Available online at www.sciencedirect.com

SCIENCE DIRECT®

International Journal of Non-Linear Mechanics 40 (2005) 1088–1101

INTERNATIONAL JOURNAL OF NON-LINEAR MECHANICS

www.elsevier.com/locate/nlm

Non-linear systems under parametric white noise input: Digital simulation and response

A. Pirrotta

Dipartimento di Ingegneria Strutturale e Geotecnica, Università degli Studi di Palermo, Via delle Scienze, 90128, Palermo, Italy

Received 9 September 2003; received in revised form 14 January 2004; accepted 4 March 2005

Abstract

Monte Carlo technique is constituted of three steps. Therefore, improving such technique in practice means, improving the procedure used in one of the three following steps: (i) sample paths of the stochastic input process, (ii) calculation of the outputs corresponding to the generated input samples by using methods of classical dynamics and (iii) estimating statistics of the output process from sample outputs related to the previous step. For linear and non-linear systems driven by parametric impulsive inputs such as normal or non-normal white noises, a general integration method requires a considerable reduction of the integration step when the impulse occurs, treating the impulse as a physical one, by means of a window function of finite duration. This makes Monte Carlo simulation very prohibitive from a computational time point of view. While knowing the exact jump value of the response at impulse occurring that is expressed by a numerical series, the aforementioned problem is overcome because there is no need to reduce the integration step saving computational time, reliability being equal as shown by means of a numerical example.

Keywords: Parametric impulse; Normal white noise; Poissonian white noise; α-stable white noise

1. Introduction

Random vibration problems correspond to evaluating the solution of deterministic differential equations driven by random process inputs and constitute an extension of classical dynamic problems to the case in which the randomness of the external or parametric excitation can be modeled either directly or indirectly by Gaussian or non-Gaussian processes. Hereinafter the case in which the input is a normal or non-normal white noise process will be investigated.

A current research trend relates to the development of efficient methods [1–6] that can provide differential equations governing the evolution of the response statistics by using the classical Itô differential rule and its extension for solving systems driven by both external [7–11] and parametric [12–19] non-normal white noise excitations. Accuracy and efficiency aspects of these alternative methods are confirmed by a comparison with Monte Carlo simulation technique, based on

E-mail address: pirrotta@stru.diseg.unipa.it.

0020-7462/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijnonlinmec.2005.04.001

very effective algorithms for solving wide variety of problems.

As is well known, the problem with this latter technique is that is computationally expensive, because it requires a very high sample size of the process input and consequently a large number of deterministic analyses. However, the recent advent of high-speed digital computers has made Monte Carlo (MC) technique competitive with the aforementioned methods than ever before [20]; such that in parallel to perform new techniques, giving directly the output statistics, an attempt to enhance the efficiency of MC technique is desirable.

Basically MC technique is constituted of three steps. Therefore, to improve such technique in practice means enhancing the procedure used in one of the three following steps: (i) sample paths of the stochastic input process, (ii) calculating the outputs corresponding to the generated input samples by using methods of classical dynamics and (iii) estimating statistics of the output process from sample outputs related to the previous step.

Pertaining the first step, there are several alternative approaches to the classical one currently under development, either regarding the random digits [21] or concerning the importance sampling technique for Poissonian processes [22,23]. However, very little is found in improving the second step. Regarding the third step, MC approach becomes time consuming for non-linear systems that need higher-order statistics. In [24] an attempt is made. An alternative approach uses MC method to calculate the first few moments of the response process with a small number of input samples, and then calculating higher-order statistics becomes an easy task, because they are solution of linear differential equations, once the first few response moments are introduced.

This paper aims at contributing to speed up the calculation involved in the second step, the main issue is addressed for normal or non-normal parametric white noise input, since in these circumstances, it may be considered that the input is impulsive and a correct integration scheme at each impulse occurrence is needed for. For linear and non-linear systems driven by parametric impulsive inputs such as normal or non-normal white noises, a general integration method requires a considerable reduction of the integration step when the impulse occurs, treating the impulse as a physical one, by means of a window function of finite duration. This makes MC simulation very prohibitive from a computational time point of view. While knowing the exact jump value of the response at occurring of impulse, that is expressed by a numerical series [25–27], the aforementioned problem is overcome because there is no need to reduce the integration step saving computational time, reliability being equal as shown by means of a numerical example.

2. Preliminary concepts

The Poisson white noise process, the generator of all white noise stochastic processes, here labeled $W_p(t)$, consists of a train of Dirac's delta impulses $\delta(t - T_k)$ occurring in time according to the realization T_k of a random variable with Poisson law distribution and with random amplitude Y having assigned distribution (independent of the random time arrivals). This Poisson process is thus represented in the form

$$W_p(t) = \sum_{k=1}^{N(t)} Y_k \delta(t - T_k), \tag{1}$$

where $N(t)$ is a Poisson counting process giving the number of impulses in the time interval $[0, t)$, Y_k is the kth realization of the random variable Y, T_k is the kth realization of the random variable T and $\delta(\cdot)$ is the Dirac's delta function. The stochastic process (1) is termed as *Poisson white noise* since its correlation function is a Dirac's delta function:

$$\begin{aligned} k_2[W_p(t_1)W_p(t_2)] &= R_{W_p}(t_1, t_2) \\ &= \lambda E[Y^2]\delta(t_1 - t_2) \end{aligned} \tag{2}$$

where k_j is the jth cumulant, λ the mean number of impulses per unit time and $E[\cdot]$ means ensemble average. From Eq. (2) it may be recognized that the power spectral density function (PSD) is constant at overall frequency range and this explains the nomenclature of white noise. On the other hand the process (1) is not normal because cumulants of order > 2, are different from zero

$$\begin{aligned} &k_j[W_p(t_1), W_p(t_2), \ldots, W_p(t_j)] \\ &\quad = \lambda E[Y^j]\delta(t_1 - t_2)\cdots\delta(t_1 - t_j) \end{aligned} \tag{3}$$

from Eq. (3) it is recognized that the jth cumulant is the product of Dirac's deltas and for this reason the

Poisson white noise is also termed *delta correlated process*.

Moreover, the Poisson white noise $W_p(t)$ can be considered as the formal derivative of the so-called *homogeneous compound Poisson* process $C(t)$, defined by

$$C(t) = \sum_{k=1}^{N(t)} Y_k U(t - t_k), \tag{4}$$

where $U(t)$ is the unit step function. If λ tends to infinity and at the same time $\lambda E[Y^2]$ keeps a constant value, then the Poisson white noise $W_p(t)$ tends towards the normal white noise $W_n(t)$. Since $W_n(t)$ is normal, then the correlation function remains the same form, as the Poisson process $W_p(t)$, but higher-order cumulants are zero. The normal white noise $W_n(t)$ may be thought as the formal derivative of the so-called *Brownian motion $B(t)$*. At the limit, starting from the Compound Poisson process, as aforementioned, if λ tends to infinity and at the same time $\lambda E[Y^2]$ remains constant, the compound process itself becomes a Brownian motion. It will be emphasized that sample functions of Poisson process and sample functions of normal white noise are quite different, since in the former case, well spaced impulses of finite amplitude appear, while in the latter, impulses with infinitesimal amplitude occur in a dense temporal space. In order to assess the validity of these statements increments of the process $B(t)$ and $C(t)$ will be compared. $B(t)$ and $C(t)$ have, both, independent increments: $\mathrm{d}B(t) = B(t + \mathrm{d}t) - B(t)$ represents the area of the infinitesimal impulse in the time interval $(t, t + \mathrm{d}t)$, while $\mathrm{d}C = C(t + \mathrm{d}t) - C(t)$ may be either a finite quantity or zero, according to whether the impulse falls into the interval $(t, t + \mathrm{d}t)$ or not. For this reason it may be easily shown that

$$E[(\mathrm{d}B(t))^2] = q_2\, \mathrm{d}t, \tag{5a}$$

$$E[(\mathrm{d}C(t))^2] = \lambda E[Y^2]\, \mathrm{d}t. \tag{5b}$$

From Eqs. (5) it is evident that increments of $B(t)$ or $C(t)$ may have identical value (for instance setting $q_2 = \lambda E[Y^2]$). However since $B(t)$ is Gaussian and $C(t)$ is not Gaussian, substantial differences arise from higher-order statistics, as

$$E[(\mathrm{d}B(t))^k] = \mathrm{O}(\mathrm{d}t^{k/2}), \quad \forall k \geqslant 2, \tag{6a}$$

$$E[(\mathrm{d}C(t))^k] = \lambda E[Y^k]\, \mathrm{d}t; \quad \forall k > 2. \tag{6b}$$

