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Multiplicative cases from additive cases: Extension of Kolmogorov–Feller equation to parametric Poisson white noise processes

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Abstract

In this paper the response of nonlinear systems driven by parametric Poissonian white noise is examined.

As is well known, the response sample function or the response statistics of a system driven by external white noise processes is completely defined. Starting from the system driven by external white noise processes, when an invertible nonlinear transformation is applied, the transformed system in the new state variable is driven by a parametric type excitation. So this latter artificial system may be used as a tool to find out the proper solution to solve systems driven by parametric white noises. In fact, solving this new system, being the nonlinear transformation invertible, we must pass from the solution of the artificial system (driven by parametric noise) to that of the original one (driven by external noise, that is known). Moreover, introducing this invertible nonlinear transformation into the Itô's rule for the original system driven by external input, one can derive the Itô's rule for systems driven by a parametric type excitation, directly. In this latter case one can see how natural is the presence of the Wong–Zakai correction term or the presence of the hierarchy of correction terms in the case of normal and Poissonian white noise, respectively. Direct transformation on the Fokker–Planck and on the Kolmogorov–Feller equation for the case of parametric input are found.

Keywords: Stochastic differential calculus; Poisson input; Itô's calculus; Fokker–Planck equation; Kolmogorov–Feller equation; Parametric forces

1. Introduction

The problem of predicting the response statistics of linear or nonlinear dynamical systems under white noise processes is a very important task because the applications in many areas such as physics, engineering, astrophysics and so on. Many books (see e.g. [1–7]) have been devoted to this subject, termed as stochastic differential calculus (SDC) and the corresponding differential equations are termed as stochastic differential equations (SDE).

In dealing with linear and nonlinear systems driven by external white noise the literature is exhaustive and the response statistics are given by applying the main tool of Itô's calculus [8]. In the case of external Poisson white noise the response statistics are easily performed by properly extending the Itô's differential rule [9–12].

If the excitation is modulated by a function of the response, the system is labelled as driven by *parametric or multiplicative excitation*, and the problem of predicting the solution response is not trivial. So far, the response of a system driven by a parametric normal white noise has been evaluated by simply modifying the drift term accounting for the Wong–Zakai or Stratonovich (WZ–S) corrective term, for writing the differential equation in incremental form [13, 14]. At the beginning, some controversy arose about the presence of this corrective term [15–17], but at present this is widely used. The common motivation of this extra term, in passing from the original equation to the Itô type stochastic differential equation, is related to the local irregularity of the Brownian motion process. However, the necessity of the extra term may be easily explained by considering that increments $\mathrm{d}B(t)$ of a Brownian process $B(t)$ are of the order $(\mathrm{d}t)^{1/2}$ $(0(\mathrm{d}B) = \mathrm{d}t^{1/2})$.

Di Paola and Falsone [18–20] dealt with general non-Gaussian, delta correlated processes, which also include the Poisson impulse process; for the latter they proposed a series of corrective terms in passing from the original differential equation to the Itô's one. Since the response of

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such systems is framed into stochastic differential calculus, the latter formulation has been susceptible of many interpretations more or less in agreement. Intuitively, when dealing with Poisson input, that is the formal derivative of a Compound process $C(t)$, characterized by having the expectations of order $\mathrm{d}t\,(0(E[\mathrm{d}C(t)]^r) = \mathrm{d}t, (r = 1, 2, \ldots))$, the corrective terms should be present more naturally than the case of normal white noise. Also because, reminding that the normal white noise is a limiting case of Poisson input, one might wonder how could the response for normal parametric white noise be a limiting case of a Poissonian one, if not even one corrective term is taken into account [21–36]. However, the aforementioned system could be considered as a system driven by several parametric impulses. This problem, framed in a deterministic field, has been solved by a formulation, which has been proved from a mathematical point of view [37–41]. Just this formulation coalesces with that obtained from stochastic differential calculus [18–20], and stresses the need of all these corrective terms.

The aim of this paper is to get the same formulation by using a different and simple methodology. The response sample function or the response statistics of a system driven by external white noise processes is completely defined. Starting from this system driven by external white noise processes, when an invertible nonlinear transformation is applied, the transformed system in the new state variable is driven by a parametric type excitation, that is the talking point. So this latter system is artificial because it represents a tool to find out the proper solution to solve systems driven by parametric white noises. In fact, solving this new system, being the nonlinear transformation invertible, we must pass from the solution of the artificial system (driven by parametric noise) to that of the original one (driven by external noise, that is known). So this is a tool to verify if the solution provided for the system driven by parametric white noises is definitely correct or not. Well, this simple idea is developed throughout this paper, demonstrating that the need for these corrective terms is apparent, otherwise we cannot find the right solution coming back to the original system. Moreover, it is shown that by using the classical relationship between the probability density function (PDF) of the response of the system driven by external white noise and the PDF of the new state variable obtained by the nonlinear transformation, the Fokker–Planck (FP) equation and the Kolmogorov–Feller (KF) equation for the case of parametric input are readily found. Since both FP and KF equations are entirely deterministic, the presence of the corrective terms is now unequivocal. Furthermore they will be developed in an easy way to find out the solution in terms of moment, of probabilistic density and characteristic functions, and since the nonlinear transformation doesn't appear at the end of writing the evolution of function response in terms of moments, of probabilistic density, of characteristic functions, these are the probabilistic function descriptors of linear or nonlinear systems driven by parametric Poisson input. Obviously the response for systems driven by either external Poisson input, or normal white noise are limiting cases of the previous one.

