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ORIGINAL PAPER

A comparison index for interval ordering based on generalized Hukuhara difference

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Abstract Interval methods is one option for managing uncertainty in optimization problems and in decision management. The precise numerical estimation of coefficients may be meaningless in real-world applications, because data sources are often uncertain, vague and incomplete. In this paper we introduce a comparison index for interval ordering based on the generalized Hukuhara difference; we show that the new index includes the commonly used order relations proposed in literature. The definition of a risk measure guarantees the possibility to quantify a worst-case loss when solving maximization or minimization problems with intervals.

Keywords Interval-valued functions · Generalized Hukuhara difference · Interval analysis · Interval optimization

1 Introduction

In the history of mathematical programming, the model coefficients have mostly been treated as constant values; however, an optimization problem is often defined in a real-world framework and the values of the coefficients, both in the objective function and the constraints, are imprecise and uncertain, so they have to be modelled in a proper way when approaching a decision problem.

Two main approaches can be identified in order to model uncertainty. Stochastic programming was introduced with Dantzig's book published in the early 1960s (Dantzig 1963) as the extension of linear and nonlinear programming to decision models when coefficients behave as random variables with known probability distributions.

A second approach is fuzzy programming, where the constraints and objective function are modelled as fuzzy sets, supposed that their membership functions are specified. Dubois analyzes the involved uncertainty theories in Dubois (2010), where he underlines that fuzzy set theory was not introduced by Zadeh (1965) in order to replace probability theory, but rather to be engaged synergistically in order to model the information in natural language. Optimal decision making under fuzzy and possibilistic uncertainty has over forty years of research starting in 1970 with the paper by Bellman and Zadeh (1970) and developed with the paper of Tanaka et al. (1974) with a presentation by Zimmermann (1987). Inuiguchi and Ramik (2000) contribute this field with seminal papers, for example, the authors review some fuzzy mathematical programming methods and compare them with stochastic programming in portfolio selection problems.

A well-established setting to model uncertainty and imprecision is based on interval analysis, introduced by Moore (1979) and further developed by many papers and books. Furthermore, interval analysis is a step in handling fuzzy arithmetic, via the well known LU representation of Negoita and Ralescu (1975). A real (compact) interval A consists of a nonempty set of real numbers (eventually a single value) and is represented either by a lower-upper notation $A = [a^-, a^+]$, where $a^- = \min A$ and $a^+ = \max A$, or by a midpoint-radius notation $A = (\widehat{a}; \overline{a})$ in terms of

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its *center* (or *midpoint*) $\widehat{a} = \frac{a^- + a^+}{2}$ and *spread* (or *radius*) $\overline{a} = \frac{a^+ - a^-}{2} \geq 0$.

A starting point in interval optimization is the ranking of intervals and there exists an extended literature on this topic [see e.g. the presentation in chapter 2 of Sengupta and Pal (2009) and Wang and Kerre (2001a, b) for the fuzzy case]. Our work will start with the order relations proposed and analyzed by Ishibuchi and Tanaka (1990); they consider the coefficients in mathematical programming problems as intervals and introduce five order relations for ranking two intervals A and B (in minimization and in maximization problems), based on the comparison of the lower and upper values a^-, a^+, b^-, b^+ and of the midpoint and radius values $\widehat{a}, \overline{a}$ and $\widehat{b}, \overline{b}$.

Optimization problems in which the coefficients of the objective function and the constraints are interval numbers have been investigated in a seminal paper by Tong (1994); the interval of the solution is deduced by taking the maximum value range and minimum value range inequalities as constraint conditions. Sengupta and Pal (2000), studied the same problem and proposed the concept of the acceptability index; see also Sengupta and Pal (2009), an extended presentation of many contributions around the main theme.

In Jiang et al. (2008), a nonlinear interval programming problem is studied when coefficients are uncertain and the key methodology adopted to solve it is to convert the interval single-objective problem into a two-objective problem, which considers both of the average value and the robustness of the design. Ramik extensively worked on the topic and in Ramik (2007) he introduces a class of fuzzy optimization problems with objective function depending on fuzzy parameters.

A recent extended overview of the different approaches reported in the literature to deal with uncertainty in multiple objective linear models through interval programming, can be found in Oliveira and Henggeler Antunes (2007).

In optimization problems, the ordering of intervals is a central aspect and the contribution of the paper goes into the direction to define an appropriate comparison index that indicates the possible relative positions of the two intervals A, B; the proposed index is based on the so called generalized Hukuhara difference for intervals, introduced in Markov (1977, 1979) (called the *inner difference*) and in other papers and authors from different settings [the π-difference in Plotnikova (2005) and Chalco-Cano et al. (2011); the gH-difference in Stefanini (2010)]. As we will see, the ratio $\gamma_{A,B} = \frac{\overline{a} - \overline{b}}{\widehat{a} - \widehat{b}}$ (a real number when $\widehat{a} \neq \widehat{b}$ and possibly $+\infty$ or $-\infty$ when $\widehat{a} = \widehat{b}$) characterizes the five order relations proposed in Ishibuchi and Tanaka (1990) and is useful to define a measure of risk connected with the choice of one of two intervals A and B when they are partially overlapping.

The paper is organized as follows: in Sect. 2 the basic elements of interval mathematics are introduced. The comparison index based on the gH-difference is defined in Sect. 3 and its basic properties are illustrated. Its application to optimization problems with interval coefficients are shown In Sect. 4, we show that the comparison index allows the definition of a risk measure and some examples are given. In Sect. 5, we discuss the use of the comparison index in the context of interval inequalities and we introduce a new partial order for intervals in terms of two parameters that control two kinds of possible risks when the intervals overlap. Conclusions and some challenging ideas are collected in the final section.

2 Interval arithmetic

In the mathematics of intervals (see Dubois and Prade 1980, 2000; Zadeh 1965; Moore 1979; Moore et al. 2009), the ranking of intervals is deduced with an order relation. Given an interval $A = [a^-, a^+]$ with $a^- \leq a^+$, it is possible to represent A in terms of the following values:

$$\widehat{a} = \frac{a^+ + a^-}{2}, \quad \overline{a} = \frac{a^+ - a^-}{2} \tag{1}$$

and we obtain the so called midpoint-radius representation $A = (\widehat{a}; \overline{a})$. It holds $\overline{a} \geq 0$ and we have that the following equalities are true:

$$a^- = \widehat{a} - \overline{a}, \quad a^+ = \widehat{a} + \overline{a}. \tag{2}$$

In the paper, the two notations $A = [a^-, a^+]$ or $A = (\widehat{a}; \overline{a})$ are used for the same interval A. The set of real compact intervals is usually denoted by $\mathbb{IR}$ or simply by $\mathbb{I}$.

The fundamentals of interval arithmetic are given for $A = [a^-, a^+] = (\widehat{a}; \overline{a})$ and $B = [b^-, b^+] = (\widehat{b}; \overline{b})$:

$$A + B = [a^- + b^-, a^+ + b^+] = (\widehat{a} + \widehat{b}; \overline{a} + \overline{b})$$
$$A - B = [a^- - b^+, a^+ - b^-] = (\widehat{a} - \widehat{b}; \overline{a} - \overline{b})$$

If λ is a scalar then:

$$\lambda A = [\min\{\lambda a^-, \lambda a^+\}, \max\{\lambda a^-, \lambda a^+\}] = (\lambda \widehat{a}; |\lambda| \overline{a}).$$

The central notion in the paper is the generalized Hukuhara difference, introduced by several authors with different names [*inner difference* in Markov (1977, 1979); *gH-difference* in Stefanini (2010); *π-difference* in Plotnikova (2005), Chalco-Cano et al. (2011)] and defined as

$$A \ominus_{gH} B = [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}] = (\widehat{a} - \widehat{b}; |\overline{a} - \overline{b}|).$$

The gH-difference satisfies several properties:

1. $A \ominus_{gH} A = \{0\}$;
2. (i) $(A+B) \ominus_{gH} B = A$; (ii) $A \ominus_{gH} (A - B) = B$; (iii) $A \ominus_{gH} (A+B) = -B$;
3. $A \ominus_{gH} B$ exists if and only if $B \ominus_{gH} A$ and $(-B) \ominus_{gH} (-A)$ exist and $A \ominus_{gH} B = (-B) \ominus_{gH} (-A) = -(B \ominus_{gH} A)$;
4. In general, $B - A = A - B$ does not imply $A = B$; but $A \ominus_{gH} B = B \ominus_{gH} A = C$ if and only if $C = -C$ and $C = \{0\}$ if and only if $A = B$;
5. If $B \ominus_{gH} A$ exists then either $A + (B \ominus_{gH} A) = B$ or $B - (B \ominus_{gH} A) = A$ and both equalities hold if and only if $B \ominus_{gH} A$ is a singleton set;
6. If $B \ominus_{gH} A = C$ exists, then for all D either $(B + D) \ominus_{gH} A = C + D$ or $B \ominus_{gH} (A + D) = C - D$.