From these equations it may be recognized that an increment of the Wiener process is of order $\mathrm{d}t^{1/2}$((O $\mathrm{d}B(t) = \mathrm{d}t^{1/2}$), while the order of $\mathrm{d}C(t)$ may not be established since moments of an increment of Poisson process are all of the same order.

The last class of white noise is represented by the *α-stable Lévy white noise* process $W_\alpha(t)$. In analogy to the definition of the previous white noises, an α-stable Lévy white noise may be defined as the formal derivative of a corresponding *α-stable Lévy motion $L_\alpha(t)$*. Increment of the Lévy motion $\mathrm{d}L_\alpha(t)$ are independent (like $\mathrm{d}B(t)$ and $\mathrm{d}C(t)$), and they are defined through the characteristic function (CF) $\phi_{\mathrm{d}L_\alpha}(\vartheta)$:

$$\phi_{\mathrm{d}L_\alpha}(\vartheta) = \exp(-\mathrm{d}t|\vartheta|^\alpha); \quad 0 < \alpha \leqslant 2. \tag{7}$$

An α-stable Lévy motion is non-normal and the moments do not exist, unless the moments $E[|\mathrm{d}L|^p]$ with $p < \alpha$ and $\alpha \neq 2$. For $\alpha = 2$, the α-stable Lévy white noise reverts to the Gaussian white noise. In general, the smaller the α, the greater is the departure of the α-stable Lévy white noise from the Gaussian one.

2.1. Generation of sample functions of white noise processes

Each sample function of $W(t)$ is easily performed for Poissonian white noise, simply by generating samples of the random variable Y with the assigned distribution of the amplitudes and independent times distributed according to Poisson law. Each sample function is generated by attributing at each time say T_k the kth realization of Y say Y_k.

The normal white noise may be generated by subdividing the time axis $(0, \bar{T})$, where $\bar{T}$ is the observation period which is usually divided into small intervals usually of equal length, Δt, and at each temporal step it is attributed a realization of a random variable with unit variance and hence the realization of increment $\Delta B_r(t) = B(t_r + \Delta t) - B(t_r)$ is defined by

$$\Delta B_r(t) = \Delta t^{1/2} G_r, \tag{8}$$

where G_r are independent realizations of a zero mean normal variable having variance q_2.

For the Lévy white noise the procedure is analogous to the case of increments of Brownian motion, increments of the Lévy motion are defined by

$$\Delta L_\alpha(t_r) = \Delta t^{1/\alpha} X_r, \tag{9}$$

where X_r are independent realizations of an α-stable random variable generated as follows [28]:

$$X_r = \frac{\sin \alpha U_r}{(\cos U_r)^{1/\alpha}} \left[\frac{\cos(1-\alpha)U_r}{V_r} \right]^{(1-\alpha)/\alpha}, \tag{10}$$

where U_r and V_r are realizations of independent random variables. Specifically, U has uniform distribution on $[-\pi/2, \pi/2]$, and V has exponential distribution with unit mean.

From Eq. (10) the impulsive nature of the α-stable Lévy white noise may be deduced. Recognize that, for $\alpha \neq 2$, the α-stable random variable X in Eq. (10) has no moments. Samples of X, in fact, do diverge as $U \to \pm\pi/2$ for $1 \leqslant \alpha < 2$, and as $U \to \pm\pi/2$ or $V \to 0$ for $\alpha < 1$. That is, in a given interval Δt samples of X may be of order $1/\Delta t^{1/\alpha}$ with non-zero probability.

In order to compare the order of magnitude of $\Delta t^{1/\alpha} X_r$, it is well known that for any α-stable variable with zero skewness and shift, $\text{Prob}\{|X| > \rho\} \sim D_\alpha \rho^{-\alpha}$ as $\rho \to \infty (D_\alpha = \Gamma(\alpha)\sin(\alpha\pi/2)/\pi)$. Hence $\text{Prob}\{|X| > \Delta t^{-1/\alpha}\} \sim D_\alpha \Delta t$ as $\Delta t \to 0$. Moreover, it may be also shown that

$$\begin{aligned} E[\mathrm{d}L_\alpha^j] &= \lim_{\Delta t \to 0} \Delta t^{j/\alpha} \int_{-\Delta t^{-1/\alpha}}^{\Delta t^{-1/\alpha}} x^j p_X(x)\,\mathrm{d}x \\ &= k_j\,\mathrm{d}t, \end{aligned} \tag{11}$$

where k_j depends on α. As an example if X is a Cauchy random variable ($\alpha = 1$), the PDF of X is $p_X(x) = 1/\pi(1+x^2)$ and $k_{2j} = 2/\pi(2j-1)$, while $k_{2j+1} = 0,)$ $(j=1,2,3,\ldots)$. If $(\alpha=0.5)(\beta=1)$ i.e. X is a Lèvy random variable, $p_X(x) = x^{-3/2}\exp(-1/2x)/\sqrt{2\pi}$, then $E[\mathrm{d}L_{0.5}^j] = k_j\,\mathrm{d}t$, k_j being a constant depending on j that may be evaluated easily by MATHEMATICA using Eq. (11). From these considerations, by comparing Eq. (6b) with Eq. (11) it appears that an α-stable Lèvy process behaves like a Poissonian one. The only difference is that for Poissonian white noise in each temporal interval $\mathrm{d}C(t) = 0$ unless an impulse occurs, while for the Lèvy white noise in each temporal interval, impulses of order of amplitudes ranging from $\Delta t^{1/\alpha}$ and Δt may occur depending on the realization of X_r. In the limit case $\alpha = 2$, since $D_\alpha = 0$, the amplitude of impulses is governed by $\Delta t^{1/2}$ and then the order of magnitude of impulses in each temporal interval is $\Delta t^{1/2}$, and $E[\mathrm{d}L_2^{2j}]$ is an infinitesimal quantity of $\mathrm{d}t^j$.

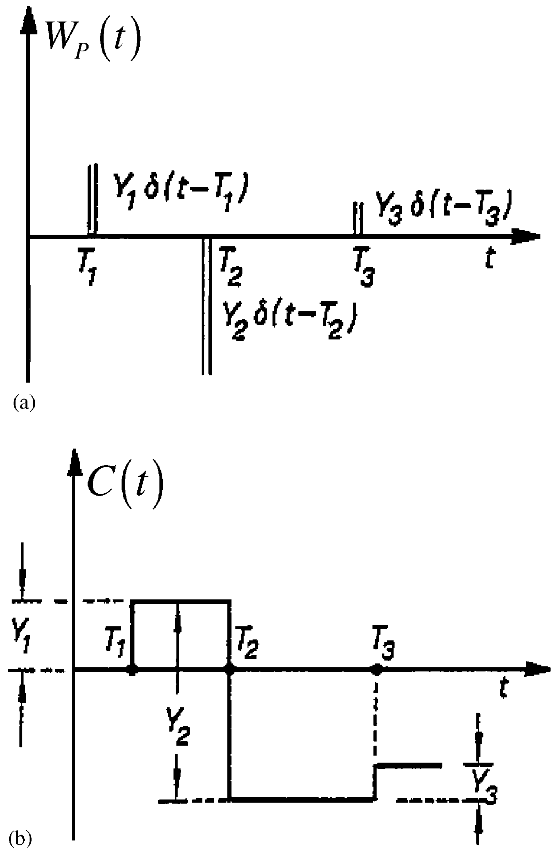

Fig. 1. (a) Poissonian white noise process and (b) Compound Poisson process.

