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DEFINING RELATIONS OF MINIMAL DEGREE OF THE TRACE ALGEBRA OF 3×3 MATRICES

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Abstract

The trace algebra C_{nd} over a field of characteristic 0 is generated by all traces of products of d generic $n \times n$ matrices, $n, d \geq 2$. Starting with the generating set of C_{3d} given by Abeasis and Pittaluga in 1989, we have shown that the minimal degree of the set of defining relations of C_{3d} is equal to 7 for any $d \geq 3$. We have determined all relations of minimal degree. For $d = 3$ we have also found the defining relations of degree 8.

Key words and phrases: generic matrices, matrix invariants, trace algebras, defining relations

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Introduction. Let K be any field of characteristic 0 and let $X_i = \left(x_{pq}^{(i)}\right)$, $p, q = 1, \ldots, n$, $i = 1, \ldots, d$, be d generic $n \times n$ matrices. The pure (or commutative) trace algebra C_{nd} is generated by all traces of products $\text{tr}(X_{i_1} \cdots X_{i_k})$ and coincides with the algebra of invariants of the general linear group $GL_n = GL_n(K)$ acting by simultaneous conjugation on d matrices of size $n \times n$. The algebra C_{nd} is finitely generated by traces $\text{tr}(X_{i_1} \cdots X_{i_k})$ of degree $k \leq N$. The exact value of $N = N(n)$ is known for $n \leq 4$ only. Namely, $N(2) = 3$, $N(3) = 6$, and $N(4) = 10$. A description of the defining relations of C_{nd} is given by the Razmyslov-Procesi theory [14,15] in the language of ideals of the group algebras of symmetric groups. For a background on algebras of matrix invariants see, e.g. [9,11] and for computation aspects of the theory see [8].

Explicit minimal sets of generators of C_{nd} and the defining relations between them are found in few cases only. The picture is completely clear for $n = 2$ and any d, see e.g. [9] for details. The only other case when the defining relations of C_{nd} are explicitly given is $n = 3$, $d = 2$, see the comments below. For $n = 3$, $d \geq 3$ and $n \geq 4$ and $d \geq 2$,

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nothing is known about the concrete form of the defining relations with respect to fixed minimal systems of generators.

Teranishi [16] found a system of 11 generators of C_{32} and showed that C_{32} can be defined by a single relation of degree 12. The explicit form of the relation was found by Nakamoto [13], over $\mathbb{Z}$, with respect to a slightly different system of generators. Abeasis and Pittaluga [1] found a system of generators of C_{3d}, for any $d \geq 2$, in terms of representation theory of the symmetric and general linear groups, in the spirit of its usage in theory of PI-algebras. Aslaksen, Drensky and Sadikova [2] gave the defining relation of C_{32} with respect to the set found in [1]. Their relation is much simpler than that in [13]. For C_{42}, a set of generators was found by Teranishi [16, 17] and a minimal set by Drensky and Sadikova [10], in terms of the approach in [1]. Djoković [6] gave another minimal set of 32 generators of C_{42} consisting of traces of products only. He also found a minimal set of 173 generators of C_{52}.

The determination of generators and defining relations is simpler, if one knows the Hilbert (or Poincaré) series of the algebra. Again, the only completely understood case is $n = 2$. The other cases, when the Hilbert series are explicitly given, are $n = 3$, $d = 2$ (Teranishi [16]) and $d = 3$ (Berele and Stembridge [5]), $n = 4$, $d = 2$ (Teranishi [17] (with some typos) and corrected by Berele and Stembridge [5]). Recently Djoković [6] has also calculated the Hilbert series of C_{52} and C_{62}.

