



A multidimensional critical factorization theorem[☆]

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Abstract

The Critical Factorization Theorem is one of the principal results in combinatorics on words. It relates local periodicities of a word to its global periodicity. In this paper we give a multidimensional extension of it. More precisely, we give a new proof of the Critical Factorization Theorem, but in a weak form, where the weakness is due to the fact that we lose the tightness of the local repetition order. In exchange, we gain the possibility of extending our proof to the multidimensional case. Indeed, this new proof makes use of the Theorem of Fine and Wilf, that has several classical generalizations to the multidimensional case.

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Keywords: Combinatorics on words; Periodicity; Multidimensional words

1. Introduction

As Professor M.P. Schutzenberger wrote in 1983, “Periodicity is an important property of words that is often used in applications of combinatorics on words. The main results concerning it are the theorem of Fine and Wilf [. . .] and the Critical Factorization Theorem” (cf. [27]). Since then, combinatorics on words has expanded a great deal and now other results may be added to the previous ones. Among them even an older result that became important because the well-known Sturmian words represent an extremal case for it, the

[☆] Partially supported by MIUR National Project PRIN “Linguaggi Formali e Automi: teoria ed applicazioni” and University of Palermo Research Project (ex 60%) “Tecniche per lo sviluppo e l’analisi di algoritmi e strutture dati”.

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Morse-Hedlund Theorem (cf. [35] and see also [28,17,37,23] and references therein). Another important result is a periodicity theorem proved in [31] (cf. also for instance [11,26]), which is a “tight evolution” of a combinatorial lemma firstly proved in [20], used in the design and analysis of a famous string matching algorithm described in the same article (cf. also [13]).

In this paper, we will focus on one of the results mentioned by Schutzenberger, the critical factorization theorem, found out by Césari and Vincent in [10] and developed by Duval in [15]. It relates local periodicities of a word to its global periodicity. For a reference on it and on other combinatorial results, see also [11, Chapter 6], [28, Chapter 8], [7, Chapter 3], [26,25], and references therein.

For what concerns Computer Science and the theory of Combinatorics on Words, this problem has a long tradition and it is an active research field (cf. for instance [11,27,7,9,15,16,28,26,31,36] and references therein). Among the applications of the critical factorization theorem in this field we recall a famous string matching algorithm, described in [12], and an algorithm that finds a short superstring (cf. [8]). This theorem is also deeply linked with longest unbordered factors of a word and a recently solved famous conjecture by Duval (cf. [22,27,28,33] and references therein).

Concerning other fields, the same problem appears also in the theory of long-range order for discrete structures in \mathbb{R}^d . Indeed, the recent discovery and study of quasicrystals (cf. [41] and [14]) raises the question: what geometric and physical conditions force a structure to be crystalline rather than quasi-crystalline. Some answers to previous question have been given in [14] where some conditions on “local rules” imply a “global” crystalline structure. (cf. also [14,41,34,38,39,5,24] and references therein).

The same problem appears also in [6] with applications in Biology and in the study of spatial structure of proteins. The solution proposed in this paper has been applied in several research studies.

The aim of this paper is to give a first generalization of the critical factorization theorem to the multidimensional domain. Up to our knowledge, no such generalization exists in the scientific literature.

The techniques used here for proving the first generalization to multidimensional words come from the theory of multidimensional periodicities, that was introduced in their seminal and fundamental work by Amir and Benson in [1–3] (cf. also [4,19,21,40,43,17]). In particular our techniques make use of some results developed in [21].

Recently, we have been able to give a proof of a weak form of the Critical Factorization Theorem, by using the Theorem of Fine and Wilf. For this last theorem, which is the tight version of the periodicity lemma, there exist generalizations to the multidimensional case (cf. [21,32,40,42]). This fact has suggested us to try to extend our proof to the multidimensional case.

The paper is organized as follows. In the next section we give some basic definitions and state some fundamental theorems in the unidimensional case. Moreover, we state and prove a weak form of the critical factorization theorem. This proof, as already stressed, will be essential for the demonstration of the main result of the paper.

In Section 3, we examine the multidimensional case. Firstly, we give some fundamental definitions and we extend some unidimensional lemmas to this case. Afterwards, we prove our main theorem in the bidimensional case. In fact the proof in this case is simpler to read

and the one in the d -dimensional case is only a trivial generalization of it that follows the same steps.

In Section 4, we describe some research directions that are related to the results of this paper and we state some open problems.

2. Basic definitions and the unidimensional case

For any notation not explicitly defined in this paper we refer to [21,27–29].

We begin this paper by considering unidimensional case for words, because it is in some sense the simpler and most deeply studied case and it can help the intuition. Indeed, we will state a weak form of the critical factorization theorem.

The alphabet we are considering here is any set, but it is usually finite. We want just to observe that the main results of this paper are alphabet-independent.

A bi-infinite word w is a function from \mathbb{Z} to the alphabet A . Therefore $w(i)$ is a letter of A that is the image of i by the function w . The set of all bi-infinite words is $A^{\mathbb{Z}}$.

Definition 1. Let w be a bi-infinite word. A factor of length n (or block of n consecutive letters, or subword of length n) of w is a word of the form $x = w(l)w(l+1) \cdots w(u)$, such that $u - l = n - 1$. We accept l to be $-\infty$ and u to be $+\infty$, and not just integers, i.e. we consider also, as factors of w , infinite words.