Once we define the procedure to generate white noises, $W_p(t)$, $W_\alpha(t)$ and $W_n(t)$, we observe three sample functions of them represented are as in Figs. 1–3. In Fig. 1a the non-normal white noise $W_p(t)$ is represented and the corresponding compound Poisson process is represented in Fig. 1b. From the first figure one can observe that the sample function of $W_p(t)$ is regular in all the time axis, except for the time T_k in which the impulses occur. In Fig. 2a the non-normal white noise $W_\alpha(t)$ is depicted as long as the impulses have finite magnitude, and the corresponding α-stable Lévy motion $L_\alpha(t)$ is reported in Fig. 2b. From Fig. 2a one can see that in each selected Δt there is an impulse, but these are not comparable in magnitude. In Fig. 3a, a sample process $W_n(t)$ is represented and the corresponding Brownian motion process is represented in Fig. 3b and it appears that $B(t)$ is continuous.

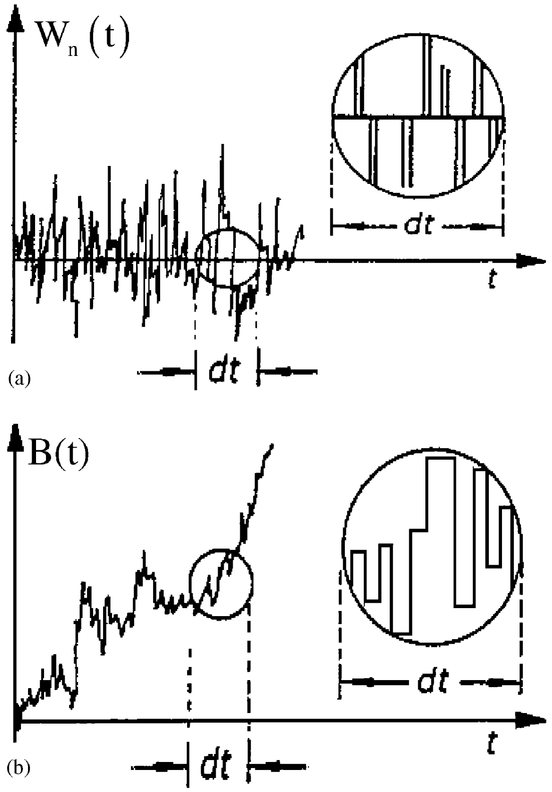

Fig. 2. (a) Normal white noise and (b) Wiener process.

In the next section it will be shown that the problem of evaluating the response by using step-by-step integration method for the three aforementioned cases is exactly the same.

2.2. Monte Carlo simulation of non-linear systems excited by samples of parametric white noise

In this section the digital simulation will be presented for both normal and non-normal white noise processes. Because it is easier to understand the case of non-normal starting from Poissonian white noise, this case will be presented first.

2.3. Non-linear systems excited by samples of Poissonian white noise

The Monte Carlo approach consists in generating several sample functions of the input, in evaluating

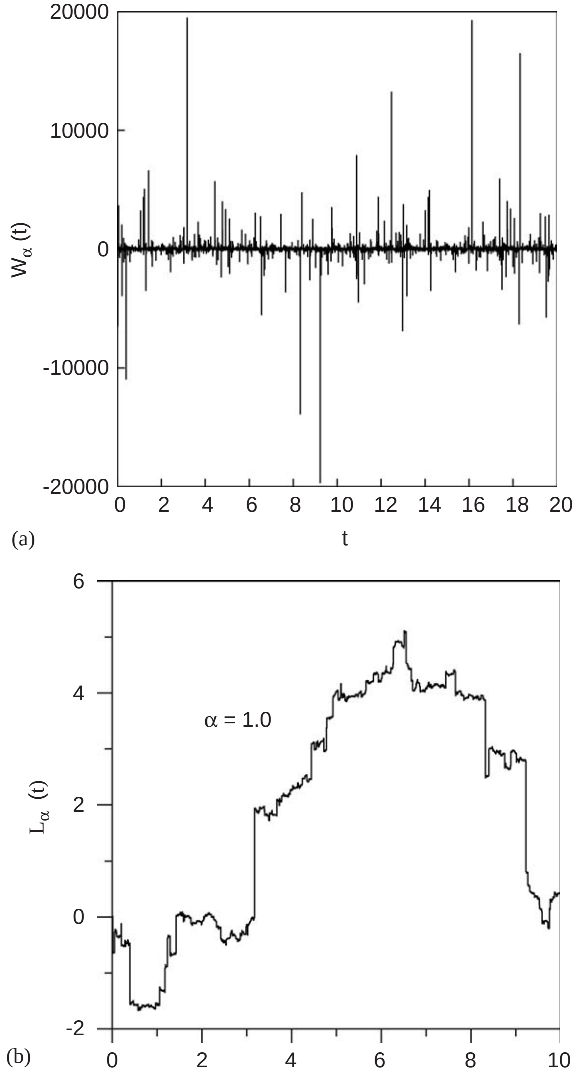

Fig. 3. (a) α-stable Lévy white noise process and (b) α-stable Lévy motion.

the response for each sample function and then in performing averages and other response statistics.

Let the dynamical system be given in the form

$$\dot{Z}(t) = f(Z,t) + g(Z,t)W_p(t), \quad Z(t_0) = Z_0, \quad (12)$$

where $f(Z,t)$ and $g(Z,t)$ are deterministic non-linear functions of Z and t, $W_p(t)$ is a sample function of a Poissonian white noise, and Z_0 is the initial condition that can be either deterministic or random whose distribution is known and independent of the Dirac's delta occurrences. Once the sample process is generated as above, intensities and locations of spikes are known and then the problem is deterministic. Between two subsequent deltas occurring at times T_{k-1}, T_k we have to solve a non-linear problem of the form

$$\dot{Z}(t) = f(Z,t), \quad \forall t: T_{k-1}^{+} \leqslant t < T_k^{-}. \tag{13}$$

The initial condition for this problem is $Z(T_{k-1}^{+})$, where the apex + means immediately after. By solving this problem we can arrive at $Z(T_k^{-})$, the value of $Z(t)$ immediately before the spike in T_k. When the spike occurs in T_k, the problem of predicting the response immediately after is a very hard task in principle, and it is governed by

$$\dot{Z}(t) = f(Z,t) + g(Z,t)Y_k\delta(t-T_k), \quad \forall t: T_k^{-} \leqslant t < T_k^{+} \tag{14}$$

with initial condition $Z(T_k^{-})$ already known. Eq. (14) can be rewritten in the form

$$\mathrm{d}Z(t) = f(Z,t)\,\mathrm{d}t + g(Z,t)Y_k\,\mathrm{d}U(t-T_k), \quad \forall t: T_k^{-} \leqslant t < T_k^{+}, \tag{15}$$

where $U(\cdot)$ is the unit step function. The integral over the time interval $T_{k-\varepsilon}/T_{k+\varepsilon}$, with ε arbitrary small, can be written by selecting any partition $t_1, t_2, \dots, t_j, \dots t_n$ of the time interval $T_{k-\varepsilon}/T_{k+\varepsilon}$, as follows:

$$Z(T_{k+\varepsilon}) - Z(T_{k-\varepsilon}) = \int_{T_{k-\varepsilon}}^{T_{k+\varepsilon}} f(Z,t)\,\mathrm{d}t + Y_k \int_{T_{k-\varepsilon}}^{T_{k+\varepsilon}} g(Z,t)\,\mathrm{d}U(t-T_k). \tag{16}$$