2. Nonlinear differential equations driven by external Poisson white noise

In this section some preliminary well-known concepts on the nonlinear differential equations driven by external Poisson white noise are briefly summarized for clarity sake as well as to introduce appropriate symbologies.

Let the equation of motion of a nonlinear half oscillator be given in the form:

$$\dot{X}(t) = f(X, t) + W_p(t) \tag{1}$$

where $W_p(t)$ is a *non-normal Poisson white noise* process characterized by having cumulants of order j, $K_j[\cdot]$ expressed as [42]

$$K_j[W_p(t_1), W_p(t_2), \ldots, W_p(t_j)] = \lambda E[Z^j]\delta(t_1 - t_2)\delta(t_1 - t_3)\ldots\delta(t_1 - t_j) \tag{2}$$

being $E[\cdot]$ a stochastic average, $\delta(\cdot)$ the Dirac's delta function, λ the mean number of impulses per unit time, Z a random variable describing the impulse amplitudes occurring at random times T; the former and the latter are independent random variables. In particular the distribution of Z is assigned and T is distributed according to Poisson law. Explicit expression of $W_p(t)$ is usually cast in the form:

$$W(t) = \sum_{k=1}^{N(t)} Z_k \delta(t - T_k) \tag{3}$$

where $N(t)$ is the so called *Poisson counting process* giving the total number of impulses in the time interval $[0, t)$ regardless their amplitude and with initial condition $N(0) = 0$ with probability one.

The Poisson process may be obtained as the formal derivative of the so called *Compound Poisson* process $C(t)$ defined as:

$$C(t) = \sum_{k=1}^{N(t)} Z_k U(t - T_k). \tag{4}$$

$U(t)$ being the unit step function. Increments of the Compound Poisson process $C(t)$ are mutually independent and the probabilistic descriptors of $\mathrm{d}C(t)$ are given as:

$$E[\mathrm{d}C^j(t)] = K_j[\mathrm{d}C(t)] = \lambda E[Z^j]\mathrm{d}t. \tag{5}$$

Eq. (1) may be transformed into an incremental form as:

$$\mathrm{d}X(t) = f(X, t)\mathrm{d}t + \mathrm{d}C(t). \tag{6}$$

Let $\varphi(X(t), t)$, be any scalar real valued function of X and t, continuously differentiable on t and belonging to the class C_∞ with respect to X, (the class of infinite times differentiable on X), then the rule of differentiation of composite function is given as:

$$\mathrm{d}\varphi(X(t), t) = \frac{\partial\varphi(X(t), t)}{\partial t}\mathrm{d}t + \frac{\partial\varphi(X(t), t)}{\partial X}\mathrm{d}X + \sum_{j=2}^{\infty}\frac{1}{j!}\frac{\partial^j\varphi(X(t), t)}{\partial X^j}(\mathrm{d}X)^j \tag{7}$$

the presence of the summation at the right hand side of Eq. (7) is explained by the observation that $(\mathrm{d}X(t))^j$ contains terms $(\mathrm{d}C(t))^j$ whose statistics are all of order $\mathrm{d}t$ (see Eq. (5)).

Properly selecting $\varphi(X(t),t)$, equations governing the evolution of moments ($\varphi(X,t) = X^k$), or the equation of the characteristic function ($\varphi(X,t) = \exp(\mathrm{i}\vartheta X)$, being ϑ a real parameter) may easily be obtained. In order to get these equations we have to take into account that $\mathrm{d}C(t) = C(t+\mathrm{d}t) - C(t)$ represents the future excitation that is independent of the past history of the excitation and then the so called non-anticipating property of Itô's calculus holds true. That is, for any nonlinear function $\rho(X,t)$ of the response process $X(t)$, $E[\rho(X,t)(\mathrm{d}C(t))^k] = E[\rho(X,t)]E[(\mathrm{d}C(t))^k]$.

By inserting $\varphi(X,t) = \exp(\mathrm{i}\vartheta X)$ into Eq. (7), making the stochastic average and dividing by $\mathrm{d}t$, the following equation in terms of characteristic function is expressed in two equivalent forms:

$$\frac{\partial\phi_X(\vartheta,t)}{\partial t} = \mathrm{i}\vartheta E[f(X,t)\exp(\mathrm{i}\vartheta X)] + \lambda\phi_X(\vartheta,t)(\phi_Z(\vartheta)-1) \tag{8a}$$

$$\frac{\partial\phi_X(\vartheta,t)}{\partial t} = \mathrm{i}\vartheta E[f(X,t)\exp(\mathrm{i}\vartheta X)] + \lambda\phi_X(\vartheta,t)\sum_{j=1}^{\infty}\frac{(\mathrm{i}\vartheta)^j}{j!}E[Z^j] \tag{8b}$$

where $\phi_X(\vartheta,t) = E[\exp(\mathrm{i}\vartheta X)]$ is just the characteristic function (CF).