A bi-infinite word w has period $p \in \mathbb{N}$, $p > 0$, or equivalently, p is a period of w if for any $i \in \mathbb{Z}$, $w(i) = w(i + p)$. A finite word x has period p if it is a factor of a bi-infinite word having period p , or, equivalently, if $w(i) = w(i + p)$, for any $i \in \mathbb{Z}$ such that both $w(i)$ and $w(i + p)$ belong to x . If the bi-infinite word w has a period, the smallest of all its periods is called the *period* of w . If w has no periods, w is called aperiodic and its period is considered to be $+\infty$.

Note that for a finite word x , the above definition of periodicity differs from the classical one by the fact that any p greater than the length of x is considered to be a period of x , while it is not in the classical definition.

If x is a finite word, we denote by $x(i)$ the i th letter of x . Thus we can write $x = x(1) \cdots x(n)$, where n is the length of x . We define the numbers from 1 up to $n - 1$ to be the positions of x . Roughly speaking, position i lies between letters $x(i)$ and $x(i + 1)$. The set of positions of a bi-infinite word is \mathbb{Z} .

We now define *repetitions* and *central local periods*.

Definition 2. Let w be a finite or infinite word. A factor of length n (or block of n consecutive letters) of w , $w(j)w(j+1) \cdots w(j+n-1)$, is a repetition of order α , with $\alpha \geq 1$ a real number, if there exists a natural number p , $0 < p \leq n$ such that $w(i) = w(i + p)$ for $i = j, \dots, j + n - 1 - p$ and such that $n/p \geq \alpha$. The number p is called a period of the repetition. The smallest of all periods is called *the* period of the repetition.

Note that if a factor of length n is a repetition of order α then it is trivially also a repetition of order α' for any $1 \leq \alpha' \leq \alpha$ of the same period, because the definition of repetition involves an

inequality. Any word of length n and period $p \leq n$ has order n/p . Note also that a repetition can have more than one period. Consequently, if a factor x of length n is a repetition of order α and period p and if it has also period q , then factor x is a repetition of order $\alpha(p/q)$ and period q .

As an example let us consider the word $w = abaaba$. It is a repetition of length 6 and has order 1 with period 6, order $\frac{6}{5}$ with period 5 and order 2 with period 3. In particular the length of the repetition is always a period and, so, any factor is a repetition of order 1 and period its length.

We refer to [28] for a formal definition of local period $c_\alpha(w, i)$ of order α in position i of the word w .

Roughly speaking $c_\alpha(w, i)$ is the period of the smallest repetition of order α that matches w when the center of the repetition is placed in position i . The match can be virtual, in the sense that it is considered valid also when there are no more letters to match in w .

Definition 3. A position i is α -critical (or just *critical*, when α is fixed) if $c_\alpha(w, i)$ coincides with the period of the whole word w .

Above definition slightly differs from the corresponding one given in [28], where a critical position represents the maximum of the local periods.

Remark 1. Note that if w has period p (where p can also be $+\infty$) then for any α and for any $i \in \mathbb{Z}$, $c_\alpha(w, i) \leq p$. This inequality can be strict for all i , i.e. it may be that there are no α -critical points. For instance, if w is the periodic word with period 4 $w = \dots 1000100010001000 \dots$ and if $\alpha = \frac{4}{3}$, then no position is α -critical. Indeed the period of w is 4, while w has in every position i a central repetition of order greater than or equal to $\frac{4}{3}$ and period smaller than or equal to 3. In fact, if i is between two 0s, then the central repetition in position i of order greater than or equal to $\frac{4}{3}$ is 00, that has order 2 and period equal to 1. If i has a 1 immediately to the right (resp. left), then the central repetition in position i of order greater than or equal to $\frac{4}{3}$ is 0010 (resp. 0100), that has order exactly equal to $\frac{4}{3}$ and period 3. For another more complex example, we refer to Chapter 8 of [28] where it is shown that if w is the Fibonacci bi-infinite word (that has period $+\infty$), then for any α , $1 \leq \alpha < 2$ and for any i one has that $c_\alpha(w, i) \leq k_\alpha(w)$, where $k_\alpha(w)$ is a constant depending on α .

In this section, we give a new proof of a weak form of the critical factorization theorem. The weakness is due to the fact that we loose the tightness of the local repetition order. In exchange, we gain the possibility of extending our proof to two dimensions. Indeed, this new proof makes use of the Theorem of Fine and Wilf, that has several classical generalizations to the bidimensional case.

Let us, firstly, state two lemmas and the Theorem of Fine and Wilf, that will be helpful for our new proof (see [28, Lemma 8.1.2, Lemma 8.1.3, Theorem 8.1.4]).

Lemma 1. Let u, v, w be words such that uv and vw have period p and $|v| \geq p$. Then the word uvw has period p .

Lemma 2. *Suppose that w has period q and that there exists a factor v of w with $|v| \geq q$ that has period r , where r divides q . Then w has period r .*

Theorem 1 (Fine and Wilf [18]). *Let w be a word having periods p and q , being $q \leq p$ two positive integers. If $|w| \geq p + q - \gcd(p, q)$, then w has also period $\gcd(p, q)$.*

Before stating and proving the weak form of the critical factorization theorem, let us give a definition that we will use in our proof.

Definition 4. Let w be a word and i, j be two positions of w . We define $P_\alpha(i, j) = \sup\{c_\alpha(w, l) \mid i \leq l \leq j\}$. If i and j coincide, respectively, with the first and last positions in w (including the case $i = -\infty$ and $j = +\infty$), we denote $P_\alpha(w) = P_\alpha(i, j)$.

The following theorem is, for $\alpha = 2$, the critical factorization theorem. We will prove it for $\alpha = 4$, because under this hypothesis we will be able to use the Theorem of Fine and Wilf.

