Chapter 15

THE PROMETEIA MODEL FOR MANAGING INSURANCE POLICIES WITH GUARANTEES

ANDREA CONSIGLIO
University of Palermo, Italy

FLAVIO COCCO
Prometeia, Bologna, Italy

STAVROS A. ZENIOS
Department of Public and Business Administration, University of Cyprus, 1678 Nicosia, Cyprus
and
Senior Fellow, The Wharton School, University of Pennsylvania, USA
e-mail: zenioss@ucy.ac.cy

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Abstract

Insurance products become increasingly more innovative in order to face competitive pressures. Insurance policies today come with guarantees on the minimum rate of return, bonus provisions, and surrender options. These features make them attractive for investors who seek not only insurance but also investment vehicles. However, new policies are much more complex to price and fund than traditional insurance products. In this chapter we discuss the development of a scenario-based optimization model for asset and liability management for the participating policies with guarantees and bonus provisions offered by Italian insurers. The changing landscape of the financial services in Italy sets the backdrop for the development of this system which was the result of a multi-year collaborative effort between academic researchers, the research staff at Prometeia in Bologna, and end-users from diverse Italian insurers. The model is presented, its key features are discussed in detail, and several extensions are briefly introduced. The resulting system allows the analysis of the tradeoffs facing an insurance firm in structuring its policies as well as the choices in covering their cost. It is applied to the analysis of policies offered by Italian insurance firms. While the optimized model results are in general agreement with current industry practices, inefficiencies are still identified and potential improvements are suggested. Extensive numerical experiments provide significant insights on features of the participating guaranteed policies.

Keywords

risk management, asset-liability management, insurance products with guarantee

JEL classification: C61, G22, G32

1. Introduction

The last decade brought about a phenomenal increase of consumer sophistication in terms of the financial products they buy. This trend is universal among developed economies, from the advanced and traditionally liberal economies of North America to the increasingly deregulated economies of the European Union and pre-accession States, and the post-Communist countries.

The numbers are telling: In the 1980s almost 40% of the US consumer financial assets were in Bank deposits. By 1996 bank deposits accounted for less than 20% of consumers' financial assets with mutual funds and insurance/pension funds absorbing the difference (Harker and Zenios, 2000, Ch. 1). Similar trends are observed in Italy. The traded financial assets of Italian households more than doubled in the 5-year period from 1997, and the bulk of the increase was absorbed by mutual funds and asset management; see Table 1.

The increase in traded financial assets comes with increased diversification of the Italian household portfolio, similar to the one witnessed in the US a decade earlier. Figure 1 shows a strong growth of mutual funds and equity shares at the expense of liquid assets and bonds. Today one third of the total revenues of the Italian banking industry is originated by asset management services.

These statistics reveal the *outcome* of a changing behavior on the part of consumers. What are the changing characteristics of the consumers, however, that bring about this new pattern of investment? The annual *Household Savings Outlook* carried out by Prometeia—a Bologna based company established in 1981 to carry out economic research and analysis, and provide consulting services to major financial institutions and government agencies in Italy—in collaboration with Eurisko—a Milan based company conducting research on consumption, communications, and social transformation—provides important insights. First, the traditional distinction between *delegation* of asset management to a pension fund or an insurance firm by the majority of consumers, and *autonomy* in the management of assets by wealthy investors, no

Table 1
Traded financial assets by Italian households during 1997–2002 in billions of ITL

	1997	1998	1999	2000	2001	2002
Household total	944.853	1427.999	1781.996	2124.102	2488.154	2877.773
% of household's assets	23.6	31.4	34.6	38.3	41.9	44.8
Mutual funds	368.432	720.823	920.304	1077.360	1237.964	1386.519
Asset management	375.465	542.205	673.500	781.300	880.450	956.970
Life and general insurance	165.000	202.300	257.400	329.600	433.400	574.000

Source: ISVAP, the board of regulators for Italian insurers.

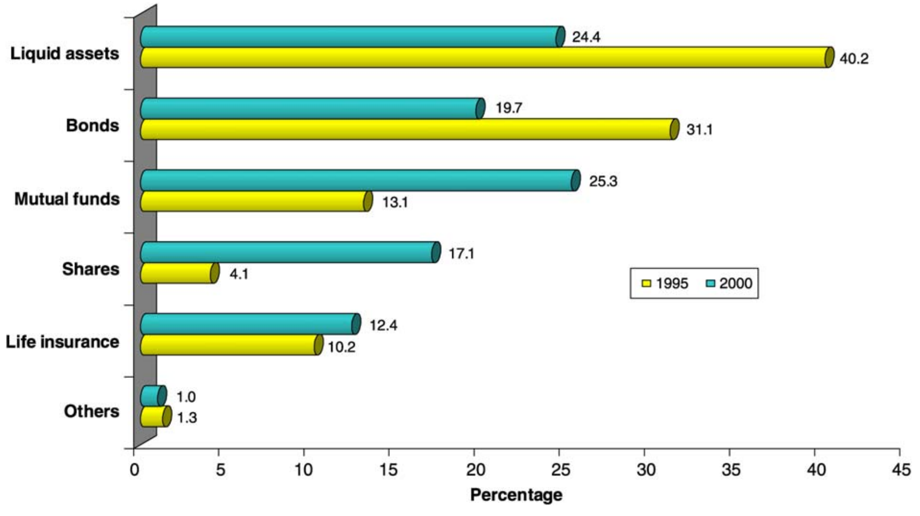

Fig. 1. The evolution of Italian household portfolios.

longer appears to be valid. Both attitudes are present in the behavioral patterns of private savers.

Second, the trend in behavioral profiles is towards higher levels of autonomy, and there is an increased propensity towards innovative instruments as manifested in the data of Figure 1. The group of Italian households classified as “innovators” grew steadily from 6.7% in 1991 to 22.6% by 2001. Each percentage point increase added a further 200,000 households to this category. Today this segment numbers 4.3 million Italian households. Households in this category adopt a very professional approach to questions of finance. They are able—or at least they feel so—to manage their financial affairs, and they rely on integrated delivery channels for doing so, using on-line information and conducting business by phone.

Third, an analysis of the influence of quantitative variables on the savings habits of households shows that awareness of financial indicators, and in particular the performance of managed asset returns, is influencing household behavior. Investors in older age groups are more aware of such indicators than the younger generations. The survey also reveals that the trend towards increased diversification of assets under management will continue unabated during the next three years. The investors’ favorites are insurance and portfolio management. (The survey was conducted just prior to the stalling of the world-wide bull markets so the projection of a continued favor towards portfolio management can be questioned.)

In this environment the Italian insurance industry has come under increasing pressure. The statistics of Table 1 reveal that assets invested in life and general insurance increased by 99% in the period of interest while assets in mutual funds increased by 190%, and those under asset management by 110%. Insurance companies trail the competition in claiming a share of the household's wallet. The industry expects to reverse this trend by 2002. By that time Italian households are expected to increase their traded assets by 200%, with the insurance policies increasing their share by 250%, mutual funds by 280%, and asset managers by 150%. The main competitive weapon in the arsenal of the insurance firms are innovative policies that offer both traditional insurance and participation in the company's profits. These policies combine features of traditional insurance from actuarial risks and of investment vehicles such as mutual funds.

Insurance products with minimum guaranteed rate of return and bonus provisions play today a key role in the insurers' business portfolio. Such products were first offered by insurance companies in the inflationary seventies. In order to compete with the high yields of Treasury bonds of that time, insurance policies were enhanced with both a minimum guaranteed rate of return and a bonus provision when asset fund returns exceed the minimum guarantee. The right to surrender the product at any time before maturity is also often given to policyholders. Such policies, known as unit-linked or index-linked, are prevalent among continental European insurance companies, but they are also encountered in the UK, United States and Canada. In the low-inflation 1990s insurance companies still could not abandon these products due to the competitive pressures outlined above.

With the historically low interest rates prevailing currently the management of such policies is becoming more challenging. Reliance on fixed-income assets is unlikely to yield the guaranteed rate of return. For instance, Italian guaranteed rates after 1998 are at 3%. The difference between the guaranteed rate and the ten-year yield is only 1%, which is inadequate for covering the firm's costs. In Germany the guaranteed rates after 1998 are at 3.5% differing from the ten-year yield only by 0.5%. Danish products offered guarantees of 3% until 1999, which were reduced to 2% afterwards. In Japan Nissan Mutual Life failed on a \$2.56 billion liability arising from a 4.7% guaranteed policy.

In response to the challenges facing Italian insurers, Prometeia developed an asset and liability management system for participating insurance policies with guarantees (Consiglio, Cocco and Zenios, 2000, 2001). The system utilizes recent advances in financial engineering, with the use of scenario-based optimization models, to integrate the insurer's asset allocation problem with that of designing competitive policies. The competing interests of shareholders, policyholders and regulators are cast in a common framework so that efficient tradeoffs can be reached. In this chapter we discuss the model and illustrate its performance. In particular, it is shown that traditional methods are inadequate and innovative models are needed to address the complexities of these products. The resulting model allows the insurer to address asset allocation issues both locally and internationally in a way that is consistent with the offering of competitive products and the shareholders' interests, while satisfying the regulators. Section 2 discusses the Italian insurance industry and describes the characteristics of modern insurance products. Section 3 describes the model and Section 4 reports on model performance from the perspective of the shareholders, the policyholders and the regulators.

2. The Italian insurance industry

The Italian insurance industry is regulated and supervised by ISVAP, Istituto per la Vigilanza sulle Assicurazioni, established by law in 1982. The supervisory framework aims at the stability of the market of insurance undertakings, and at the solvency and efficiency of insurance market participants. ISVAP ensures that the technical, financial and accounting management of institutions under its supervision complies with the laws, regulations and administrative provisions in force.

In the performance of its duties ISVAP may require supervised undertakings to disclose data, management practices and other related information. This supervision monitors the undertaking's financial position, with particular regard to the existence of sufficient solvency margins and adequate technical provisions to ensure that adequate assets are available to cover the entire business.

Progress of the Italian legal framework over the last twenty years—the ISVAP web page lists 51 regulatory provisions—lead supervisors to devote increasing attention to data processing and real-time analysis of data. With solid preventive supervision in place ISVAP can intervene on a timely fashion in any risky situation. The availability of sophisticated safeguards, and the increased financial activity of the last decade driven by the changing nature of the Italian consumer described above, brought the creation of numerous and complex groups of insurance undertakings. These undertakings offer more innovative products, in response to market pressures, and they also take a more active role in the management of their assets and market risks in delivering quality products to clients. The average composition of the portfolios for life insurance, for instance, has been evolving towards more aggressive positions with increasing holdings in equity and high-quality corporate bonds as shown in Table 2. During the same period the industry has been promoting novel insurance policies with guarantees and participation in the profits.

Table 2
The structure of portfolios of Italian life insurers in percentage of total assets held in the major asset categories

Year	Titoli di Stato (Govt. bonds)	Azioni (stocks)	Obbligazioni (bonds)	Titoli in valuta (intnl. investment)
1995	65.2	7.8	14.8	11.4
1996	65.2	7.6	13.6	13.2
1997	60.2	9.1	14.7	15.0
1998	55.2	10.0	16.6	16.4

Source: ISVAP, the board of regulators for Italian insurers.

2.1. Guaranteed products with bonus provisions

Financial products with guarantees on the minimum rate of return come in two distinct flavors: *maturity guarantees* and *multi-period guarantees*. In the former case the guarantee applies only to maturity of the contract, and returns above the guarantee occurring before maturity offset shortfalls at other periods. In the later case the time to maturity is divided into subperiods—quarterly or biannually—and the guarantee applies at the end of each period. Hence, excess returns in one sub-period cannot be used to finance shortfalls in other sub-periods. Such guaranteed products appear in insurance policies, guaranteed investment contracts, and some pension plans, see, e.g., Hansen and Miltersen (**?**).

Policyholders *participate* in the firm's profits, receiving a *bonus* whenever the return of the firm's portfolio exceeds the guarantee, creating a *surplus* for the firm. Bonuses may be distributed only at maturity, at multiple periods until maturity, or using a combination of distribution plans. Another important distinction is made according to the bonus distribution mechanism. In particular, some products distribute bonuses using a *smoothing* formula such as the average portfolio value or portfolio return over some time period, while others distribute a pre-specified fraction of the portfolio return or portfolio value net any liabilities. The earlier *unit-linked* policies would pay a benefit—upon death or maturity—which was the greater of the guaranteed amount and the value of the insurer's reference portfolio. These were simple maturity guarantees with bonus paid at maturity as well. At the other extreme of complexity we have the modern UK insurance policies. These policies declare at each subperiod a fraction of the surplus, estimated using a smoothing function, as *reversionary* bonus which is then guaranteed. The remaining surplus is managed as an *investment reserve*, and is returned to customers as terminal bonus if it is positive at maturity or upon death; see Ross (1989) and Chadburn (1997). These policies are multi-period guarantees with bonuses paid in part at intermediate times and in part at maturity. Further discussion on the characteristics of products with guarantees is found in Kat (2001) and the papers cited below.

The Prometeia model described here considers multi-period guarantees with bonuses that are paid at each subperiod and are subsequently guaranteed. The bonus is contractually determined as a fraction of the portfolio excess return above the guaranteed rate during each subperiod. The guaranteed rate is also contractually specified. To illustrate the nature of this product, we graph in Figure 2 the growth of a liability that participates by 85% in a given portfolio while it guarantees a return of at least 3% in each period. The liability is *lifted* every time a bonus is paid and the minimum guarantee applies to the increased liability: what is given cannot be taken away. This feature creates a complex nonlinear interaction between the rate of return of the portfolio and the total return of the liability.