An inverse Fourier transform yields the two equivalent expressions of the Kolmogorov–Feller (KF) equation governing the evolution of the PDF $p_X(x,t)$:

$$\frac{\partial p_X(x,t)}{\partial t} = -\frac{\partial}{\partial x}(f(x,t)p_X(x,t)) - \lambda p_X(x,t) + \lambda\int_{-\infty}^{\infty} p_X(x-\xi,t)p_Z(\xi)\mathrm{d}\xi \tag{9a}$$

$$\frac{\partial p_X(x,t)}{\partial t} = -\frac{\partial}{\partial x}(f(x,t)p_X(x,t)) + \lambda\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}E[Z^j]\frac{\partial^j}{\partial x^j}[p_X(x,t)]. \tag{9b}$$

If λ tends to infinity and at the same time $\lambda E[Y^2]$ keeps a constant value say q_2, then the Poisson white noise reverts to a Gaussian white noise $W_0(t)$, moreover $\mathrm{d}C(t)$ reverts to the increment of the Brownian motion $B(t)$ with intensity q_2, and in differential rule (7) only the second term in the summation at the r.h.s. must be retained since the order of $\mathrm{d}B(t)$ is $(\mathrm{d}t)^{1/2}$. Then, in this case, if $E[Z] = 0$, the classical Fokker–Planck equation is fully restored.

$$\frac{\partial p_X(x,t)}{\partial t} = -\frac{\partial}{\partial x}(f(x,t)p_X(x,t)) + \frac{q_2}{2}\frac{\partial^2 p_X(x,t)}{\partial x^2}. \tag{10}$$

3. Nonlinear transformation into stochastic differential equation

As mentioned before, starting from a system driven by external white noise processes, when an invertible nonlinear transformation is applied, the transformed system in the new state variable is driven by a parametric type excitation. So this latter system is artificial because, throughout this paper, it is used as a tool to find out the proper solution to elucidate systems driven by parametric white noises. In fact, solving this new system, being the nonlinear transformation invertible, we must pass from the solution of the artificial system to that of the original one (that is already known). This concept is developed and the FP and KF equations for the case of parametric excitation will be derived by means of nonlinear transformation on SDE.

Now let us suppose that a new state variable $Y(t)$ is constructed as a nonlinear invertible transformation of the stochastic response $X(t)$ of the SDE (1) as follows:

$$Y(t) = u(X(t)); \qquad X(t) = v(Y(t)). \tag{11a,b}$$

Being $u(\cdot)$ and $v(\cdot)$ deterministic nonlinear functions.

According to classical rules of derivative composite functions $\dot{Y}(t)$ may be written as:

$$\dot{Y}(t) = \frac{\partial u(X(t))}{\partial X}\dot{X} = G(Y,t)\dot{X} \tag{12}$$

where $G(Y,t)$ is $[\partial u(X(t))/\partial X]$ evaluated in $X = v(Y)$. By multiplying Eq. (1) by $G(Y,t)$, the following differential equation is easily found:

$$\dot{Y} = F(Y,t) + G(Y,t)W_p(t) \tag{13}$$

where:

$$F(Y,t) = G(Y,t)f(v(Y),t). \tag{14}$$

From Eq. (13) it is evident that, starting from SDE driven by external load, when nonlinear transformation occurs, the new SDE governing the evolution of the new state variable $Y(t)$ is driven by a parametric type excitation.

3.1. Nonlinear transformation for a system with normal excitation

It will be convenient to start from the system excited by normal white noise and then examine the system under Poisson white noise. If in Eq. (13) the driving process $W_p(t)$ tends towards a normal white noise $W_0(t)$, as has been previously stated in Eq. (7), the summation will be extended retaining only two terms. Selecting $\varphi(X(t),t) = u(X(t))$ we get:

$$\mathrm{d}u(X(t)) = \sum_{j=1}^{2}\frac{1}{j!}\frac{\partial^j u(X(t))}{\partial X^j}(\mathrm{d}X(t))^j \tag{15}$$

where $\mathrm{d}X(t)$ is given in Eq. (6), in which $\mathrm{d}C(t)$ is substituted by $\mathrm{d}B(t)$, and introducing the following relationships:

$$u(X(t)) = Y(t); \qquad \frac{\partial u(X(t))}{\partial X} = G^{(1)}(Y,t) = G(Y,t);$$
$$\frac{\partial^2 u(X(t))}{\partial X^2} = \frac{\partial G^{(1)}(Y,t)}{\partial Y}G(Y,t) = G^{(2)}(Y,t) \tag{16}$$

into Eq. (15), it leads to:

$$dY(t) = F(Y,t)dt + G(Y,t)dB(t) + \frac{1}{2}G(Y,t)(dB(t))^2. \quad (17)$$

That exactly coalesces with the classical Itô equation associated with Eq. (13) for the case of parametric normal white noise. Moreover, in Eq. (17) the Wong–Zakai or Stratonovich correction term explicitly appears. It will be emphasized that since in Eq. (17) the nonlinear transformation does not appear we may state that when we directly start from Eq. (13) the differential rule associated with Eq. (13) (for $W_p(t) = W_0(t)$) is that reported in Eq. (17).

