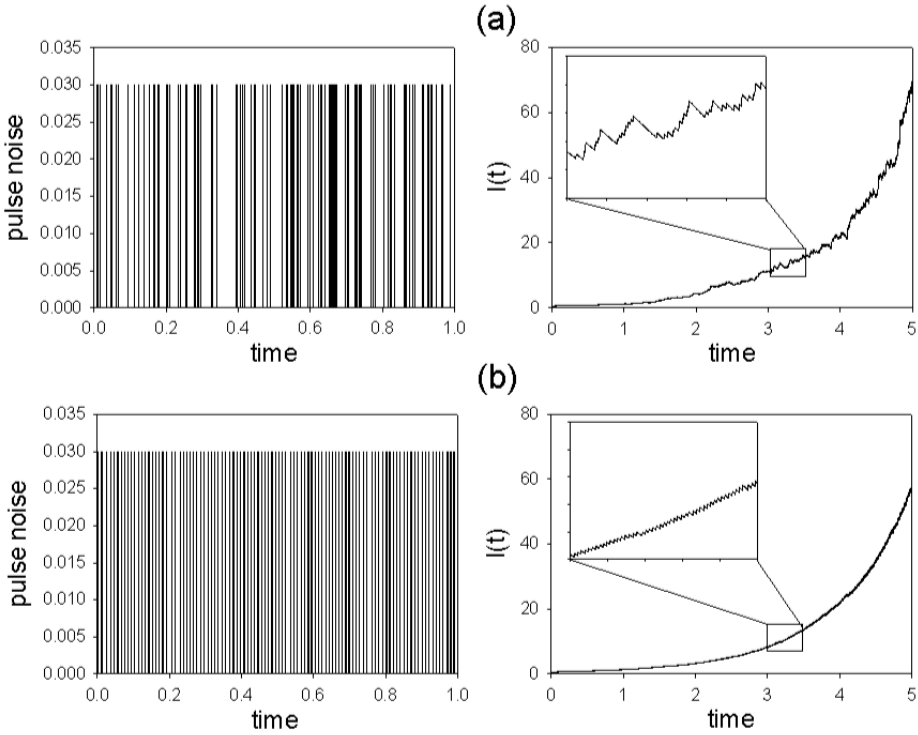


Fig. 2. Pulse noise  $\xi(t)$  and corresponding time series of the number of infected people  $I(t)$  vs time, for two values of the delay  $\zeta_0$ . The values of the parameters are:  $T = \langle \zeta \rangle = 10\tau$ ,  $\tau = 10^{-3}$ ,  $f = 3 \cdot 10^{-2}$ ,  $I(0) = 0.5$ , and  $a = -2$ . (a) white noise source, with  $\zeta_0 = 0$ ; (b) correlated noise source, with  $\zeta_0 = 8\tau$ .

#### 4. Simple Dynamics in the Presence of Noise with Memory

We start to analyze the role of the noise with memory considering a simple system described by the equation

$$\frac{dI}{dt} = \xi(t) I + a I, \tag{12}$$

where  $\xi(t)$  is the noise with memory,  $a$  is a negative constant, and  $I(t)$  can be a species density in population dynamics or the number of infected people in epidemics. We interpret this stochastic differential equation in the Ito sense. This equation has been previously investigated in Refs. [14,15].

The formal solution of Eq. (12) is

$$I(t) = I(0) \exp \left[ at + \int_0^t \xi(t') dt' \right]. \tag{13}$$

By numerical integration of Eq. (12) we find that the noise amplitude parameter  $f$  has a critical value  $f_c$ . For  $f > f_c$  an instability occurs and the illness exhibits a