2.2. Current asset and liability management practices

The shift from actuarial to financial pricing of insurance liabilities (Embrechts, 2000; Babbel, 2001) and widely perceived problems (highlighted by the Nissan bankruptcy

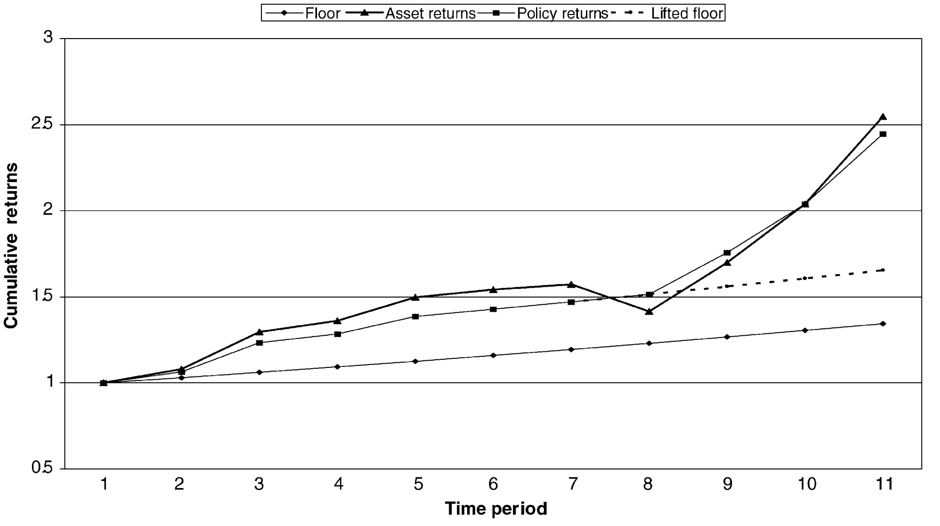

Fig. 2. Typical returns of the asset portfolio and a participating policy with a multi-period guaranteed return of 3% and participation rate of 85%. The guarantee applies to a liability that is lifted every time a bonus is paid as illustrated at period seven. The asset portfolio experienced substantial losses at period seven while the liability grew at the 3% guaranteed rate. Subsequent superior returns of the assets allowed the firm to recover its losses by the tenth period and achieve a positive net return at maturity.

case) brought about an interest in applying the theory of financial asset pricing to the analysis of insurance policies with guarantees and bonus provisions; see, e.g., Giraldi et al. (**?**). Single period guarantees have payoffs that resemble those of a European-type option, as the policyholder receives at maturity the maximum between the guaranteed amount and the value of the bonus. Multi-period guarantees may have features, such as a surrender options, that makes their payoff identical to American type options. Hence, option pricing could be applied to the pricing of these policies.

The pricing of the option embedded in the early products with guarantees was addressed in the seminal papers of Brennan and Schwartz (1976) and Boyle and Schwartz (1977). They analyzed unit-linked maturity guarantee policies. Perhaps the most complete analysis of modern life insurance contracts—complete in the sense that it prices in an integrated framework several components of the policy—is due to Grosen and Jørgensen (2000). They decompose the liability of modern participating policies with guarantees into a risk-free bond (the minimum guarantee), a bonus option, and a surrender option. The first two taken together are a European contract and all three together are an American contract, and the authors develop numerical techniques for pricing both. Hansen and Miltersen (2002) extend this model to the pricing of contracts with a smoothing surplus distribution mechanism of the form used by most Danish life-insurance companies and pension plans. They use the model to study different methods for funding these products, either by charging the customers directly or by keeping a share of the surplus. Similarly, Bacinello (2001) develops pricing models that permit her to study the interplay between the volatility of the underlying asset portfolio, the participation level for determining bonuses, and the guaranteed rate. Boyle and Hardy (1997) take this line of inquiry in a different direction by analyzing alternative reserving methods for satisfying the guarantee. More practical aspects of the problem are studied by Giraldi et al. (**?**) and Siglienti (2000).

It is worth noting that current literature assumes the asset side is given *a priori* as a well-diversified portfolio which evolves according to a given stochastic process. For instance, Brennan–Schwartz, Grosen–Jørgensen and Bacinello assume a geometric Browning motion, while Miltersen and Persson (1999) rely on the Heath–Jarrow–Morton framework and price multi-period guaranteed contracts linked either to a stock investment or the short-term interest rate. There is nothing wrong with these approaches, of course, except that part of the problem of the insurance companies is precisely to determine the structure of the asset portfolio. Indeed, all of the above references carry out simulations for different values of the volatility of the assets. Brennan and Schwartz (1979) devote a section to the analysis of "misspecification of the stochastic process". Bacinello goes on to suggest that the insurance company should structure several reference portfolios according to their volatility and offer its customers choices among different triplets of guaranteed rate, bonus provision, and asset portfolio volatility. To this suggestion of endogenizing the asset decision we subscribe. It is a prime example of *integrated financial product management* advocated by Holmer and Zenios (1995).

Independently of the literature that prices the option embedded in the liabilities we have seen an interest in the use of portfolio optimization models for asset and liability management for insurance companies. The most prominent example is for a Japanese insurance firm—not too surprising given what has transpired in the Japanese financial markets—the Yasuda Kasai model developed by the Frank Russel Company. This model received coverage not only in the academic literature but also in the press, see Carinõ and Ziemba (1998). Other successful examples include the Towers Perrin model of Mulvey and Thorlacius (1998), the CALM model of Consigli and Dempster (1998) and the Gjensidige Liv model of Høyland (1998). These models have been successful in practical settings but their application does not cover participating policies with guarantees. One reason is that insurance firms pursued integrated asset and liability management strategies for those products they understood well. This has been the case for policies that encompass mostly actuarial risk such as the fire and property insurance of the Yasuda Kasai model. Another reason is that the technology of scenario optimization through large-scale stochastic programming has only recently been developed into computable models, see, e.g., Censor and Zenios (1997).

Finally, the combination of a guarantee with a bonus provision introduces nonlinearities which complicate the model. Traditional approaches such as the mean-variance analysis are inadequate as they fails to capture some important characteristics of the problem. There is nothing efficient about efficient portfolios when the nonlinearity of the embedded options is properly accounted for. Novel models are needed to integrate the asset management problem with the characteristics of liabilities with minimum guarantee. Such a model was developed through a multi-year collaborative effort between academic researchers, the research staff at Prometeia in Bologna, and end-users from diverse Italian insurers. It is presented next.

3. The scenario optimization model

We develop in this section the model for asset and liability management for multi-period participating policies with guarantees. It is a mathematical program that models stochastic variables using discrete scenarios. All portfolio decisions are made at $t = 0$ in anticipation of an uncertain future. At the end of the planning horizon the impact of these portfolio decisions in different scenarios is evaluated and risk aversion is introduced through a utility function. Portfolio decisions optimize the expected utility over the specified horizon.

3.1. Features of the model

In the model we consider three accounts: (i) a liability account that grows according to the contractual guaranteed rate and bonus provision, (ii) an asset account that grows according to the portfolio returns, net any payments due to death or policy surrenders, and (iii) a shortfall account that monitors lags of the portfolio return against the guarantee. In the base model shortfall is funded by equity but later we introduce alternative reserving methods.

The multi-period dynamics of these accounts are conditioned on discrete scenarios of realized asset returns and the composition of the asset portfolio. Within this framework a regulatory constraint on leverage is imposed. At maturity the difference between the asset and the liability accounts is the surplus realized by the firm after it has fulfilled its contractual obligations. In the policies considered here this surplus remains with the shareholders. This surplus is a random variable, and a utility function is introduced to incorporate risk aversion.

3.2. Notation

We let Ω denote the index set of scenarios $l = 1, 2, \ldots, N$, indicating realizations of random variables, $\mathcal{U}$ the universe of available asset instruments, and $t = 1, 2, \ldots, T$, discrete points in time from today ($t = 0$) until maturity T. The data of the problem are as follows:

- r_{it}^{l} rate of return of asset i during the period $t-1$ to t in scenario l.
- r_{ft}^{l} risk free rate during the period $t-1$ to t in scenario l.
- g minimum guaranteed rate of return.
- β participation rate indicating the percentage of portfolio return paid to policyholders.
- ρ regulatory equity to debt ratio.
- Λ_t^{l} probability of abandon of the policy due to lapse or death at period t in scenario l.

The variables of the model are defined as follows:

x_i	percentage of initial capital invested in the ith asset.
y^l_{At}	expenses due to lapse or death at time t in scenario l.
z^l_t	shortfall below the guaranteed rate at time t in scenario l.
A^l_t	asset value at time t in scenario l.
E^l_t	total equity at time t in scenario l.
L^l_t	liability value at time t in scenario l.
R^l_{Pt}	portfolio rate of return during the period $t-1$ to t in scenario l.
y^{+l}_t	excess return over g at time t in scenario l.
y^{-l}_t	shortfall return under g at time t in scenario l.

3.3. Variable dynamics and constraints

We invest the premium collected (L_0) and the equity required by the regulators ($E_0 = \rho L_0$) in the asset portfolio. Our initial endowment $A_0 = L_0(1+\rho)$ is allocated to assets in proportion x_i such that $\sum_{i\in\mathcal{U}} x_i = 1$, and the dynamics of the portfolio return are given by

$$R^l_{Pt} = \sum_{i\in\mathcal{U}} x_i r^l_{it}, \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{1}$$

The investment variables are nonnegative so that short sales are not allowed.

We now turn to the modeling of the liability account. Liabilities will grow at a rate which is at least equal to the guarantee. Excess returns over g are returned to the policyholders according to the participation rate β. The dynamics of the liability account are given by

$$L^l_t = \left(1 - \Lambda^l_t\right) L^l_{t-1} \left(1 + \max\left[\beta R^l_{Pt}, g\right]\right), \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{2}$$

The max operator introduces a discontinuity in the model. To circumvent this difficulty we introduce variables y^{+l}_t and y^{-l}_t to measure the portfolio excess return over the guaranteed rate, and the shortfall below the guarantee, respectively. They satisfy

$$\beta R^l_{Pt} - g = y^{+l}_t - y^{-l}_t, \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega, \tag{3}$$

$$y^{+l}_t \geqslant 0, \quad y^{-l}_t \geqslant 0, \quad y^{+l}_t y^{-l}_t = 0, \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{4}$$

Only one of these variables can be nonzero at any given time and in a given scenario.

The dynamics for the value of the liability are rewritten as

$$L^l_t = \left(1 - \Lambda^l_t\right) L^l_{t-1} \left(1 + g + y^{+l}_t\right), \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{5}$$

Liabilities grow at least at the rate of g. Any excess return is added to the liabilities and the guarantee applies to the lifted liabilities.

At each period the insurance company makes payments due to policyholders abandoning their policies because of death or lapse. Payments are equal to the value of the liability times the probability of abandonment, i.e.,

$$y_{At}^l = \Lambda_t^l L_{t-1}^l \left(1 + g + y_t^{+l}\right), \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{6}$$

Whenever the portfolio return is below the guaranteed rate we need to infuse cash into the asset portfolio in order to meet the final liabilities. The shortfall account is modeled by the dynamics

$$z_t^l = y_t^{-l} L_{t-1}^l, \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{7}$$

In the base model shortfalls are funded through equity. We assume that equity is reinvested at the risk-free rate and is returned to the shareholders at the end of the planning horizon. (This is not all the shareholders get; they also receive dividends.) The dynamics of the equity are given by

$$E_t^l = E_{t-1}^l \left(1 + r_{ft}^l\right) + z_t^l, \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{8}$$

By assuming the risk free rate as the alternative rate at which the shareholders could invest their money, we analyze the *excess* return offered to shareholders by the participating contract modeled here, over the benchmark risk free investment. In principle, one could use the firm's internal rate of return as the alternative rate, and analyze the excess return offered by the policy modeled here, over the firm's other lines of business. In this setting, however, the problem would not be to optimize the asset allocation to maximize shareholder value, since this would already be endogenous in the internal rate of return calculations. Instead we could determine the most attractive features for the policyholders—g and β—that will make the firm indifferent in offering the new policy or maintaining its current line of business. This approach deserves further investigation. For the purpose of optimizing alternative policies for the shareholders, while satisfying the contractual obligations to the policyholders, the estimation of excess return over the risk free rate is a reasonable benchmark. In Sections 4.3.1 and 4.3.3 we consider other alternatives for funding the shortfalls through long-term debt or short-term borrowing.

We now have the components needed to model the asset dynamics, taking into account the cash infusion that funds shortfalls, z_t^l, and the outflows due to actuarial events y_{At}^l, i.e.,

$$A_t^l = A_{t-1}^l \left(1 + R_{Pt}^l\right) + z_t^l - y_{At}^l, \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{9}$$

In order to satisfy the regulatory constraint the ratio between the equity value and liabilities must exceed ρ. That is,

$$\frac{V_{ET}^l}{L_T^l} \geqslant \rho, \quad \text{for all } l \in \Omega, \tag{10}$$

where V_{ET}^l is the value of equity at the end of the planning horizon T. If the company sells only a single policy the value of its equity will be equal to the final asset value—which includes the equity needed to fund shortfall—minus the final liability due to the policyholders, and we have

$$V_{ET}^l = A_T^l - L_T^l. \tag{11}$$

Having described the assets and liability accounts in a way that the key features of the policy—guaranteed rate and bonus provisions—are accounted for, we turn to the choice of an appropriate objective function. We model the goal of a for-profit institution to maximize the return on its equity, and, more precisely in this case, to maximize any excess return on equity after all liabilities are paid for. Since return on equity is scenario dependent we maximize the expected value of the utility of excess return. This expected value is converted into a certainty equivalent for easy reference. The objective function of the model is to compute the maximal Certainty Equivalent Excess Return on Equity (CEexROE) given by

$$\text{CEexROE} \doteq U^{-1}\left\{\operatorname*{Max}_x \frac{1}{N}\sum_{l\in\Omega} U\left\{\frac{A_T^l - L_T^l}{E_T^l}\right\}\right\}, \tag{12}$$

where $U\{\cdot\}$ denotes the decision maker's utility function and $A_T^l - L_T^l$ is the shareholder's reward in scenario l. We assume a power utility function with constant relative risk aversion of the form $U(V) = \frac{1}{\gamma}V^\gamma$, where $V \geqslant 0$, and $\gamma < 1$. In the base model we assume $\gamma = 0$ in which case the utility function is the logarithm corresponding to growth-optimal policies for the firm. In Section 4.5.1 we study the effect of changing the risk aversion parameter.

As a byproduct of our model we calculate the cost of funding the guaranteed product. Every time the portfolio return drops below the guaranteed rate, we counterbalance the erosion of our assets by infusing cash. This cost can be charged either to the policyholders, as soon as they enter the insurance contract, or covered through shareholder's equity or by issuing debt. These choices entail a tradeoff between the return to shareholders and return to policyholders. We study in the next section this tradeoff.