If $\varepsilon \to 0$, the first integral is zero, while the second one is not a Riemann–Stieltjies integral because the expression:

$$Y_k \int_{T_{k-\varepsilon}}^{T_{k+\varepsilon}} g(Z,t)\,\mathrm{d}U(t-T_k) = Y_k \lim_{\substack{n\to\infty \\ \Delta t_{\max}\to 0}} \times \sum_{j=1}^{n} g(Z(\bar{t}_j),\bar{t}_j)[U(t_j-T_k) - U(t_{j-1}-T_k)] \tag{17}$$

depends on the intermediate point selected $\bar{t}_j$, since $Z(t)$ exhibits a jump in $T_{k-\varepsilon}/T_{k+\varepsilon}$. In expression (17), $Z(\bar{t}_j)$ is the response at an intermediate point $\bar{t}_j$ between t_{j-1} and t_j and $\Delta t_{\max}$ is the maximum amplitude of the intervals into which $[T_{k-\varepsilon}, T_{k+\varepsilon}]$ has been subdivided. Assuming $\bar{t}_j = t_{j-1}$, one obtains a *forward integral*, also called Itô integral (1951), and the sum in Eq. (17) gives

$$Z(T_k^{+}) - Z(T_k^{-}) = Y_k g\,(Z(T_k^{-}), T_k). \tag{18}$$

Performing the summation in Stratonovich sense (1951), i.e. the same as using a classical trapezoidal rule, one gets

$$Z(T_k^{+}) - Z(T_k^{-}) = Y_k g\left(\frac{Z(T_k^{+}) + Z(T_k^{-})}{2}, T_k\right). \tag{19}$$

From the non-linear relationship (19), one can obtain $Z(T_k^{+})$ evaluated in Stratonovich sense [13]. However at this stage, people can be perplexed, because in the Dirac's delta occurrence one can predict an approximate value of the jump and not the exact one. Indeed, recently Càddemi and Di Paola [25,26], Di Paola and Pirrotta [27], give a correct answer for the response to parametric impulses. The main steps for obtaining the jump in the Dirac's delta occurrence are here summarized. They obtain the exact solution after the impulse, simply, neglecting the first integral in Eq. (16) (as an infinitesimal quantity with respect to the finite jump in (T_k^{-}, T_k^{+})) and then expanding the jump in a series,

$$\Delta Z_k = Z(T_k^{+}) - Z(T_k^{-}) = \mathrm{d}Z(t)]_{T_k^{-}} + \frac{1}{2!}\mathrm{d}^2 Z(t)\Big]_{T_k^{-}} + \frac{1}{3!}\mathrm{d}^3 Z(t)\Big]_{T_k^{-}} + \cdots . \tag{20}$$

The formal expansion (20) is not meaningless since $Z(t)$ is left continuous and the differentials are evaluated in T_k^{-}. Moreover, they showed that the series can be written as follows:

$$\Delta Z_k = \sum_{j=1}^{\infty} \frac{g^{(j)}(Z(T_k^{-}), T_k)}{j!}, \tag{21}$$

where $g^{(j)}$ can be evaluated in recursive form as follows:

$$g^{(j)}(Z(t),t) = \frac{\partial g^{(j-1)}(Z(t),t)}{\partial Z} g^{(1)}(Z(t),t); \quad g^{(1)}(Z(t),t) = Y_k g(Z(t),t). \tag{22}$$

From Eq. (22) one evaluates the coefficient of the jump, to know the jump immediately after the impulse.

As an example we want to evaluate the jump for the differential equation

$$\dot{Z}(t) = f(Z,t) + Z\,Y_k\delta(t - T_k). \tag{23}$$

Let $Z(t_k^-)$ be the solution before the Dirac's delta in T_k. The jump according to Eq. (21) can be written as

$$\Delta Z_k = \sum_{j=1}^{\infty} \frac{Z(T_k^-)Y_k^j}{j!} = Z(T_k^-)[\exp(Y_k) - 1]. \tag{24}$$

This jump prediction is exactly coincident with that evaluated using MATHEMATICA program. Analogous coincidence may be found for $g(Z,t) = Z^2$ for which MATHEMATICA gives the jump value and series (21) is the expansion of the jump.

From this example one realizes that the jump depends on the value immediately before the Dirac's occurrence and on the intensity Y_k of the impulse. Retaining the first term in the series we obtain the jump evaluated in the Itô sense. The first two terms give an approximation of the jump evaluated in Stratonovich sense. Such approximation gives accurate results depending on the intensity of the Dirac's delta Y_k and on the value of the response immediately before the spike $Z(T_k^-)$ as shown in Fig. 4. It is to be remarked that if the function $g(Z,t)$ in Eq. (12) does not depend on Z, that if the case is external excitation, the rule (21) is not in contrast with the classical result, as in fact only the first term in the summation is necessary.

Once the correct rule of jump evaluation is defined, let us go back to the problem of integrating sample function of SDE driven by Poissonian parametric impulse that will be performed straightforward. In fact the step-by-step integration technique starts at $t = 0$, with initial condition Z_0 and $Z(T_1) - Z(t_0) = f(Z_0)\Delta t$ (the probability of an impulse occurrence in $t = 0$ is zero w.p. 1), and follows up to the first time T_1 in which the impulse of amplitude Y_1 occurs. The value of $Z(T_1^-)$ is already known and the value $Z(T_1^+)$ is given by Eq. (24). This value is the initial condition in the time interval (T_1, T_2), where T_2 is the realization of the random variable T obeying the condition $T_1 < T_2 < \cdots < T_r$.

For the case of 2-stable or α-stable white noise process, at each temporal step an impulse occurs whose amplitude is given by Eqs. (8) and (9), respectively.

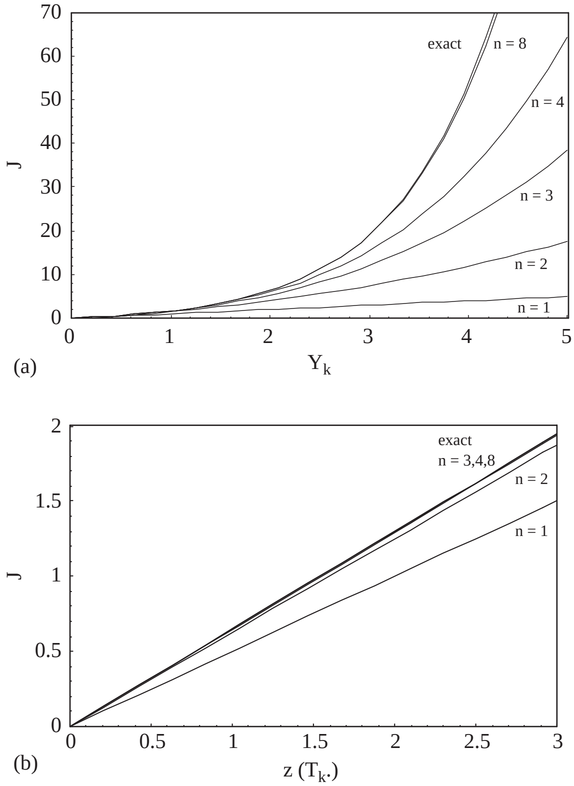

Fig. 4. (a) Jump versus Y_k for fixed value of $Z(T_k^-) = 1$ and different number of terms included in the summation (21) and result of Eq. (24) (exact jump) and (b) Jump versus $Z(T_k^-)$ for fixed value of $Y_k = 0.5$ and different number of terms included in the summation (21) and result of Eq. (24) (exact jump).