The minimal generating set of C_{3d} given in [1] consists of

$$g = g(d) = \frac{1}{240}d(5d^5 + 19d^4 - 5d^3 + 65d^2 + 636)$$

homogeneous trace polynomials $u_1, \ldots, u_g$ of degree ≤ 6. Hence, C_{3d} is isomorphic to the factor algebra $K[y_1, \ldots, y_g]/I$. Defining $\deg(y_i) = \deg(u_i)$, the ideal I is homogeneous. For $d = 3$, the comparison of the Hilbert series of $C_{33} \cong K[y_1, \ldots, y_g]/I$ and $K[y_1, \ldots, y_g]$ surprisingly gives that any homogeneous minimal system of generators of the ideal I contains no elements of degree ≤ 6, three elements of degree 7 and 30 elements of degree 8. The purpose of the present paper is to find the defining relations of minimal degree for C_{3d} and any $d \geq 3$ with respect to the generating set in [1]. It has turned out that the minimal degree of the relations is equal to 7 for all $d \geq 3$ and there are a lot of relations of degree 7. (Compare with the single relation of degree 12 in the case $d = 2$.) The dimension of the vector space of relations of degree 7 is equal to

$$r_7 = r_7(d) = \frac{2}{7!}(d+1)d(d-1)(d-2)(41d^3 - 86d^2 + 114d - 360).$$

For $d = 3$ we have computed also the homogeneous relations of degree 8. The defining relations are given in the language of representation theory of GL_d. There is a simple algorithm which gives the explicit form of all relations of degree 7, and of degree 8 for $d = 3$. The proofs involve basic representation theory of GL_d and develop further ideas of [2, 10] and our recent paper [3] combined with computer calculations with Maple. Our methods are quite general and we believe that they can be successfully used for further investigation of generic trace algebras and other algebras close to them.

The complete proofs of the paper will be published elsewhere. They are posted as the preprint [4] at the preprint server of Cornell University.

1. Preliminaries. We fix $n = 3$ and $d \geq 3$ and denote by $X_1, \ldots, X_d$ the d generic 3×3 matrices. It is a standard trick to replace the generic matrices with generic traceless matrices. We express X_i in the form

$$X_i = \frac{1}{3}\text{tr}(X_i)e + x_i, \quad i = 1, \ldots, d,$$

where e is the identity 3×3 matrix and

$$x_i = \begin{pmatrix} x_{11}^{(i)} & x_{12}^{(i)} & x_{13}^{(i)} \\ x_{21}^{(i)} & x_{22}^{(i)} & x_{23}^{(i)} \\ x_{31}^{(i)} & x_{32}^{(i)} & -(x_{11}^{(i)}+x_{22}^{(i)}) \end{pmatrix} \qquad (1)$$

is a generic traceless matrix. Then

$$C = C_{3d} \cong K[\mathrm{tr}(X_1),\dots,\mathrm{tr}(X_d)] \otimes_K C_0, \qquad (2)$$

where the algebra C_0 is generated by the traces of products $\mathrm{tr}(x_{i_1}\cdots x_{i_k})$, $k\le 6$. Hence, the problem for defining relations of C can be replaced by a similar problem for C_0.

Let $C_0^+ = \omega(C_0)$ be the augmentation ideal of C_0. It consists of all trace polynomials $f(x_1,\dots,x_d)\in C_0$ without constant terms. Any minimal system of generators of C_0 lying in C_0^+ forms a basis of the vector space C_0^+ modulo $(C_0^+)^2$. The algebra C_{3d} is $\mathbb{Z}$-graded assuming that the trace $\mathrm{tr}(X_{i_1}\cdots X_{i_k})$ is of degree k, and this grading is inherited by C_0. It has also a more precise $\mathbb{Z}^d$-multigrading counting the degree in any X_i. The numbers $g_1,g_2,\dots,g_6$ of elements of degree $1,2,\dots,6$, respectively, in any homogeneous minimal system of generators is an invariant of C_{3d}. Any homogeneous minimal system $\{f_1,\dots,f_h\}$ of generators of C_0 consists of $g_2,\dots,g_6$ elements of degree $2,\dots,6$. Hence,

$$C_0 \cong K[z_1,\dots,z_h]/J,$$

with isomorphism defined by $z_j + J \to f_j$, $j=1,\dots,h=g_2+\dots+g_6$. If $u_j(z_1,\dots,z_h)$, $j=1,\dots,r$, is a system of generators of the ideal J, then

$$u_j(f_1,\dots,f_h)=0, \quad j=1,\dots,r,$$

is a system of defining relations of C_0 with respect to the system of generators $\{f_1,\dots,f_h\}$. We denote by r_k the number of elements of degree k in such a system. Clearly, r_k is the dimension of the homogeneous component of degree k of the vector space $J/JK[z_1,\dots,z_h]^+$.