The cost of the guarantee is the expected present value of reserves required to fund shortfalls due to portfolio performances below the guarantee. The dynamic variable E_t^l models precisely the total funds required up to time t, valued at the risk-free rate. However, E_t^l also embeds the initial amount of equity required by the regulators. This is not a cost and it must be deducted from E_t^l. Thus, the cost of the guarantee is given as the expected present value of the final equity E_T^l adjusted by the regulatory equity, that is,

$$\overline{O}_G = \frac{1}{N}\sum_{l=1}^{N}\left(\frac{E_T^l}{\prod_{t=1}^{T}(1 + r_{ft}^l)} - \rho L_0\right). \tag{13}$$

$\overline{O}_G$ is the expected present value of the reserves required to fund this product. This can be interpreted as the cost to be paid by shareholders in order to benefit from the upside potential of the surplus. A more precise interpretation of $\overline{O}_G$ is as the *expected downside risk* of the policy. This is not the risk-neutral price of the participating policies with guarantees that would be obtained under an assumption of complete markets for trading the liabilities arising from such contracts. This is the question addressed through an options pricing approach in the literature cited above, Brennan–Schwartz, Boyle–Schwartz, Bacinello, Grosen–Jørgensen, Hansen–Miltersen, Miltersen–Persson.

3.4. Linearly constrained optimization model

The model defined in the previous section is a nonlinearly constrained optimization model and is computationally intractable for large scale applications. However, the nonlinear constraints (5)–(9) are definitional constraints which determine the value of the respective variables at the end of the horizon. We solve these dynamic equations analytically (see Appendix A) to obtain end-of-horizon analytic expressions for A_T^l, L_T^l, and E_T^l. These expressions are substituted in the objective function to obtain the equivalent linearly constrained nonlinear program below. The regulatory constraint (10), however, cannot be linearized. For solution purposes the regulatory constraint is relaxed and its validity is tested *ex post*. Empirical results later on demonstrate that the regulatory constraint is not binding for the policies considered here and for the generated scenarios of asset returns. However, there is no assurance that this will always be the case, and we may need to resort to nonlinearly constrained optimization for solving this model.

$$\begin{aligned}
\underset{x\geqslant 0}{\text{Maximize}}\ \frac{1}{N}\sum_{l\in\Omega} U\Bigg\{\Bigg[&(1+\rho)\prod_{t=1}^{T}\big(1+R_{Pt}^l\big)+\sum_{t=1}^{T}\big(y_t^{-l}-\Lambda_t^l\big(1+g+y_t^{+l}\big)\big)\\
&\times\prod_{\tau=t+1}^{T}\big(1+R_{P\tau}^l\big)\prod_{\tau=1}^{t-1}\big(1+g+y_\tau^{+l}\big)\big(1-\Lambda_\tau^l\big)\\
&-\prod_{t=1}^{T}\big(1-\Lambda_t^l\big)\big(1+g+y_t^{+l}\big)\Bigg]\\
&\Bigg/\Bigg[\rho\prod_{t=1}^{T}\big(1+r_{ft}^l\big)+\sum_{t=1}^{T}y_t^{-l}\phi(t,T)\prod_{\tau=1}^{t-1}\big(1-\Lambda_\tau^l\big)\big(1+g+y_\tau^{+l}\big)\Bigg]\Bigg\}
\end{aligned}\tag{14}$$

$$\text{s.t.}\quad \sum_{i\in\mathcal{U}} x_i = 1, \tag{15}$$

$$\beta R_{Pt}^l - g = y_t^{+l} - y_t^{-l}, \quad \text{for } t=1,2,\ldots,T, \text{ and for all } l\in\Omega, \tag{16}$$

$$R_{Pt}^l = \sum_{i\in\mathcal{U}} x_i r_{it}^l, \quad \text{for } t=1,2,\ldots,T, \text{ and for all } l\in\Omega. \tag{17}$$

The inverse of the utility function, U^{-1}, of the optimal objective value of this problem is the CEexROE.

3.5. Surrender option

The probability of abandon Λ_t^l is determined from both actuarial events (death) and economic considerations (surrendering the policy). The actuarial component is readily obtained from mortality tables. However, the lapse behavior of policyholders needs to be modeled taking into account the economic incentive to surrender the policy and invest into competing products. This dimension is modeled here.

Modeling the lapse behavior serves as a sensitivity analysis of the model for studying errors introduced due to various sources of model risk. For instance, in recent years many actuaries have pointed out that the aging of the population has introduced a modeling risk in the actuarial framework. The *longevity risk* affects the probability of survival for sectors of the population in their retirement years. Pension fund managers will then face higher liabilities than those planned. On the contrary, life insurance products benefit from longevity risk since the payments due to death are reduced. The modeling of lapse undertaken here is but one example of the additional sources of uncertainty that could be incorporated in the model if data are available.

We discuss here two assumptions about policy lapse which can be embedded into the model.

Fixed lapse: Under this assumption the probability of surrendering the policy (Λ_t) is constant throughout the life of the contract. This assumption is quite realistic. For instance, an analysis of a panel of British households shows that the percentage of lapse is constant over the period 1994–1997 and it averages to 1.4% (see the Personal Investment Authority report, 1999). An estimate from rough data available to us for Italian households indicate a modest lapse rate of the order of 2%.

Variable lapse: Under this assumption the policyholders' decision to surrender their policy is affected by economic factors. For instance, in the analysis of mortgage backed securities (see Kang and Zenios, ?) prepayment models are calibrated to describe household attitude towards market factors, such as the prevailing mortgage refinancing rates, and social factors such as age of the household and demographics. Similarly we can link the dynamics of Λ_t^l to economic variables. If we assume that lapse is driven by the minimum guarantee level g, then the lapse probability is a function of the spread between g and the rate on other investments offered in the capital markets

$$\Lambda_t^l = f\left(r_{It}^l - g\right), \tag{18}$$

where r_{It}^l is a suitable benchmark of the return offered by competing products; this can be, for instance, the return on the 10-year Government bond index. The surrender probability is now indexed by scenario as it depends on the competitors' rate r_{It}^l. We expect policyholders to surrender their policies when alternative investments provide a return higher than the guarantee g.

Perhaps the most significant factor affecting lapse is the bonus policy followed by the company. Evidence to this is provided for some similar products—single premium deferred annuities—by Asay, Bouyoucos and Marciano (1993). If the insurance company's crediting rate is significantly lower than that of the competition then lapse rates will be high. In participating policies the credit rate is determined by the performance of the portfolio. Thus, an integrative asset and liability management approach is essential in accurately capturing the lapse rates of these products.

Assuming that the competitors offer rates equal to the relevant market benchmark, we express lapse rates in the form

$$\Lambda_t^l = f\left(r_{It}^l - \left(g + \varepsilon_t^{+l}\right)\right). \tag{19}$$

(Recall that $g + \varepsilon_t^{+l}$ is the rate credited to policyholders and it reflects both the guarantee and the bonus policy.) This formula embodies the complex games facing the insurer: large minimum guarantees subdue the effects of the competition but come at a large cost or low CEexROE. This will also be demonstrated in Section 4 where the model is validated.

A convenient general form for function $f(\cdot)$ governing the surrender behavior has been studied by Asay, Bouyoucos and Marciano (1993). In this study the lapse probability is given by

$$\Lambda_t^l = a + b\tan^{-1}\left[m\left(r_{It}^l - i_t^l - y\right) - n\right]. \tag{20}$$

The variable i_t^l is the company's credit rate which can be modeled as a constant (Eq. (18)) or as a variable determined by policy and market performance (Eq. (19)), r_{It}^l is the rate offered by the competitors, and y is a measure of policyholders' inertia in exercising the surrender option. The parameters a, b, m, n are chosen to give lapse rates that fit historically observed data. For instance, the model should fit the lowest and highest lapse rates that have been observed under extremely favorable and unfavorable conditions, and the lapse rates observed when the insurance product was offering the same credit rates as the benchmark.

Figure 3 shows different lapse curves when varying the parameter to fit maximum and minimum values and different average lapse rates. Lapse rates will be, on the average, lower when there are large penalties for early surrender of the policy. The different curves shown in the figure could fit, for instance, the historically observed lapse rates of policies with different surrender charges. We observe that lapse rates may differ substantially when the company is offering a credit rate which is less than the competitors' rate. This situation occurs when assets perform poorly with respect to the rest of the industry. Careful modeling of the lapse behavior is needed in these cases to avoid igniting a vicious circle which could lead to bankruptcy.

3.6. *Model extensions*

We point out possible extensions of this model. Periodic premia can readily be incorporated into Eq. (9). Bonus policies based on averaging portfolio performance can also be included in the model. The liability equation (2) must be modified to include average

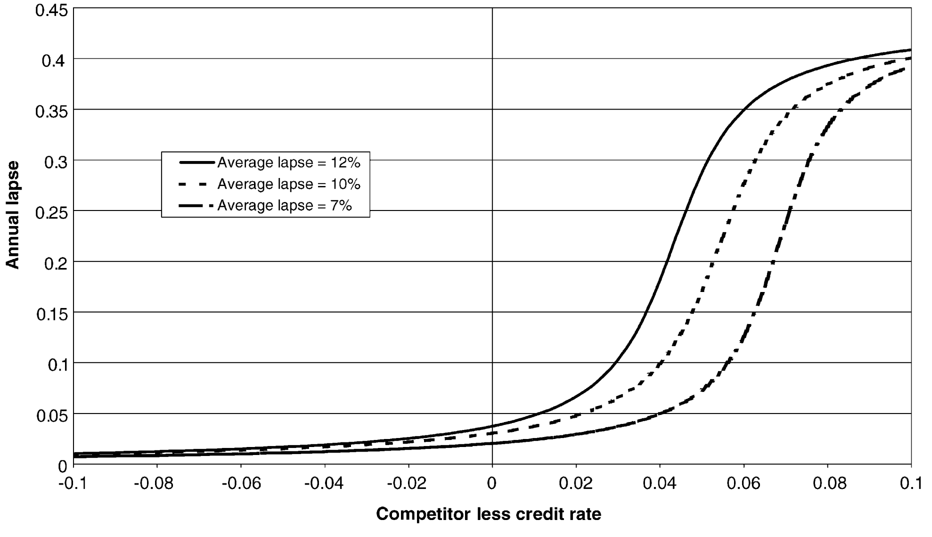

Fig. 3. Typical lapse functions with average lapse rates ranging from 7 to 12%.

portfolio performance over the history of interest (say the last t_h periods) as follows

$$L_t^l = \left(1 - \Lambda_t^l\right) L_{t-1}^l \left(1 + \max\left[\beta \sum_{\tau=t-t_h}^{t} R_{P\tau}^l, g\right]\right), \tag{21}$$

for $t = 1, 2, \ldots, T$, and for all $l \in \Omega$.

Guaranteed rates and bonus rates that are exogenously given functions of time, g_t and β_t, are easy to incorporate. Similarly, we can incorporate liabilities due to lapse, although a lapse model must first be built and calibrated as discussed above. Incorporating participation rates that are functions of the asset returns—as is the case with the UK insurance policies—complicates the model since the participation rate β_t is a variable; see, e.g., Consiglio, Saunders and Zenios (2003, 2006). The split of bonus into reversionary bonus, which is guaranteed, and an investment reserve which is returned as a bonus at maturity, if nonnegative, introduces significant modifications to the model. These issues are discussed in Section 3.7.

The base model developed here funds shortfalls through equity. Extensions to deal with the funding of shortfalls through long- or short-term debt are given in Sections 4.3.1 and 4.3.3, respectively. Furthermore, unlimited access to equity for funding shortfalls is assumed in the base model. We could do away with this assumption by imposing additional constraints, but this would complicate the model rendering it computationally intractable. The probability of insolvency is analyzed through postoptimality analysis in Section 4.3.2, and is used to guide the debt structure in funding shortfalls using a combination of equity and debt.

3.7. Reversionary and terminal bonuses

Some policies use a smoothing mechanism to estimate bonuses, disbursing higher bonuses when market conditions are favorable, and decreasing bonuses when the insurer's portfolio is under-performing. Changes are autoregressive so that big swings are avoided, as those are viewed unfavorably by policyholders. The policies offered in the UK are the best known example with these characteristics. The bonus philosophy of the UK insurers is based on regulatory requirements that bonus distribution should accord with the policyholders' reasonable expectations of the company's behavior (Ross, 1989; Chadburn, 1997).

To satisfy the policyholders' expectations UK insurers offer *reversionary bonuses* that, once announced, are subsequently guaranteed. In addition, they deliver a *terminal bonus*, that is, a function of the excess asset value upon maturity. In general, the reversionary bonus and the guaranteed rates of the UK insurers are lower than those offered by their Italian colleagues. However, policyholders receive the lion's share of any excess asset value, while in the Italian case the insurer's shareholders benefit from the terminal excess asset value. Italian insurers offer a big bird at hand, but nothing in the bush; UK insurers offer a small bird at hand, and ten in the bush.

To model these policies we introduce variable RB_t^l to denote the reversionary bonus disbursed at period t in scenario l. This variable evolves according to the autoregressive equation

$$RB_t^l = 0.5RB_{t-1}^l + \Delta B_t^l, \tag{22}$$

where the constant 0.5 ensures that policyholders are not too unpleasantly surprised by downward swings of their bonuses, and ΔB_t^l is the change in the bonus. This may be positive or negative and is computed as follows

$$\Delta B_t^l = 0.5 \max\left[\frac{r_{It}^l - g}{1+g}, 0\right] - 0.25 \max\left[\frac{L_t^l - A_t^l}{A_t^l}, 0\right]. \tag{23}$$

The first term on the right of this equation is positive whenever some benchmark return r_{It}^l exceeds the guarantee, otherwise it is zero. The benchmark return is taken in the UK to be the yield on long risk free securities. The second term is positive whenever the asset value is less than the liability value, otherwise it is zero. With this formula the bonus rate is increased whenever the market rates increase, but it is decreased whenever the insurer faces the prospect of insolvency.

The variable dynamics and constraints of policies with smoothed reversionary and terminal bonuses can now be formulated, building on the base model of Section 3.3.

The dynamics of the liability account are given by

$$L_t^l = \left(1 - \Lambda_t^l\right) L_{t-1}^l (1+g)\left(1 + \max\left[RB_{t-1}^l, 0\right]\right), \tag{24}$$

for $t = 1, 2, \ldots, T$, and for all $l \in \Omega$.