By denoting as ΔS_k the impulse occurring in $(t_k, t_k + \Delta t)$ ($\Delta S_k = \Delta t^{1/2} G_k$ for Gaussian and $\Delta S_k = \Delta t^{1/\alpha} X_k$ for α-stable white noise), the integral in the case of parametric input is given by

$$\Delta Z(t_k) = Z(t_k + \Delta t) - Z(t_k) = f(Z(t_k), t_k)\Delta t + \sum_{j=1}^{\infty} \frac{g^{(j)}(Z(t_k), t_k)}{j!}(\Delta S_k)^j. \tag{25}$$

It is obvious that if $\Delta S_k = \Delta t^{1/2} G_k$, only the first two terms in Eq. (25) need to be inserted for very small Δt since other terms in the summation (25) are of higher order than Δt. In passing we note that the second order is just the Wong–Zakay or Stratonovich correction term.

On the contrary for α-stable Lévy white noises process, the number of terms is strictly related to the realization of the random variable in the step. For example, if $\alpha = 0.5$, by looking at Eq. (9), it seems that $\Delta L_\alpha(t)$ is of order Δt^2, and hence the summation in (25) may be disregarded (for Δt very small). The problem is that α-stable random variables have long tails in the probability density function. For such we have already discussed the probabilities that the realizations X_k of the random variable X may assume very high value, e.g. of order $1/\Delta t$, $1/(\Delta t)^2$, etc. It follows that depending on the order of amplitude of X_k less or more terms in the summation (25) are needed to have a correct jump prevision in $(t_k, t_k + \Delta t)$.

Summing up, in the case of normal white noise processes two terms in the summation (25) will be included, while for the α-stable white noise or for Poissonian white noise, more and more terms are necessary for evaluating the response at the end of each step depending on the realization of the α-stable random variable in the step or by the value of the realization of the amplitude for the Poissonian case.

It is worth remarking that the Poissonian white noise has been defined and generated as a true white noise, since it is constituted as a sequence of mathematical impulses (Dirac's delta). On the contrary, the Gaussian white noise and the Lèvy white noise, whose generation is made according to the above-mentioned procedure, are only a band limited white noise, with cutoff frequency $\omega_c = 1/2\pi\,\Delta t$, since the selected time interval Δt is a finite quantity. However, if the frequency content of the system is low with respect to the cutoff frequency of the input, the system does not distinguish between the ideal white noise and the band limited white noise.

3. Multi-degree of freedom non-linear systems

The previous concepts may be easily extended for the case of MDOF systems. As aforementioned the crucial point to perform the step-by-step analysis for parametric white noise input process is related to the jump evaluation in each temporal step. The analysis for MDOF systems is based on the same procedure. Let an n-degree of freedom non-linear system be given in the form

$$\dot{\mathbf{Z}}(t) = \mathbf{f}(\mathbf{Z}, t) + \mathbf{G}(\mathbf{Z}, t)\mathbf{W}(t); \quad \mathbf{Z}(t_0) = \mathbf{Z}_0, \tag{26}$$

where $\mathbf{Z}$ is the n-state space variable vector, $\mathbf{f}(\mathbf{Z}, t)$ is an n vector of non-linear functions of $\mathbf{Z}$ and t, $\mathbf{G}(\mathbf{Z}, t)$ is an $(n \times m)$ matrix of non-linear functions.

The first step consists in finding the response in the sth interval. Let us suppose that the $\mathbf{W}(t) = \mathbf{Y}\delta(t - T_k)$, where $\mathbf{Y}$ is an m-vector whose components are the strength of the Dirac's delta in the interval $t_s \leqslant T_k \leqslant t_s + \Delta t$.

The increment of the vector $\mathbf{Z}(t)$ in the interval $((t_s/t_s) + \Delta t)$ is then given by

$$\Delta\mathbf{Z}(t_s) = \mathbf{f}(\mathbf{Z}(t_s), t_s)\Delta t + \sum_{j=1}^{\infty} \frac{\mathbf{g}^{(j)}(\mathbf{Z}(t_s), t_s)}{j!} \tag{27}$$

where,

$$\begin{aligned} &\mathbf{g}(\mathbf{Z}(t), t) = \mathbf{G}(\mathbf{Z}(t), t)\mathbf{Y}, \\ &\mathbf{g}^{(j)}(\mathbf{Z}(t), t) = (\nabla_Z \mathbf{g}^{(j-1)}(\mathbf{Z}(t), t))\mathbf{g}^{(1)}(\mathbf{Z}(t), t), \\ &\mathbf{g}^{(1)}(\mathbf{Z}(t), t) = \mathbf{g}(\mathbf{Z}(t), t), \end{aligned} \tag{28}$$

and $\nabla_Z \mathbf{g}^{(j-1)}(\mathbf{Z}(t), t)$ is the gradient operator of the vector, i.e.

$$\nabla_Z \mathbf{g}^{(j)}(\mathbf{Z}(t), t) = \begin{bmatrix} \frac{\partial g_1^{(j)}}{\partial Z_1} & \frac{\partial g_1^{(j)}}{\partial Z_2} & & \frac{\partial g_1^{(j)}}{\partial Z_n} \\ \frac{\partial g_2^{(j)}}{\partial Z_1} & \frac{\partial g_2^{(j)}}{\partial Z_2} & & \frac{\partial g_2^{(j)}}{\partial Z_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_n^{(j)}}{\partial Z_1} & \frac{\partial g_n^{(j)}}{\partial Z_2} & & \frac{\partial g_n^{(j)}}{\partial Z_n} \end{bmatrix}. \tag{29}$$

4. Numerical applications

Accuracy and efficiency aspects are asserted by means of a non-linear system driven at first by parametric Poissonian white noise, then by an α-stable Lévy white noise and at last by a normal white noise. The governing equation of motion is expressed in the form:

$$\dot{Z} = aZ + bZ^3 + \gamma Z^2 W_p(t); \quad Z(0) = 1. \tag{30}$$

When no impulse occurs, the increment of the response is simply given by

$$\Delta Z_s = Z(t_s + \Delta t) - Z(t_s) = (aZ(t_s) + bZ^3(t_s))\Delta t; \quad \forall t_s \neq T_k \tag{31}$$

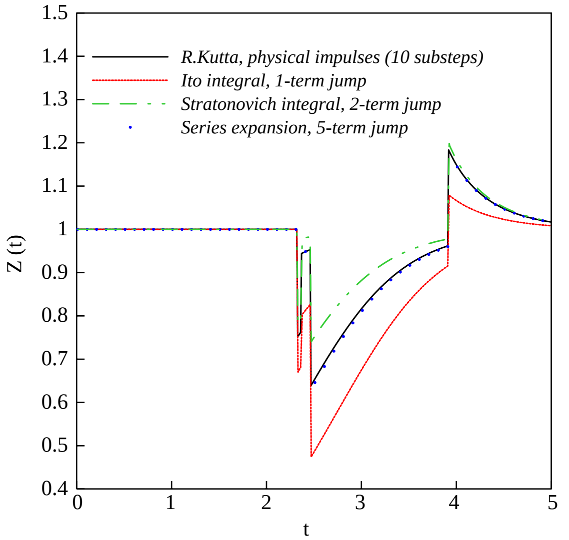

Fig. 5. Response sample function to Poissonian white noise process.