Now we summarize the necessary background on representation theory of GL_d, see [12] for general facts, and [7] for applications in the spirit of the problems considered here. The irreducible polynomial representations of GL_d are indexed by partitions $\lambda=(\lambda_1,\dots,\lambda_d)$, $\lambda_1\ge\dots\ge\lambda_d\ge 0$. We denote by $W(\lambda)=W_d(\lambda)$ the corresponding irreducible GL_d-module. The group GL_d acts in the natural way on the d-dimensional vector space $K\cdot x_1+\dots+K\cdot x_d$ and this action is extended diagonally on the free associative algebra $K\langle x_1,\dots,x_d\rangle$.

The module $W(\lambda)\subset K\langle x_1,\dots,x_d\rangle$ is generated by a unique, up to a multiplicative constant, homogeneous element w_λ of degree λ_j with respect to x_j, called the highest weight vector of $W(\lambda)$. If W_i, $i=1,\dots,m$, are m isomorphic copies of the GL_d-module $W(\lambda)$ and $w_i\in W_i$ are highest weight vectors, then the highest weight vector of any submodule $W(\lambda)$ of the direct sum $W_1\oplus\dots\oplus W_m$ has the form $\xi_1w_1+\dots+\xi_mw_m$ for some $\xi_i\in K$. It is convenient to work with an explicit copy of $W(\lambda)$ in $K\langle x_1,\dots,x_d\rangle$ obtained in the following way. Let

$$s_k(x_1,\dots,x_k)=\sum_{\sigma\in S_k}\mathrm{sign}(\sigma)x_{\sigma(1)}\cdots x_{\sigma(k)}$$

be the standard polynomial of degree k. (Clearly, $s_2(x_1,x_2)=x_1x_2-x_2x_1=[x_1,x_2]$ is the commutator of x_1 and x_2.) If the lengths of the columns of the diagram of λ are, respectively, $k_1,\dots,k_p$, $p=\lambda_1$, then

$$w_\lambda = w_\lambda(x_1,\dots,x_{k_1}) = s_{k_1}(x_1,\dots,x_{k_1})\cdots s_{k_p}(x_1,\dots,x_{k_p}) \qquad (3)$$

is the highest weight vector of a submodule $W(\lambda)\subset K\langle x_1,\dots,x_d\rangle$.

If W is a GL_d-submodule or a factor module of $K\langle x_1,\ldots,x_d\rangle$, then W inherits the $\mathbb{Z}^d$-grading of $K\langle x_1,\ldots,x_d\rangle$. Recall that the Hilbert series of W with respect to its $\mathbb{Z}^d$-multigrading is defined as the formal power series

$$H(W,t_1,\ldots,t_d)=\sum_{k_i\geq 0}\dim(W^{(k_1,\ldots,k_d)})t_1^{k_1}\cdots t_d^{k_d}$$

with coefficients equal to the dimensions of the homogeneous components $W^{(k_1,\ldots,k_d)}$ of degree $(k_1,\ldots,k_d)$. It plays the role of GL_d-character of W: If

$$W\cong\sum_{\lambda}m(\lambda)W(\lambda),$$

then

$$H(W,t_1,\ldots,t_d)=\sum_{\lambda}m(\lambda)S_\lambda(t_1,\ldots,t_d),$$

where $S_\lambda=S_\lambda(t_1,\ldots,t_d)$ is the Schur function associated with λ, and the multiplicities $m(\lambda)$ are determined by $H(W,t_1,\ldots,t_d)$.