Given an interval $A = [a^-, a^+] = (\widehat{a}; \overline{a})$, we define the (modified) p-norm $\|A\|_p = (|\widehat{a}|^p + \overline{a}^p)^{\frac{1}{p}}$, $\|A\|_\infty = \max\{|\widehat{a}|, \overline{a}\}$ and $d_H(A,B) = \|A \ominus_{gH} B\|_\infty$.

In order to compare intervals, several (partial) orders have been introduced for intervals $A = [a^-, a^+] = (\widehat{a}; \overline{a})$ and $B = [b^-, b^+] = (\widehat{b}; \overline{b})$ [see Ishibuchi and Tanaka (1990) and, e.g., Jiang et al. (2008); Sengupta and Pal (2009)]:

– Upper versus Lower $\leq_{UL}$ order:

$$A \leq_{UL} B \iff a^+ \leq b^-;$$

this order relation requires that A and B be essentially separated, i.e. $a \leq b$ for all $a \in A$ and all $b \in B$; clearly, the $\leq_{UL}$ order does not present problems in its interpretation: in minimization any possible value in A is preferred to all values of B and, in maximization any value of B is preferred to all values of A.
Some attention is required if the (internal parts of the) intervals overlap: in this situation, the comparison is not immediate and several order relations may be considered.

– Lower and Upper $\leq_{LU}$ order:

$$A \leq_{LU} B \iff a^- \leq b^- \quad \text{and} \quad a^+ \leq b^+ \tag{3}$$

– Center and Max-Width $\leq_{CW_M}$ order:

$$A \leq_{CW_M} B \iff \widehat{a} \leq \widehat{b} \quad \text{and} \quad \overline{a} \geq \overline{b} \tag{4}$$

– Center and min-Width $\leq_{CW_m}$ order:

$$A \leq_{CW_m} B \iff \widehat{a} \leq \widehat{b} \quad \text{and} \quad \overline{a} \leq \overline{b} \tag{5}$$

– Lower and Center $\leq_{LC}$ order:

$$A \leq_{LC} B \iff \widehat{a} \leq \widehat{b} \quad \text{and} \quad a^- \leq b^- \tag{6}$$

– Upper and Center $\leq_{UC}$ order:

$$A \leq_{UC} B \iff \widehat{a} \leq \widehat{b} \quad \text{and} \quad a^+ \leq b^+. \tag{7}$$

When the five inequalities are strict then they can be defined by adding the condition $A \neq B$ (i.e. $\widehat{a} \neq \widehat{b}$ or $\overline{a} \neq \overline{b}$).

The following properties hold.

Proposition 1 *Let $A = [a^-, a^+] = (\widehat{a}; \overline{a})$ and $B = [b^-, b^+] = (\widehat{b}; \overline{b})$ be two intervals. Then*

1. *$(A \leq_{LU} B$ and $B \leq_{CWm} A)$ if and only if $A = B$;*
2. $(A \leq_{LU} B$ and $B \leq_{CWM} A)$ if and only if $A = B$;
3. *$A \leq_{LC} B$ if and only if $(A \leq_{LU} B$ or $A \leq_{CWM} B)$;*
4. *$A \leq_{UC} B$ if and only if $(A \leq_{LU} B$ or $A \leq_{CWm} B)$;*
5. *If $A \leq_{CW_M}$ then $A \leq_{LC} B$;*
6. *If $A \leq_{CW_m}$ then $A \leq_{UC} B$;*
7. *$A \leq_{LU} B$ if and only if $(A \leq_{LC} B$ and $A \leq_{UC} B)$.*

Remark 2 If $\widehat{a} = \widehat{b}$ then it is impossible to have $A <_{LU} B$. If $\widehat{a} < \widehat{b}$ and $\overline{a} = \overline{b}$, then $A <_{LU} B$.

It is well known that the partial order relations above (3)–(7), each define a complete lattice structure $(\mathbb{I}, \leq_O)$ on the space of real intervals $\mathbb{I}$, where $O \in \{LU, LC, UC, CW_M, CW_m\}$ (see e.g. Kaburlasos 2006; Kehagia 2011; Papadakis and Kaburlasos 2010) (Fig. 1). Several authors have introduced interval-based comparison indices to help in decision making with interval imprecision or uncertainty; a comparison index is designed as a tool to help in choosing one of two or more intervals, representing the uncertain or imprecisely defined outcome of a decision problem. For a recent presentation and an extended overview of this topic, see Sengupta and Pal (2009, in particular, chapters 1 and 2).

3 A comparison index based on the gH-difference

A promising feature for a new comparison index for intervals is that it includes the commonly used order relations. We suggest a comparison index (a preliminary study is in Guerra and Stefanini 2011) based on the generalized Hukuhara difference, where the notation for the corresponding interval $A \ominus_{gH} B$ is

$$[(A \ominus_{gH} B)^-, (A \ominus_{gH} B)^+]$$

in the standard interval notation, or

$$\left(\widehat{(A \ominus_{gH} B)}; \overline{(A \ominus_{gH} B)}\right)$$

in the midpoint-radius notation.

A good property for the gH-difference is that it always exists for any pairs of intervals A, B and is useful to

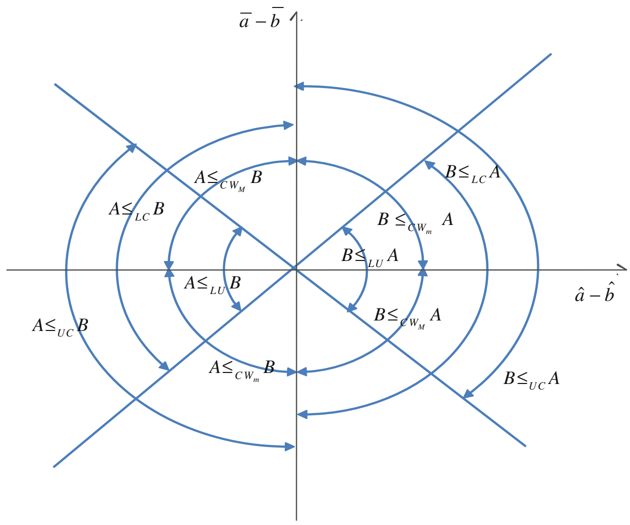

Fig. 1 Representation of the five (partial) order relations in terms of the mid-point and radius of the intervals. The *horizontal axis* represents the midpoint difference $\widehat{a}-\widehat{b}$, while the *vertical axis* represents the radius difference $\overline{a}-\overline{b}$. In the $(\widehat{a}-\widehat{b},\overline{a}-\overline{b})$ plane, the regions where the five orders $A\leq_O B$ are valid, with $O\in\{LU,LC,UC,CW_M,CW_m\}$ are bounded by the axes $\widehat{a}-\widehat{b}$ and $\overline{a}-\overline{b}$ and the bissectrices $\overline{a}-\overline{b}=\widehat{a}-\widehat{b}$ and $\overline{a}-\overline{b}=-(\widehat{a}-\widehat{b})$. Consider that, in all cases, $A\leq_O B$ requires that $\widehat{a}-\widehat{b}\leq 0$ (*left of vertical axis*). In the figure, also the inverse (dual) orders $A\geq_O B \iff B\leq_O A$ are represented (*right of the vertical axis*)

analyze the basic order relations in terms of arithmetic interval operations. Some properties relating the orders and the gH-difference are immediate to prove.

Proposition 3 *Consider two intervals A and B; then*

1. $A\leq_{LU} B \iff \left(A\ominus_{gH} B\right)^{+}\leq 0;$
2. *If* ($A\leq_{CW_M} B$, *or* $A\leq_{CW_m} B$, *or* $A\leq_{LC} B$, *or* $A\leq_{UC} B$) *then* $\left(A\ominus_{gH} B\right)^{-}\leq 0$ *and* $A\,\widehat{\ominus_{gH}}\,B\leq 0$.

We suggest the following comparison index, based on gH-difference:

Definition 4 Given two distinct intervals $A\neq B$, the gH-comparison index of order $p>0$ is defined as

$$CI_p(A,B)=\frac{A\,\widehat{\ominus_{gH}}\,B}{\left\|A\ominus_{gH} B\right\|_p} \tag{8}$$

where $A\ominus_{gH} B$ is the gH-difference, $\forall$ A, B.

The main properties of the index are the following.