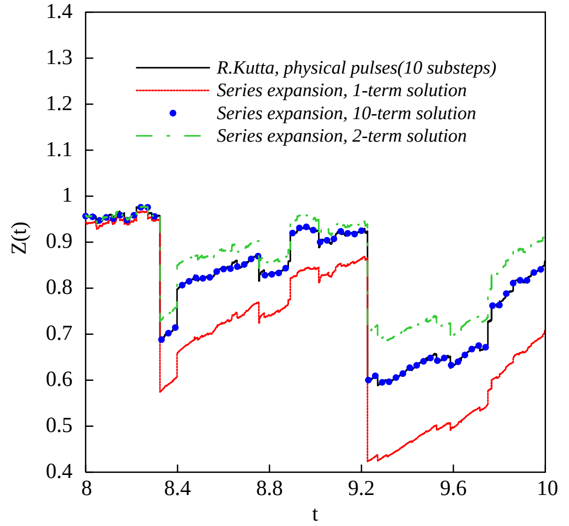

Fig. 6. Response sample function to α-stable Lévy white noise process.

at impulse occurrence, Eq. (24) is particularized as

$$\Delta Z_k = Z(T_k^+) - Z(T_k^-) = (aZ(T_k^-) + bZ^3(T_k^-))\Delta t + \sum_{j=1}^{5} \gamma^j Z^{(j+1)}(T_k^-) Y_k^j; \quad t = T_k, \tag{32}$$

where $a=1$, $b=-1$ and $\gamma=0.3$. Regarding the Poisson white noise process $W_p(t)$, this is characterized by the mean number of impulse per unit time, $\lambda = 1$, and the distribution of the random variable Y is Gaussian with a zero mean and unit variance.

In Fig. 5 a sample function of response is depicted showing the results performed by the proposed MC by using 5 terms in the expression (32), and compared with those obtained by a classical MC method. The latter is performed by considering each impulse as a physical one, distributed over a finite time interval equal to $\Delta t = 0.01$ s and by further subdividing this time interval in 10 parts (see Appendix A).

As shown in Fig. 5 the results are totally overlapped, highlighting the efficiency of the proposed method, since there is a computational time saving power of ten times, at least. Moreover in the same figure we also report the Itô result, which considers only one term of the series, and the Stratonovich result, which considers two terms of the series. For these two last cases, the results in terms of response sample function are quite different and this also influences the response statistics in a very sensible way.

Analogous analysis has been performed for the same system as before driven by a parametric 1-stable Lévy white noise

$$\dot{Z} = aZ + bZ^3 + \gamma Z^2 W_1(t); \quad Z(0) = 1 \tag{33}$$

with $\gamma = 0.3$. Particularizing Eq. (25) one gets

$$\Delta Z(t_k) = Z(t_k + \Delta t) - Z(t_k) = (a\, Z(t_k) + bZ^3(t_k))\Delta t + \sum_{j=1}^{10} \gamma^j (Z(t_k))^{j+1} (\Delta t X_k)^j. \tag{34}$$

In Fig. 6, a sample function of response is depicted and it shows the results obtained by the proposed MC using 10 terms in the expression (34), and compares it with those obtained by a classical MC method. The latter is performed by considering each impulse as a physical one, distributed over a finite time interval equal to $\Delta t = 0.001$ s and by further subdividing this time interval in 10 parts. Also in this case there is a computational time saving power, and the results totally overlap and the proposed method is reliable.

Similar results and the inferred considerations for the last example represented by the same dynamical

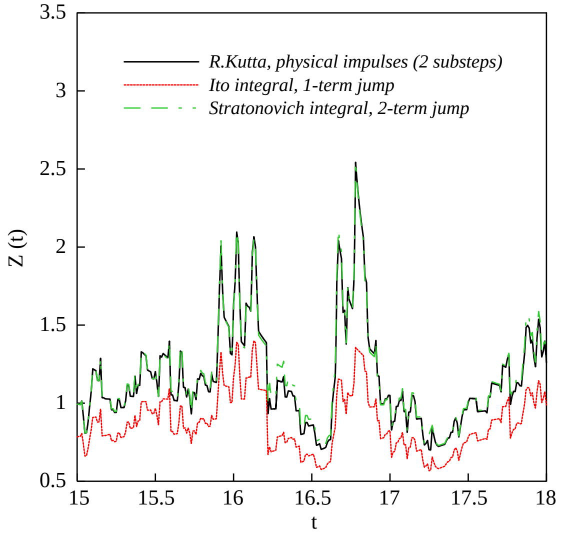

Fig. 7. Response sample function to normal white noise process.

system driven by a normal white noise or a 2-stable Lévy white noise are given taking $\gamma = 0.8$. In this case an increment of the response function is given in the form

$$\Delta Z(t_k) = Z(t_k + \Delta t) - Z(t_k) = (aZ(t_k) + b\,Z^3(t_k))\Delta t + \sum_{j=1}^{2} \gamma^j (Z(t_k))^{j+1} (\Delta t^{1/2} G_k)^j \tag{35}$$

and the results are depicted in Fig. 7 having adopted a $\Delta t = 0.01$ s and for classical MC this interval has been subdivided into two parts.

5. Conclusions

In this paper the problem of Monte Carlo simulation of non-linear systems under Gaussian and non-Gaussian white noise processes has been extensively examined . The three types of white noise processes (normal, Poissonian, α-stable) experience a common feature: at each time interval they exhibit impulse occurrence. The main difference between them is in the amplitude of impulse. For the Poissonian white noise, for each sample function in the generic time interval the amplitude of the impulse is either zero or a finite quantity (the latter situation occurs when the impulse is present in the generic interval). For Gaussian white noise at each interval an impulse of order of magnitude $\Delta t^{1/2}$ occurs. While for the α-stable process, the impulse is governed by the value of realization of the random variable. It follows that if the realization of the α-stable random variable assumes a finite quantity, then the impulse is of order $\Delta t^{1/\alpha}$, but in some intervals because the probability density function of an α-stable random variable has heavy tails, there is a finite probability that the order of magnitude of the impulse is $(1/\Delta t)^{1/\alpha}$. Keeping this machinery in mind, the only problem in integrating non-linear differential equation for each sample of white input is in evaluating response for impulsive input. In the case of external excitation the problem in any case is trivial. In the case of parametric excitation the integration at each time interval has to be treated with care. We have two possibilities, the first one is in considering in each time interval the impulse as a physical one (i.e. the input is constant in the interval) but having the same area of the original Dirac's delta occurrence. In this case the interval will be subdivided into smaller and smaller steps and then any integration rule may be used for. Otherwise at each interval we know the realization of the input and arrive at the asymptotic solution by using the series for the jump evaluation of the response.

In the paper the two ways have been compared showing the usual step-by-step integration method.

Acknowledgements

The author is deeply grateful to Prof. M. Di Paola for the invaluable suggestions.

Appendix A. An advise on Monte Carlo simulation for Non-linear systems under parametric impulse

It will be worth remarking that, when dealing with response of systems under parametric impulsive loads, the Monte Carlo simulation has to be treated with care. To better explain this consideration one may consider a dynamical system under a single parametric impulse:

$$\dot{z} = f(z,t) + \gamma g(z,t)\delta(t - \bar{t}); \quad z(0) = z_0, \tag{A.1}$$

where z is the state variable, $f(z,t)$ and $g(z,t)$ are non-linear functions of the response, upper dot means time derivative, $\delta(.)$ is the Dirac's delta function and

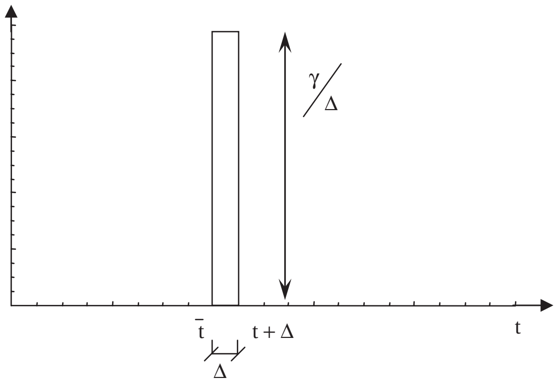

Fig. A.1. Physical impulse.