An interval of positions $[i, j]$, $i \leq j$ of a word w is the set of integers $\{i, \dots, j\}$ that are also positions of w . Its size is its cardinality, that is $j - i + 1$.

Theorem 2. *Let w be a (finite or infinite) word, $[i, j]$ be an interval of positions in w , $i, j \in \mathbb{Z} \cup \{-\infty, +\infty\}$ and $\alpha = 4$.*

- (1) *Every interval $[t_1, t_2] \subseteq [i, j]$, $t_1, t_2 \in \mathbb{Z}$, of size $t_2 - t_1 + 1 = \max(1, P_\alpha(i, j) - 1)$ contains a position l such that $c_\alpha(w, l) = P_\alpha(i, j)$.*
- (2) *If*

$$i' = \inf \left\{ l - \left\lceil \frac{\alpha}{2} \cdot c_\alpha(w, l) \right\rceil \mid i \leq l \leq j, \right. \\ \left. \text{where } l \text{ is such that } c_\alpha(w, l) = P_\alpha(i, j) \right\}, \\ j' = \sup \left\{ l + \left\lfloor \frac{\alpha}{2} \cdot c_\alpha(w, l) \right\rfloor - 1 \mid i \leq l \leq j, \right. \\ \left. \text{where } l \text{ is such that } c_\alpha(w, l) = P_\alpha(i, j) \right\},$$

then w has period $P_\alpha(i, j)$ in $[i'', j'']$, where

$$i'' = \begin{cases} i' & \text{if } i' \text{ is a position in } w, \\ \text{the first position in } w & \text{otherwise,} \end{cases} \\ j'' = \begin{cases} j' & \text{if } j' \text{ is a position in } w, \\ \text{the first position in } w & \text{otherwise.} \end{cases}$$

Proof. The proof is by induction on $P_\alpha(i, j)$.

- If $P_\alpha(i, j) = 1$, $c_\alpha(w, l) = 1$ whenever in $[i, j]$ and then there is only one letter labelling all the positions in this interval. Thus, w has period $P_\alpha(i, j) = c_\alpha(w, l) = 1$ in the whole $[i, j]$ and the assertion is trivially true.
- Let us suppose that both assertions of the theorem hold true for all intervals $[\hat{i}, \hat{j}]$ such that $P_\alpha(\hat{i}, \hat{j}) < n$. Let $[i, j]$ be an interval such that $P_\alpha(i, j) = n$. We shall prove that both statements hold true even in $[i, j]$.

Let us assume that there exists, by contradiction, an interval $[t'_1, t'_2] \subseteq [i, j]$ of size equal to $\max(1, P_\alpha(i, j) - 1) = n - 1$ that does not contain any position k such that $c_\alpha(w, k) = n$. Let us slide this interval to the left or to the right until we find $[t_1, t_2]$ of size $t_2 - t_1 + 1 = n - 1$ such that the first position $k', i \leq k' \leq j$ to the left or to the right of $[t_1, t_2]$ verifies $c_\alpha(w, k') = n$ and $P_\alpha(t_1, t_2) = q < n$. This k' must exist because $P_\alpha(i, j) = n$. Let us suppose that k' is to the left (the proof being analogous in the other case) of the interval, i.e. $k' = t_1 - 1$.

By the inductive hypothesis we know that each subinterval of size $\max(1, q - 1)$ contains a position l such that $c_\alpha(w, l) = P_\alpha(t_1, t_2) = q$. In particular, there exists at least one such l among the first $\max(1, q - 1)$ positions of $[t_1, t_2]$. Let \underline{l} be the smallest among them.

By the inductive hypothesis, on the interval $[\underline{i}'', t_2]$, with

$$\underline{i}'' = \begin{cases} \underline{i}' & \text{if } \underline{i}' \text{ is a position in } w, \\ \text{the first position in } w & \text{otherwise} \end{cases}$$

and

$$\underline{i}' = \inf \left\{ l - \left\lceil \frac{\alpha}{2} \cdot c_\alpha(w, l) \right\rceil \mid i \leq l \leq j, \text{ where } l \text{ is such that } c_\alpha(w, l) = P_\alpha(i, j) \right\},$$

w is periodic of period q .

Claim. Position \underline{i}' belongs to the domain of the word w and, therefore, $\underline{i}'' = \underline{i}'$.

Proof of the Claim. Let us suppose, on the contrary, that the claim is false. Since \underline{i}'' is the first position in w and $\underline{l} \geq t_1 > k'$, there exists in position k' an α -local repetition of period q , contradicting the fact that $c_\alpha(w, k') = n$ and the claim is proved.

By hypothesis, $\underline{i}'' = \underline{l} - \lceil \alpha/2 \cdot c_\alpha(w, \underline{l}) \rceil = \underline{l} - 2q$. Thus, interval $[\underline{i}'', t_2]$ has size $t_2 - \underline{i}'' + 1 = t_2 - (\underline{l} - 2q) + 1 \geq t_2 - (t_1 + q - 1 - 2q) + 1 = t_2 - t_1 + q + 2 \geq n - 2 + q + 2 = n + q$. Hence the word $x = w(\underline{i}'') \dots w(t_2)$ has length greater than or equal to $n + q$. But x has also period n because it is contained in the repetition of order α centered in k' . Therefore, we can apply the Theorem of Fine and Wilf to x and we conclude that the word x has period an integer d that divides both n and q . By Lemma 2 this implies that the whole word centered in k' has period d , which is absurd because of the minimality of $c_\alpha(w, k') = n$.