The action of GL_d on $K\langle x_1,\ldots,x_d\rangle$ is inherited by the algebras C_{3d} and C_0. Now we discuss the approach of Abeasis and Pittaluga [1] for the special case $n=3$. (Pay attention that the partitions in [1] are given in "Francophone" way, i.e., transposed to ours.) The algebra C_{3d} has a system of generators of degree ≤ 6. Let U_k be the subalgebra of C_{3d} generated by all traces $\mathrm{tr}(X_{i_1}\cdots X_{i_l})$ of degree $l\leq k$. Clearly, U_k is also a GL_d-submodule of C_{3d}. Let $C_{3d}^{(k+1)}$ be the homogeneous component of degree $k+1$ of C_{3d}. Then the intersection $U_k\cap C_{3d}^{(k+1)}$ is a GL_d-module and has a complement G_{k+1} in $C_{3d}^{(k+1)}$, which is the GL_d-module of the "new" generators of degree $k+1$. We may assume that G_{k+1} is a submodule of the GL_d-module spanned by traces of products $\mathrm{tr}(X_{i_1}\cdots X_{i_{k+1}})$ of degree $k+1$. The GL_d-module of the generators of C_{3d} is

$$G=G_1\oplus G_2\oplus\cdots\oplus G_6.$$

Proposition 1. (Abeasis and Pittaluga [1]) The GL_d-module G of the generators of C_{3d} decomposes as

$$G=W(1)\oplus W(2)\oplus W(3)\oplus W(1^3)\oplus W(2^2)\oplus W(2,1^2)$$
$$\oplus W(3,1^2)\oplus W(2^2,1)\oplus W(1^5)\oplus W(3^2)\oplus W(3,1^3).$$

Each module $W(\lambda)\subset G$ is generated by the "canonical" highest weight vector $\mathrm{tr}(w_\lambda(X_1,\ldots,X_d))$, where w_λ is given in (3).

2. The symmetric algebra of the generators. We consider the symmetric algebra $S=K[G_2\oplus\cdots\oplus G_6]$ of the GL_d-module of the generators of the algebra C_0, with the grading and GL_d-module structure induced by those of C_0. The defining relations of the algebra C_0 are in the square of the augmentation ideal $\omega(S)$ of S. Since we are interested in the defining relations of degree 7 for C_0 for any $d\geq 3$ and of degree 8 for $d=3$, we decompose the homogeneous components of degree 7 and 8 of the Hilbert series of the ideal $\omega^2(S)$ into a sum of Schur functions. This gives the decomposition of the homogeneous components of degree 7 and 8 of $\omega^2(S)$ into a sum of irreducible GL_d-modules. Then we list those explicit generators of the irreducible components which participate in the relations.

Proposition 2. The homogeneous components $(\omega^2(S))^{(k)}$ of degree $k\leq 7$ of the square $\omega^2(S)$ of the augmentation ideal of the symmetric algebra of $G_2\oplus\cdots\oplus G_6$ decomposes as

$$(\omega^2(S))^{(4)}=W(4)\oplus W(2^2),$$

$$(\omega^2(S))^{(5)} = W(5) \oplus W(4,1) \oplus W(3,2) \oplus W(3,1^2) \oplus W(2,1^3),$$

$$(\omega^2(S))^{(6)} = 2W(6) \oplus 3W(4,2) \oplus 2W(4,1^2)$$

$$\oplus 2W(3,2,1) \oplus 2W(3,1^3) \oplus 3W(2^3) \oplus W(2^2,1^2) \oplus W(2,1^4),$$

$$(\omega^2(S))^{(7)} = W(7) \oplus W(6,1) \oplus 3W(5,2) \oplus 3W(5,1^2)$$

$$\oplus W(4,3) \oplus 5W(4,2,1) \oplus 3W(4,1^3) \oplus 3W(3^2,1) \oplus 4W(3,2^2)$$

$$\oplus 6W(3,2,1^2) \oplus 2W(3,1^4) \oplus 2W(2^3,1) \oplus 3W(2^2,1^3) \oplus 2W(2,1^5).$$