γ is a real constant, z_0 is the relevant initial condition. Generally applying the Monte Carlo simulation the impulse is considered as a physical one, i.e. represented by a window function of finite duration Δ and amplitude γ/Δ in the interval $\bar{t}$–$\bar{t}+\Delta$ (see Fig. A.1).

The jump evaluation is given by

$$z(\bar{t}+\Delta) - z(\bar{t}^-) = \frac{\gamma}{\Delta}\int_{\bar{t}}^{\bar{t}+\Delta} g(z,\tau)\,\mathrm{d}\tau. \tag{A.2}$$

From this equation it may be recognized that the jump for a physical impulse depends on γ and on the total area of $g(z,\tau)$ during the time at which the impulse is present. From this consideration it appears that, due to the strong variations of $g(z,t)$, during the impulse occurrence, assuming the initial value $g(z,\bar{t}^-)$ is an unacceptable approximation, in fact we must subdivide the interval Δ into several substeps, no matter the amplitude of Δ, although this latter is very small, because the fundamental thing is the value of γ. After subdividing Δ into several substeps, only inside a substep we can use what ever method one likes even a forward difference integration scheme.

These considerations will be apparent by use of a trivial but effective example given by

$$\dot{z} = \gamma z \delta(t-1); \quad z(0) = 0.5 \tag{A.3}$$

with chosen value of $\gamma = 10$. This system is solved in closed form using for instance “MATHEMATICA” software as

$$z(t) = z(0)\exp(\gamma U(t-1)) \tag{A.4}$$

depicted in Fig. A.2 and being $U(.)$ is the unit step function.

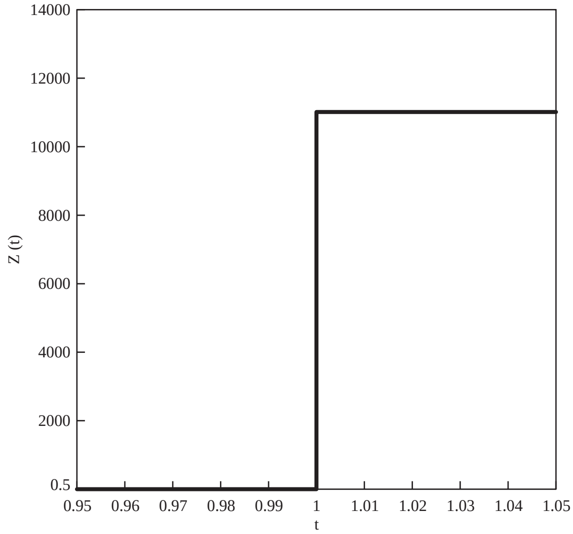

Fig. A.2. Exact solution of Eq. (A.3).

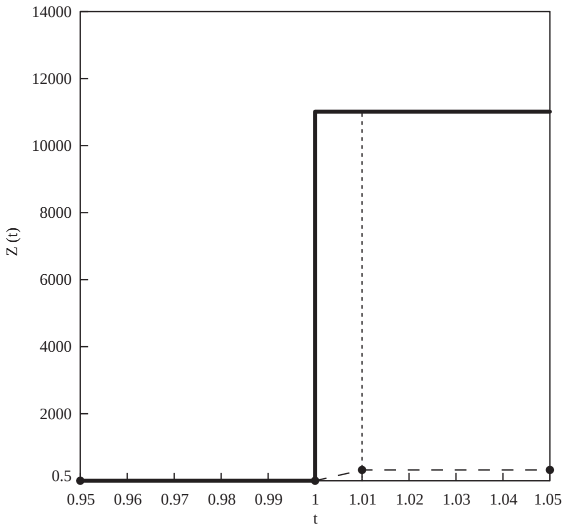

Fig. A.3. Comparison between exact solution of Eq. (A.3) and the response by using the physical impulse concept, choosing an interval $\Delta = 0.01$ and not subdividing it into substeps.

Evaluating the response by using the physical impulse concept, choosing an interval $\Delta = 0.01$ and not subdividing it into substeps there is a substantial difference between this latter and the exact value, as depicted in Fig. A.3 and reported in Table 1. In these tables are also reported the results using the series (21)

Table 1

Time	Physical impulse $\Delta = 0.01$	Exact (MATHEMATICA)	Eq. (A.5)		
			2 terms	10 terms	25 terms
1.00	0.5	0.5	0.5	0.5	0.5
1.01	322	11013	30.5	6421	11013

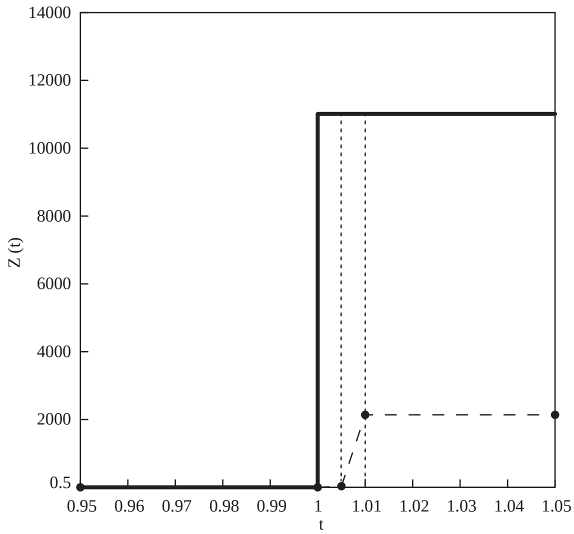

Fig. A.4. Comparison between exact solution of Eq. (A.3) and the response by using the physical impulse concept, choosing an interval $\Delta = 0.01$ and subdividing it into two substeps.

such that the solution is given by

$$z(t) = z(0)\left(\sum_{j=1}^{\infty} \frac{\gamma^j}{j!} U(t-1) + 1\right) \tag{A.5}$$

where $U(.)$ is the unit step function.

Moreover subdividing the interval from 1.00 to 1.01 into two substeps, the solution is not acceptable, as stressed by results in Fig. A.4 and Table 2.

A reliable solution is obtained subdividing the interval into 100 substeps as stressed by the results reported in Fig. A.5 and Table 3.

At this point one may think to overcome this problem, just reducing much the value of the window Δ. But this is totally wrong. For instance choosing an interval $\Delta = 0.0001$ and subdividing it into 100 substeps, the results are totally the same as before as the comparison between Tables 3 and 4 or between Figs. A.5 and A.6 stresses. That means we absolutely need to subdivide the window Δ into several substeps, because it is not a matter of the value of Δ, but of γ.

Identical considerations may be made for other forms of non-linearity such as $g(z, t) = z^2$ or

Table 2

Time	Physical impulse $\Delta = 0.01$	Exact (MATHEMATICA)	Eq. (A.5)		
			2 terms	10 terms	25 terms
1.00	0.5	0.5	0.5	0.5	0.5
1.005	32.68	11013	30.5	6421	11013
1.01	2136	11013	30.5	6421	11013

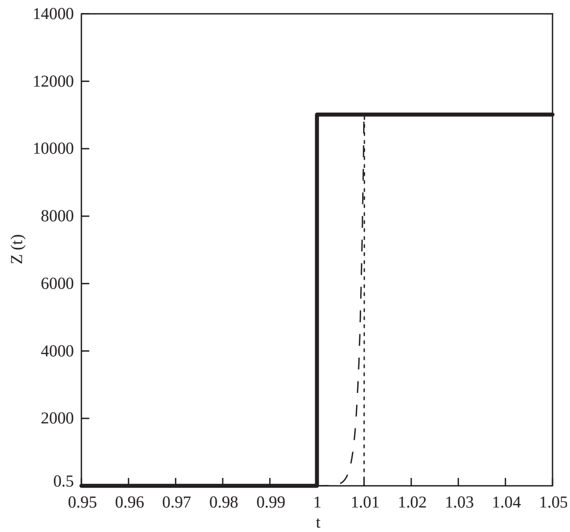

Fig. A.5. Comparison between exact solution of Eq. (A.3) and the response by using the physical impulse concept, choosing an interval $\Delta = 0.01$ and subdividing it into 100 substeps.