Liability payments are exactly as in the base model

$$y_{At}^l = \Lambda_t^l L_{t-1}^l \left(1 + g + y_t^{+l}\right), \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{25}$$

The asset dynamics take into account outflows due to actuarial events but, unlike the base model, there is no cash infusion. The asset value is allowed to go below the liability value and this will have an effect on the reversionary bonus.

$$A_t^l = A_{t-1}^l\big(1 + R_{Pt}^l\big) - y_{At}^l, \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{26}$$

The equity equation from the base model (8) is split into two equations: One that models the dynamics of the shareholder equity growing at the risk free rate, and one that models shortfalls so that the total shortfall (if any) at maturity can be assessed. The shareholder equity follows the dynamics

$$E_t^l = E_{t-1}^l\big(1 + r_{ft}^l\big), \quad \text{for } t = 1, 2, \ldots, T, \text{ and for all } l \in \Omega. \tag{27}$$

The lag of assets against the liabilities is given by

$$E_t'^l = \max\big[(1+\rho)L_t^l - A_t^l, 0\big], \tag{28}$$

and whenever the lag increases the total shortfall, z_t^l, increases according to the dynamics

$$z_t^l = z_{t-1}^l\big(1 + r_{ft}^l\big) + \max\big[\big(E_t'^l - E_{t-1}'^l\big), 0\big], \tag{29}$$

for $t = 1, 2, \ldots, T$, and for all $l \in \Omega$.

With these dynamics the terminal bonus paid to policyholders at maturity T is given by

$$TB_T^l = \gamma \max\big[A_T^l - L_T^l, 0\big], \tag{30}$$

and the return on equity to shareholders is given by

$$\mathrm{ROE}^l = \frac{A_T^l - L_T^l - TB_T^l}{E_T^l}. \tag{31}$$

4. Model testing and validation

We now turn to the testing of the model. We start first with the application of the traditional portfolio diversification approach based on the mean-variance optimization. We show that the standard application of the mean-variance optimization fails to capture some important characteristics of the problem. There is nothing efficient about efficient portfolios when the nonlinearity of the embedded options is properly accounted for. We show that the novel model based on scenario optimization adds value to the risk management process for these complex insurance products. The value of integrated financial product management is extensively argued in practice, see, e.g., Stulz (**?**) but case studies showing that an integrative perspective adds value are scant; see Holmer and Zenios (1995) for some examples.

Second, we show that the model quantifies the tradeoffs between the different targets of the insurance firm: providing the best products for its policyholders, providing the highest excess return to its shareholders, satisfying the guarantee at the lowest possible cost and with high probability. Some interesting insights are obtained on the structure of the optimal portfolios as the tradeoffs vary across the spectrum.

Third, we analyze alternative debt structures whereby the cost of the guarantee is funded through equity or through debt with either long or short maturities.

Fourth, we study some additional features of the model: the effects of the choice of a utility function, the effects of using international asset classes and corporate bonds and the effects of policy surrender options (lapse).

Finally, we will see from the empirical results that the Italian insurance industry operates at levels which are close to optimal but not quite so. There is room for improvement either by offering more competitive products or by generating higher excess returns for the benefit of the shareholders. How are the improvements possible? The answer is found in the comparison of the optimal portfolios generated by our model with benchmark portfolios. We will see that the benchmark portfolios generate tradeoffs in the space of cost of guarantee vs. net excess return on equity that are inefficient. The optimized portfolios lead to policies with the same cost but higher excess return on equity.

The basic asset classes considered in our study are 23 stock indexes of the Milano Stock Exchange, and three Salomon Brother indexes of Italian Government bonds (Appendix B). Italian insurers are also allowed to invest up to 10% of the value of their portfolio in international assets. We report results with the inclusion of international asset classes: the Morgan Stanley stock indices for USA, UK and Japan and the J.P. Morgan Government bond indices for the same countries.

We employ a simple approach for generating scenarios using only the available data without any mathematical modeling, by bootstrapping a set of historical records. Each scenario is a sample of returns of the assets obtained by sampling returns that were observed in the past. Dates from the available historical records are selected randomly, and for each date in the sample we read the returns of all assets classes realized during the previous month. These samples are scenarios of *monthly* returns. To generate scenarios of returns for a long horizon—say 10 years—we sample 120 monthly returns from different points in time. The compounded return of the sampled series is one scenario of the 10-year return. The process is repeated to generate the desired number of scenarios for the 10-year period. With this approach the correlations among asset classes are preserved.

Additional scenarios could also be included, although methods for generating them should be specified. Model-based scenario generation methods for asset returns are popular in the insurance industry—e.g., the Wilkie (1995) model or the Towers Perrin model (Mulvey and Thorlacius, 1998)—and could be readily incorporated into the scenario optimization model. Alternatively, one could use expert opinion or "scenario proxies" as discussed in Dembo et al. (2000).

For the numerical experiments we bootstrap monthly records from the ten year period January 1990 to February 2000. The monthly returns are compounded to yearly returns. For each asset class we generate 500 scenarios of returns during a 10 year horizon ($T =$

120 months). We consider an initial liability $L_0 = 1$ for a contract with participation rate $\beta = 85\%$ and equity to liability ratio $\rho = 4\%$. The model is tested for guarantees ranging from 1 to 15%.

In our experiments we set lapse probabilities to zero and the probability that a policyholder abandons the policy is the mortality rate which we obtain from the Italian mortality tables. For each model run we determine the net annualized after-tax CEexROE

$$\left(\sqrt[T]{\text{CEexROE}} - 1\right)(1 - \kappa), \tag{32}$$

where κ is the tax rate set at 51%.

4.1. The value of integrative asset and liability management

In this section we compare the scenario optimization asset and liability management model with traditional asset allocation using the mean-variance analysis. We demonstrate that the integrative approach adds value to the asset and liability management process.

4.1.1. Traditional approach using mean-variance asset allocation

Diversified portfolios of stocks and bonds for an Italian insurance firm are built using the mean-variance optimization. Using the asset classes of the Italian stock and bond markets, we obtain the efficient frontier of expected return vs. standard deviation illustrated in Figure 4. Should an insurance firm offering a minimum guarantee product choose portfolios—based on its appetite for risk—from the set of efficient portfolios? On the same figure we plot each one of the efficient portfolios in the space of shareholder's reward (CEexROE) versus the firm's risk (cost of the guarantee). There is nothing efficient about efficient portfolios when the liability created by the minimum guarantee policy is accounted for. Portfolios from A to G are on the mean-variance frontier that lies below the capital market line. It is not surprising that they are not efficient in the CEexROE vs. cost-of-guarantee space. However, the tangent portfolio G is also inefficient. A more aggressive portfolio strategy is needed in order to achieve the minimum guaranteed return and deliver excess return to shareholder. And still this increasing appetite for higher but risky returns is not monotonic. As we move away from portfolio G towards the most risky portfolio B, we see at first the cost of the guarantee declining and CEexROE improving. But as we approach B the shareholder's value erodes, just as Siglienti found out from his simulations. For these very volatile portfolios the embedded option is deep in-the-money, and shareholders' money are used to compensate for the shortfalls without realizing any excess returns.

This first step of our analysis has shown that it is important to take an integrative view of the asset allocation problem of firms issuing products with guarantees. Properly accounting for the cost of the guarantee is important, if the firm is to avoid unnecessary risk exposures and destroy the shareholder's value.

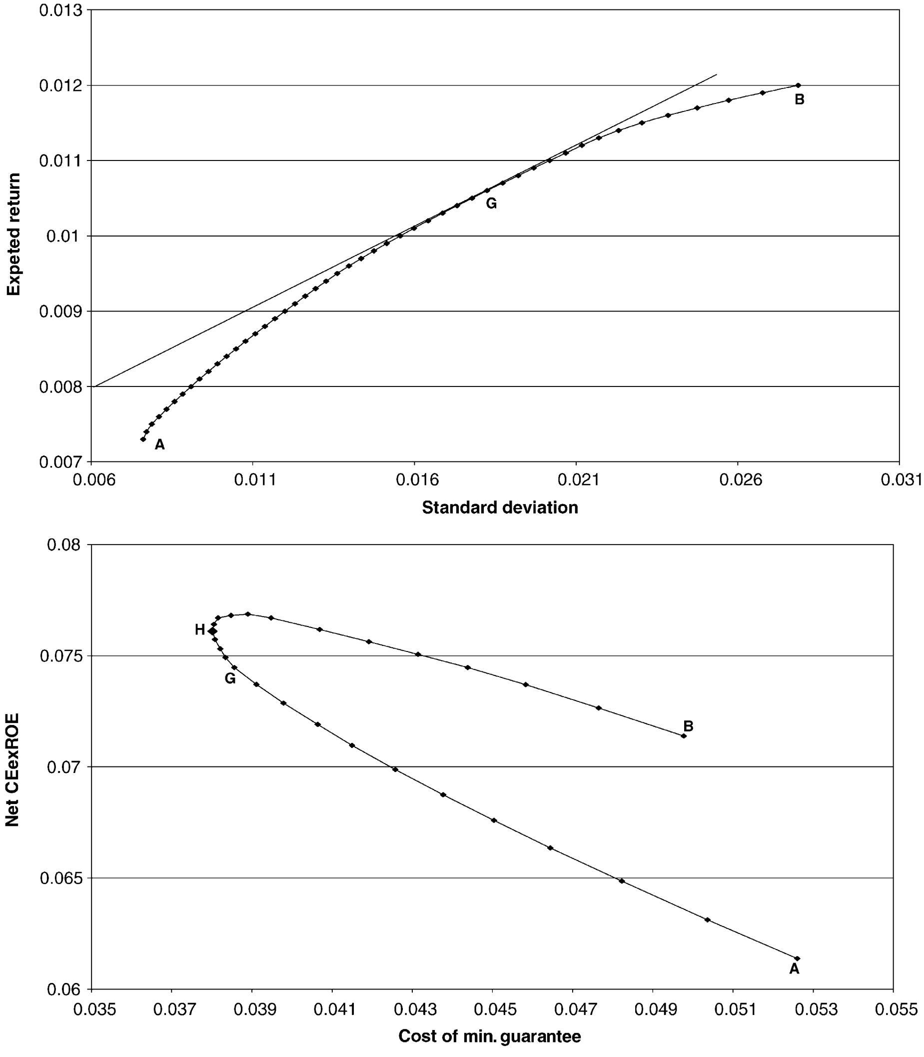

Fig. 4. Mean-variance efficient portfolios of Italian stocks and bonds and the capital market line (top) and the corresponding certainty equivalent excess return of equity (CEexROE) to shareholders vs. cost of the minimum guarantee for each portfolio (bottom).

In a nutshell the management of minimum guarantee products is a balancing act. Too much reliance on bonds and the guarantee is not met. Excessive reliance on stocks and shareholder's value is destroyed.

Is it possible to incorporate the random liability in a mean-variance model, and develop efficient portfolios in the CEexROE vs. cost-of-guarantee space? Unfortunately, the return of the liability depends on the return of the asset portfolio and this is not known without determining simultaneously the structure of the asset portfolio. The return of the liability is endogenous to the portfolio selection model. Furthermore, the liability return has a floor—the minimum guarantee. This creates nonlinearities in the model, and highly asymmetric returns that are not conducive to mean-variance type of modeling. While semi-variance or other risk measures could be used to handle the asymmetric returns, the problem that the return of the liability is endogenous to the portfolio selection model remains. The integrative model developed earlier is essential.

4.1.2. *Integrative asset and liability modeling*

The results in Figure 5 show the tradeoff between upside potential versus the downside risk achieved when using the models of this paper. Each point on this figure corresponds to an optimal asset portfolio for each level of minimum guarantee. On the same figure we plot the tradeoff between CEexROE and cost of the guarantee from the portfolios of Figure 4. We see that even portfolio H is dominated by the portfolios obtained by an integrative model. The traditional approach of portfolio diversification—Figure 4 (top)—followed by a post optimality analysis to incorporate the minimum guarantee liability and its cost—Figure 4 (bottom)—yields suboptimal results. The integrative approach adds value. The analysis carried out here with market data for a real policy shows that the added value can be substantial.

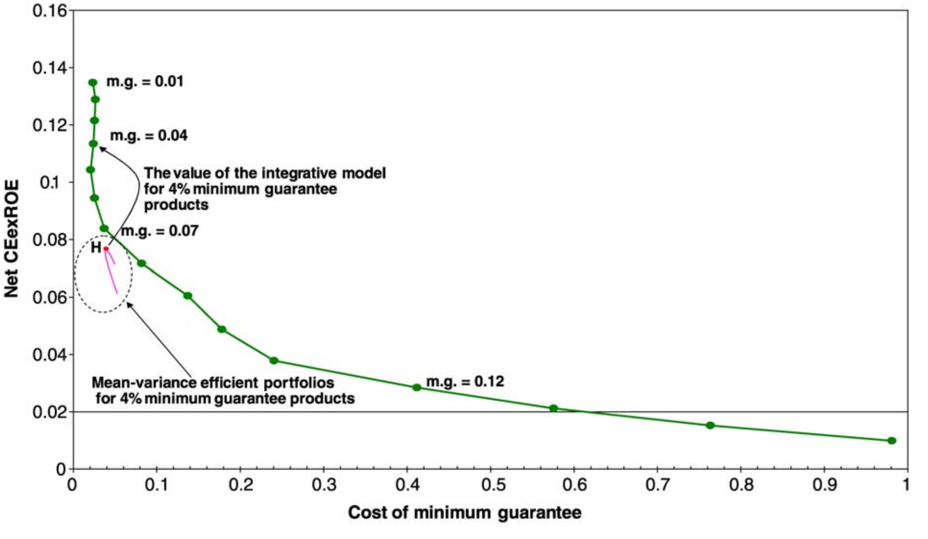

Fig. 5. Certainty equivalent excess return of equity (CEexROE) to shareholders vs. cost of the minimum guarantee for the integrated portfolios at different levels of minimum guarantee, and for the mean-variance efficient portfolios (insert).

4.2. Analysis of the tradeoffs

We now turn to the analysis of the tradeoffs between the guaranteed rate of return offered to policyholders and the net CEexROE on shareholders' equity. This is shown in Figure 6, where the optimal asset allocation among the broad classes of bonds and stocks is also shown for the different guaranteed returns.