The differential rule for any scalar real valued function $\varphi(Y,t)$ is written as:

$$d\varphi(Y,t) = \frac{\partial\varphi(Y,t)}{\partial t}dt + \frac{\partial\varphi(Y,t)}{\partial Y}dY + \frac{1}{2}\frac{\partial^2\varphi(Y,t)}{\partial Y^2}(dY)^2 \quad (18)$$

and by selecting $\varphi(Y,t) = \exp(i\vartheta Y)$, making stochastic average, taking into account the nonanticipating property of Itô calculus and dividing by dt, the equation governing the evolution of the CF is readily found:

$$\frac{\partial\phi_Y(\vartheta,t)}{\partial t} = i\vartheta E[F(Y,t)\exp(i\vartheta Y)]dt + \frac{q_2}{2}i\vartheta E\left[G(y)\frac{\partial G(y)}{\partial y}\exp(i\vartheta Y)\right] + \frac{q_2}{2}(i\vartheta)^2 E[G^2(y)\exp(i\vartheta Y)] \quad (19)$$

and then an inverse Fourier transform exactly restores the FP equation for the case of parametric noise:

$$\frac{\partial p_Y(y)}{\partial t} = -\frac{\partial}{\partial y}\left(F(y,t)p_Y(y) + \frac{q_2}{2}G(y)\frac{\partial G(y)}{\partial y}p_Y(y)\right) + \frac{q_2}{2}\frac{\partial^2(p_Y(y)G^2(y))}{\partial y^2}. \quad (20)$$

Summing up: when a nonlinear transformation is applied in a nonlinear system driven by external normal white noise, the equation in the new state variable is driven by a parametric white noise. By using the classical Itô rule given in Eq. (15) we get the Itô differential equation in which the Wong–Zakai or Stratonovich correction term explicitly appears, and the FP equation for parametric normal white noise is obtained.

A different way for finding Eq. (20) may be pursued by considering that since the two processes $X(t)$ and $Y(t)$ are related to each other by invertible nonlinear transformations the fundamental relationship:

$$p_X(x)dx = p_Y(y)dy \quad (21)$$

holds true. This equation is entirely deterministic and since $Y(t) = u(X(t))$; $X(t) = v(Y(t))$, Eq. (21) may be rewritten in the form:

$$p_X(x) = p_Y(y)G(y) \quad (22)$$

and then the following identities hold true:

$$\frac{\partial p_X(x)}{\partial t} = G(y)\frac{\partial p_Y(y)}{\partial t} \quad (23a)$$

$$\frac{\partial}{\partial x}(f(x,t)p_X(x)) = G(y)\frac{\partial}{\partial y}(F(y,t)p_Y(y)) \quad (23b)$$

$$\frac{\partial}{\partial x}(p_X(x)) = G(y)\frac{\partial}{\partial y}(G(y)p_Y(y)) \quad (23c)$$

$$\frac{\partial^2}{\partial x^2}(p_X(x)) = G(y)\frac{\partial}{\partial y}\left(G(y)\frac{\partial}{\partial y}(G(y)p_Y(y))\right) \quad (23d)$$

By directly transforming Eq. (10) we exactly get the FP equation for parametric white noise. The latter observation is really important because Eqs. (21)–(23) are entirely deterministic, no irregular stochastic processes like the white noise appear and then it is evident that the equation for parametric excitation may be simply obtained by the case of external excitation. On this solid ground the extension to the case of Poissonian white noise is immediate.

3.2. Nonlinear transformation for a system with Poisson excitation

Let us start from Eq. (7) in which now all the terms have to be retained because, according to Eq. (5) all power of the statistics of the increment of the process $C(t)$ are all infinitesimal and of the same order dt. We now perform the nonlinear transformations given in Eq. (11) and we extend: Eq. (16)

$$u(X(t)) = Y(t); \quad \frac{\partial u(X(t))}{\partial X} = G^{(1)}(Y,t) = G(Y,t); \quad \frac{\partial^j u(X(t))}{\partial X^j} = \frac{\partial G^{(j-1)}(Y,t)}{\partial Y}G(Y,t) = G^{(j)}(Y,t) \quad (24)$$

into Eq. (15); extended to all terms, it leads to:

$$dY(t) = \sum_{j=1}^{\infty}\frac{1}{j!}G^{(j)}(Y,t)(f(X,t)dt + dC)^j. \quad (25)$$

By neglecting higher order terms than dt, the following equation ruling the evolution of $Y(t)$ is obtained as:

$$dY(t) = F(Y,t)dt + \sum_{j=1}^{\infty}\frac{1}{j!}G^{(j)}(Y,t)(dC)^j. \quad (26)$$

This equation exactly coincides with that proposed by Di Paola and Falsone [18–20] on the basis of difference between increments and differentials.

Looking at Eq. (26), some observations may be pointed out:

(i) In passing from Eq. (13) enforced by parametric impulses, the incremental rule (15) gives a hierarchy of correction terms;
(ii) since in Eq. (26) the nonlinear transformation does not explicitly appear, then the rule in passing from Eq. (13) to the rule (26) is always valid, also if the original system with external excitation is unknown;

(iii) all the terms in the summation (26) are of the same order ($\mathrm{d}t$), then they cannot be neglected;
(iv) if $W_p(t) \to W_0(t)$ that is the case of normal white noise $\mathrm{d}C(t) \to \mathrm{d}B(t)$ and then only the first two terms of the summation (26) appear, and the second one coincides with the Wong–Zakai or Stratonovich [13,14] correction term.