Table 3

Time	Physical impulse $\Delta = 0.01$	Exact (MATHEMATICA)	Eq. (A.5)		
			2 terms	10 terms	25 terms
1.00	0.5	0.5	0.5	0.5	0.5
1.0001	0.5526	11013	30.5	6421	11013
1.0002	0.6107	11013	30.5	6421	11013
...	...	...	...	...	...
1.0099	9965	11013	30.5	6421	11013
1.01	11013	11013	30.5	6421	11013

Table 4

Time	Physical impulse $\Delta = 0.0001$	Exact (MATHEMATICA)	Eq. (A.5)		
			2 terms	10 terms	25 terms
1.00	0.5	0.5	0.5	0.5	0.5
1.000001	0.5526	11013	30.5	6421	11013
1.000002	0.6107	11013	30.5	6421	11013
...	...	...	...	...	...
1.000099	9965	11013	30.5	6421	11013
1.0001	11013	11013	30.5	6421	11013

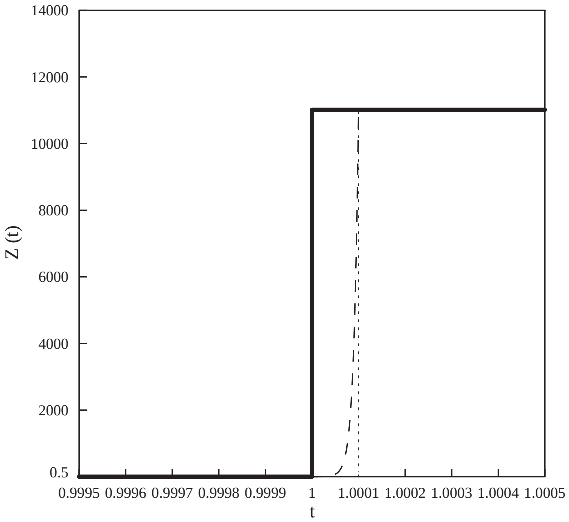

Fig. A.6. Comparison between exact solution of Eq. (A.3) and the response by using the physical impulse concept, choosing an interval $\Delta = 0.0001$ and subdividing it into 100 substeps.

$g(z,t) = z^3$ because the MATHEMATICA also gives the exact solution and the value of the jump always coincides with the value obtained by using series (21).

References

[1] R.L. Stratonovich, Topics in the Theory of Random Noise, Gordon and Breach, New York, 1963.
[2] R.A. Ibrahim, Parametric Random Vibration, Research Studies Press, 1985.
[3] C.W. Gardiner, Handbook of Stochastic Methods, Springer, Berlin, 1990.
[4] Y.K. Lin, Probabilistic Theory of Structural Dynamics, McGraw-Hill, New York, 1967.
[5] L. Arnold, Stochastic Differential Equations: Theory and Applications, Wiley, New York, 1974.
[6] T.T. Soong, M. Grigoriu, Random Vibration of Mechanical Structural Systems, Prentice-Hall, Englewood Cliffs, NJ, 1993.
[7] L.D. Lutes, S.L. Hu, Non normal stochastic response of linear systems, J. Appl. Mech. 112 (2) (1986) 127–141.
[8] L.D. Lutes, Cumulants of stochastic response for linear systems, J. Eng. Mech. ASCE 112 (10) (1986) 1062–1075.
[9] J.B. Roberts, System response random impulses, J. Sound Vibrat. 24 (1972) 23–34.
[10] W. Feller, On the integro-differential equations of completely discontinuous Markov processes, Trans. Amer. Math. Soc. 48 (3) (1948).
[11] R. Iwankiewicz, S.R.K. Nielsen, P. Christensen, Dynamic response of non-linear systems to Poisson distributed pulse train: Markov approach, in: F. Casciati, I. Elishakoff, J.B. Roberts, Non-linear Structural Systems under Random Condition, 1990, pp. 223-228.
[12] M. Di Paola, G. Falsone, A. Pirrotta, Stochastic response analysis of non-linear systems under Gaussian inputs, Probab. Eng. Mech. 7 (1992) 15–21.
[13] M. Di Paola, G. Falsone, Stochastic dynamics of non-linear systems driven by non normal delta correlated process, J. Appl. Mech. 60 (1993) 141–148.
[14] C. Proppe, Stochastic linearization of dynamical systems under parametric Poisson white noise excitation, Int. J. Non-Linear Mech. 38 (4) (2003) 557–564.
[15] C. Proppe, The Wong-Zakai theorem for dynamical systems with parametric Poisson white noise excitation, Int. J. Eng. Sci. 40 (10) (2002) 1165–1178.
[16] R. Iwankiewicz, Dynamical systems with multiplicative Poisson impulse process excitation, Proceedings of Iutam Symposium on Non-linear Stochastic Dynamics, University of Illinois at Urbana-Champaign, 26–30 August, 2002.
[17] R. Iwankiewicz, Dynamic response of non-linear systems to random trains of non-overlapping pulses, Meccanica 37 (1) (2002) 167–178.
[18] R. Iwankiewicz, S.R.K. Nielsen, Dynamic response of non-linear systems to Poisson distributed random impulses, J. Sound Vibrat. 156 (3) (1992) 407–423.
[19] L.D. Lutes, Integral representations of increments of stochastic processes, Meccanica 37 (1) (2002) 193–206.
[20] H.J. Pradlwarter, G.I. Schueller, On advanced Monte Carlo simulation procedures in stochastic structural dynamics, Int. J. Non-linear Mech. 32 (1997) 735–744.
[21] K. Entacher, T. Schell, A. Uhl, Evolutionary optimization of random number generators, Proceedings of the International Conference on Monte Carlo Simulation, Monte Carlo, 18–21 June, 2001.
[22] H. Tanaka, Importance sampling simulation for Compound Poisson processes, Proceedings of the International Conference on Monte Carlo Simulation, Monte Carlo, 18–21 June, 2000
[23] M. Grigoriu, Applied Non-Gaussian Processes. Example, Theory, Simulation, Linear Random Vibration and MATLAB Solutions, Prentice-Hall, Englewood Cliffs, NJ, 1995.
[24] P. Cacciola, G. Muscolino, G. Ricciardi, MCS in the pdf evaluation of non-linear systems under Poisson white

noiseinput, Proceedings of the International Conference on Monte Carlo Simulation, Monte Carlo, 18–21 June, 2000.
[25] S. Caddemi, M. Di Paola, Non-linear system response for impulsive parametric input, J. Appl. Mech. 64 (1997) 642–648.
[26] S. Caddemi, M. Di Paola, Ideal and Physical White Noise in Stochastic Analysis, Int. J. Non-linear Mech. 31 (5) (1995) 581–590.
[27] M. Di Paola, A. Pirrotta, Non-linear systems under impulsive parametric input, Int. J. Non-linear Mech. 34 (1999) 843–851.
[28] G. Samorodnitsky, M.S. Taqqu, Stable non-Gaussian random processes, Stochastic Models with Infinite Variance, Chapman and Hall, London, 1994.