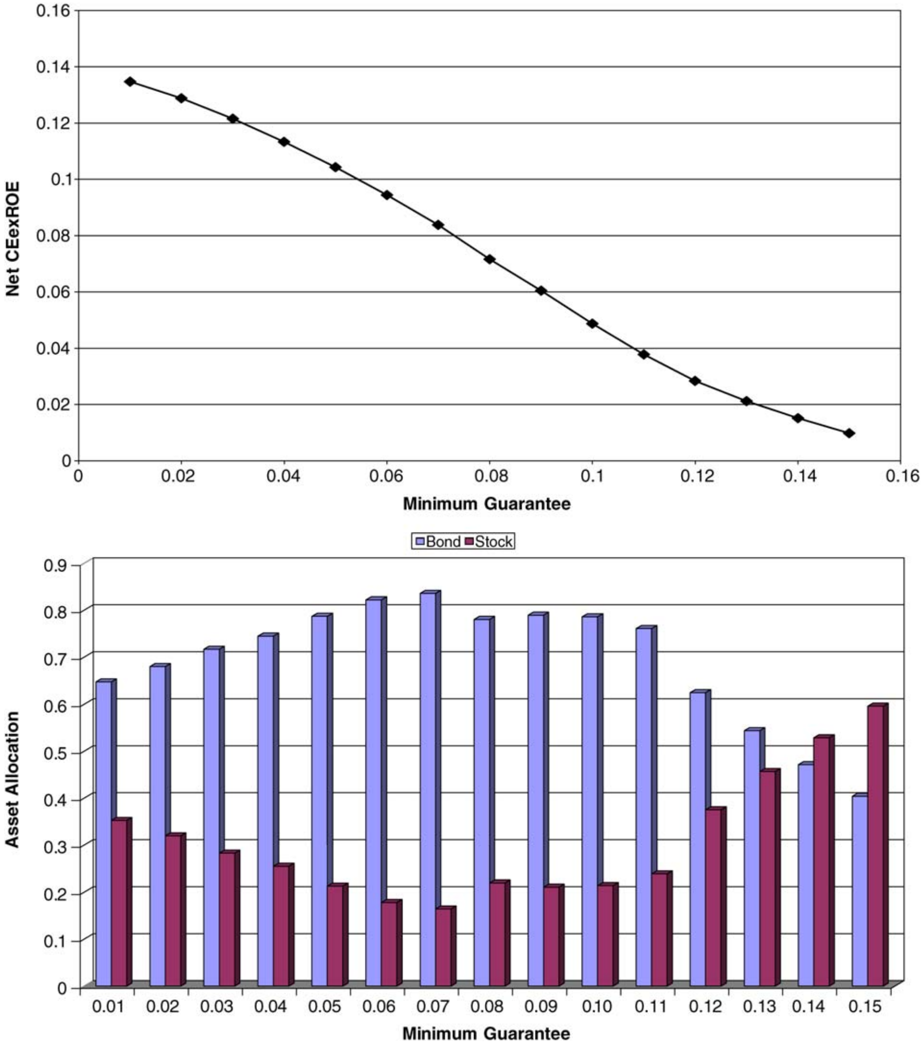

Fig. 6. Net CEexROE (annualized) for different levels of the guarantee (top) and the corresponding broad asset allocations (bottom).

At first glance the portfolio structures appear puzzling. One expects that as the guarantee increases the amount of stock holdings should grow. However, we observe that for low guarantees (less than 7% for the market sectors we consider) the holdings in stock increase with lower guarantees. For low g the embedded option is far out of the money, even when the asset portfolio is mostly equity and very volatile. The asset allocation strategy maximizes CEexROE by taking higher risks in the equities market. A marginal increase of the shortfall cost allows higher CEexROE. This is further clarified in Figure 5, showing the tradeoff between cost of the guarantee and net annualized CEexROE. At values of g less than 7% the option embedded in the liability is out-of-the-money and any excess return is passed on to the shareholders thus improving CEexROE. As the guarantee increases above 7%, the option goes deeper into the money, the cost of the guarantee increases significantly and CEexROE erodes. Note from Figure 6 that higher values of the guarantee must be backed by aggressive portfolios with high equity content, but in this case the portfolio volatility is not translated into high CEexROE for the shareholders but into higher returns for the policyholders. This is consistent with the conclusion of Siglienti (2000) that excessive investments in equity destroy shareholder's value. However, for the guaranteed rates of 3 to 4% offered by Italian insurers it appears that the optimal portfolios consist of 20 to 25% in equities, as opposed to 15% that was obtained by Siglienti using simulations. This discrepancy could be, in part, due to the data of scenario returns used in his study and ours. However, it may also be due to the fact that with the scenario model developed here the portfolio composition is optimized.

4.3. Analysis of alternative debt structures

So far we have assumed that the cost of the guarantee is covered by shareholders. It is possible, however, that such costs are charged to policyholders or funded by issuing debt. (Note that for mutual insurance firms the policyholders are the shareholders so the point of who pays for the cost is mute. However, the issue of raising debt remains.) In either case there are advantages and disadvantages. In particular, if we let the policyholder assume the total cost, we run the risk of not being competitive, loose market share, and experience increased lapse. If we issue debt, we are liable for interest payments at the end of the planning horizon which could reduce our final return. Furthermore, companies face leverage restrictions. It may not be possible to cover all the cost of the guarantee by issuing debt because it will increase the leverage of the company beyond what is allowed by the regulators or accepted by the market.

Another important point in pursuing this question concerns the maturity of the issued debt. We start by considering long-term debt.

4.3.1. Long-term financing of shortfalls

To issue long-term debt we determine the amount of cash that we need to borrow in order to cover, with a certain probability, future expenditures due to shortfalls over all scenarios. If we indicate by α a confidence level we are searching for the α-percentile,

O_G^α, such that the cost of the guarantee O_G^l in scenario l satisfies

$$P\big(O_G^l \geqslant O_G^\alpha \mid l \in \Omega\big) = \alpha. \tag{33}$$

The cost of the guarantee in scenario l is given by Eq. (13) as

$$O_G^l = \frac{E_T^l}{\prod_{t=1}^T (1 + r_{ft}^l)} - \rho L_0. \tag{34}$$

Note that O_G^α need not to be raised through the issue of debt only. It is just the reserves needed to fund shortfalls. Strategic considerations will subdivide O_G^α among policyholder charges, C_G, issue of debt or direct borrowing from money markets, D_G, and/or equity supplement, E_S. Thus, we have

$$O_G^\alpha = C_G + D_G + E_S. \tag{35}$$

Given the debt structure implied in (35) we determine the final income I_T^l, for each scenario $l \in \Omega$, as

$$I_T^l = A_T^l - L_T^l - D_G(1 + r_f + \delta)^T + (C_G - J_S) \prod_{t=1}^T \big(1 + r_{ft}^l\big), \tag{36}$$

where J_S are the fixed costs (in percentage of the initial liability) and δ is a spread over the risk free rate so that $r_f + \delta$ is the borrowing interest rate. Debt structures for which at $I_T^l < 0$ for some scenario $l \in \Omega$ should be discarded as leading the firm into insolvency, even if the probability of such events is very low.

The net Return-on-Equity (ROE) corresponding to a given debt structure in each scenario is given by

$$ROE^l = \frac{I_T^l(1 - \kappa)}{\rho L_0 + E_S}. \tag{37}$$

This is not the *ex ante* excess return on equity optimized with the base model, but the *ex post* realized total return on equity achieved when the structure of debt has also been specified. This measure can be used to analyze the probability of insolvency when the cost of the guarantee is funded by shareholders instead of being charged, at least in part, to policyholders.

We report some results with the analysis described here. Tables are generated to study the tradeoffs between leverage, policyholder charges, and shareholder returns. Table 3 summarizes data that assist the decision maker to take a position according to her strategic views and constraints. If no entries are displayed these choices cannot be implemented, either because some I_T^l are negative (this occurs when charges to policyholders are very low and high debt levels yield a negative final income), or because the amount of money necessary to cover shortfalls is fully covered by the policyholder charges. This implies a negative debt level at maturity of the product.

For example, by choosing a leverage level equal to 0.5, the highest yearly net CEROE is 0.183. Note that, if the firm wishes to achieve higher performance level, the leverage

Table 3
Net CEROE for different combinations of leverage and policyholder charges. The table is built for a guarantee $g = 4\%$ at a confidence level $\alpha = 1\%$

Leverage levels	Policyholder charges									
	0	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0	0.121125	0.124595	0.128295	0.132256	0.136515	0.141118	0.146123	0.151602	0.15765	0.164391
0.125	0.123946	0.127684	0.131656	0.135891	0.14043	0.145317	0.150612	0.156387	0.16274	0.169795
0.25	0.126654	0.13064	0.13486	0.139346	0.144137	0.14928	0.154834	0.160873	0.167495	0.174827
0.375	0.12926	0.133474	0.137923	0.142638	0.147659	0.153033	0.158821	0.165097	0.17196	0.179538
0.5	0.13177	0.136197	0.140857	0.145783	0.151014	0.156599	0.162599	0.169089	0.176169	0.183968
0.625	0.134193	0.138817	0.143673	0.148794	0.154219	0.159997	0.16619	0.172875	0.180151	0.188151
0.75	0.136533	0.141343	0.146381	0.151682	0.157285	0.163242	0.169612	0.176475	0.183932	0.192114
0.875	0.138798	0.143781	0.148989	0.154458	0.160227	0.166348	0.172882	0.179909	0.18753	0.195879
1	0.140991	0.146137	0.151505	0.15713	0.163053	0.169327	0.176013	0.183191	0.190964	0.199468
1.125	0.143118	0.148417	0.153935	0.159706	0.165774	0.172189	0.179016	0.186335	0.19425	0.202896
1.25	0.145182	0.150626	0.156285	0.162194	0.168396	0.174944	0.181903	0.189353	0.197399	0.206177
1.375	0.147188	0.152769	0.15856	0.164599	0.170928	0.177601	0.184682	0.192255	0.200423	
1.5	0.149138	0.154849	0.160766	0.166927	0.173375	0.180165	0.187362	0.19505	0.203333	
1.625	0.151037	0.156871	0.162907	0.169183	0.175744	0.182644	0.18995	0.197745		
1.75	0.152886	0.158837	0.164986	0.171371	0.178039	0.185044	0.192452	0.200349		
1.875	0.154688	0.160751	0.167007	0.173497	0.180265	0.187369	0.194875			
2	0.156446	0.162616	0.168974	0.175562	0.182427	0.189624	0.197222			
2.125		0.164433	0.17089	0.177572	0.184528	0.191814				
2.25		0.166207	0.172757	0.179529	0.186571	0.193942				
2.375		0.167938	0.174577	0.181435	0.188561					
2.5		0.16963	0.176354	0.183294	0.190499					
2.625			0.178089	0.185108						
2.75			0.179785	0.186879						
2.875			0.181443							
3			0.183064							
3.125										
3.25										
3.375										

Table 4
The relation between net CEROE, policyholder charges and guarantee

Policyholder charges	Minimum guarantee							
	0.01	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**
0	0.144564	0.139163	0.135832	0.13177	0.130433	0.120909	0.110348	0.099402
0.01	0.148057	0.142648	0.139726	0.136197	0.136011	0.126397	0.115193	0.102442
0.02	0.151703	0.146281	0.143803	0.140857	0.141965	0.132226	0.120278	0.105562
0.03	0.155517	0.150077	0.148086	0.145783	0.148361	0.138457	0.125641	0.10877
0.04	0.15952	0.154056	0.152599	0.151014	0.155289	0.145166	0.131326	0.112075
0.05	0.163732	0.158239	0.157375	0.156599	0.162863	0.152453	0.13739	0.115487
0.06	0.168182	0.162651	0.162452	0.162599	0.171238	0.16045	0.143903	0.119017
0.07	0.1729	0.167323	0.167876	0.169089	0.180626	0.169337	0.150957	0.122676
0.08	0.177925	0.172291	0.173703	0.176169	0.191338	0.17937	0.15867	0.126479
0.09	0.183304	0.177599	0.180007	0.183968		0.190924	0.167202	0.130443
0.1	0.189093	0.183304	0.18688	0.192664			0.176778	0.134586

The table is built with confidence level $\alpha = 1\%$ and leverage (debt-to-equity ratio) equal to 0.5.

should also increase. Also, observe the inverse relation between leverage and policyholder charges. The greater the amount we charge to the policyholder, the lower is the leverage required to achieve a given annualized net CEROE.

The model can generate similar tables to study the many interactions of endowment with guarantee. For example, we could be interested in investigating the effect of different guarantee levels to the policyholder charges and yearly returns. We first estimate, at a given confidence level α, the cost of the guarantee O_G^α, and then apportion this cost to policyholders (C_G in Eq. (35)) and fund the rest through debt or equity surcharge. Depending on C_G we observe a change in the CEROE to shareholders. Table 4 shows this relationship. We observe the same behavior we had seen between $\overline{O}_G$ and net CEexROE in Figure 5. The model chooses more aggressive strategies for low g because it is then possible to achieve higher levels of CEexROE at little cost. Recall that we are working with percentiles and the impact of aggressive strategies is much more evident on the tails. When the guarantee is low at $g = 0.01$ we need higher policyholder charges to reach the highest return, while for $g = 0.05$ lower charges are required.

The results in Table 3 should be examined taking into account a measure of risk associated with the CEROE of every combination of policyholder charges and leverage level. The probability that excess value per share, P_{EVS}^-, will become negative is a measure of risk of the CEROE, and these probabilities corresponding to Table 3 are shown in Table 5. Observe that the upper-left entry has a P_{EVS}^- equal to 0.58. This means that there is a 58% chance that the present value of the final equity is less than the amount invested today by the shareholders, even though the net CEROE is acceptable (12%). This position is risky. The reason why this position is quite risky is due to the fact that we are asking our shareholders to fund the total α-percentile cost of the guarantee. No charges are passed on to policyholders.

Table 5
Relationship between P_{EVS}^{-}—the probability that excess value per share will fall below zero—leverage and policyholder charges

Leverage levels	Policyholder charges									
	0	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0	0.58	0.522	0.462	0.4	0.344	0.278	0.208	0.148	0.096	0.042
0.125	0.534	0.478	0.416	0.366	0.302	0.242	0.172	0.112	0.072	0.02
0.25	0.508	0.444	0.394	0.338	0.274	0.212	0.15	0.1	0.06	0.012
0.375	0.476	0.416	0.368	0.306	0.252	0.188	0.134	0.092	0.042	0.012
0.5	0.444	0.396	0.346	0.284	0.226	0.162	0.118	0.076	0.032	0.006
0.625	0.418	0.374	0.322	0.266	0.212	0.152	0.106	0.068	0.022	0.004
0.75	0.404	0.366	0.304	0.258	0.198	0.144	0.098	0.056	0.016	0.002
0.875	0.4	0.354	0.286	0.234	0.184	0.136	0.092	0.05	0.012	0.002
1	0.378	0.33	0.28	0.224	0.162	0.124	0.088	0.04	0.012	0.002
1.125	0.37	0.318	0.266	0.216	0.156	0.114	0.078	0.036	0.008	0.002
1.25	0.364	0.31	0.264	0.208	0.146	0.108	0.074	0.032	0.008	0.002
1.375	0.356	0.296	0.254	0.2	0.146	0.104	0.07	0.026	0.004	
1.5	0.35	0.286	0.24	0.196	0.142	0.098	0.062	0.026	0.004	
1.625	0.332	0.282	0.234	0.188	0.136	0.096	0.06	0.02		
1.75	0.322	0.276	0.224	0.178	0.132	0.094	0.054	0.016		
1.875	0.316	0.266	0.22	0.162	0.126	0.092	0.052			
2	0.314	0.266	0.214	0.156	0.122	0.086	0.05			
2.125		0.264	0.214	0.15	0.118	0.084				
2.25		0.264	0.208	0.148	0.116	0.08				
2.375		0.26	0.202	0.146	0.112					
2.5		0.244	0.202	0.146	0.11					
2.625			0.198	0.146						
2.75			0.196	0.144						
2.875			0.196							
3			0.19							
3.125										
3.25										
3.375										

The table is built for a guarantee $g = 4\%$ and confidence level $\alpha = 1\%$.