Therefore, the distance between two different positions l_1 and l_2 such that $c_\alpha(w, l_1) = c_\alpha(w, l_2) = P_\alpha(i, j) = n$ is at most $n - 1$ and the first assertion of the theorem is proved. By Lemma 1 the word w has period n in the interval $[\underline{i}'', j'']$, and this concludes the proof. \square

Recall that a critical position l is defined, in this paper, to be a position where the local period $c_\alpha(w, l)$ is equal to the global period $p(w)$ of the word w . The first part of previous theorem tell us that in every interval of length $\max(1, P_\alpha(w) - 1)$ there is a critical position, while the second part implies that $P_\alpha(w)$ is equal to $p(w)$. Therefore, the following corollary is straightforward.

Corollary 1. Every interval $[t_1, t_2]$, with $t_1, t_2 \in \mathbb{Z}$ positions in w , of size $t_2 - t_1 + 1 = \max(1, p(w) - 1)$ contains a critical position.

3. The multidimensional case and the critical factorization theorem

Let us consider, now, the multidimensional case. Analogously to the unidimensional case we may define a d -dimensional word w as a function from a subset $X \subseteq \mathbb{Z}^d$ to an alphabet A . The set X is called the shape of w and it is denoted by $sh(w)$. A factor v of a multidimensional word w is any restriction of w to any subset $Y \subseteq X$ and clearly $sh(v) = Y$.

A word w is called an n -cubic word or simply a cubic word if its shape $sh(w)$ is either an hypercube of \mathbb{Z}^d , i.e. it is of the form $\{j_1, \dots, j_1 + n - 1\} \times \dots \times \{j_d, \dots, j_d + n - 1\}$ for some natural number n , or a translate of a quadrant, or the whole \mathbb{Z}^d .

Let us give the definition of multidimensional periodicity for words.

Definition 5. Let H be an additive subgroup of \mathbb{Z}^d , different from $\langle 0_d \rangle = \{0_d\}$ (the zero subgroup). A multidimensional word w having shape \mathbb{Z}^d has period H , or equivalently H is a period of w , if for any $i \in \mathbb{Z}^d$ and for any element g of the subgroup H one has that $w(i) = w(i + g)$.

A multidimensional word w' has period H if it is a factor of a multidimensional word w having shape \mathbb{Z}^d that has period H .

If w has no periods, w is called aperiodic and its period is considered to be $+\infty$.

Note that for $d = 1$, the above definition of periodicity is equivalent to the one given in the unidimensional case, if we consider equivalent the two properties of having period p and having period the subgroup $H = \langle p \rangle$ generated by $p > 0$ (recall that all subgroups of \mathbb{Z} can be generated by only one element, that is unique up to a factor ± 1).

The non-zero subgroups of \mathbb{Z}^d can only have dimension an integer d' , with $1 \leq d' \leq d$.

For instance the word w from \mathbb{Z}^2 to $A = \mathbb{Z}$ defined by $w(i_1, i_2) = i_2$ has period H_1 , where H_1 is generated by the element $(1, 0)$ (in short $H_1 = \langle (1, 0) \rangle$) and has dimension 1.

The word w from \mathbb{Z}^2 to $A = \{0, 1, 2, 3\}$ defined by $w(i_1, i_2) = x$, where x is the decimal corresponding to the binary number composed of two digits, the first being $i_1 \pmod{2}$ and the second $i_2 \pmod{2}$, has period H_2 , where H_2 is generated by the elements $(2, 0)$ and $(0, 2)$ (in short $H_2 = \langle (2, 0), (0, 2) \rangle$). Subgroup H_2 has dimension 2.

The subgroups of \mathbb{Z}^d of dimension d are called *lattices* or *fully dimensional* subgroups. They are the only subgroups that have finite index. If a factor has period a lattice H , it is called *fully periodic*.

Another remarkable class of subgroups is that of subgroups of dimension 1. They are generated by only one element. Trivially, if $H = \langle q \rangle$ then $H = \langle -q \rangle$, and q and $-q$ are the only generators of H . Suppose that H has dimension 1, q is its generator and it is also a period of a factor v . The element q is called *periodicity vector* of v . It is easy to see that $q \in \mathbb{Z}^d$ is a periodicity vector of v if and only if for any $z \in \mathbb{Z}$ and for any $i \in sh(v)$, such that $i + zq \in sh(v)$, one has that $v(i) = v(i + zq)$.

Remark 2. If a factor v has period H and if H' is a subgroup of H then v has also period H' . In particular, if v has period H , for any $q \in H$ one has that q is a periodicity vector of v . In the unidimensional case this is equivalent to say that if v has period p then every multiple of p is a period.

Note that if w is an application from \mathbb{Z}^d to A and has period H and if g is an application from A to any set B , the word $g \cdot w$, composition of w and g , from \mathbb{Z}^d to B has also period H . In other terms, projections preserve the period.

Recall that any subgroup H of dimension $d > 1$ has infinitely many bases, where a basis of H is any set consisting of a minimal number of generators.

We give now the classical definition of transversal of a subgroup.

Definition 6. Given a subgroup H , a transversal T_H of H is a subset of \mathbb{Z}^d such that for any element $i \in \mathbb{Z}^d$, there exists a unique element $j \in T_H$ such that $i - j \in H$.

Remark 3. Note that if T_H is a transversal of H then for any $q \in \mathbb{Z}^d$ the set $q + T_H$ is also a transversal of H . Indeed for any element $i \in \mathbb{Z}^d$, let us consider $i - q$. Since T_H is a transversal there exists a unique element $j' \in T_H$ such that $i - q - j' = i - (q + j') \in H$. The unique element $j = q + j'$ belongs to $q + T_H$, and so $q + T_H$ satisfies previous definition.