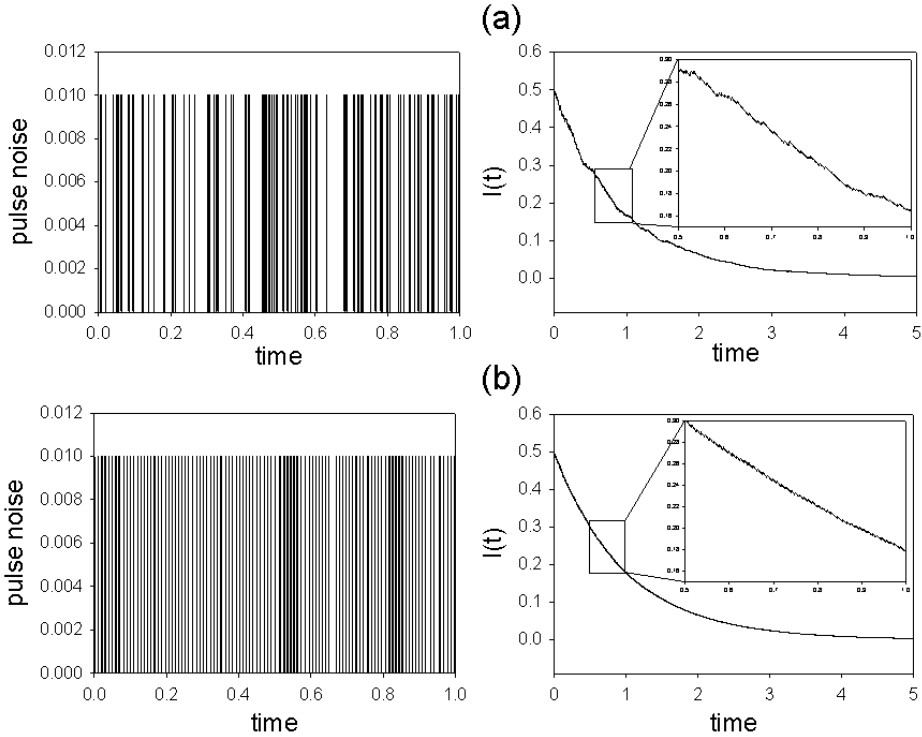


Fig. 3. Pulse noise  $\xi(t)$  and corresponding time series of the number of infected people  $I(t)$  vs time, for two values of the delay  $\zeta_0$ . Here  $f = 10^{-2}$ , and all other parameter values are the same of Fig. 2. (a) white noise source, with  $\zeta_0 = 0$ ; (b) correlated noise source, with  $\zeta_0 = 8\tau$ .

divergent behavior. By setting constant the value of the average period  $T = \langle \zeta \rangle = 10\tau$ , time series of the infection density are obtained for different values of the delay  $\zeta_0$ , which represents the time before a new pulse can occur. We report the results in Fig. 2, together with the pulse noise source. The parameter setting is:  $I(0) = 0.5$ ,  $f = 3 \cdot 10^{-2}$ ,  $a = -2$  and  $\tau = 10^{-3}$ . In Fig. 3 we show the same quantities  $\xi(t)$  and  $I(t)$ , for the same parameter setting with  $f = 10^{-2}$ . In both figures an exponential behavior of  $I(t)$  appears with fluctuations more pronounced, when the noise source is white ( $\zeta_0 = 0$ ) (see Figs. 2(a) and 3(a)). For correlated noise we have a quasi-periodical process. The exponential behavior is decreasing (see Fig. 2(b)) or increasing (see Fig. 3(b)) depending on the value of parameter  $f$  with respect to the critical value  $f_c$ . We observe clearly that the noise memory affects directly the fluctuations of the number of infected people  $I(t)$ . Using noise with zero memory we find that the critical value is  $f_c = 2 \cdot 10^{-2}$ . The noise memory also affects the exact value of  $f_c$ . In particular, when  $f$  is near the critical value with zero memory, the time behavior of the average number of infected people changes from increasing to decreasing exponential behavior, depending on the value of the noise memory. In Fig. 4 we report the time behavior of the average ensemble  $\langle I(t) \rangle$  for different values of noise parameters  $f$  and  $\zeta_0$ . Namely for (a)  $\zeta_0 = 0$ ,  $f = 2.01 \cdot 10^{-2}$ ,

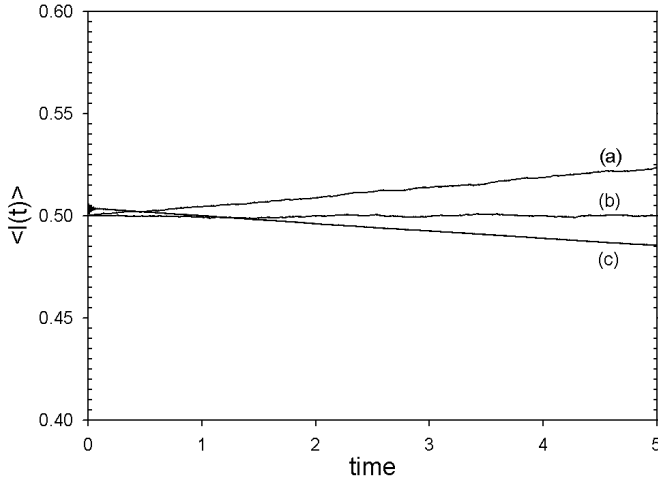


Fig. 4. Time behavior of the ensemble average of the number of infected people  $I(t)$  series, for different values both of the delay  $\zeta_0$  and the noise amplitude  $f$ . All the other parameter values are the same of Fig. 2. (a) white noise source ( $\zeta_0 = 0$ ) with  $f = 2.01 \cdot 10^{-2}$ ; (b) white noise source ( $\zeta_0 = 0$ ) with  $f = 2.00 \cdot 10^{-2}$ ; (c) correlated noise source with  $\zeta_0 = 8\tau$  and  $f = 2.01 \cdot 10^{-2}$ . The number of numerical realizations is  $10^4$ .