Proposition 3. The following elements of $S = K[G_2 \oplus \cdots \oplus G_6]$ are highest weight vectors:
For $\lambda = (4,1^3)$:

$$w_1 = (\mathrm{tr}(s_3(x_1,x_2,x_3)(x_1x_4 + x_4x_1)) - \mathrm{tr}(s_3(x_1,x_2,x_4)(x_1x_3 + x_3x_1))$$

$$+\mathrm{tr}(s_3(x_1,x_3,x_4)(x_1x_2 + x_2x_1)) + 3\mathrm{tr}(s_3(x_2,x_3,x_4)x_1^2))\mathrm{tr}(x_1^2)$$

$$+5(-\mathrm{tr}(s_3(x_1,x_2,x_3)x_1^2)\mathrm{tr}(x_1x_4)$$

$$+\mathrm{tr}(s_3(x_1,x_2,x_4)x_1^2)\mathrm{tr}(x_1x_3) - \mathrm{tr}(s_3(x_1,x_3,x_4)x_1^2)\mathrm{tr}(x_1x_2)),$$

$$w_2 = (\mathrm{tr}(s_3(x_1,x_2,x_3)x_4) - \mathrm{tr}(s_3(x_1,x_2,x_4)x_3) + \mathrm{tr}(s_3(x_1,x_3,x_4)x_2)$$

$$+3\mathrm{tr}(s_3(x_2,x_3,x_4)x_1))\mathrm{tr}(x_1^3) + 4(-\mathrm{tr}(s_3(x_1,x_2,x_3)x_1)\mathrm{tr}(x_1^2x_4)$$

$$+\mathrm{tr}(s_3(x_1,x_2,x_4)x_1)\mathrm{tr}(x_1^2x_3) - \mathrm{tr}(s_3(x_1,x_3,x_4)x_1)\mathrm{tr}(x_1^2x_2)),$$

$$w_3 = (\mathrm{tr}(s_3(x_2,x_3,x_4))\mathrm{tr}(x_1^2) - \mathrm{tr}(s_3(x_1,x_3,x_4))\mathrm{tr}(x_1x_2)$$

$$+\mathrm{tr}(s_3(x_1,x_2,x_4))\mathrm{tr}(x_1x_3) - \mathrm{tr}(s_3(x_1,x_2,x_3))\mathrm{tr}(x_1x_4))\mathrm{tr}(x_1^2).$$

For $\lambda = (3,2^2)$:

$$w_1 = \sum_{\sigma \in S_3} \mathrm{sign}(\sigma)\mathrm{tr}(s_3(x_1,x_2,x_3)x_{\sigma(1)}x_{\sigma(2)})\mathrm{tr}(x_1x_{\sigma(3)}),$$

$$w_2 = \mathrm{tr}(s_3(x_1,x_2,x_3)x_1)\mathrm{tr}(s_3(x_1,x_2,x_3)),$$

$$w_3 = \mathrm{tr}([x_1,x_2]^2)\mathrm{tr}(x_1x_3^2) + \mathrm{tr}([x_1,x_3]^2)\mathrm{tr}(x_1x_2^2) + \mathrm{tr}([x_2,x_3]^2)\mathrm{tr}(x_1^3)$$

$$-\mathrm{tr}([x_1,x_2][x_1,x_3])\mathrm{tr}(x_1(x_2x_3 + x_3x_2))$$

$$+2\mathrm{tr}([x_1,x_2][x_2,x_3])\mathrm{tr}(x_1^2x_3) - 2\mathrm{tr}([x_1,x_3][x_2,x_3])\mathrm{tr}(x_1^2x_2).$$