We now define the order of periodicity of factors of w that have an “hypercube” as “shape”. These factors will be used to define what is a local periodicity in the multidimensional case of words.

Definition 7. An n -cubic factor v is a repetition of order α , with $\alpha \geq 1$ a real number, (or an α -repetition), if

- (1) v is L periodic, where L is a full dimensional subgroup of \mathbb{Z}^d ;
- (2) n is such that $n/h_L \geq \alpha$, where h_L is the smallest integer such that every hypercube of side length h_L contains a transversal of L .

The lattice L is called a period of the α -repetition v .

Definition 8. The word w has a central repetition v of order α in position $j = (j_1, \dots, j_d) \in \mathbb{Z}^d$, if v is a $2n$ -cubic word that is a repetition of order α and shape $\{-n+1, \dots, n-1, n\}^d$ that matches w when its center is placed in position j , i.e., more formally, if for any $i \in sh(v)$ such that $j + i \in sh(w)$ one has that $w(j + i) = v(i)$. We say also that the repetition v is centered in position j .

If w has at least a central repetition of order α and period L in position $j \in \mathbb{Z}^d$, the set \mathcal{H} of all h_L such that every hypercube of side length h_L contains a transversal of L is non-empty. We will denote by $c_\alpha(w, j)$ the minimum of this set \mathcal{H} .

If w has no central repetitions of order α in position j , we set $c_\alpha(w, j) = +\infty$.

Note that if an n -cubic factor is a repetition of order α then it is also a repetition of order α' for any $1 \leq \alpha' \leq \alpha$ of same period because we used an inequality in the definition of repetition.

Note also that a factor, and, as well, repetitions can have more than one period.

Note also that it is possible to give geometrical properties of the basis vectors of L to satisfy condition 2. For instance, in dimension $d = 1$, $L = \langle p \rangle$ and the property is that $h_L \geq p$. For dimension $d = 2$, $L = \langle (a, b), (c, d) \rangle$, with $a > 0$ and $c > 0$ and the requirement is that $h_L \geq a + c$ and $h_L \geq |b| + |d|$ (cf. [1]).

We have now all notions necessary to state and prove the main result of the paper. We only need some more “tools” that will play an important role in the proof of it. Let us start by stating a result that is analogous for the multidimensional case to the Periodicity Lemma (cf. [30]). It is an immediate consequence of the generalization to the multidimensional case of the Theorem of Fine and Wilf given in [21].

Let H be a subgroup of \mathbb{Z}^d , q an element of \mathbb{Z}^d and v a word and let $sh(v)$ be its shape.

Theorem 3. *Suppose that v is H -periodic and that q is a periodicity vector for v . If $sh(v) \cap (sh(v) + q)$ contains a transversal of H then v is also H' -periodic, where $H' = \langle H, q \rangle$ is the lattice generated by H and q .*

Next two lemmas are the d -dimensional generalizations, respectively of Lemmas 1 and 2. They will be necessary for the proof of Theorem 4, as the unidimensional corresponding ones were necessary for proving Theorem 2.

Lemma 3. *Let v_1 and v_2 be two factors of same word w of \mathbb{Z}^d that have both period the subgroup H . If $sh(v_1) \cap sh(v_2)$ contains a transversal of H then the factor v that has shape $sh(v) = sh(v_1) \cup sh(v_2)$ has also period H .*

Proof. We have to prove that for any $i, j \in sh(v)$, if $i - j \in H$ then $v(i) = v(j)$. Previous equality trivially holds if both i, j belong to $sh(v_1)$ or to $sh(v_2)$. We can suppose now, without loss of generality, that $i \in sh(v_1)$ and $j \in sh(v_2)$. Since $sh(v_1) \cap sh(v_2)$ contains a transversal of H , there exists $i' \in sh(v_1) \cap sh(v_2)$ such that $i' - i \in H$. Since v_1 has period H and also $i' \in sh(v_1)$, $v(i) = v_1(i) = v_1(i') = v(i')$.

But i' belongs also to $sh(v_2)$, and, moreover $i' - j = i' - i + i - j$ that belongs to H because both $i' - i$ and $i - j$ belong to H . Since v_2 has period H and also $i' \in sh(v_2)$, $v(i') = v_1(i') = v_1(j) = v(j)$ and the lemma is proved. \square

Lemma 4. *Let v_1 and v_2 be two factors of the same word w on \mathbb{Z}^d with $sh(v_2) \subseteq sh(v_1)$. Suppose that v_1 has period H_1 and that v_2 has period H_2 , with H_1 a subgroup of H_2 , and that $sh(v_2)$ contains a transversal of H_1 . Under these hypotheses v_1 has period H_2 .*

Proof. We have to prove that for any $i, j \in sh(v_1)$, if $i - j = h \in H_2$ then $v_1(i) = v_1(j)$. Since $sh(v_2)$ contains a transversal of H_1 , there exist $i', j' \in sh(v_2)$ such that $i' - i = l_i \in H_1$ and $j' - j = l_j \in H_1$. Since $sh(v_2) \subseteq sh(v_1)$ and since v_1 has period H_1 , $v_1(i) = v_1(i')$ and $v_1(j) = v_1(j')$. But $i' - j' = i' - i + i - j + j - j' = l_i + l_j + h$. Since H_1 is a subgroup of H_2 , $l_i + l_j + h \in H_2$. Since $i', j' \in sh(v_2)$, $i' - j' \in H_2$ and v_2 has period H_2 , $v_1(i') = v_2(i') = v_2(j') = v_1(j')$ and the lemma is proved. \square

From now on we will deal with bidimensional words and we will state and prove the main result of this paper in that case. In fact, the proof is in this case simpler to read. The proof in the d -dimensional case follows the same steps and it will be not reported here.