Proposition 5 *Given two distinct intervals* $A\neq B$, *we have* $\forall\ p>0$

1. $CI_p(A,B)\in[-1,1]$,
2. $CI_p(A,B)=-CI_p(B,A)$,
3. $CI_p(A,B)=0\iff \widehat{a}=\widehat{b}$,
4. $\left|CI_p(A,B)\right|=1\iff(\overline{a}=\overline{b}$ *and* $\widehat{a}\neq\widehat{b})$,
5. $CI_p(A,B)\geq 0\iff \widehat{a}\geq\widehat{b}$,
6. *An invariance of scale holds:*
$$CI_p(kA,kB)=\begin{cases}CI_p(A,B) & \text{if } k>0\\ -CI_p(A,B) & \text{if } k<0,\end{cases}$$
7. $CI_p(A+C,B+C)=CI_p(A,B)$.

Proof Properties from (1) to (6) are immediate. For (7) it is sufficient to consider that the following equality holds: $(A+C)\ominus_{gH}(B+C)=\left(A\ominus_{gH} B\right)$ (see Stefanin 2010).□

In this paper, we will investigate the comparison index with $p=2$; we denote it by $CI(A,B)$, and we write, from Definition 4,

$$CI(A,B)=\frac{A\,\widehat{\ominus_{gH}}\,B}{\left\|A\ominus_{gH} B\right\|_2}=\frac{\widehat{a}-\widehat{b}}{\sqrt{\left(\widehat{a}-\widehat{b}\right)^2+\left(\overline{a}-\overline{b}\right)^2}}.$$

Assuming that $\widehat{a}\neq\widehat{b}$, we can define the ratio

$$\gamma_{A,B}=\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} \tag{9}$$

and write the following expression for $CI(A,B)$:

$$CI(A,B)=\begin{cases}\frac{1}{\sqrt{1+(\gamma_{A,B})^2}} & \text{if } \widehat{a}>\widehat{b}\\ 0 & \text{if } \widehat{a}=\widehat{b}\\ \frac{-1}{\sqrt{1+(\gamma_{A,B})^2}} & \text{if } \widehat{a}<\widehat{b}\end{cases}. \tag{10}$$

Remark 6 We can define the parameter $\gamma_{A,B}$ also in the case where $\widehat{a}=\widehat{b}$, by assuming $\gamma_{A,B}=+\infty$ if $\overline{a}>\overline{b}$ and $\gamma_{A,B}=-\infty$ if $\overline{a}<\overline{b}$; finally, if $A=B$, we define $\gamma_{A,B}=0$.

Remark 7 For the parameter $\gamma_{A,B}$ the following properties are immediate:

1. an invariance of scale holds:
$$\gamma_{kA,kB}=\begin{cases}\gamma_{A,B} & \text{if } k>0\\ -\gamma_{A,B} & \text{if } k<0,\end{cases}$$
2. $\gamma_{A+C,B+C}=\gamma_{A,B}$.

The parameter $\gamma_{A,B}$ can be determined for all the possible positions of two intervals $A=(\widehat{a};\overline{a})=[a^-,a^+]$ and $B=(\widehat{b};\overline{b})=[b^-,b^+]$ (see figure 2.1 in Sengupta and Pal 2009) and in particular it characterizes how the two intervals A and B overlap (Fig. 2).

Case 1 is an unambiguous one and the strict dominance is verified: $B\leq_{LU} A$ and $\widehat{a}-\widehat{b}\geq\overline{a}+\overline{b}$. Here we have $\left|\gamma_{A,B}\right|\leq 1$.

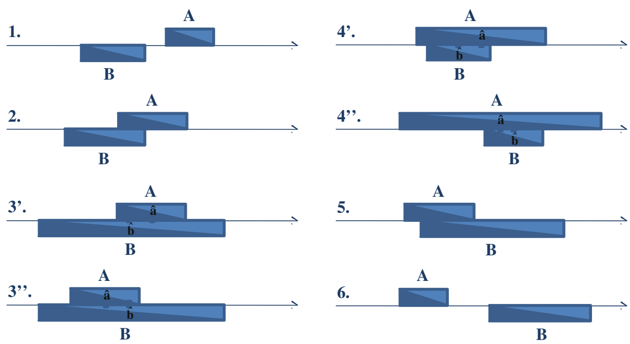

Fig. 2 The eight possible positions of two real intervals A and B. In the extreme cases *1* and *6*, the two intervals do not overlap; cases *2* and *5* present a partial overlapping; cases *3* and *4* present full inclusion ($A \subset B$ in case *3* and $B \subset A$ in case *4*). Cases *3* and *4* are divided into two subcases to distinguish when $\widehat{a}$ is less than $\widehat{b}$ or $\widehat{a}$ is greater than $\widehat{b}$

In cases from 2 to 5 the role of uncertainty is crucial. In case 2 we have $b^- \leq a^-$ and $b^+ \leq a^+$ or equivalently $B \leq_{LU} A$.

Case 3 has to be split up into the two sub-cases $3'$ and $3''$ depending on the relative positions of the midpoints. In case $3'$ we have $\widehat{a} > \widehat{b}$ and $1 + \gamma_{A,B} \leq 0$. In case $3''$ we have $\widehat{a} < \widehat{b}$ and $1 - \gamma_{A,B} \leq 0$.

Also the case 4 has to be split up into the two sub-cases $4'$ and $4''$ depending on the relative positions of the midpoints. In case $4'$ we have $\widehat{a} > \widehat{b}$ and $1 + \gamma_{A,B} \leq 0$. In case $4''$ we have $\widehat{a} < \widehat{b}$ and $1 - \gamma_{A,B} \leq 0$.

In case 5 we have $a^- \leq b^-$ and $a^+ \leq b^+$ or equivalently $A \leq_{LU} B$.

Case 6 is again an unambiguous case and the strict dominance is verified: $A \leq_{LU} B$ and $\widehat{b} - \widehat{a} \geq \overline{a} + \overline{b}$. Here we have again, as in case 1, $|\gamma_{A,B}| \leq 1$.

In conclusion we state then when $\widehat{a} \neq \widehat{b}$ and $|\gamma| \leq 1$ then the decision can be based on the values $\widehat{a}$ and $\widehat{b}$ because no risk is produced; on the other hand, when $|\gamma| > 1$ then a more careful analysis has to be carried out because a risky situation arises.

In terms of $\gamma_{A,B}$, the five order relations of Sect. 2 can be characterized as follows:

Proposition 8 *Let* $A = [a^-, a^+] = (\widehat{a}; \overline{a})$ *and* $B = [b^-, b^+] = (\widehat{b}; \overline{b})$ *be two intervals and suppose that* $\widehat{a} < \widehat{b}$. *Then it holds that*

1. $A <_{LU} B \iff \gamma_{A,B} \in [-1, 1]$,
2. $A <_{CW_M} B \iff \gamma_{A,B} \leq 0$,
3. $A <_{CW_m} B \iff \gamma_{A,B} \geq 0$,
4. $A <_{LC} B \iff \gamma_{A,B} \leq 1$,
5. $A <_{UC} B \iff \gamma_{A,B} \geq -1$.

Proof For (1) we have $A \leq_{LU} B$ if and only if $\widehat{a} - \overline{a} \leq \widehat{b} - \overline{b}, \widehat{a} + \overline{a} \leq \widehat{b} + \overline{b}$, i.e. $\widehat{a} - \widehat{b} \leq \overline{a} - \overline{b}, \overline{a} - \overline{b} \leq \widehat{b} - \widehat{a}$; considering that $\widehat{a} - \widehat{b} < 0$, we obtain $\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} \leq 1$, $-\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} \leq 1$. On the other hand, if $-1 \leq \frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} \leq 1$ and $\widehat{a} - \widehat{b} < 0$, we have $\overline{a} - \overline{b} \leq \widehat{b} - \widehat{a}, \overline{a} - \overline{b} \geq \widehat{a} - \widehat{b}$ and we obtain $a^+ \leq b^+, b^- \geq a^-$.

For (2) we have $A \leq_{CWM} B$ if and only if $\widehat{a} \leq \widehat{b}, \overline{a} \geq \overline{b}$; considering that $\widehat{a} - \widehat{b} < 0$, this is equivalent to $\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} \leq 0$.

For (3) we have $A \leq_{CWm} B$ if and only if $\widehat{a} \leq \widehat{b}, \overline{a} \leq \overline{b}$; considering that $\widehat{a} - \widehat{b} < 0$, this is equivalent to $\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} \geq 0$.

For (4) we have $A \leq_{LC} B$ if and only if $\widehat{a} - \overline{a} \leq \widehat{b} - \overline{b}, \widehat{a} \leq \widehat{b}$; considering that $\widehat{a} - \widehat{b} < 0$, we obtain $\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} \leq 1$. On the other hand, if $\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} \leq 1$ and $\widehat{a} - \widehat{b} < 0$, we have $\overline{a} - \overline{b} \geq \widehat{a} - \widehat{b}$ and we obtain $b^- \geq a^-$.