Once Eq. (26) is derived, the differential rule for any scalar real valued function $\varphi(Y, t)$ is written as:

$$\begin{aligned}\mathrm{d}\varphi(Y,t) &= \frac{\partial\varphi(Y,t)}{\partial t}\mathrm{d}t + \sum_{k=1}^{\infty}\frac{\partial^k\varphi(Y,t)}{\partial Y^k}(\mathrm{d}Y)^k \\ &= \frac{\partial\varphi(Y,t)}{\partial t}\mathrm{d}t + \sum_{k=1}^{\infty}\frac{\partial^k\varphi(Y,t)}{\partial Y^k} \\ &\quad\times\left(F(Y,t)\,\mathrm{d}t + \sum_{j=1}^{\infty}\frac{G^{(j)}(Y)}{j!}(\mathrm{d}C)^j\right)^k. \end{aligned} \tag{27}$$

It is worth remarking that if one leaves out the terms in the summation (26) for $j > 2$ an entirely different and simpler rule of differentiation of composite functions emerges.

By putting $\varphi(Y, t) = \exp(\mathrm{i}\vartheta Y)$, the Eq. (27), neglecting higher infinitesimals than $\mathrm{d}t$ specifies into:

$$\begin{aligned}\mathrm{d}\exp(\mathrm{i}\vartheta Y) &= \mathrm{i}\vartheta\exp(\mathrm{i}\vartheta Y)F(Y,t)\mathrm{d}t + \sum_{k=1}^{\infty}(\mathrm{i}\vartheta)^k \\ &\quad\times\left(\sum_{j=1}^{\infty}\frac{G^{(j)}(Y)}{j!}(\mathrm{d}C)^j\right)^k\exp(\mathrm{i}\vartheta Y)\end{aligned} \tag{28}$$

making stochastic average and using the non-anticipating property:

$$\begin{aligned}\mathrm{d}\phi_Y(\vartheta,t) &= \mathrm{i}\vartheta E[\exp(\mathrm{i}\vartheta Y)F(Y,t)]\mathrm{d}t + \sum_{k=1}^{\infty}(\mathrm{i}\vartheta)^k \\ &\quad\times E\left[\exp(\mathrm{i}\vartheta Y)\left(\sum_{j=1}^{\infty}\frac{G^{(j)}(Y)}{j!}(\mathrm{d}C)^j\right)^k\right]\end{aligned} \tag{29}$$

performing the stochastic average in the second term of right hand side of Eq. (29) is a very hard task. In order to obtain explicit expression it should be stressed that since the processes $X(t)$ and $Y(t)$ are related to each another by invertible nonlinear transformations the fundamental relationship (21) remains valid.

Since $Y(t) = u(X(t))$, $X(t) = v(Y(t))$, it may be rewritten Eq. (21) in the form:

$$p_X(x) = p_Y(y)G(y) \tag{30}$$

and then

$$\begin{aligned}&\frac{\partial p_X(x)}{\partial t} = G(y)\frac{\partial p_Y(y)}{\partial t};\\ &\frac{\partial}{\partial x}(f(x,t)p_X(x)) = G(y)\frac{\partial}{\partial y}(F(y,t)p_Y(y))\end{aligned} \tag{31}$$

$$\frac{\partial^k}{\partial x^k}(p_X(x)) = G(y)\Theta_k[G(y),\{p_Y(y)\}]. \tag{32}$$

The operator at the right hand side of Eq. (32) is given as:

$$\Theta_k[G(y),\{p_Y(y)\}] = \left\{\underbrace{\frac{\partial}{\partial y}\left(G(y)\frac{\partial}{\partial y}\left(G(y)\cdots\frac{\partial}{\partial y}(G(y)p_Y(y))\right)\right)}_{k-\text{fold}}\right\}. \tag{33}$$

Taking into account Eqs. (30), (33) and (9b) the Kolmogorov–Feller equation extended to the case of parametric type excitation is obtained simply:

$$\begin{aligned}\frac{\partial p_Y(y)}{\partial t} &= -\frac{\partial}{\partial y}(F(y,t)p_Y(y)) \\ &\quad+\lambda\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\Theta_k[G(y),\{p_Y(y)\}]E[(Z)^k].\end{aligned} \tag{34}$$

It coincides with the equation proposed in [24] and obtained starting directly from Eq. (6) and making an inverse Fourier transform of Eq. (29) with lengthy manipulations. It should be remarked that since in Eq. (34) no explicit dependence on the original nonlinear transformation is present, it may be stated that the Kolmogorov–Feller equation for parametric type excitation associated with a differential equation of the form $\dot{Y} = F(Y,t) + G(Y,t)W_p(t)$ may be given by Eq. (34). On the other hand, in some papers, [34,35] when parametric type excitation appears the rule of differentiation of composite function used has been treated simply by writing the SDE in terms of increment in the form:

$$\mathrm{d}Y(t) = F(Y,t)\mathrm{d}t + G(Y,t)\mathrm{d}C(t) \tag{35}$$

that is neglecting the terms $\sum_{j=2}^{\infty}G^j(Y,t)(\mathrm{d}C)^j/j!$; using Eq. (27) the differential equation (29) reduces to:

$$\begin{aligned}\frac{\partial\phi_Y(\vartheta,t)}{\partial t} &= \mathrm{i}\vartheta E[\exp(\mathrm{i}\vartheta Y)F(Y,t)] \\ &\quad+\lambda\sum_{k=1}^{\infty}\frac{(\mathrm{i}\vartheta)^k}{k!}E[Z^k]E[\exp(\mathrm{i}\vartheta Y)(G(Y))^k]\end{aligned} \tag{36}$$

and the Kolmogorov–Feller equation, using Eq. (35) for increments is then rewritten as:

$$\begin{aligned}\frac{\partial p_Y(y)}{\partial t} &= -\frac{\partial}{\partial y}(F(y,t)p_Y(y)) \\ &\quad+\lambda\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\partial^k}{\partial y^k}[G(Y)p_Y(y,t)]E[(Z)^k].\end{aligned} \tag{37}$$