For $\lambda = (3,2,1^2)$:

$$w_1 = (\mathrm{tr}(s_3(x_1,x_2,x_3)(x_2x_4 + x_4x_2)) - \mathrm{tr}(s_3(x_1,x_2,x_4)(x_2x_3 + x_3x_2))$$

$$+4\mathrm{tr}(s_3(x_2,x_3,x_4)(x_1x_2 + x_2x_1)) + 2\mathrm{tr}(s_3(x_1,x_3,x_4)x_2^2))\mathrm{tr}(x_1^2)$$

$$+(-\mathrm{tr}(s_3(x_1,x_2,x_3)(x_1x_4 + x_4x_1)) + \mathrm{tr}(s_3(x_1,x_2,x_4)(x_1x_3 + x_3x_1))$$

$$-6\mathrm{tr}(s_3(x_1,x_3,x_4)(x_1x_2 + x_2x_1)) - 8\mathrm{tr}(s_3(x_2,x_3,x_4)x_1^2))\mathrm{tr}(x_1x_2)$$

$$+5(-\mathrm{tr}(s_3(x_1,x_2,x_3)(x_1x_2 + x_2x_1))\mathrm{tr}(x_1x_4)$$

$$+\mathrm{tr}(s_3(x_1,x_2,x_4)(x_1x_2 + x_2x_1))\mathrm{tr}(x_1x_3))$$

$$+10(\mathrm{tr}(s_3(x_1,x_2,x_3)x_1^2)\mathrm{tr}(x_2x_4)-\mathrm{tr}(s_3(x_1,x_2,x_4)x_1^2)\mathrm{tr}(x_2x_3)$$
$$+\mathrm{tr}(s_3(x_1,x_3,x_4)x_1^2)\mathrm{tr}(x_2^2)),$$
$$w_3=\mathrm{tr}([x_1,x_2]^2)\mathrm{tr}(s_3(x_1,x_3,x_4))-\mathrm{tr}([x_1,x_2][x_1,x_3])\mathrm{tr}(s_3(x_1,x_2,x_4))$$
$$+\mathrm{tr}([x_1,x_2][x_1,x_4])\mathrm{tr}(s_3(x_1,x_2,x_3)),$$
$$w_4=-2\mathrm{tr}(s_3(x_2,x_3,x_4)x_2)\mathrm{tr}(x_1^3)+2(\mathrm{tr}(s_3(x_1,x_3,x_4)x_2)$$
$$+\mathrm{tr}(s_3(x_2,x_3,x_4)x_1))\mathrm{tr}(x_1^2x_2)-2\mathrm{tr}(s_3(x_1,x_2,x_4)x_2)\mathrm{tr}(x_1^2x_3)$$
$$+2\mathrm{tr}(s_3(x_1,x_2,x_3)x_2)\mathrm{tr}(x_1^2x_4)-2\mathrm{tr}(s_3(x_1,x_3,x_4)x_1)\mathrm{tr}(x_1x_2^2)$$
$$+\mathrm{tr}(s_3(x_1,x_2,x_4)x_1)\mathrm{tr}(x_1(x_2x_3+x_3x_2))-\mathrm{tr}(s_3(x_1,x_2,x_3)x_1)\mathrm{tr}(x_1(x_2x_4+x_4x_2)),$$
$$w_6=(\mathrm{tr}(x_1^2)\mathrm{tr}(x_2^2)-\mathrm{tr}(x_1x_2)^2)\mathrm{tr}(s_3(x_1,x_3,x_4))$$
$$+(-\mathrm{tr}(x_1^2)\mathrm{tr}(x_2x_3)+\mathrm{tr}(x_1x_2)\mathrm{tr}(x_1x_3))\mathrm{tr}(s_3(x_1,x_2,x_4))$$
$$+(\mathrm{tr}(x_1^2)\mathrm{tr}(x_2x_4)-\mathrm{tr}(x_1x_2)\mathrm{tr}(x_1x_4))\mathrm{tr}(s_3(x_1,x_2,x_3)).$$

For $\lambda=(2^3,1)$:

$$w_1=\mathrm{tr}(s_3(x_2,x_3,x_4)[x_2,x_3])\mathrm{tr}(x_1^2)$$
$$-(\mathrm{tr}(s_3(x_1,x_3,x_4)[x_2,x_3])+\mathrm{tr}(s_3(x_2,x_3,x_4)[x_1,x_3]))\mathrm{tr}(x_1x_2)$$
$$+(\mathrm{tr}(s_3(x_1,x_2,x_4)[x_2,x_3])+\mathrm{tr}(s_3(x_2,x_3,x_4)[x_1,x_2]))\mathrm{tr}(x_1x_3)$$
$$-\mathrm{tr}(s_3(x_1,x_2,x_3)[x_2,x_3])\mathrm{tr}(x_1x_4)+\mathrm{tr}(s_3(x_1,x_3,x_4)[x_1,x_3])\mathrm{tr}(x_2^2)$$
$$-(\mathrm{tr}(s_3(x_1,x_2,x_4)[x_1,x_3])+\mathrm{tr}(s_3(x_1,x_3,x_4)[x_1,x_2]))\mathrm{tr}(x_2x_3)$$
$$+\mathrm{tr}(s_3(x_1,x_2,x_3)[x_1,x_3])\mathrm{tr}(x_2x_4)+\mathrm{tr}(s_3(x_1,x_2,x_4)[x_1,x_2])\mathrm{tr}(x_3^2)$$
$$-\mathrm{tr}(s_3(x_1,x_2,x_3)[x_1,x_2])\mathrm{tr}(x_3x_4),$$
$$w_2=(-3(\mathrm{tr}(s_3(x_1,x_2,x_3)x_4)+\mathrm{tr}(s_3(x_2,x_3,x_4)x_1))+\mathrm{tr}(s_3(x_1,x_3,x_4)x_2)$$
$$+\mathrm{tr}(s_3(x_2,x_3,x_4)x_1))-\mathrm{tr}(s_3(x_1,x_2,x_4)x_3)$$
$$+\mathrm{tr}(s_3(x_2,x_3,x_4)x_1)))\mathrm{tr}(s_3(x_1,x_2,x_3))$$
$$+4\mathrm{tr}(s_3(x_1,x_2,x_3)x_3)\mathrm{tr}(s_3(x_1,x_2,x_4))$$
$$-4\mathrm{tr}(s_3(x_1,x_2,x_3)x_2)\mathrm{tr}(s_3(x_1,x_3,x_4))$$
$$+4\mathrm{tr}(s_3(x_1,x_2,x_3)x_1)\mathrm{tr}(s_3(x_2,x_3,x_4)).$$

For $\lambda=(2^2,1^3)$:

$$w_2=\sum_{\sigma\in S_5}\sum_{\tau\in S_2}\mathrm{sign}(\sigma\tau)\mathrm{tr}(s_3(x_{\tau(1)},x_{\sigma(1)},x_{\sigma(2)}))(\mathrm{tr}(s_3(x_{\sigma(3)},x_{\sigma(4)},x_{\sigma(5)})x_{\tau(2)})$$
$$+\sum_{\sigma\in S_5}\sum_{\tau\in S_2}\mathrm{sign}(\sigma\tau)\mathrm{tr}(s_3(x_{\tau(1)},x_{\sigma(1)},x_{\sigma(2)}))(\mathrm{tr}(s_3(x_{\tau(2)},x_{\sigma(3)},x_{\sigma(4)})x_{\sigma(5)}).$$

For $\lambda=(2,1^5)$:

$$w_1=\sum_{\sigma\in S_6}\mathrm{sign}(\sigma)\mathrm{tr}(s_5(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)},x_{\sigma(5)}))\mathrm{tr}(x_1x_{\sigma(6)}),$$

$$w_2 = \sum_{\sigma \in S_6} \mathrm{sign}(\sigma)\mathrm{tr}(s_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})x_1))\mathrm{tr}(s_3(x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)}))$$

$$+ \sum_{\sigma \in S_6} \mathrm{sign}(\sigma)\mathrm{tr}(s_3(x_1, x_{\sigma(1)}, x_{\sigma(2)})x_{\sigma(3)}))\mathrm{tr}(s_3(x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)})).$$

The notation in the above proposition follows the complete version [4] of our paper. Some of the partitions λ of 7 and some of the polynomials w_i from [4] are not included here because they do not appear in the statements of the main results.