For (5) we have $A \leq_{UC} B$ if and only if $\widehat{a} \leq \widehat{b}, \widehat{a} + \overline{a} \leq \widehat{b} + \overline{b}$; considering that $\widehat{a} - \widehat{b} < 0$, we obtain $\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} \geq -1$. On the other hand, if $-1 \leq \frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}}$ and $\widehat{a} - \widehat{b} < 0$, we also have $\overline{a} - \overline{b} \leq \widehat{b} - \widehat{a}$, i.e., $a^+ \leq b^+$. $\square$

In the case $p = 2$ and assuming $\widehat{a} \neq \widehat{b}$, the comparison index in (8) is such that

$$CI^2(A, B) = \frac{1}{1 + (\gamma_{A,B})^2} \in [0, 1]$$

and, for any interval A and B,

$$(1 + \gamma_{A,B}^2) CI^2(A, B) = 1.$$

Remark 9 It is possible to define a comparison index for fuzzy intervals, using the α-cut representation. If u and v are two fuzzy intervals with α-cuts (with the obvious meaning of the symbols applied level-wise) $[u]_\alpha = [u_\alpha^-, u_\alpha^+] = (\widehat{u}_\alpha, \overline{u}_\alpha)$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+] = (\widehat{v}_\alpha, \overline{v}_\alpha)$, respectively, we consider

$$CI_{p,\alpha}(u, v) = CI_p([u]_\alpha, [v]_\alpha) \quad p > 0, \ \alpha \in [0, 1]$$

and, e.g., any possibilistic average

$$CI_p(u, v) = \int_0^1 \varphi(\alpha) CI_{p,\alpha}(u, v) \mathrm{d}\alpha, \quad p > 0$$

can be used with a weighting function $\varphi : [0, 1] \longrightarrow [0, 1]$ such that

$$\int_0^1 \varphi(\alpha) \mathrm{d}\alpha = 1.$$

Remark 10 It is interesting to observe that the squared comparison index $CI^2(A,B)$ is a Cauchy type function. If $\widehat{a}-\widehat{b}$ and $\overline{a}-\overline{b}$ are uncorrelated random variables from a normal distribution $N(0, \sigma^2)$, then the marginal probability density function of the ratio $\gamma_{A,B}=\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}}$ is a Cauchy probability density function with median equal to zero and shape factor equal to one, i.e., $\frac{1}{\pi}CI_2^2(A,B)=p(\gamma_{A,B};1)$ where

$$p(x;a)=\frac{a}{\pi(a^2+x^2)}$$

is the Cauchy density function. Using the properties of $p(x;\ a)$, it follows that $\frac{1}{2}$ is the marginal probability to obtain $\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}}\in[-1,1]$, i.e., the probability to have $A\leq_{LU}B$, when $\widehat{a}-\widehat{b},\overline{a}-\overline{b}\in N(0,\sigma^2)$ are randomly generated; analogously, the probability to have $A\leq_{LC}B$ is $\frac{3}{4}$; the probability to have $A\leq_{UC}B$ is $\frac{3}{4}$; the probability to have $A\leq_{CW_M}B$ is $\frac{1}{2}$ and the probability to have $A\leq_{CW_m}B$ is $\frac{1}{2}$.

4 Comparison index and optimization

We consider the choice between two intervals $A=[a^-,a^+]=(\widehat{a},\overline{a})$ and $B=[b^-,b^+]=(\widehat{b},\overline{b})$ and we will prefer to choose A instead of B if "A is smaller than B" (for a minimization) or if "A is greater than B" (for a maximization).

The notions of "smaller than" and "greater than" are strictly related to the order relation we have in mind to rank intervals; in particular, with respect to the (partial) order we have selected, two intervals may not be comparable and in such situations it is not immediate to chose what interval will be the best one.

This is true, in particular, if the internal parts of the intervals overlap. In fact, if we are minimizing and $a^+\leq b^-$, then interval A (as a whole) is smaller than interval B because $a\leq b$ for all possible values $a\in A$ and $b\in B$; in this case, A is commonly chosen for the minimum, or B for the maximum. If the intervals overlap then the choice will depend on their relative position and the availability of some criterion is necessary to help for a final decision.

The comparison index $CI(A,B)$, and in particular the ratio $\gamma_{A,B}=\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}}$, can be helpful in the framework of a typical optimization problem with interval-valued objective function.

In the rest of the paper, we make use of the following simple equalities (if $\widehat{a}\neq\widehat{b}$):

$$\begin{aligned}a^- - b^- &= (\widehat{a}-\widehat{b})-(\overline{a}-\overline{b})\\ &=(\widehat{a}-\widehat{b})\left(1-\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}}\right)\\ &=(\widehat{a}-\widehat{b})(1-\gamma_{A,B})\end{aligned}$$

and

$$\begin{aligned}a^+ - b^+ &= (\widehat{a}-\widehat{b})+(\overline{a}-\overline{b})\\ &=(\widehat{a}-\widehat{b})(1+\gamma_{A,B}).\end{aligned}$$

4.1 The case of minimization

Using the interval orders described above and generally adopted in interval minimization, we will in advance choose A with respect to B if $\widehat{a}<\widehat{b}$; as we have discussed, possible ambiguities appear when the two intervals overlap.

We can use the ratio $\gamma_{A,B}=\frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}}$ to define a "risk" measure, when we choose A instead of B only on the basis of $\widehat{a}<\widehat{b}$. With respect to the midpoint values $\widehat{a}$ and $\widehat{b}$, the quantity $\widehat{b}-\widehat{a}>0$ is called the *mid-gain* associated to the choice of A. If the difference $\widehat{b}-\widehat{a}$ is high, then we "expect" a good choice, but clearly all depends on how the two intervals eventually overlap.

Assume a positive mid-gain $\widehat{b}-\widehat{a}>0$ and consider the whole intervals; we distinguish the following cases:

1. $a^-\leq b^-$; in this case, we have both $A<_{LC}B$ and $A<_{CWM}B$ and for any value $b\in B$ there exist elements $a\in A$ such that $a<b$, i.e. any element in B is worse than some elements of A. In this case, we have
$$\frac{a^- - b^-}{\widehat{a}-\widehat{b}}=1-\gamma_{A,B}>0.$$
2. $a^->b^-$; in this case, the relations $A<_{LC}B$ and $A\leq_{CW_M}$ are not valid and some values $b\in B$ are better than all elements of A; the positive difference $a^- - b^-$ measures the possible worst-case loss, and it is interesting to compare it with the mid-gain. The ratio of worst-case loss to mid-gain is given by
$$\frac{a^- - b^-}{\widehat{a}-\widehat{b}}=1-\gamma_{A,B}<0.$$
3. $a^+\leq b^+$; in this case, we have both $A<_{UC}B$ and $A\leq_{CW_m}$ and some value $b\in B$ is greater than all elements $a\in A$. In this case, we have
$$\frac{a^+ - b^+}{\widehat{a}-\widehat{b}}=1+\gamma_{A,B}>0.$$
4. $a^+>b^+$; in this case, the relations $A<_{UC}B$ and $A\leq_{CW_m}$ are not valid and some values $a\in A$ are worse than all elements of B; the positive difference $a^+ - b^+$

measures the possible worst-case loss, and we compare it with the mid-gain. The ratio worst-case loss to mid-gain is now given by

$$\frac{a^+ - b^+}{\widehat{a} - \widehat{b}} = 1 + \gamma_{A,B} < 0.$$

Considering all the possible situations, we deduce that a positive worst-case loss appears in cases 2 and 4, i.e., when $1 - \gamma_{A,B} < 0$ or when $1 + \gamma_{A,B} < 0$.

If $1 - \gamma_{A,B} < 0$, as soon as $a^- - b^-$ becomes more negative, the term $1 - \gamma_{A,B}$ gives the relative amount of the possible loss with respect to the mid-gain; for example, if $1 - \gamma_{A,B} = -1$ (i.e. $\gamma_{A,B} = 2$) the possible loss equals the mid-gain. If the parameter $\gamma_{A,B}$ increases then the possible relative loss increases.

If $1 + \gamma_{A,B} < 0$, as soon as $a^+ - b^+$ becomes more positive, the term $1 + \gamma_{A,B}$ gives the relative amount of the possible loss with respect to the mid-gain; for example, if $1 + \gamma_{A,B} = -1$ (i.e. $\gamma_{A,B} = -2$) the possible loss equals the mid-gain.

In conclusion, we have the following interpretation of the parameter $\gamma_{A,B}$, and consequently of the comparison index $CI(A, B)$:

- If $\widehat{a} < \widehat{b}$ and $-1 \le \gamma_{A,B} \le 1$, no worst-case loss appears and we have $A <_{LU} B$.
- If $\widehat{a} < \widehat{b}$ and $\gamma_{A,B} > 1$, a worst-case loss appears on the left side of the intervals (i.e. some values of B are better than all values of A); the quantity $1 - \gamma_{A,B} < 0$ gives a measure of this first kind of risk.
- If $\widehat{a} < \widehat{b}$ and $\gamma_{A,B} < -1$, a worst-case loss appears on the right side of the intervals (i.e. some values of A are worse than all values of B); the quantity $1 + \gamma_{A,B} < 0$ gives a measure of this second kind of risk.