That is entirely different from Eq. (34). However, if Eq. (37) is used, starting from a SDE driven by external input making the nonlinear transformation described above, then $p_X(x)\mathrm{d}x \neq p_Y(y)\mathrm{d}y$ and consequently the correspondent results obtained by Eq. (37) are incorrect.

Moreover, if $W_p(t) \to W_0(t)$ using Eq. (34) the classical FPK is restored, and we use Eq. (37), the Wong–Zakai correction term disappears and this is meaningless.

4. Moment equation approach

Previous sections have dealt with the response statistics of a system driven by parametric white noises either normal or Poissonian. However, it is also important to provide response statistics through the moment equation approach. In order to use this for investigating external Poisson pulses, (Eq. (1)) two different strategies may be pursued. The first one involves the differential rule (7) by putting $\varphi(X,t) = X^k(t)$; by so doing the differential equation governing the evolution of $E[X^k]$ may easily be obtained in the form:

$$\dot{E}[X^k] = kE[X^{k-1}f(x,t)] + \lambda \sum_{j=1}^{k} \frac{k(k-1)\cdots(k-j+1)}{j!} E[Z^j]E[X^{k-j}]. \tag{38}$$

This is a linear set of a differential equation in the unknown moments. The problem of moments is connected with the fact that the means $E[X^{k-1}f(x,t)]$ are still unknown but they may be computed if the CF or PDF is known. In the case of polynomial nonlinearities that is $f(x,t) = \sum_{j=1}^{r} a_j X^j$, Eq. (38) constitutes an infinite hierarchy in the sense that in the moment equation of order k, moments of order $E[X^{k+r-1}]$ appear and then, unless the case is of $r = 1$ linear case, higher order moments appear that require some closure for approximate solutions.

The second way is to consider the nonlinear transformation $X^k = Y$, averaging Eq. (26); the first order moment of Y is simply obtained, and may be seen to coincide with Eq. (38). It should be stressed that in passing between $\dot{Y} = F(Y,t) + G(Y,t)W_p(t)$ and $\mathrm{d}Y(t)$ if the latter is taken as $\mathrm{d}Y(t) = F(Y,t)\mathrm{d}t + G(Y,t)\mathrm{d}C(t)$, that is by neglecting in the summation at the right hand side of Eq. (26) all the terms with $j \geq 2$ (see e.g. [34,35]); then the moment equations obtained directly by using the rule are valid for external Poisson white noise and those obtained by making the nonlinear transformation do not coincide. Such an example is:

$$\dot{X}(t) = -\rho X(t) + W_p(t); \quad (\rho > 0) \tag{39}$$

then the moment equations up to the fourth order, using the rule (7), are:

$$\dot{E}[X] = -\rho E[X] + \lambda E[Z] \tag{40a}$$

$$\dot{E}[X^2] = -2\rho E[X^2] + 2\lambda E[X]E[Z] + \lambda E[Z^2] \tag{40b}$$

$$\dot{E}[X^3] = -3\rho E[X^3] + 3\lambda E[X^2]E[Z] + 3\lambda E[X]E[Z^2] + \lambda E[Z^3] \tag{40c}$$

$$\dot{E}[X^4] = -4\rho E[X^4] + 4\lambda E[X^3]E[Z] + 6\lambda E[X^2]E[Z^2] + 4\lambda E[X]E[Z^3] + \lambda E[Z^4]. \tag{40d}$$

Now considering the nonlinear transformation $Y = X^2$, then $X = Y^{1/2}$, $G(Y) = 2\sqrt{Y}$, $F(Y) = -2\rho Y$ and the parametric differential equation ruling $Y = X^2$ is then:

$$\dot{Y}(t) = -2\rho Y(t) + 2\sqrt{Y}W_p(t). \tag{41}$$

It follows that $G^{(1)}(Y) = 2\sqrt{Y}$; $G^{(2)}(Y) = 2$; $G^{(j)}(Y) = 0$; $\forall j > 2$, and then the Itô differential equation connected with Eq. (41) Becomes:

$$\mathrm{d}Y(t) = -2\rho Y(t)\mathrm{d}t + 2\sqrt{Y}\mathrm{d}C(t) + (\mathrm{d}C(t))^2. \tag{42}$$

Making stochastic averages and dividing by dt the following equation ruling the evolution of E[Y] is obtained:

$$\dot{E}[Y] = -2\rho E[Y] + 2\lambda E[Y^{1/2}]E[Z] + \lambda E[Z^2] \tag{43}$$

but $E[Y] = E[X^2]$ and then Eq. (43) exactly coincides with Eq. (40b); it is worth noting that if Eq. (36) is used, that is $\mathrm{d}Y(t) = -2\rho Y(t)\mathrm{d}t + 2\sqrt{Y}\mathrm{d}C(t)$, then making the average the last term in the Eq. (43) disappears and E[Y] does not coincide with $E[X^2]$. Moreover, for the solution $X = -Y^{1/2}$, it has been obtained the same result.