Moreover, we will focus on squared words. Our result will be proved only in this case.

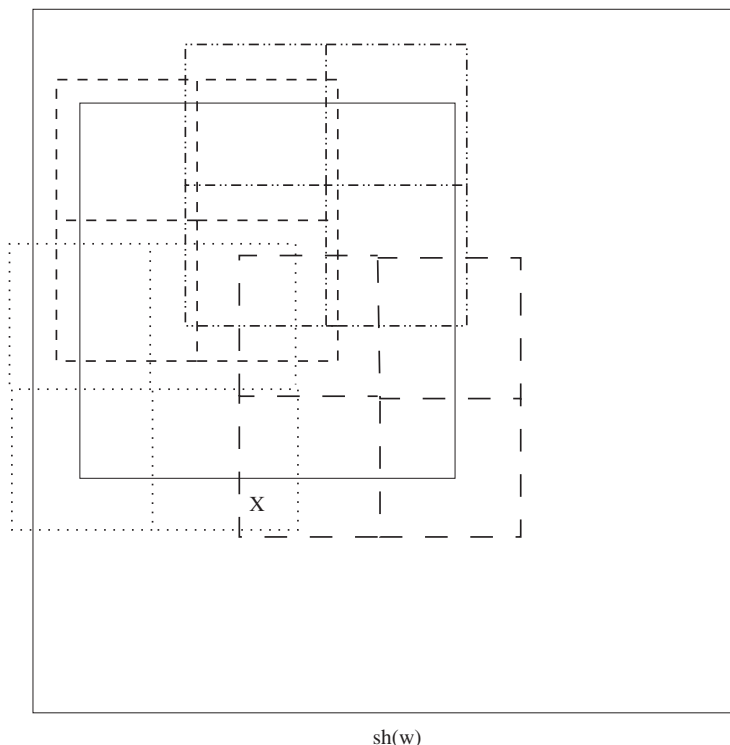


Fig. 1. α -repetitions such that $c_\alpha(w, l) = P_\alpha(X)$, for $\alpha = 2$.

Definition 9. Let w be a word and X be a square contained in $sh(w)$. We define $P_\alpha(X) = \sup\{c_\alpha(w, l) \mid l \in X\}$. We call a position l *critical* if there exists a period L of w such that every square of side length $c_\alpha(w, l)$ contains a transversal of L .

Remark 4. We notice that lattice L in above definition depends only on w and $c_\alpha(w, l)$ and not directly on α . Roughly speaking, a position is critical if a local period is also a global period.

Theorem 4. Let w be a (finite or infinite) squared bidimensional word, X be a square contained in the shape of w and $\alpha = 4$.

- (1) Every square $T \subseteq X$, of side length $sd(T) = \max(1, P_\alpha(X) - 1)$ contains a position l such that $c_\alpha(w, l) = P_\alpha(X)$.
- (2) Let v be the factor of w the shape of which is the intersection of $sh(w)$ and the union of the shapes of the α -repetitions centered in positions $l \in X$ such that $c_\alpha(w, l) = P_\alpha(X)$. Then v has period L , where L is a subgroup such that every square of side length $P_\alpha(X)$ contains a transversal of L (Figs. 1, 2).

Proof. The proof is by induction on $P_\alpha(X)$. Let X' be the union of the shapes of the α -repetitions centered in position $l \in X$. If $P_\alpha(X) = 1$, there is only one letter labelling all the positions in X' and the assertions are trivially true.

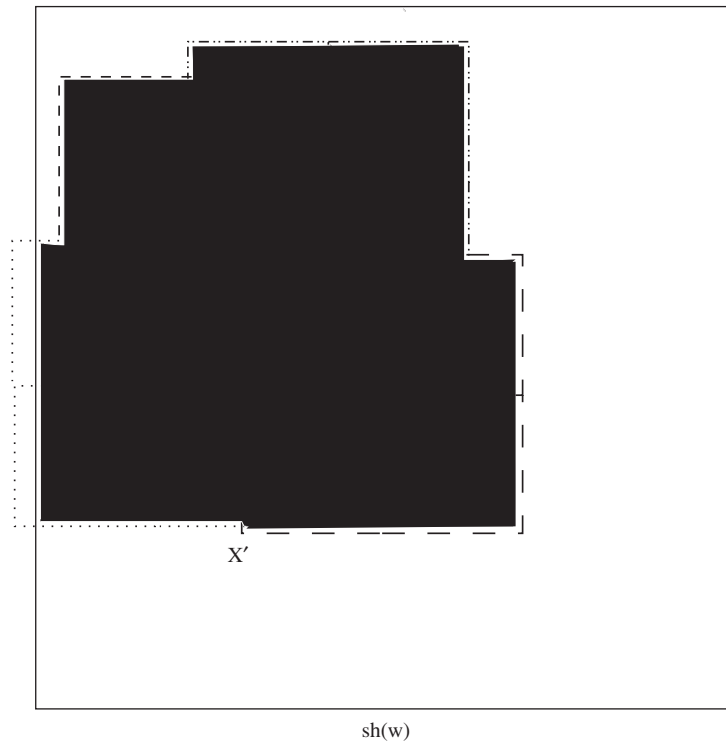


Fig. 2. Intersection of $sh(w)$ and the union of the shapes of the α -repetitions centered in position $l \in X$.