4.2 The case of maximization

In an analogous way as in minimization, we will essentially choose A with respect to B if $\widehat{a} > \widehat{b}$ and the ratio $\gamma_{A,B} = \frac{\overline{a} - \overline{b}}{\widehat{a} - \widehat{b}}$ can be used to define a "risk" measure, when we choose A instead of B on the basis of the inequality $\widehat{a} > \widehat{b}$.

The results for a maximization problem can be rephrased; with respect to the midpoint values $\widehat{a}$ and $\widehat{b}$, the quantity $\widehat{a} - \widehat{b} > 0$ is called the *mid-gain* associated to the choice of A. In some sense, the results are dual with respect to the ones obtained in the case of minimization.

Assuming a positive mid-gain $\widehat{a} - \widehat{b} > 0$ we distinguish the following cases:

1. $a^- < b^-$; in this case, there are possible values $a \in A$ that are worse than all values $b \in B$, i.e., there exists a worst-case loss $a^- - b^- < 0$. The ratio of worst-case loss to mid-gain is
$$\frac{a^- - b^-}{\widehat{a} - \widehat{b}} = 1 - \gamma_{A,B} < 0.$$
2. $a^- \ge b^-$; in this case, some values $b \in B$ are worse than all elements of A; there is no worst-case loss and
$$\frac{a^- - b^-}{\widehat{a} - \widehat{b}} = 1 - \gamma_{A,B} > 0.$$
3. $a^+ < b^+$; in this case, all elements $a \in A$ are worse than some elements of B; the negative difference $a^+ - b^+$ measures the possible worst-case loss, and the ratio of worst-case loss to mid-gain is given by
$$\frac{a^+ - b^+}{\widehat{a} - \widehat{b}} = 1 + \gamma_{A,B} < 0.$$
4. $a^+ \ge b^+$; in this case, there are elements $a \in A$ that are better than all values in B and there is no worst-case loss; we have
$$\frac{a^+ - b^+}{\widehat{a} - \widehat{b}} = 1 + \gamma_{A,B} > 0.$$

We deduce that a positive worst-case loss appears in cases 1 and 3, i.e., when $1 - \gamma_{A,B} < 0$ or when $1 + \gamma_{A,B} < 0$. We have the following interpretation of the term $\gamma_{A,B}$:

- If $\widehat{a} > \widehat{b}$ and $-1 \le \gamma_{A,B} \le 1$, no worst-case loss appears.
- If $\widehat{a} > \widehat{b}$ and $\gamma_{A,B} > 1$, a worst-case loss appears on the left side of the intervals (i.e. some values of A are worse than all values of B); the quantity $1 - \gamma_{A,B} < 0$ gives a measure of this kind of risk.
- If $\widehat{a} > \widehat{b}$ and $\gamma_{A,B} < -1$, a worst-case loss appears on the right side of the intervals (i.e. some values of B are better than all values of A); the quantity $1 + \gamma_{A,B} < 0$ gives a measure of this kind of risk.

4.3 Definition of a risk measure and some examples

The discussion for the minimization and maximization problems brings out some considerations: we may possibly face two kinds of risk, due to the possibility of a worst-case loss if we choose on the basis of the midpoint values $\widehat{a}$ and $\widehat{b}$.

Definition 11 A *type I risk* is defined to be the possible worst-case loss when we choose A instead of B, and there exist elements in B which are better than all elements of A; this happens in a minimization problem, when $1 - \gamma_{A,B} < 0$; and happens in a maximization problem when $1 + \gamma_{A,B} < 0$.

Definition 12 A *type II risk* is defined to be the possible worst-case loss when we choose A instead of B, and there exist elements in A which are worse than all elements of B; this happens in a minimization problem, when $1 + \gamma_{A,B} < 0$; and happens in a maximization problem when $1 - \gamma_{A,B} < 0$.

The two definitions are illustrated in Fig. 3: the two types of risk for optimization problems are represented from top to bottom.

Definition 13 Given two intervals $A = [a^-, a^+] = (\widehat{a}, \overline{a})$ and $B = [b^-, b^+] = (\widehat{b}, \overline{b})$ with $\widehat{a} \neq \widehat{b}$, we define the following *risk measure* $R(A, B)$ relative to the comparison of A and B:

$$R(A,B) = \min\{(1-\gamma_{A,B})_-, (1+\gamma_{A,B})_-\} \tag{11}$$

where $(x)_-$ is the negative part of $x \in \mathbb{R}$, defined by

$$(x)_- = \left\langle \begin{array}{ll} -x & \text{if } x<0 \\ 0 & \text{if } x \geq 0. \end{array} \right.$$

We always have $R(A, B) = R(B, A) \geq 0$. As a function of $\gamma_{A,B}$, the risk measure $R(A, B)$ is null if and only if $\gamma_{A,B} \in [-1, 1]$, and increases as soon as $\gamma_{A,B}$ goes far from $[-1,1]$.

Proposition 14 *Consider two intervals A and B such that $\widehat{a} \neq \widehat{b}$. With reference to the defined comparison index $CI(A, B)$, we have*

$$R(A,B) = 0 \text{ if and only if } CI(A,B) \in \left]-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right[.$$

Proof We have seen that $CI^2(A,B) = \frac{1}{1+\gamma_{A,B}^2}$ so that $\gamma_{A,B} \in [-1, 1]$ if and only if $CI^2(A,B) \in [\frac{1}{2}, 1]$ and the conclusion follows immediately. □

Remark 15 It is important to remark that, for the same minimization (or maximization) problem, the two types of risk cannot coexist; in fact, it is impossible that both $1 + \gamma_{A,B} < 0$ and $1 - \gamma_{A,B} < 0$ hold simultaneously for the same intervals A and B.

With the definition of the risk measure $R(A, B)$ and taking into account that only one of the two values $1 + \gamma_{A,B}$ or $1 - \gamma_{A,B}$ can be negative, we have the following result for the total worst-case loss, when we choose A or B in terms of the values $\widehat{a}$ and $\widehat{b}$.

For a minimization, we choose A if $\widehat{a} < \widehat{b}$ and, if $R(A, B) > 0$, the worst case loss is

$$L_{\min}(A,B) = (\widehat{b} - \widehat{a})R(A,B) > 0 \tag{12}$$

resulting from one of two possible worst-cases losses

- a type-I loss, if $1 - \gamma_{A,B} < 0$;
- a type-II loss, if $1 + \gamma_{A,B} < 0$.

For a maximization, we choose A if $\widehat{a} > \widehat{b}$ and, if $R(A, B) > 0$, the worst case loss is

$$L_{\max}(A,B) = (\widehat{a} - \widehat{b})R(A,B) > 0 \tag{13}$$

resulting from one of two possible worst-cases losses

- a type-I loss, if $1 + \gamma_{A,B} < 0$;
- a type-II loss, if $1 - \gamma_{A,B} < 0$.

Given then two intervals A and B, it is preliminary possible to state from the value of $\gamma_{A,B}$ how relevant may be the risk connected with the choice.

A possible meaningful interpretation of the worst case loss $L_{\min}(A,B) = (\widehat{b} - \widehat{a})R(A,B) > 0$ can be given in a problem of a investment choice between two stocks that have different uncertainties. As usual, in financial markets, the favorable source of uncertainty is the volatility. Suppose A is a stock and the analysis of the last two years time series of prices enable us to say that its returns move in the interval $[-3\,\%, 15\,\%]$. The same analysis for a stock B shows that the returns move in a range $[-8\,\%, 24\,\%]$ revealing uncertainty in the period under consideration. A decision between the two stocks has to be taken. The mid-point forms are $A = (6;9)$, $B = (8;16)$.