(b)  $\zeta_0 = 0$ ,  $f = f_c = 2.00 \cdot 10^{-2}$ , and (c)  $\zeta_0 = 8\tau$ ,  $f = 2.01 \cdot 10^{-2}$ . The average is performed on  $10^4$  numerical experiments. We see that for  $f = 2.01 \cdot 10^{-2} \simeq f_c$  (critical value for white noise with zero memory), by increasing the noise memory from zero to  $\zeta_0 = 8\tau$ , an instability-stability transition occurs. The average number of infected people changes from increasing ((a) in Fig. 4) to decreasing ((b) in Fig. 4) exponential behavior.

## 5. Conclusions

The noise with memory described above is a very useful tool for modelling quasi-periodical processes in nature. The parameter of periodicity allows to distinguish deterministic and random processes, and to adjust model to a real process accordingly. When this noise is included into the equation as a multiplicative noise, we can see qualitatively different behavior of the system in dependence on the noise amplitude and the periodicity parameter of the noise source. The role of the noise memory is to induce an enhancement of fluctuations of the number of infected people  $I(t)$  and to shift the critical value. When the noise parameter  $f$  is near the critical value with zero memory, a stability-instability transition occurs. For epidemic dynamics we find that, for pulse amplitude parameter less than the critical value ( $f < f_c$ ), we obtain quick recovering. Moreover we find that the role of the noise memory, that is the incubation period, is to reduce the amplitude of the fluctuations of the infected people. This causes a more regular time behavior of the illness.

Finally concerning population dynamics of some species, this noise can represent situations in which the species density growth is a quick process, and the decreasing

process is continuous in time. The delay  $\zeta_0$  should represent the time taken by the individuals of the species to become adult. So we could estimate the dependence of population dynamics on this parameter, with all other ones fixed.

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### References

- [1] See the special section on “*Complex Systems*”, *Science* **284** (1999) 79–107.
- [2] O. N. Bjornstad and B. T. Grenfell, *Noisy Clockwork: Time Series Analysis of Population Fluctuations in Animals*, *Science* **293** (2001) 638–643.
- [3] See the special section on “*Ecology Through Time*”, *Science* **293** (2001) 623–657.
- [4] W. J. Blake, M. Kaern, C. R. Cantor and J. J. Collins, *Noise in eukaryotic gene expression*, *Nature* **422** (2003) 633–637.
- [5] E. M. Ozbudak, M. Thattai, I. Kurtser, A. D. Grossman and A. van Oudenaarden, *Regulation of noise in the expression of a single gene*, *Nat. Genet.* **1** (2002) 69–73.
- [6] P. Turchin, L. Oksanen, P. Ekerholm, T. Oksanen and H. Henttonen, *Are lemmings prey or predators?* *Nature* **405** (2000) 562–565.
- [7] B. Spagnolo, A. Fiasconaro and D. Valenti, *Noise induced phenomena in Lotka-Volterra systems*, *Fluctuation and Noise Letters* **3** (2003) L177–L185.
- [8] D. Valenti, A. Fiasconaro and B. Spagnolo, *Stochastic resonance and noise delayed extinction in a model of two competing species*, *Physica A* **331** (2004) 477–486.
- [9] B. Spagnolo, D. Valenti and A. Fiasconaro, *Noise in ecosystems: A short review*, *Mathematical Biosciences and Engineering* **1** (2004) 185–211.
- [10] H. C. Tuckwell and E. Le Corfec, *A stochastic model for early HIV-1 population dynamics*, *J. Theor. Biol.* **195** (1998) 451–463.
- [11] J. L. W. Gielen, *A stochastic model for epidemics based on the renewal equation*, *J. Biological Systems* **8** (2000) 1–20.
- [12] B. Spagnolo, M. Cirone, A. La Barbera and F. de Pasquale, *Noise induced effects in population dynamics*, *J. Physics: Condensed Matter* **14** (2002) 2247–2255.
- [13] R. Stratonovich, *Topics in the Theory of Random Noise*, Vol. **1**, Gordon and Breach, New York – London (1963).
- [14] Ya. B. Zeldovich, *Intermittency in random media*, *Sov. Phys. Usp.* **30** (1987) 353–369.
- [15] I. M. Sokolov and V. V. Belik, *Competition between Lévy jumps and continuous drift*, *Physica A* **330** (2003) 46–52.
- [16] C. W. Gardiner, *Handbook of Stochastic Methods for Physics* cap. 4, Springer-Verlag, Berlin (1985).