4.3.2. Insolvency risks

So far we analyzed alternative decisions based only on the net CEexROE and market constraints (policyholder charges, leverage, etc.). Our analysis is missing a measure of risk of the ROE. It is not yet clear how alternative guarantees and debt allocations according to Eq. (35) affect the risk of ROE in Eq. (37). One could argue that the risk aversion of the decision maker is embedded in the utility function of the optimization model. This is true, but the utility function was used only to guide decisions on the asset side, and the estimation of net total CEROE from (37) does not incorporate risk aversion when choosing a debt structure. Furthermore, the utility function ensures the solvency of the fund by covering shortfalls with infusion of equity. However, under certain conditions no external sources of equity will be available. The analysis we carry out here compensates for these omissions. It considers the risk of insolvency when structuring the issue of debt, thus incorporating risk aversion in structuring the debt in addition to structuring the asset portfolio.

Define $\overline{R}_I$ as the expected excess return over the risk free rate for this line of business and $\bar{r}_f$ as the expected risk free rate. The rate at which we must discount the final income I_G^l is given by $R_\mu = \bar{r}_f + \overline{R}_I$. For our shareholders I_G^l represents the value of the equity at the end of the planning period and they are willing to stay in this business if the discounted value of this equity is not less than the initial capital invested. The shareholders will keep their shares if the *Excess Value per Share* (*EVS*) is greater than zero with a high probability. Recalling that the initial amount of equity is $\rho L_0 + E_S$ (E_S could be equal to zero) the *EVS* in each scenario is given by

$$EVS^l = \frac{I_G^l\,(1-\kappa)}{(1+R_\mu)^T} - (\rho L_0 + E_S). \tag{38}$$

The risk related to a specific debt allocation is given by the probability that *EVS* is less than zero, i.e., $P_{EVS}^- = P(EVS^l < 0 \mid l \in \Omega)$. This is the probability of insolvency and can be determined by calculating the EVS^l for each $l \in \Omega$, order from the lowest to the highest, and look for the rank of the first EVS^l that is negative, i.e.,

$$P_{EVS}^- = \frac{\text{rank}(EVS^l < 0)}{N}. \tag{39}$$

The *EVS* can be used to determine the amount of policyholder charges required to make P_{EVS}^- equal to a given confidence level. Recall that I_G^l, and consequently EVS^l, is a function of C_G, E_S, and D_G. If we fix E_G then I_G is a function of C_G (D_G is determined from Eq. (35)). Through a linesearch we can determine C_G^* such that

$$P\big[EVS(C_G^*) < 0\big] = \alpha. \tag{40}$$

In our experiments we set $\overline{R}_I = 6\%$ and the probability of insolvency $\alpha = 1\%$. Figure 7 shows the results of the linesearch which solves Eq. (40) for different values of equity supplement E_S. We observe that for guarantees higher than 6% the CEROE increases. How is it possible that higher guarantees can yield higher returns? The puzzle is resolved if we note that the increase in returns is accompanied by a significant increase of policyholder charges. The increases in the policyholder charges fund the guarantee and preserve equity from falling below its present value.

In practice significant increase in policyholder charges would be unacceptable, and would lead to increased lapses. Our analysis can be used as a demarcation criterion between "good" and "bad" levels of the guarantee. For instance, the Italian insurance industry offers products with guarantees in the range 3 to 4%. Our analysis shows that they could consider increasing the guarantee up to 6% without significant increase of

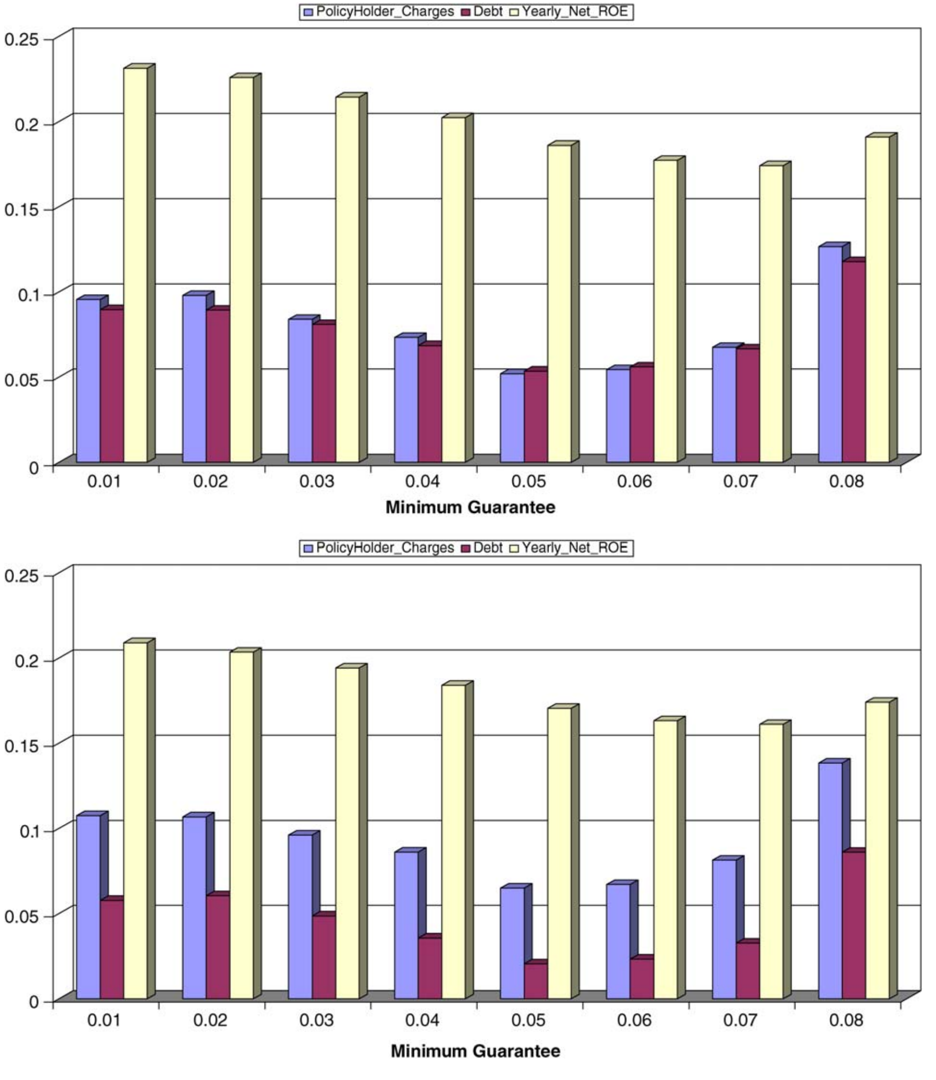

Fig. 7. The levels of policyholder charge, debt and net CEROE such that the probability of insolvency is $P[EVS(C_G^*) < 0] = 1\%$, for equity supplement $E_S = 0$ (top) and $E_S = 0.02$ (bottom).

charges to policyholders or reduction of CEROE. (One may justify the difference from the operating guarantee of 4% to the peak optimized value of 6% as the cost of running the business. If so this cost is high.) For guarantees above 6% we note a substantial increase to policyholder charges at a marginal improvement in CEROE, and this is clearly unacceptable to both policyholders and shareholders.

4.3.3. Short term financing of shortfalls

To this point the analysis has determined the cost of the shortfalls O_G^α and funded it through a combination of debt D_G, charges to policyholders C_G, and equity E_S. Now, let us fix policyholder charges and equity and let the debt fluctuate according to the shortfall O_G^l realized in each scenario. Thus we consider funding part of the shortfall through short-term financing. Instead of issuing a bond for a notional equal to D_G and maturity T, we will borrow money when a shortfall occurs. The debt for each scenario is given by

$$D_G^l = O_G^l - C_G - E_S. \tag{41}$$

We assume that it is possible to borrow money at a spread δ over the risk-free rate. The definition of the final income becomes

$$I_T^l = A_T^l - L_T^l - D_G^l \prod_{t=1}^{T} \left(1 + r_{ft}^l + \delta\right) + (C_G - J_S) \prod_{t=1}^{T} \left(1 + r_{ft}^l\right). \tag{42}$$

We can apply the analysis of the previous section to determine policyholder charges C_G, and estimate the distribution of D_G^l. We solve Eq. (40) and display in Figure 8 the C_G^* for different levels of the guarantee and for $\delta = 2\%$. Note that policyholder charges C_G^* are substantially lower than those obtained by solving (40) in the previous section as reported in Figure 7. This is expected as short-term financing of the cost in a dynamic strategy, as opposed to the fixed strategy of issuing long-term debt. These findings are

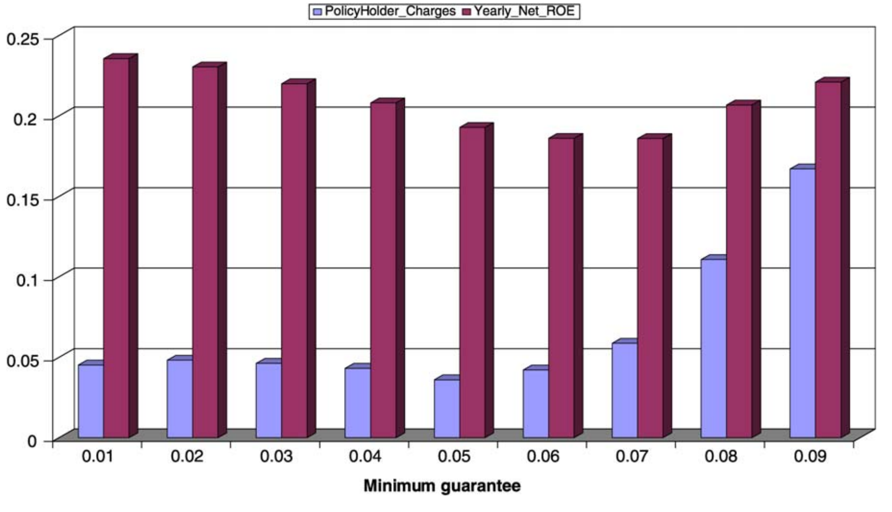

Fig. 8. The levels of policyholder charges and net CEROE for different guarantee such that $P[EVS(C_G^*) < 0] = 1\%$.

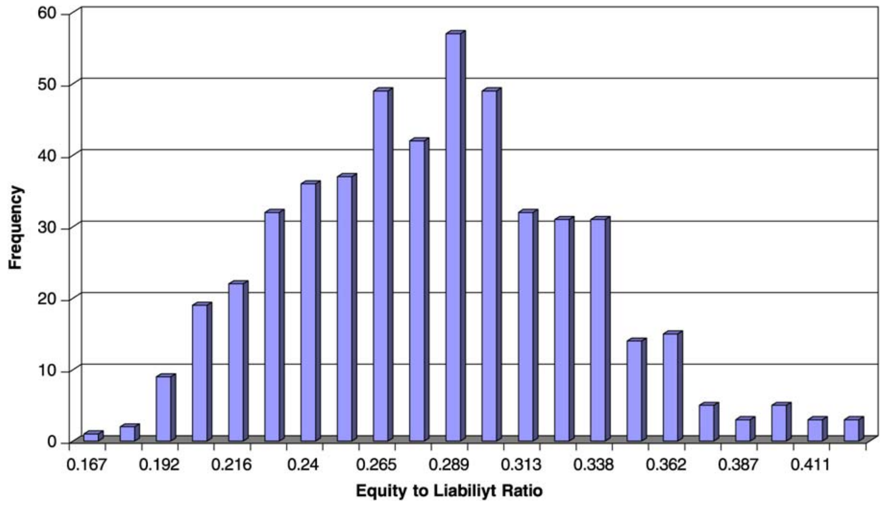

Fig. 9. Distribution of equity-to-liability ratio at the end of the planning horizon for a guarantee of 5%.

consistent with the comparison of the two reserving methods in Boyle–Hardy. Since D_G^l is scenario-dependent, it compensates for those scenarios with high shortfalls, while it is low (or null) for those scenarios with low shortfalls.

4.4. The view from the regulator's desk

We show in Figure 9 the distribution of the equity to liability ratio (cf. Eq. (10)) for a guarantee of 5%. Similar figures were obtained for guarantees ranging from 1 to 10%. This figure shows that for different values of the guarantee the minimum ratio of equity to liability is greater than the regulatory requirement. For the type of policies analyzed here using a logarithmic utility function, and for the scenarios sampled from the past ten bullish years, the regulatory constraint is satisfied without explicitly including it in the model.

4.5. Additional model features

We study now some additional features of the model, namely the effects of the choice of a utility function, the effects of international diversification and investments in corporate bonds, and the effects of the policy surrender option.