The following value can be computed: $1 - \gamma_{A,B} = 1 - \frac{\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} = \frac{\widehat{a}-\widehat{b}-\overline{a}-\overline{b}}{\widehat{a}-\widehat{b}} = \frac{a^- - b^-}{\widehat{a}-\widehat{b}} = \frac{-3+8}{6-8} = \frac{-5}{2} < 0$ and it can be

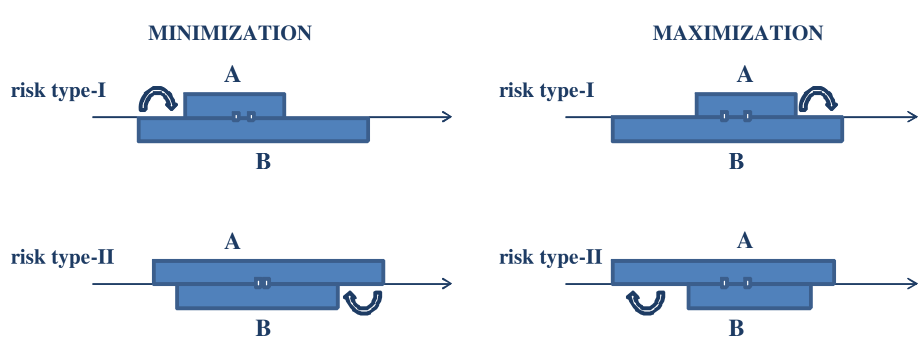

Fig. 3 In a decision process we indicate the interval values that produce *risk type-I* or *risk type-II* in a minimization or maximization problem

interpreted as the possibility to choose A instead B loosing 2 % in the midpoint value of the returns in order to avoid a worst case risk type-I loss of -5 %. In fact, all values of B between -8 % and -3 % are returns that we prefer to avoid also if the mid-point $\widehat{a}$ is smaller than $\widehat{b}$.

The value of $\gamma_{A,B}$ contributes in the possibility to take a good decision, depending on my personal aversion or propensity to risk or on my actual believes or feelings.

4.4 The case of intervals with equal midpoint values

As we have seen, the parameter $\gamma_{A,B}$ is not well defined when we have $\widehat{a} = \widehat{b}$ and $\overline{a} \neq \overline{b}$, i.e., when we have to compare A and B on the basis of the uncertainty induced by the difference in the radius. In the following discussion, we will see that the parameter $\gamma_{A,B}$ may help in the analysis of this critical case. We will see the risk associated to the choice of A against B, when it is supposed we have a preference for A.

In a minimization problem, the choice of A can be viewed as the result of the comparison of B with a (small) modification $A'_\varepsilon = (\widehat{a} - \varepsilon, \overline{a})$ obtained by reducing the midpoint value of A with a small $\varepsilon > 0$. For all values of $\varepsilon > 0$, the ratio

$$\gamma'_\varepsilon = \gamma_{A'_\varepsilon,B} = \frac{\overline{a} - \overline{b}}{-\varepsilon}$$

is well defined and we have the following risk factors

$$1 - \gamma'_\varepsilon = \frac{\overline{a} - \overline{b} + \varepsilon}{\varepsilon}, \quad 1 + \gamma'_\varepsilon = \frac{\overline{b} - \overline{a} + \varepsilon}{\varepsilon}$$

It follows that, for small $\varepsilon > 0$,

$$(1 - \gamma'_\varepsilon < 0 \text{ and } 1 + \gamma'_\varepsilon > 0) \text{ if and only if } \overline{a} < \overline{b}$$
$$(1 - \gamma'_\varepsilon > 0 \text{ and } 1 + \gamma'_\varepsilon < 0) \text{ if and only if } \overline{a} > \overline{b}$$

and we conclude that, if we choose A in minimization, then a risk of type I appears if $\overline{a} < \overline{b}$ and a risk of type II appears if $\overline{a} > \overline{b}$.

In a maximization problem, the choice of A can be viewed as the result of the comparison of B with a (small) modification $A''_\varepsilon = (\widehat{a} + \varepsilon, \overline{a})$ obtained by increasing the midpoint value of A with a small $\varepsilon > 0$. The ratio $\gamma''_\varepsilon = \gamma_{A''_\varepsilon,B} = \frac{\overline{a} - \overline{b}}{\varepsilon}$ is well defined and, for small $\varepsilon > 0$, we obtain

$$(1 - \gamma''_\varepsilon > 0 \text{ and } 1 + \gamma''_\varepsilon < 0) \text{ if and only if } \overline{a} < \overline{b}$$
$$(1 - \gamma''_\varepsilon < 0 \text{ and } 1 + \gamma''_\varepsilon > 0) \text{ if and only if } \overline{a} > \overline{b}$$

and we conclude that, if we choose A in maximization, then a risk of type I appears if $\overline{a} < \overline{b}$ and a risk of type II appears if $\overline{a} > \overline{b}$.

The following example is taken from Sengupta and Pal (2009).

Example 16 Compare $A = (7;5)$ to $B_1 = (13;1)$, $B_2 = (10;1)$, $B_3 = (7;1)$, $B_4 = (4;1)$, $B_5 = (1;1)$ for minimization or maximization. Denoting $\gamma_{A,B\_i} = \gamma_i$; we obtain $\gamma_1 = -\frac{2}{3}, \gamma_2 = -\frac{4}{3}, \gamma_3 = \nexists, \gamma_4 = \frac{4}{3}, \gamma_5 = \frac{2}{3}$ so that the risk factors are $1 - \gamma_1 = \frac{5}{3} > 0, 1 + \gamma_1 = \frac{1}{3} > 0, 1 - \gamma_2 = \frac{7}{3} > 0, 1 + \gamma_2 = -\frac{1}{3} < 0, 1 - \gamma_4 = -\frac{1}{3} > 0, 1 + \gamma_4 = \frac{7}{3} > 0, 1 - \gamma_5 = \frac{1}{3} > 0, 1 + \gamma_5 = \frac{5}{3} > 0$ and the comparisons are in the following Table 1.

Remark 17 In the case $\widehat{a} = \widehat{b}$ (and $\overline{a} \neq \overline{b}$) any choice A or B for minimization or maximization has a risk. A risk of type I appears if we choose the smallest interval (and some elements in the biggest interval are better); a risk of type II appears if we choose the biggest interval.

5 The comparison index for interval inequalities

The comparison index can be used also in situations where a variable interval of the form Ax is compared with a fixed interval B.

Example 18 We consider $x \geq 0$ and we compare $Ax = [10x, 20x] = (15x; 5x)$ to $B = [5, 35] = (20; 15)$; the risk factors are

$$1 - \gamma_{Ax,B} = \frac{2x - 1}{3x - 4} \quad \text{and} \quad 1 + \gamma_{Ax,B} = \frac{4x - 7}{3x - 4};$$

If we are minimizing, we have $\widehat{ax} < \widehat{b}$ when $3x - 4 < 0$, i.e., when $x \in [0, \frac{4}{3}[$; if $0 \leq x < \frac{1}{2}$ we have $1 - \gamma_{Ax,B} > 0$; if

Table 1 Comparison of intervals for the selection of minimum and maximum

Comparison	*minimization*		$L_{\min}$	*mid-gain*	*maximization*		$L_{\max}$
A to B_1	A	*no risk*	0	6	B_1	*no risk*	0
A to B_2	A	*risk II*	1	3	B_2	*risk I*	1
A to B_3	{ A	*risk II*	4	0	{ B_3	*risk II*	4
	B_3	*risk I*	4	0	A	*risk I*	4
A to B_4	B_4	*risk I*	1	3	A	*risk II*	1
A to B_5	B_5	*no risk*	0	6	A	*no risk*	0

$\frac{1}{2} < x < \frac{4}{3}$ we have $1 - \gamma_{Ax,B} < 0$; and $1 + \gamma_{Ax,B} > 0$ for all $x \in [0, \frac{4}{3}[$. It follows that a risk of type I appears for values of $x \in \left]\frac{1}{2}, \frac{4}{3}\right[$ with $L_{\min} = 10x - 5$; and there is no risk for all $x \in [0, \frac{1}{2}[$. If we are maximizing, we have $\widehat{a}x > \widehat{b}$ when $3x - 4 > 0$, i.e., when $x \in \left]\frac{4}{3}, +\infty\right[$; if $\frac{4}{3} < x < \frac{7}{4}$ we have $1 + \gamma_{Ax,B} < 0$; if $x > \frac{7}{4}$ we have $1 + \gamma_{Ax,B} > 0$; and $1 - \gamma_{Ax,B} > 0$ for all $x \in \left]\frac{4}{3}, +\infty\right[$. It follows that a risk of type I appears for values of $x \in \left]\frac{4}{3}, \frac{7}{4}\right[$ with $L_{\max} = 10x - 5$; and there is no risk for all $x \in \left]\frac{7}{4}, +\infty\right[$.