Let us suppose that the statements of the theorem hold true for all squares T such that $P_\alpha(T) < n$. Let X be a square such that $P_\alpha(X) = n$.

Let us assume that there exists, on the contrary, a square $T' \subseteq X$ having side length equal to $\max(1, P_\alpha(X) - 1) = n - 1$ that does not contain any position k such that $c_\alpha(w, k) = n$. Let us slide this square in any possible direction until we find a square T such that there exists a position $k \in X$ just out, at distance one (by using the distance induced by the *sup* norm) from the perimeter of T , such that $c_\alpha(w, k) = n$. This k must exist because $P_\alpha(X) = n$. Notice that $P_\alpha(T)$ is smaller than n . We denote this value $P_\alpha(T)$ by q .

By the inductive hypothesis we know that v has period L_2 , where v is the factor of w the shape of which is the intersection of $sh(w)$ and the union T' of the shapes of the α -repetitions centered in position $l \in T$ such that $c_\alpha(w, l) = P_\alpha(T) = q$ and L_2 is such that every square of side length q contains a transversal of it.

Again by the inductive hypothesis, the square C , that has side length $sd(T) + 2(q + 1)$ and centre T (see Fig. 3) is contained in T' .

We know that the factor of w that has shape $C \cap sh(w)$ has period L_2 . We know that $sh(w)$ contains both T and the position k . We want to prove that $C \cap sh(w)$ encloses a square of side length at least $sd(T) + q + 1$. \square

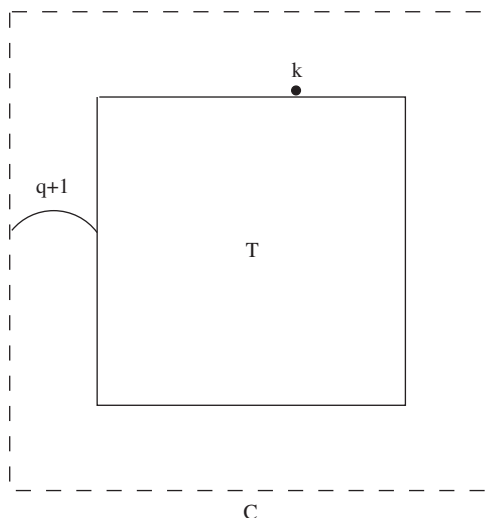


Fig. 3. Square C , that has side length $sd(T) + 2(q + 1)$ and center T .

Claim 1. $sh(w)$ is not contained in C .

Proof of the Claim 1. Let us suppose, on the contrary, that the claim is false. Since $sh(w)$ is contained in $C \cap sh(w)$, it has period L_2 . Therefore, there exists in position k an α -local repetition of period L_2 , where L_2 is such that every square of side length q contains a transversal of it. This contradicts the fact that $c_\alpha(w, k) = n > q$ and the claim is proved.

The shape $sh(w)$ is not contained in C , but encloses T . Hence, it extends at least beyond one side of C . Therefore the side length of $sh(w)$ must be greater than $sd(T) + q + 1$. Any intersection of C and a square of side length greater than $sd(T) + q + 1$ that contains T must contain a square of side length at least $sd(T) + q + 1$. This implies that the intersection of $sh(w)$ and C contains a square C_1 of side length at least $sd(T) + q + 1$ (see Fig. 4). Notice that C_1 can be chosen in such a way that it contains position k .

Let us consider the intersection C_2 between C_1 and the shape of the central α -repetition in position k . This is a square of side length $4n$, because $\alpha = 4$ and k is such that $c_\alpha(w, k) = n$. Since k belongs to C_1 and since $sd(T) \geq n$, this intersection is a square of side length greater than or equal to $n + q + 1$.

This square has periods both L and L_2 . The scheme is the same as in the unidimensional case, even if a bit more complicated. We want to apply Theorem 3 in order to prove that this square has a period lattice $L_4 = \langle L, L_2 \rangle$, generated by both L and L_2 and this is proved in the following claim.

Claim 2. C_2 has period $L_4 = \langle L, L_2 \rangle$.

Proof of the Claim 2. Since every square of side length n contains a transversal of L , there exists a basis $\{b_1, b_2\}$ of L , $L = \langle b_1, b_2 \rangle$, such that every coordinate of both b_1 and b_2 has