4.5.1. Choice of utility function

The decision maker's risk aversion specifies unique asset portfolio to back each guaranteed policy. Clearly increased risk aversion will lead to more conservative portfolios

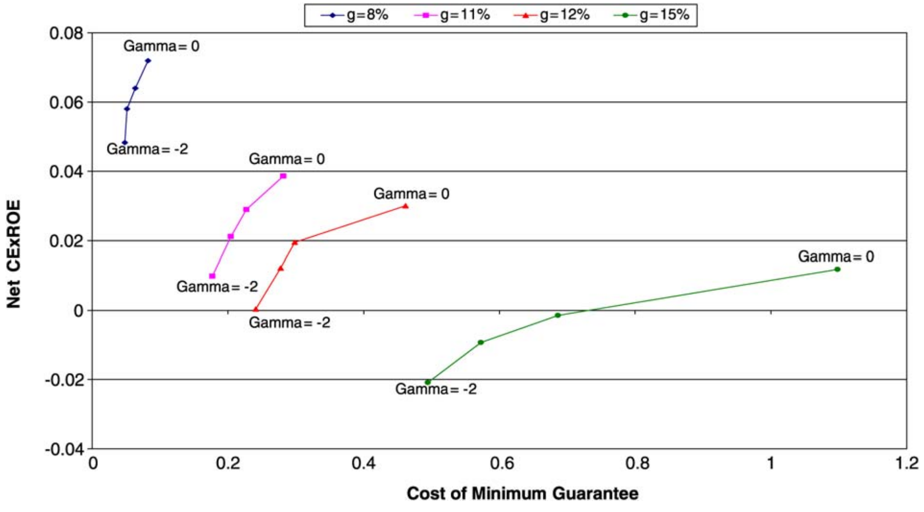

Fig. 10. Tradeoff of CEexROE against cost of the guarantee with varying risk aversion for target guarantees 8 (left), 11, 12 and 15% (right).

with higher contents of fixed income. The result will be a simultaneous reduction in both the CEexROE to shareholders and the cost of shortfalls required to fund the policy. Figure 10 illustrates the tradeoff as the risk aversion parameter γ varies from 0 (base case) to -2 (increased risk aversion) for five different target guarantees.

For low target guarantees we note that an increased appetite for risk results in higher CEexROE, with only a marginal increase in cost of the guarantee. For higher target guarantees (e.g., 15%) we note a substantial increase in the cost of the guarantee as the embedded option goes deep in the money when we increase the risk tolerance and invest into volatile assets. These results confirm our expectations on model performance and are consistent with the results of Figure 5. The model allows users to generate efficient tradeoffs that are consistent with the contractual obligations and the firm's risk tolerance.

4.5.2. *International diversification and credit risk exposures*

We extended the analysis to incorporate other assets permitted by regulations such as corporate bonds and international sovereign debt. Italian insurers are allowed to invest up to 10% of the value of their portfolio in international assets. We run the base model for a guarantee of 4%, and allowing investments in the Morgan Stanley stock indices for USA, UK and Japan and the J.P. Morgan Government bond indices for the same countries. Figure 11 illustrates the tradeoff of CEexROE against cost of the guarantee for international portfolios and portfolios with credit risky securities. The internationally diversified portfolio achieves CEexROE of 0.14 at a cost of the guarantee of 0.02. By

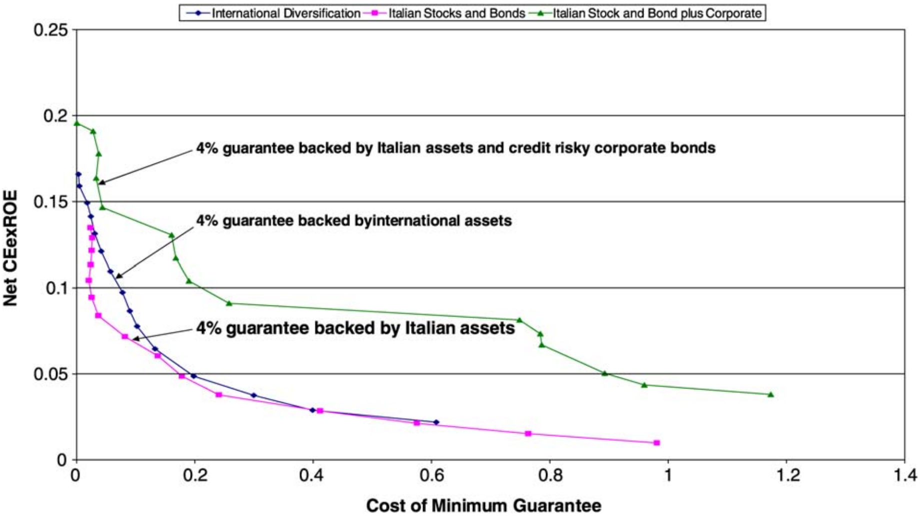

Fig. 11. Tradeoff of CEexROE against cost of the guarantee for internationally diversified portfolios and portfolios with exposure to the corporate bond markets.

contrast, we note that domestic investments in the Italian markets fund the guarantee at the same cost but yield a CEexROE of only 0.11. Similarly, investments in the US Corporate bond market improve further the CEexROE to 0.16, but at an increase of the cost to 0.033.

4.5.3. *Impact of the surrender option*

In our testing so far we took into account only the actuarial risk of the liabilities. Using the various assumptions on lapse behavior discussed in Section 3.5 we study the effect of the surrender option on the cost of the guarantee.

Figure 12 (top) illustrates the effect of lapse on the cost of the guarantee for different levels of the minimum guarantee. It is worth noting that the difference between no lapse at all and fixed lapse is significant for high levels of the minimum guarantee ($g \geqslant 7\%$). This difference is less evident when lapse is modeled as in Eq. (20).

Figure 12 (bottom) illustrates effects of lapse on the net CEexROE. Again, differences are more evident when we switch from no lapse to fixed lapse. It is worth commenting on the effect of lapse rates on the cost of the guarantee and the net CEexROE, over a range of minimum guarantees. Differences in the net CEexROE are observed for low minimum guarantees, say $g \leqslant 6\%$. On the contrary, the alternative lapse assumptions yield substantially different costs for the guarantee for $g > 6\%$. This effect can be explained in view of the option embedded in the policy (see Grosen and Jørgensen, 2000). For low levels of the minimum guarantee the option is almost always out of the money and fixed lapse will depress the net CEexROE through a constant surrender

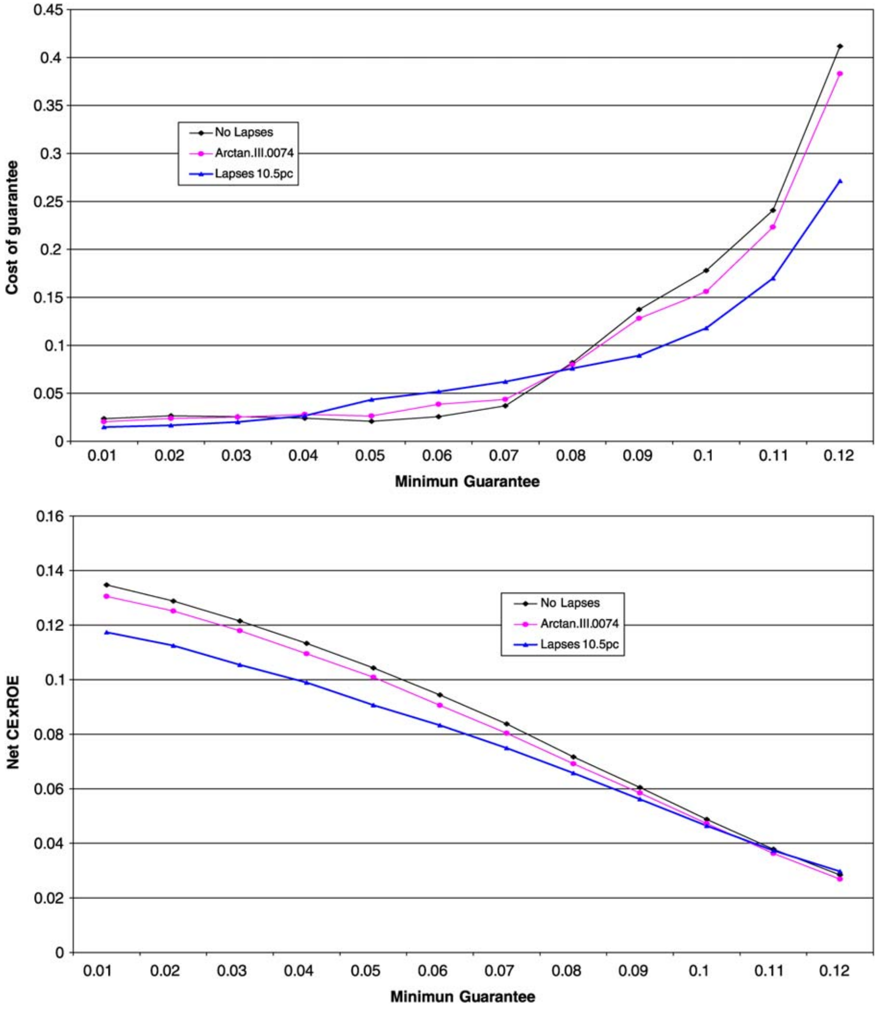

Fig. 12. The effect of lapse on cost of the minimum guarantee (top). The effect of lapse on net CEexROE (bottom).

of policies (see Eq. (6)). For larger values of the minimum guarantee, the insurance company will benefit from lapses since shortfalls are more likely and any lapsed policies relieve the company, in part, from shortfall.

4.6. Benchmarks of Italian insurance policies

In order to assess the effectiveness of the model in practice we compare the optimal portfolios with industry benchmarks. We take as benchmark a set of portfolios with a

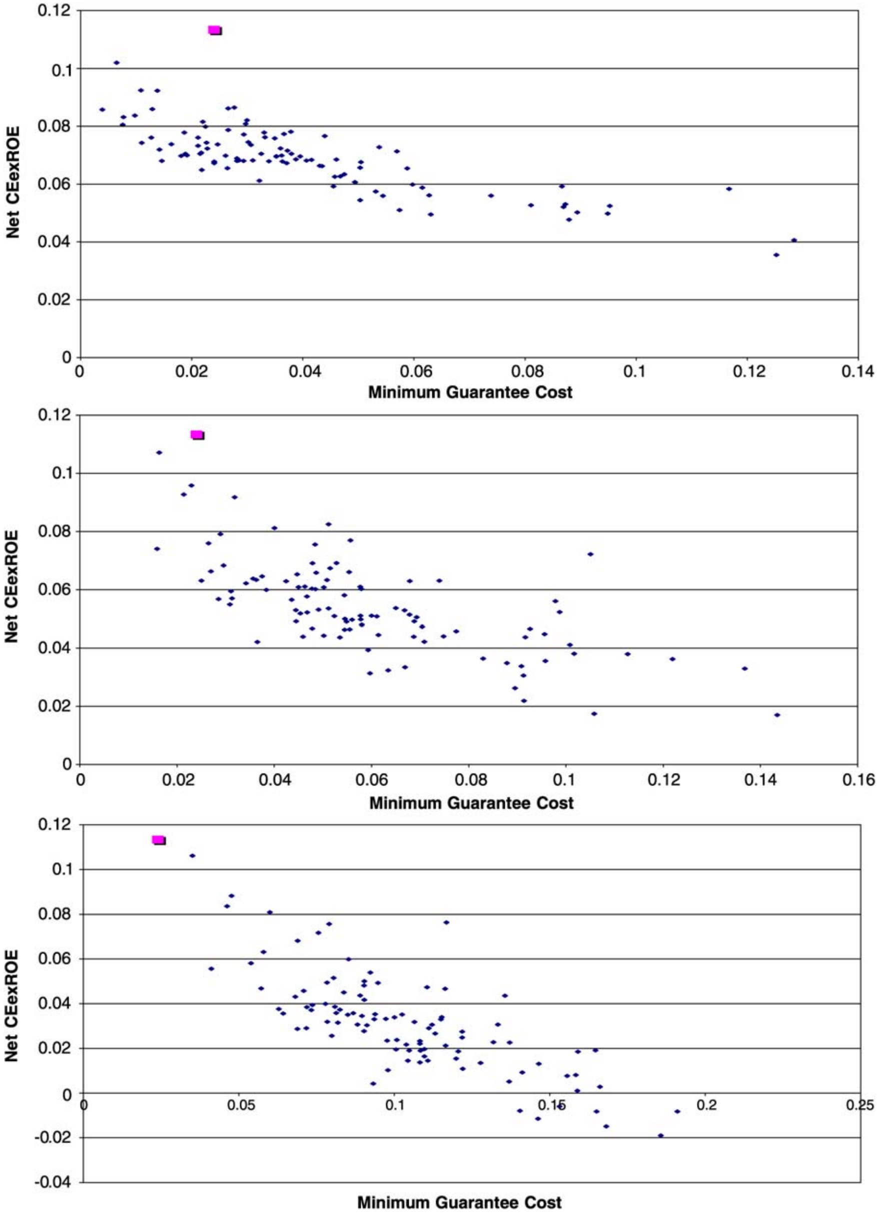

Fig. 13. Performance of benchmark portfolios (diamonds) against the optimized portfolio (square) for $g = 4\%$. Asset allocation for the benchmark portfolios is set to 90/10 (bonds/stocks), 80/20, and 70/30, respectively, from top to bottom.

fixed broad asset allocation between bonds and stocks, and random allocation among specific assets. In order to be consistent with the usual fixed-mix strategies followed by the Italian industry, we set the broad asset allocation between bonds and stocks to 90/10, 80/20, and 70/30. The results of this experiment are reported in Figure 13. Note that the optimized portfolios always dominate the benchmark portfolios in the space of cost-of-guarantee vs. CEexROE. These results justify further the integrative approach taken in this paper, whereby the insurance policy is analyzed *jointly* with the asset allocation decision instead of being analyzed for an *a priori* fixed asset portfolio. Further improvements are possible with an internationally diversified portfolio and with some exposure to credit risky securities, as analyzed in Section 4.5.2.

The results of this section are in general agreement with the current practices of Italian insurers. However, the optimized results suggest that improved policies and associated asset strategies are still possible. In particular, the findings show that the Italian insurers could increase the equity exposure of their portfolio from 20%, which is the current practice, up to 25% to 30%—see the optimal asset allocation corresponding to minimum guaranteed return $g = 3\%$ in Figure 5. This is also evident from Figure 13 where we observe that some random portfolios from the 70/30 asset allocation are closer to the optimized portfolios.

5. Conclusions

This chapter has, first-most, demonstrated that an integrative approach to the management of assets and liabilities for insurance products with guarantees and bonuses adds value. Asset structures generated with an integrative approach for specific insurance policies are efficient, as opposed to asset strategies developed in a non-integrated model.

Several interesting conclusions can be drawn from the use of the model on data from the Italian insurance industry. First, we have quantified the tradeoffs between the different targets of the insurance firm: providing the best products for its policyholders, providing the highest excess return to its shareholders, satisfying the guarantee at the lowest possible cost and with high probability. Some interesting insights are obtained on the structure of the optimal portfolios. In particular, we observe that too little equity in the portfolio and the insurer cannot meet the guarantee, while too much equity destroys shareholder value.