Example 19 We consider $x_1, x_2 \geq 0$ and we are interested to analyze the risk of the inequality "$A(x_1, x_2) < B$" (minimization), where $A(x_1, x_2) = (500x_1 + 100x_2, 20x_1 + 15x_2)$ and $B = (220;10)$. We have $\widehat{a}(x_1, x_2) = 500x_1 + 100x_2$ and $\widehat{b} = 220$; in terms of midpoints, the inequality requires $500x_1 + 100x_2 < 220$. We have $\gamma_{A(x_1,x_2),\,B} = \frac{20x_1 + 15x_2 - 10}{500x_1 + 100x_2 - 220}$, so that $1 - \gamma_{A(x_1,x_2),\,B} = \frac{480x_1 + 85x_2 - 210}{500x_1 + 100x_2 - 220}$ and $1 + \gamma_{A(x_1,x_2),\,B} = \frac{520x_1 + 115x_2 - 230}{500x_1 + 100x_2 - 220}$. It follows that there is a risk of type I with $L_{\min} = 480x_1 + 85x_2 - 210$ if $x_1, x_2 \geq 0$ belong to the polytope defined by the inequalities

$$\begin{cases} 500x_1 + 100x_2 < 220 \\ 480x_1 + 85x_2 > 210; \end{cases}$$

there is a risk of type II with $L_{\min} = 520x_1 + 115x_2 - 230$ if $x_1, x_2 \geq 0$ belong to the polytope defined by inequalities

$$\begin{cases} 500x_1 + 100x_2 < 220 \\ 520x_1 + 115x_2 > 230; \end{cases}$$

there is no risk if $x_1, x_2 \geq 0$ belong to the polytope defined by inequalities

$$\begin{cases} 500x_1 + 100x_2 < 220 \\ 480x_1 + 85x_2 \leq 210 \\ 520x_1 + 115x_2 \leq 230. \end{cases}$$

As we have seen, the two possible worst case losses are related to the value of γ. Considering for simplicity $x \geq 0$, the value of γ for the inequality $Ax < B$ is:

$$\gamma_{Ax,B} = \frac{\overline{a}x - \overline{b}}{\widehat{a}x - \widehat{b}}.$$

In order to control the extent of the possible worst case loss for the two types of risk, we can require that the value $\gamma_{Ax,B}$ be controlled for the type I risk and/or for the type II risk. To do this, we fix two values $\gamma_m < 0$ and $\gamma_M > 0$ and we require that valid values of x satisfy $\widehat{a}x < \widehat{b}$ and $\gamma_m \leq \gamma_{Ax,B} \leq \gamma_M$.

The two types of risk are eliminated as soon as $\gamma_m \in [-1, 0]$ and $\gamma_M \in [0, 1]$. The values $1 - \gamma_M$ and $1 + \gamma_m$, if negative, give the relative worst case loss with respect to $\widehat{a}x - \widehat{b}$ (see Eqs. 12, 13).

Definition 20 Given two intervals $A = [a^-, a^+] = (\widehat{a}; \overline{a})$ and $B = [b^-, b^+] = (\widehat{b}; \overline{b})$ and $\gamma_m < 0$, $\gamma_M > 0$ we define the following (strict) order relation, denoted $<_{\gamma m},\ \gamma_M$,

$$A <_{\gamma_m, \gamma_M} B \iff \begin{cases} \widehat{a} < \widehat{b} \\ \gamma_m \leq \gamma_{A,B} \leq \gamma_M \end{cases} \tag{14}$$

i.e.

$$A <_{\gamma_m, \gamma_M} B \iff \begin{cases} \widehat{a} < \widehat{b} \\ \gamma_M \left(\widehat{a} - \widehat{b} \right) \leq \overline{a} - \overline{b} \\ \gamma_m \left(\widehat{a} - \widehat{b} \right) \geq \overline{a} - \overline{b} \end{cases}$$

It is immediate to see that the relation $<_{\gamma m},\ \gamma_M$ with $\gamma_m < 0$, $\gamma_M > 0$ is antisymmetric and transitive; furthermore, there are specific values of γ_m and γ_M which make the order relation (14) equivalent to each order relations LU, LC, UC, CW_M, CW_m.

Proposition 21 Let A and B be two intervals with $\widehat{a} < \widehat{b}$, then it holds that

1. $A <_{LU} B \iff A <_{\gamma_m, \gamma_M} B$ with $\gamma_m = -1$ and $\gamma_M = 1$,
2. $A <_{CW_M} B \iff A <_{\gamma_m, \gamma_M} B$ with $\gamma_m = -\infty$ and $\gamma_M = 0$,
3. $A <_{CW_m} B \iff A <_{\gamma_m, \gamma_M} B$ with $\gamma_m = 0$ and $\gamma_M = +\infty$,
4. $A <_{LC} B \iff A <_{\gamma_m, \gamma_M} B$ with $\gamma_m = -\infty$ and $\gamma_M = 1$,
5. $A <_{UC} B \iff A <_{\gamma_m, \gamma_M} B$ with $\gamma_m = -1$ and $\gamma_M = +\infty$.

By varying the two parameters $\gamma_m < 0$, $\gamma_M > 0$, we obtain a continuum of strict (partial) order relations for intervals. The set of real intervals $\mathbb{I}$ with the order relation $\leq_{\gamma m},\ \gamma_M$ defined by

$$A \leq_{\gamma_m, \gamma_M} B \iff (A <_{\gamma_m, \gamma_M} B \text{ or } A = B) \quad \text{with } \gamma_m < 0, \gamma_M > 0$$

is a complete lattice $(\mathbb{I}, \leq_{\gamma_m, \gamma_M})$.

For a given interval $A = (\widehat{a}; \overline{a})$, consider the set of intervals

$$\begin{aligned} \mathbb{D}_{\gamma_m, \gamma_M}(A) &= \{X \in \mathbb{I} | A \leq_{\gamma_m, \gamma_M} X\} \\ &= \{(\widehat{x}; \overline{x}) | \widehat{a} < \widehat{x} \text{ and } \gamma_m \leq \gamma_{A,B} \leq \gamma_M\} \cup \{(\widehat{a}; \overline{a})\}. \end{aligned}$$

Proposition 22 For any real $\gamma_m < 0$ and $\gamma_M > 0$ and any intervals $A, B \in \mathbb{I}$, we have

1. $A \leq_{\gamma_m, \gamma_M} B$ if and only if $\mathbb{D}_{\gamma_m, \gamma_M}(B) \subseteq \mathbb{D}_{\gamma_m, \gamma_M}(A)$, and
2. $A = B$ if and only if $\mathbb{D}_{\gamma_m, \gamma_M}(A) = \mathbb{D}_{\gamma_m, \gamma_M}(B)$.

Proof We can consider $\mathbb{D}_{\gamma_m, \gamma_M}(A)$ and $\mathbb{D}_{\gamma_m, \gamma_M}(B)$ to be subsets of $\mathbb{R} \times (\mathbb{R}^+ \cup \{0\})$; in the plane $(\widehat{x}; \overline{x})$, they are defined, respectively, by the linear inequalities

$$\mathbb{D}_{\gamma_m,\gamma_M}(A):\begin{cases}\widehat{x}\geq\widehat{a}\\ \overline{x}\geq\overline{a}+\gamma_m(\widehat{x}-\widehat{a})\\ \overline{x}\leq\overline{a}+\gamma_M(\widehat{x}-\widehat{a})\\ \overline{x}\geq 0\end{cases}$$

$$\mathbb{D}_{\gamma_m,\gamma_M}(B):\begin{cases}\widehat{x}\geq\widehat{b}\\ \overline{x}\geq\overline{b}+\gamma_m(\widehat{x}-\widehat{b})\\ \overline{x}\leq\overline{b}+\gamma_M(\widehat{x}-\widehat{b})\\ \overline{x}\geq 0\end{cases};$$

the proof follows immediately by considering that $\gamma_m < 0$ and $\gamma_M > 0$. □

Given a family $\mathbb{A} = \{A_i | i \in \mathcal{I}\}$ of intervals (for any finite or infinite index set $\mathcal{I}$) the infimum and the supremum operators with respect to partial order $\leq_{\gamma_m}, \gamma_M$ respectively $C = \inf\{A \in \mathbb{A}\}$ and $D = \sup\{A \in \mathbb{A}\}$, are defined by the two intervals (in mid-point notation) $C = (\widehat{c}; \overline{c})$ and $D = (\widehat{d}; \overline{d})$

$$\widehat{c} = \frac{\gamma_M c' - \gamma_m c''}{\gamma_M - \gamma_m}, \quad \overline{c} = \frac{\gamma_M\gamma_m(c' - c'')}{\gamma_M - \gamma_m} \geq 0$$
$$\widehat{d} = \frac{\gamma_M d' - \gamma_m d''}{\gamma_M - \gamma_m}, \quad \overline{d} = \frac{\gamma_M\gamma_m(d' - d'')}{\gamma_M - \gamma_m} \geq 0$$

where $c' \leq c''$ are

$$c' = \inf\left\{\widehat{a} - \frac{\overline{a}}{\gamma_M} | A \in \mathbb{A}\right\}$$
$$c'' = \inf\left\{\widehat{a} - \frac{\overline{a}}{\gamma_m} | A \in \mathbb{A}\right\}$$

and $d' \leq d''$ are

$$d' = \sup\left\{\widehat{a} - \frac{\overline{a}}{\gamma_M} | A \in \mathbb{A}\right\}$$
$$d'' = \sup\left\{\widehat{a} - \frac{\overline{a}}{\gamma_m} | A \in \mathbb{A}\right\}.$$

Example We illustrate the order relation $\leq_{\gamma_m}, \gamma_M$ B by the following example: 100 intervals $A_i = (\widehat{a}_i; \overline{a}_i)$ are randomly generated near the interval $[-1,1] = (0;1)$ (see Fig. 4, the values $\widehat{a}_i$ are normally distributed around 0 and the positive values $\overline{a}_i$ are normal around 1).