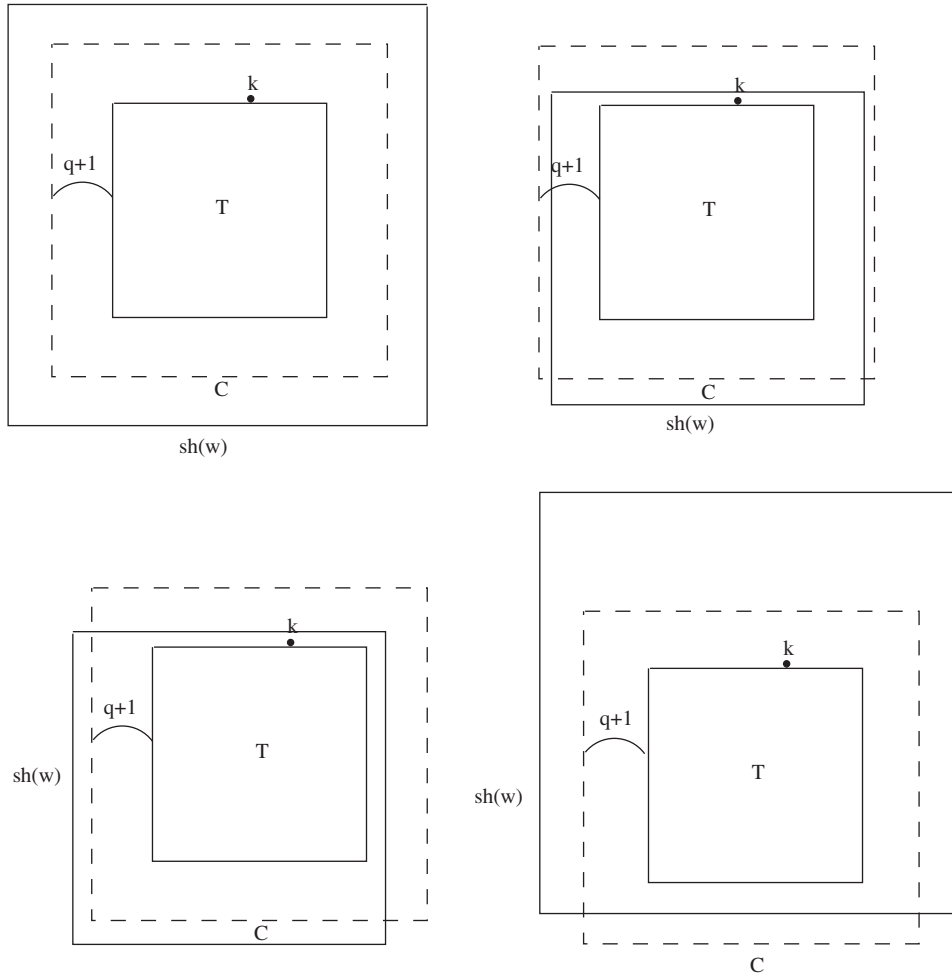


Fig. 4. All possible locations of C and $sh(w)$. All intersections between C and $sh(w)$ contain a square C_1 of side length at least $sd(T) + q + 1$.

absolute value smaller than or equal to n . This implies that both b_1 and b_2 are periodicity vectors for the restriction v of w to the shape C_2 . Hence $C_2 \cap (C_2 + b_1)$ and $C_2 \cap (C_2 + b_2)$ contain a square of side length q .

We can now apply Theorem 3 to the factor v , having shape C_2 , periodicity vector b_1 and period lattice L_2 and we obtain that v has period lattice the lattice $L_3 = \langle L_2, b_1 \rangle$. Since L_2 is a subgroup of L_3 , every square of side length q contains a transversal of L_3 . We can again apply Theorem 3 to the factor v , but this time referring to periodicity vector b_2 and period lattice L_3 . We obtain that v has lattice periodicity the lattice $L_4 = \langle L_3, b_2 \rangle = \langle L, L_2 \rangle$ and Claim 2 is proved.

Since L is a subgroup of L_4 and v contains a transversal of L , by Lemma 4 we have that the whole central α -repetition in position k has period L_4 . This is absurd because, since L_2

is a subgroup of L_4 , every square of side length q contains a transversal of L_4 , and this contradicts the minimality of $c_\alpha(w, k) = n > q$. Therefore, we have proved statement 1 of the theorem.

Let l'_1 and l'_2 be a couple of positions such that the distance (induced by the *sup* norm) between them is at most $n - 1$, such that $c_\alpha(w, l'_1) = c_\alpha(w, l'_2) = P_\alpha(X) = n$. Denote by L'_1 and L'_2 the two periods corresponding to the two central α -repetitions v'_1 and v'_2 . We want to prove that v'_1 and v'_2 are α -repetitions with the same period L' . Since the intersection of the shapes of the two central α -repetitions in positions l'_1 and l'_2 contains a square having side length at least $3n$, we can apply the reasoning used above to prove Claim 2 to conclude that this intersection has period $L' = \langle L'_1, L'_2 \rangle$. By Lemma 4 both v'_1 and v'_2 are α -repetitions with period L' .

Since every square $T \subseteq X$, of side length $sd(T) = \max(1, P_\alpha(X) - 1)$ contains a position l such that $c_\alpha(w, l) = P_\alpha(X)$, we can iteratively apply this result and conclude that there exists a lattice L' that is a period for all α -repetitions with centre in a position $l \in X$ such that $c_\alpha(w, l) = P_\alpha(X)$.

We can now iteratively apply Lemma 3 and obtain that L' is a period for the restriction of w to $sh(w) \cap X'$, where X' is the union of the shapes of the α -repetitions centered in position $l \in X$ such that $c_\alpha(w, l) = P_\alpha(X)$. \square

Corollary 2. *Every square T , with T enclosed in $sh(w)$, of side length*

$$sd(T) = \max(1, P_\alpha(sh(w)) - 1)$$

contains a critical position.

4. Conclusions and open problems

It is known that in the unidimensional case the tight value for the critical factorization theorem is $\alpha = 2$. It still remains an open problem to find the tight value of α for any dimension $d > 1$. Is it $\alpha = 2$, as in the unidimensional case?

The original critical factorization theorem was the main theoretic tool in a famous string matching algorithm (see [12]) that works with constant additional space. We hope that our results can be helpful in the design and analysis of multidimensional pattern matching algorithms.

A further problem is to extend results from the case of multidimensional words to the case of Delone sets, in particular, to extend Theorem 4 (cf. [14] and references therein).

Last, but not least, it would be interesting to see if it is possible to extend the Duval Conjecture, or Harju–Nowotka Theorem, to the multidimensional case. This problem is not trivial because it involves the generalization of unidimensional notions, such as the one of *unbordered words* (cf. [22] and references therein).

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