Second, we have analyzed different debt structures whereby the cost of the guarantee is funded through equity or through debt with either long or short maturities. The effects of these choices on the cost of the guarantee and on the probability of insolvency can be quantified, thus providing guidance to management for the selection of policies.

Third, we have seen from the empirical analysis that Italian insurers operate at levels which are close to optimal but not quite so. There is room for improvement either by offering more competitive products or by generating higher excess returns for the benefit of the shareholders and/or the policyholders.

As a caveat we add that the increase in the equity exposure suggested from the use of this model should come with an increased sophistication in the technology used to manage these assets vis-à-vis the liabilities. In particular, the asset portfolios must be carefully fine tuned with models such as the one presented here. Further analysis is needed in developing the scenarios of asset returns to be in agreement with future expectations, and to rely less on historical performance.

A significant extension for the long time horizons of the products considered would be to a multi-stage model where decisions are revised at time instances after $t = 0$ until maturity. Such dynamic stochastic programs with recourse have been developed for asset and liability management by the references given in the introduction. However, for the highly nonlinear problem we are addressing here such models are difficult to develop. The linearization of the single-stage model developed in Appendix A does not apply directly to multistage formulations. Specialized algorithms for geometric programming must be employed for the solution of multistage extensions of this model.

Acknowledgements

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Appendix A. Solving the nonlinear dynamic equations

In this section we show how to solve the nonlinear equations (5)–(9) in order to obtain the objective function (12). At time $t = 0$, the liability is the pure premium L_0. At $t = 1$ (to simplify the notation we drop the scenario superscript) we have

$$L_1 = L_0(1 - \Lambda_1)\bigl(1 + g + y_1^+\bigr). \tag{A.1}$$

At $t = 2$ we use the value of L_1 from (A.1) to obtain

$$\begin{aligned} L_2 &= L_1(1 - \Lambda_2)\bigl(1 + g + y_2^+\bigr) \\ &= L_0(1 - \Lambda_2)(1 - \Lambda_1)\bigl(1 + g + y_1^+\bigr)\bigl(1 + g + y_2^+\bigr). \end{aligned} \tag{A.2}$$

Applying this process recursively for each t we obtain the final liability as

$$L_T = L_0 \prod_{t=1}^{T} (1 - \Lambda_t)\bigl(1 + g + y_t^+\bigr). \tag{A.3}$$

For the equity dynamics we have that $E_0 = \rho L_0$. At $t = 1$

$$E_1 = \rho L_0(1 + r_{f1}) + y_1^- L_0. \tag{A.4}$$

At $t = 2$ and substituting for E_1 and L_1 from (A.4) and (A.1) we obtain

$$\begin{aligned} E_2 &= E_1(1 + r_{f2}) + y_2^- L_1 \\ &= \rho L_0(1 + r_{f1})(1 + r_{f2}) + L_0 y_1^-(1 + r_{f2}) + L_0 y_2^-(1 - \Lambda_1)\bigl(1 + g + y_1^+\bigr). \end{aligned} \tag{A.5}$$

At $t = 3$ we have

$$\begin{aligned} E_3 &= E_2(1 + r_{f3}) + y_3^- L_2 \\ &= \rho L_0(1 + r_{f1})(1 + r_{f2})(1 + r_{f3}) + L_0 y_3^-(1 + r_{f2})(1 + r_{f3}) \\ &\quad + L_0 y_3^-(1 + r_{f3})(1 - \Lambda_1)\bigl(1 + g + y_1^-\bigr) \\ &\quad + L_0 y_3^-(1 - \Lambda_2)(1 - \Lambda_1)\bigl(1 + g + y_1^+\bigr)\bigl(1 + g + y_2^+\bigr). \end{aligned} \tag{A.6}$$

Applying this process recursively for each t we obtain after some simple algebra

$$E_T = L_0\left[\rho \prod_{t=1}^{T}(1 + r_{ft}) + \sum_{t=1}^{T}\left(y_t^- \phi(t, T) \prod_{\tau=1}^{t-1}(1 - \Lambda_\tau)\bigl(1 + g + y_\tau^+\bigr)\right)\right], \tag{A.7}$$

where $\phi(t, T) = \prod_{\tau=t+1}^{T}(1 + r_{f\tau})$ is the cumulative return of the short rate from t to T.

With the same arguments it is possible to show that

$$y_{At} = L_0 \Lambda_t\bigl(1 + g + y_t^+\bigr) \prod_{\tau=1}^{t-1}(1 - \Lambda_\tau)\bigl(1 + g + y_\tau^+\bigr). \tag{A.8}$$

For the asset dynamics we have that $A_0 = L_0(1 + \rho)$. At $t = 1$

$$\begin{aligned} A_1 &= A_0(1 + R_{P1}) + y_1^- L_0 - y_{A1} \\ &= L_0(1 + \rho)(1 + R_{P1}) + y_1^- L_0 - y_{A1}. \end{aligned} \tag{A.9}$$

At $t = 2$ substituting L_1 from (A.1) we obtain

$$\begin{aligned} A_2 &= A_1(1 + R_{P2}) + y_2^- L_1 - y_{A2} \\ &= L_0(1 + \rho)(1 + R_{P1})(1 + R_{P2}) + y_1^- L_0(1 + R_{P2}) \\ &\quad - y_{A1}(1 + R_{P2}) + y_2^- L_1 - y_{A2}. \end{aligned} \tag{A.10}$$

The value of the assets at maturity is given by

$$\begin{aligned} A_T &= L_0(1 + \rho) \prod_{t=1}^{T}(1 + R_{Pt}) L_0 \sum_{t=1}^{T} y_t^- \prod_{\tau=t+1}^{T}(1 + R_{P\tau}) \prod_{\tau=1}^{t-1}\bigl(1 + g + y_\tau^+\bigr) \\ &\quad \times \prod_{\tau=1}^{t-1}(1 - \Lambda_\tau) - \sum_{t=1}^{T} y_{At} \prod_{\tau=t+1}^{T}(1 + R_{P\tau}). \end{aligned} \tag{A.11}$$

By substituting y_{At} with the expression in (A.8), we obtain

$$\begin{aligned} A_T &= L_0(1+\rho)\prod_{t=1}^{T}(1+R_{Pt}) \\ &\quad + L_0\sum_{t=1}^{T} y_t^- \prod_{\tau=t+1}^{T}(1+R_{P\tau})\prod_{\tau=1}^{t-1}\left(1+g+y_\tau^+\right)(1-\Lambda_\tau) \\ &\quad - L_0\sum_{t=1}^{T}\Lambda_t\left(1+g+y_t^+\right)\prod_{\tau=t+1}^{T}(1+R_{P\tau})\prod_{\tau=1}^{t-1}\left(1+g+y_\tau^+\right)(1-\Lambda_\tau). \end{aligned} \tag{A.12}$$

Collecting terms we obtain

$$\begin{aligned} A_T &= L_0(1+\rho)\prod_{t=1}^{T}(1+R_{Pt}) \\ &\quad + L_0\sum_{t=1}^{T}\left(y_t^- - \Lambda_t(1+g+y_t^+)\right)\prod_{\tau=t+1}^{T}(1+R_{P\tau}) \\ &\quad \times \prod_{\tau=1}^{t-1}\left(1+g+y_\tau^+\right)(1-\Lambda_\tau). \end{aligned} \tag{A.13}$$

Appendix B. Asset classes

The asset classes used in testing the base model are given in Table 6. They consist of bond indices for short-, medium-, and long-term debt of the Italian government, and stock indices of the major industrial sectors traded in the Milano stock exchange.

Table 6
Asset classes used in testing the base model

Code	Description
SBGVNIT.1-3	Salomon Brother Italian Government Bond 1–3 years
SBGVNIT.3-7	Salomon Brother Italian Government Bond 3–7 years
SBGVNIT.7-10	Salomon Brother Italian Government Bond 7–10 years
ITMSBNK	Milan Mib Historic Banks
ITMSAUT	Milan Mib Historic Cars
ITMSCEM	Milan Mib Historic Chemicals
ITMSCST	Milan Mib Historic Construction
ITMSDST	Milan Mib Historic Distribution
ITMSELT	Milan Mib Historic Electronics

(*continued on next page*)

Table 6
(*continued*)

Code	Description
ITMSFIN	Milan Mib Historic Finance
ITMSFPA	Milan Mib Historic Finance Holdings
ITMSFMS	Milan Mib Historic Finance Misc
ITMSFNS	Milan Mib Historic Finance Services
ITMSFOD	Milan Mib Historic Food
ITMSIND	Milan Mib Historic Industrials
ITMSINM	Milan Mib Historic Industrials Misc
ITMSINS	Milan Mib Historic Insurance
ITMSPUB	Milan Mib Historic Media
ITMSMAM	Milan Mib Historic MineralsMetals
ITMSPAP	Milan Mib Historic Paper
ITMSMAC	Milan Mib Historic Plants & Machine
ITMSPSU	Milan Mib Historic Pub. Util. Serv.
ITMSRES	Milan Mib Historic Real Estate
ITMSSER	Milan Mib Historic Services
ITMSTEX	Milan Mib Historic TextileClothing
ITMST&T	Milan Mib Historic Transportation & Tourism

Uncited references

(Giraldi et al., 2003)

References

Asay, M.R., Bouyoucos, P.J., Marciano, A.M., 1993. An economic approach to valuation of single premium deferred annuities. In: Zenios, S.A. (Ed.), Financial Optimization. Cambridge University Press, Cambridge, UK, pp. 100–135.

Babbel, D.F., 2001. Asset/liability management for insurers in the new era: Focus on value. Journal of Risk Finance 3 (1), 9–17, Fall.

Bacinello, A.R., 2001. Fair pricing of life insurance participating policies with a minimum guarantee. Astin Bulletin 31 (2), 275–297.

Boyle, P.P., Hardy, M.R., 1997. Reserving for maturity guarantees: Two approaches. Insurance: Mathematics & Economics 21 (2), 113–127.

Boyle, P.P., Schwartz, E.S., 1977. Equilibrium prices of guarantees under equity-linked contracts. Journal of Risk and Insurance 44, 639–660.

Brennan, M.J., Schwartz, E.S., 1976. The pricing of equity-linked life insurance policies with an asset value guarantee. Journal of Financial Economics 3, 195–213.

Brennan, M.J., Schwartz, E.S., 1979. Alternative investment strategies for the issuers of equity linked life insurance policies with an asset value guarantee. Journal of Business 52, 63–93.

Carinõ, D.R., Ziemba, W.T., 1998. Formulation of the Russel–Yasuda Kasai financial planning model. Operations Research 46 (4), 433–449.

Censor, Y., Zenios, S.A., 1997. Parallel Optimization: Theory, Algorithms, and Applications. Numerical Mathematics and Scientific Computation. Oxford University Press, New York.

Chadburn, R.G., 1997. The use of capital, bonus policy and investment policy in the control of solvency for with-profits life insurance companies in the UK, Technical report. City University, London.

Consigli, G., Dempster, M.A.H., 1998. The CALM stochastic programming model for dynamic asset and liability management. In: Ziemba, W., Mulvey, J.M. (Eds.), Worldwide Asset and Liability Modeling. Cambridge University Press, Cambridge, UK, pp. 464–500.

Consiglio, A., Cocco, F., Zenios, S.A., 2000. Asset and liability modeling for participating policies with guarantees, Working Paper 00-41-c. The Wharton Financial Institutions Center, University of Pennsylvania.

Consiglio, A., Cocco, F., Zenios, S.A., 2001. The value of integrative risk management for insurance products with minimum guarantees. Journal of Risk Finance 2 (3), 1–11.

Consiglio, A., Saunders, D., Zenios, S.A., 2003. Insurance league: Italy vs. UK. Journal of Risk Finance 4 (4), 47–54.

Consiglio, A., Saunders, D., Zenios, S.A., 2006. Asset and liability management for insurance products with minimum guarantees: The UK case. Journal of Banking and Finance 30, 645–667.

Dembo, R., Aziz, A., Rosen, D., Zerbs, M., 2000. Mark To Future—A Framework for Measuring Risk and Reward. Algorithmics Publications, May.

Embrechts, P., 2000. Actuarial versus financial pricing of insurance. Journal of Risk Finance 1 (4), 17–26.

Giraldi, C., Susinno, G., Berti, G., Brunello, J., Buttarazzi, S., Cenciarelli, G., Daroda, C., Stamegna, G., 2003. Insurance optional. In: Lipton, A. (Ed.), Exotic Options: The Cutting-Edge Collection Technical Papers Published in Risk 1999–2003. Risk Books (Chapter 35).

Grosen, A., Jørgensen, P.L., 2000. Fair valuation of life insurance liabilities: The impact of interest rate guarantees, surrender options, and bonus policies. Insurance: Mathematics & Economics 26, 37–57.

Hansen, M., Miltersen, K.R., 2002. Minimum rate of return guarantees: The Danish case. Scandinavian Actuarial Journal 4, 230–318.

Harker, P.T., Zenios, S.A. (Eds.), 2000. Performance of Financial Institutions: Efficiency, Innovation, Regulations. Cambridge University Press, Cambridge, England.

Holmer, M.R., Zenios, S.A., 1995. The productivity of financial intermediation and the technology of financial product management. Operations Research 43 (6), 970–982.

Høyland, K., 1998. Asset liability management for a life insurance company: A stochastic programming approach, PhD thesis. Norwegian University of Science and Technology, Trondheim, Norway.

Kat, H.M., 2001. Structured Equity Derivatives: The Definitive Guide to Exotic Options and Structured Notes. John Wiley & Son.

Miltersen, K.R., Persson, S., 1999. Pricing rate of return guarantees in a Heath–Jarrow–Morton framework. Insurance: Mathematics & Economics 25, 307–325.

Mulvey, J.M., Thorlacius, A.E., 1998. The Towers Perrin global capital market scenario generation system. In: Ziemba, W., Mulvey, J.M. (Eds.), Worldwide Asset and Liability Modeling. Cambridge University Press, Cambridge, UK, pp. 286–312.

Ross, M.D., 1989. Modelling a with-profit life office. Journal of the Institute of Actuaries 116, 691–716.

Siglienti, S., 2000. Consequences of the reduction of interest rates on insurance. The Geneva Papers on Risk and Insurance 25 (1), 63–77.

Wilkie, A.D., 1995. More on a stochastic asset model for actuarial use. British Actuarial Journal 1 (5), 777–964.