Then the intervals $C_{Ord} = \inf_{Ord}\{A_i\}$ and $D_{Ord} = \sup_{Ord}\{A_i\}$ are computed, for the five lattices $(\mathbb{I}, Ord)$ defined by the order relations $Ord \in \{LU, LC, UC, CW_M, CW_m\}$ and for the eight lattices defined by the order relations $\leq_{\gamma_m}, \gamma_M$, with $\gamma_m < 0$ and $\gamma_M > 0$ as in Table 2.

The eight order relations corresponding to Table 2 are denoted $G1, G2, \ldots, G8$. According to the theory, the intervals obtained for the orders $\leq_{\gamma_m}, \gamma_M$ are near to the intervals obtained for the orders LU, LC, UC, CW_M, CW_m when the pairs of parameters γ_m and γ_M are near to -1 and

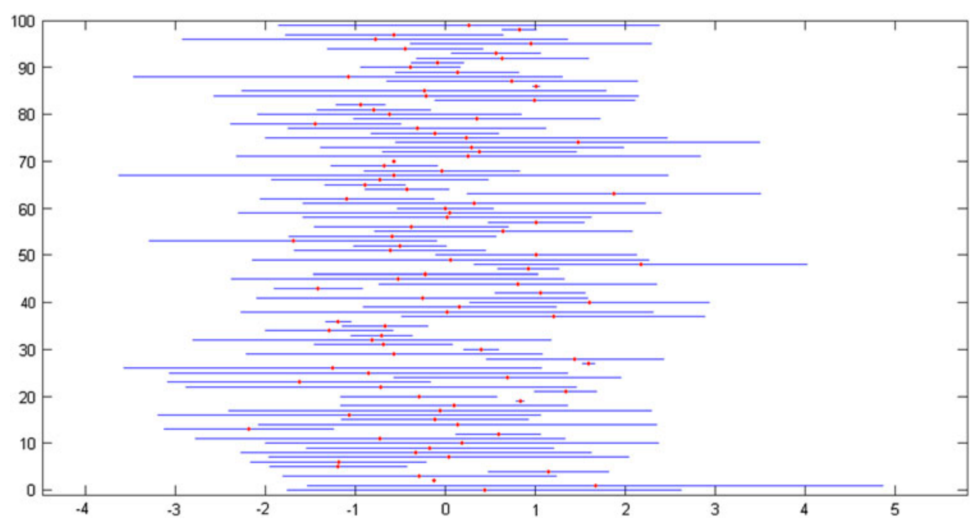

Fig. 4 100 intervals A_i randomly generated near interval $[-1,1]$

Table 2 Combination of values intervals $\gamma_m < 0$ and $\gamma_M > 0$

	1	2	3	*4*	5	6	7	8
γ_m	−1.1	−0.1	−8.0	−0.8	−8.0	−0.5	−2.0	−0.5
γ_M	0.9	8.0	0.1	8.0	0.9	0.5	0.5	2.0

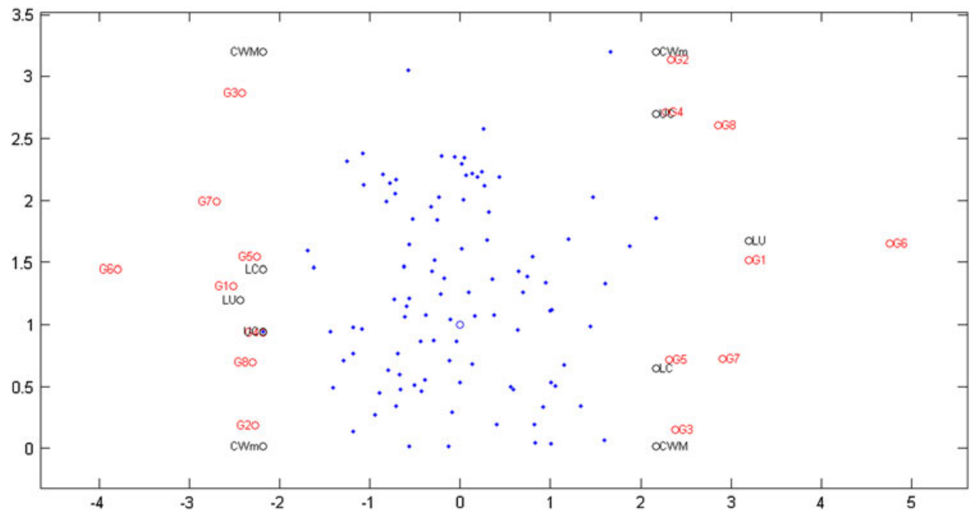

Fig. 5 Mid-point representation of 100 intervals A_i (*points*), of intervals $\inf\{A_i\}$ (*circles* on the *left of figure*) and of intervals $\sup\{A_i\}$ (*circles* on the *right of figure*) for the five orders *LU*, *UC*, *LC*, CW_m, CW_M and the eight orders $\leq \gamma_m, \gamma_M$ of Table 2, denoted *G*1,…, *G*8

1, to $-BIG$ and 1, to -1 and BIG, to $-BIG$ and 0 and to 0 and BIG, respectively (in the example, $BIG = 8$) (Fig. 5).

Remark 23 To focus on the interest for an interval ordering index, we mention that the acceptability index for inequality $A < B$, introduced by Sengupta and Pal (2000, 2009) and defined by (assuming $\overline{a} + \overline{b} > 0$)

$$Acc(A < B) = \frac{\widehat{b} - \widehat{a}}{\overline{a} + \overline{b}},$$

is successfully used to convert an interval inequality $Ax \leq B$, with $x \geq 0$, into a "crisp equivalent" form as follows

$$Ax <_{\alpha} B \iff \begin{cases} a^+ x \leq b^+ \\ Acc(B < Ax) \geq \alpha \end{cases}$$

where $\alpha \in]0, 1]$ is an assumed fixed (optimistic) threshold; substituting the expression for $Acc(B < Ax)$ we obtain

$$Ax <_{\alpha} B \iff \begin{cases} \widehat{a}x + \overline{a}x \leq \widehat{b} + \overline{b} \\ \widehat{a}x - \alpha\overline{a}x \geq \widehat{b} + \alpha\overline{b} \end{cases}.$$

This set of inequalities, being $\alpha > 0$, implies that $\widehat{a}x \geq \widehat{b} + \alpha\overline{b} + \alpha\overline{a}x > \widehat{b}$ and does not imply a control on the possible worse case losses. In fact we can see that $Ax <_{\gamma_m}, \gamma_M B$ is not equivalent to $Ax <_{\alpha} B$ in the sense that the one can not be transformed into the other.

In terms of (14), we can write:

$$Ax <_{\gamma_m, \gamma_M} B \iff \begin{cases} \widehat{a}x < \widehat{b} \\ \gamma_M \widehat{a}x - \overline{a}x \leq \gamma_M \widehat{b} - \overline{b} \\ \gamma_m \widehat{a}x - \overline{a}x \geq \gamma_m \widehat{b} - \overline{b} \end{cases}.$$

If we are minimizing and we do not accept a risk of type II, we may require that $1 + \gamma_{Ax,B} \geq 0$ (we eventually accept only a risk of type I) and we choose $\gamma_m = -1$, $\gamma_M = BIG > 0$; a risk of type II represents the possibility that we realize values in Ax that are greater than all values in B. Similarly, if we do not accept a risk of type I, then we choose $\gamma_M = 1$, $\gamma_m = -BIG < 0$; a risk of type I represents the possibility that we realize values in B that are less than all values in Ax. If $\gamma_m = -1$ and $\gamma_M = 1$ no risk of the two types is accepted. It follows that the use of the acceptability index does not avoid the two types of risk that are controlled using the order relationship (14).

6 Conclusion

We have introduced a comparison index $CI_p(A, B)$, based of the gH-difference of two intervals, and we have examined its properties, including an interesting connection with a probabilistic interpretation of its square $CI_p^2(A, B)$ when $p = 2$.

The application of the comparison index in minimization (or maximization) problems with interval-valued objective function is illustrated and two types of risk are described in terms of the new index. The preliminary results seem to encourage some additional research due to the large number of possible applications in many areas, especially in finance where the presence of uncertainty is strictly linked to the risk management.

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