For $d = 3$, the homogeneous component $(\omega^2(S))^{(8)}$ of degree 8 of the square $\omega^2(S)$ of the augmentation ideal of the symmetric algebra of $G_2 \oplus \cdots \oplus G_6$ decomposes as

$$(\omega^2(S))^{(8)} = 2W_3(8) \oplus W_3(7,1) \oplus 4W_3(6,2) \oplus 3W_3(6,1^2)$$

$$\oplus 2W_3(5,3) \oplus 6W_3(5,2,1) \oplus 4W_3(4^2) \oplus 7W_3(4,3,1) \oplus 9W_3(4,2^2) \oplus 4W_3(3^2,2).$$

We have found the corresponding highest weight vectors. For details we refer to the complete version [4] of our paper. We finish the section with the following observation obtained comparing the Hilbert series of $K[\mathrm{tr}(X_1), \mathrm{tr}(X_2), \mathrm{tr}(X_3)] \otimes_K C_0$ with the Hilbert series of S.

Proposition 4. For $d = 3$, the algebra C_0 has a minimal system of defining relations with the property that the relations of degree 7 and 8 form GL_3-modules isomorphic, respectively, to $W_3(3,2^2)$ and $W_3(4,3,1) \oplus 2W_3(4,2^2) \oplus W_3(3^2,2)$.

3. Main results. Now we present the explicit defining relations of degree 7 of the algebra C_{3d} for any $d \geq 3$ with respect to the generators of Abeasis and Pittaluga [1]. As we already mentioned, by (2) it is sufficient to give the defining relations of the algebra C_0 generated by traces $\mathrm{tr}(x_{i_1} \cdots x_{i_k})$ of products of the traceless matrices x_i. As in the previous section, we denote by S the symmetric algebra of the GL_d-module $G_2 \oplus \cdots \oplus G_6$ of generators of C_0 and call defining relations of C_0 the expressions $f = 0$, where f is an element of the kernel J of the natural homomorphisms $S \to C_0$.

Theorem 5. Let $d \geq 3$. The algebra C_0 does not have any defining relations of degree ≤ 6. The GL_d-module structure of the homogeneous defining relations of degree 7 of C_0, i.e., of the component $J^{(7)}$ in S is

$$J^{(7)} = W_d(4,1^3) \oplus W_d(3,2^2) \oplus W_d(3,2,1^2) \oplus W_d(2^3,1) \oplus W_d(2^2,1^3) \oplus W_d(2,1^5).$$

In the notation of Proposition 3, the defining relations of C_0 which are highest weight vectors are:
For $\lambda = (4,1^3)$:

$$12w_1 - 15w_2 - 20w_3 = 0.$$

For $\lambda = (3,2^2)$:

$$2w_1 - w_2 + 2w_3 = 0.$$

For $\lambda = (3,2,1^2)$:

$$-6w_1 + 10w_3 - 15w_4 + 40w_6 = 0.$$

For $\lambda = (2^3,1)$:

$$12w_1 + w_2 = 0.$$

For $\lambda = (2^2,1^3)$:

$$w_2 = 0.$$

For $\lambda = (2,1^5)$:

$$2w_1 - 5w_2 = 0.$$

The idea of the proof is the same in all cases λ. Fixing a partition λ, we find a basis $\{w_1, \dots, w_m\}$ of the vector space of highest weight vectors in $\omega^2(S)$. Then we form the linear combination

$$w = \xi_1 w_1 + \dots + \xi_m w_m$$

and require that $w(x_1, \dots, x_d) = 0$ for the generic traceless matrices from (1). The entries of $w(x_1, \dots, x_d)$ are polynomials in the entries $x_{pq}^{(i)}$ of x_i with coefficients which depend on the unknowns $\xi_1, \dots, \xi_m$. In this way, we obtain a linear homogeneous system with respect to $\xi_1, \dots, \xi_m$. The solutions of the system give the defining relations.

For $d = 3$, Proposition 4 gives that the GL_d-module structure of the homogeneous component $J^{(8)}$ of degree 8 in S is

$$J^{(8)} = W_3(4, 3, 1) \oplus 2W_3(4, 2^2) \oplus W_3(3, 2^2, 2)$$

and we have also found the explicit defining relations, see [4].

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