For the case of $Y = X^3$, $X = Y^{1/3}$ then $G(Y) = 3Y^{2/3}$, $F(Y) = -3\rho Y$. It follows that:

$$\dot{Y}(t) = -3\rho Y(t) + 3Y^{2/3}W_p(t). \tag{44}$$

The increment of this equation, according to the rule given in Eq. (26) is written in the form:

$$\mathrm{d}Y(t) = -3\rho Y(t)\mathrm{d}t + 3Y^{2/3}\mathrm{d}C(t) + 3Y^{1/3}(\mathrm{d}C(t))^2 + (\mathrm{d}C(t))^3. \tag{45}$$

By making the stochastic average and dividing by dt the equation ruling the evolution of $E[Y] = E[X^3]$ is easily obtained in the form:

$$\dot{E}[Y(t)] = -3\rho E[Y] + 3\lambda E[Y^{2/3}]E[Z] + 3\lambda E[Y^{1/3}]E[Z^2] + \lambda E[Z^3] \tag{46}$$

that exactly coincides with Eq. (40c).

Moreover by putting $Y = X^4$, $X = Y^{1/4}$ then $G(Y) = 4Y^{3/4}$, $F(Y) = -4\rho Y$. It follows that

$$\dot{Y}(t) = -4\rho Y(t) + 4Y^{3/4}W_p(t). \tag{47}$$

The increment of this equation, according to the rule given in Eq. (26) is written in the form:

$$\mathrm{d}Y(t) = -4\rho Y(t)\mathrm{d}t + 4Y^{3/4}\mathrm{d}C(t) + 6Y^{1/2}(\mathrm{d}C(t))^2 + 4Y^{1/4}(\mathrm{d}C(t))^3 + (\mathrm{d}C(t))^4. \tag{48}$$

By making the stochastic average and dividing by dt the equation ruling the evolution of $E[Y] = E[X^4]$ is written as:

$$\dot{E}[Y(t)] = -4\rho E[Y] + 4\lambda E[Y^{3/4}]E[Z] + 6\lambda E[Y^{1/2}]E[Z^2] + 4\lambda E[Y^{1/4}]E[Z^3] + \lambda E[Z^4] \tag{49}$$

that exactly coincides with Eq. (40d). Moreover, for the solution $X = -Y^{1/4}$, the same result has been obtained. However, this may happen when starting from Eq. (41) and evaluating the square value of $Y = X^2$. As a conclusion of this section it may be stated that in the case of multiplicative Poisson white noise if the rule given in Eq. (26) is not applied, results in the form of moments are totally wrong.

5. Some remarks on the direct integration of the response to sample functions of parametric Poissonian white noise

In this section some elementary considerations on the necessity of the summation given in Eq. (26) will be introduced. In the case of Poissonian white noise input the general expression for generating sample functions of such a process is given in Eq. (3), then each sample function is constituted by well-spaced impulses; then for Monte Carlo simulation two independent random variables need to be generated: (times T, giving the instants whenever an impulse occurs and another random variable Z indicates the intensity of the impulses). In the case of external excitation we have to solve a differential equation in the form:

$$\dot{X}(t) = f(X,t) + \sum_{k=1}^{N(t)} Z_k \delta(t - T_k) \tag{50}$$

supplemented by the initial condition. Let us suppose that in the generic sample function, T_{k-1} and T_k are two subsequent instants at which an impulse occurs. Supposing that we know the response immediately after the impulse occurrence T_{k-1}, then during the time lag $[T_{k-1}, T_k]$ the forcing function is always zero and the response may be easily obtained by integrating the ordinary differential equation $\dot{X}(t) = f(X,t)$ supplemented by the appropriate initial condition, that is the system response immediately after the impulse occurrence T_{k-1}. Then labelling $T_k^- = \lim_{\varepsilon\to 0} T_{k-\varepsilon}$ and $T_k^+ = \lim_{\varepsilon\to 0} T_{k+\varepsilon}$ the time immediately before and after the new impulse occurrence, all the responses up to T_k^- may be easily found by integrating $\dot{X}(t) = f(X,t)$ with initial condition T_{k-1}^+. At T_k an impulse occurs, then we have to solve the differential equation:

$$\dot{X}(t) = f(X,t) + Z_k \delta(t - T_k); \quad T_k^- \le t \le T_k^+ \tag{51}$$

and the response at T_k^+ is given as:

$$J(T_k) = X(T_k^+) - X(T_k^-) = Z_k \tag{52}$$

That is, the response exhibits a jump $J(T_k)$ whose amplitude is just the intensity of the impulse. It is worth noting that the jump does not depend on $f(X,t)$. The response immediately after the impulse is simply $X(T_k^+) = X(T_k^-) + Z_k$, that is, in the time interval $[T_k^-, T_k^+]$ the differential equation may be put in the form:

$$\dot{X}(t) = Z_k \delta(t - T_k); \quad \forall t\; T_{k-\varepsilon} \le t \le T_{k+\varepsilon}. \tag{53}$$

Let us now suppose that the nonlinear transformation is made, then in the time interval $[T_k^-, T_k^+]$ the equation becomes:

$$\dot{Y}(t) = G(Y,t) Z_k \delta(t - T_k); \quad \forall t\; T_{k-\varepsilon} \le t \le T_{k+\varepsilon} \tag{54}$$

The crucial point is: what is the jump in the case of parametric impulse? The response to this equation may be found in many previous papers [37–41], here revisited in the light of the nonlinear transformation seen before. Using the rule given in Eq. (15) (extended to all terms) we get:

$$\mathrm{d}u(X) = \sum_{j=1}^{\infty} \frac{1}{j!} \frac{\partial^j u(X)}{\partial X^j} [Z_k \mathrm{d}U(t - T_k)]^j \tag{55}$$

being $U(\cdot)$ the unit step function. Simply by taking into account Eqs. (22) and (23) and the fact that $\mathrm{d}U(t - T_k) = 1$ if $t > T_k$ we obtain:

$$J(Y(T_k)) = Y(T_k^+) - Y(T_k^-) = \sum_{j=1}^{\infty} \frac{G^{(j)}(Y(T_k^-))}{J!} Z_k^j. \tag{56}$$

That is, the correct prevision of the jump is given as a numerical series in Eq. (26). Since in Eq. (26) the nonlinear transformation equation does not explicitly appear (56) is valid for the differential equation given in Eq. (54). In passing, we note two crucial aspects:

(i) the exact jump prevision in this case depends on impulse amplitude and on the value of $G(Y)$ evaluated immediately before the impulse occurrence;

(ii) the series (56) is drastically different from other expression available in the literature [36] but agrees with some cases obtained with MATHEMATICA program in some cases (see eg. $G(Y) = Y,\ G(Y) = Y^k$).

From consideration (i) it is obvious that since the only quantities of interest for the correct evaluation of the jump are the value of $G^{(j)}(Y(T_k^-))$ immediately before the impulse, the non-anticipating property of Itô calculus is preserved.

As a conclusion of this section it is emphasized that, when parametric impulsive processes occur (as in the case of Poissonian, normal, Lèvy white noise) then Monte Carlo simulation has to be performed by properly integrating the parametric impulse using the series (56). In the case of normal white noise input, we generate sample function by subdividing the time lag into small intervals say Δt/. At each individual interval we put a spike whose area in the interval $[T_k, T_k + \Delta t]$ is $R_k \Delta t^{1/2}$ being R_k a realization of a normal random variable with assigned variance. In this case Eq. (15), neglecting higher order terms than Δt;

$$Y(T_k + \Delta t) - Y(T_k) = f(Y(T_k))\Delta t + \frac{R_k^2}{2} G^{(2)}(Y, T_k)\Delta t. \tag{57}$$

Also in this case the Wong–Zakai or Stratonovich correction term comes out in a natural way.

6. Conclusions

In this paper a different perspective on the presence of corrective terms of nonlinear stochastic differential equations enforced by parametric type normal or non-normal Poisson white noise has been introduced. The main idea is that performing some invertible nonlinear transformation on a stated variable of a nonlinear differential equation enforced by external input, is transformed into the differential equation governing the evolution of the new state variable in which parametric excitation appears. In the former case no correction terms are necessary in order to derive Fokker–Planck–Kolmogorov equation or Kolmogorov–Feller equation. In the latter case, when dealing with normal white noise, the drift term will be modified taking into account the Wong–Zakai or Stratonovich correction term. The common motivation of this extra term in passing from the original equation to the Itô type stochastic differential equation, is commonly related to the local irregularity of the Brownian motion process. In this paper it has been shown that the necessity of the extra term is related to the fact that the normal white noise may be considered as an impulsive process whose individual impulses are of the order of magnitude $\Delta t^{1/2}$.

In Poissonian white noise, well spaced impulses occur randomly distributed in time according to Poisson law with random amplitude. The problem is that for such systems, at each impulse occurrence, if the system is parametric the response exhibits a jump whose evaluation strongly influences the results. Here it has been shown in a very easy way using nonlinear transformation, that the jump prediction requires a numerical series, taking into account that for the first term results may be quite different depending on the impulse amplitude as well as on the values of the nonlinear parametric function evaluated before the impulse occurrence. The numerical series for jump prevision at each impulse occurrence agrees with some simple cases treated with Mathematica program and has been numerically tested for other cases always giving correct jump prevision. Once the jump prevision is made all the stochastic differential calculus for the case of parametric Poisson white noise may be performed in a very easy way. In this paper, however, by using nonlinear invertible transformation and assuming that the mass of probability related intervals must be equal, the Kolmogorov–Feller equation extended to the case of Poisson white noise is immediately found to agree with previous results available in the literature, but this is controversial. In this paper the extension of Kolmogorov–Feller equation is made always using deterministic concepts, starting from the Kolmogorov–Feller equation for external Poisson white noise. The main conclusion is that either working in terms of probability or in terms of moments, or at least for single impulses, in the case of parametric Poisson white noise if no extra terms are included, then the results in terms of probabilistic descriptors are incorrect.

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