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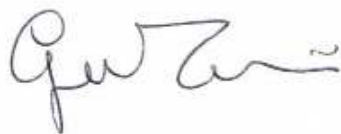
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The effects of future financing constraints on capital accumulation: Some new results on the constrained investment problem

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Abstract

In this paper, we study the effects of future constraints on current investment decisions. Unlike the standard literature on this optimizing problem, we present a model in which firms are neither always constrained nor always unconstrained. We are concerned with those cases where a firm is free from constraints at the current time but expects to face an upper bound at some later date. Using the 'no arbitrage principle' in the constrained scenario, we show how to explicitly calculate the optimal investment path switching between regimes. The analytical result shows that the effects of future financing constraints are included in the market value of the firm, and thus are captured by marginal q .

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1. Introduction

Economic debate has had much to say about the relationship between financing constraints and investment decisions. Mainly, this literature has studied the binary case of constrained versus unconstrained firms. It has often produced ambiguous results: some economists point out the ability of the q model to capture the value of constraints; others stress its inadequacy. Given this ambiguity, it is surprising that recently there has been little *theoretical* effort aimed at focusing on the conditions under which this constrained behavior arises and still less at focusing on the effects of *future* constraints on *current* investment decisions.

The purpose of this paper is to study these latter effects. Unlike the previous literature on investments and financing constraints, we present a theoretical model in which firms are neither always constrained nor always unconstrained. We are primarily concerned with those cases where a firm is free from constraints at the current time but expects to face an upper bound on financing resources at some future date. The focus of attention is on the validity of the Euler equation which drives the optimal investment path through these two alternative regimes. We explain why the optimal investment path describing the switching between regimes cannot be obtained by simply pasting together the unconstrained and the constrained parts of the trajectories. Rather, it is the result of the firm's optimal behavior.

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This approach to the constrained optimization problem extends the idea of rational expectation to the case in which restrictions will become binding at some future date: with forward-looking behavior, the firm anticipates the final outcome, implying that its optimal policy will change at the outset. We show how to calculate explicitly the investment trajectory for a firm, which will become constrained at some later date. Our main result is that the marginal value of the firm captures the effects of present and future constraints.

The difficulties encountered in studying the correlation between investment and financing constraints have prompted researchers to develop different models. Many authors have used the q framework to investigate issues deriving from models of investment with constraints. But different are the strategies followed to study this relationship. The path-breaking paper by Fazzari et al. (1988) initiates this field exploring a constrained version of the q model. It shows that financing constraints do affect investment decisions, but concludes that ‘to the extent that managers control sufficient internal funds to finance all profitable investment projects, investment demand models based on a representative firm in a perfect capital market apply’ (p. 150). The drawback of this kind of investigation is that the analysis of the correlation between financial resources and investment is restricted exclusively to periods when the constraints are binding: there is no attempt to characterize the intermediate phases when constraints are slack.

The same critique applies to models, which focus on the property of the Euler equation in presence of constraints (see for instance Whited (1992); Hubbard and Kashyap (1992); Bond and Meghir (1994), and Hubbard et al. (1995)). This kind of model assumes that so long as the firm does not come up against the constraint, it will be able to satisfy the Euler equation. In other words, constraints binding in future periods (or which have some probability of binding) have no effect on the intertemporal first order condition. Consequently, the Euler equation relating current and future marginal q value fails to hold in some periods.

This point of view has been recently challenged. Many scholars now agree that future constraints can affect current investment. Gomes (2001) argues that the value of the firm—as summarized by q —does not simply depend on the discounted value of real variables but also includes the impact of future financial constraints on current decisions. Further, Erickson and Whited (2000) using an innovative approach to the measurement error problem show that the marginal q value is a sufficient statistic to explain the investment decisions of firms even in presence of financing constraints. Unfortunately, this ‘new view’ is mainly to be found in empirical analyses where the optimization problem has not been solved explicitly. Indeed, very few efforts have been devoted to the investigation of these theoretical foundations.

As far as we know, few theoretical contributions have succeeded in explaining the relationship between current investment and future financing constraints. D’Autume and Michel (1985) show that if a firm expects a constraint on the quantity of capital goods it can buy at some future date, it will invest less in the intervening period than in the unconstrained case. However, they focus exclusively on the characteristics of the value function, without analyzing the formal conditions that guarantee the optimality of the constrained investment trajectories. To avoid these problems, Chirinko (1997) focuses on identifying a set of conditions which is sufficient to ensure that optimal behavior generates a q equation resembling the equation used in econometric work. He considers different types of financing problem, but only some of these problems imply a significant coefficient on cash flow. In several of the cases he studies, financial frictions are capitalized as part of the q value; in others, constraints affect the coefficient on cash flow. However, the paper provides few insights into the effects of future constraints on current decisions. Chatelain (1998) attempts to fill the gap between standard neoclassical investment behavior and credit constrained investment, by following a line of argument, first suggested by Whited (1992). He constructs a formal model of the way switching between financing regimes affects current investment policy. The model assumes that the regime with rationing will never be the ‘final’ one. Hence, it is unable to describe the behavior of a firm, which face rationing in the future but is currently unconstrained. Finally, Saltari and Travaglini (2001, 2003) have studied the behavior of a firm, which makes its investment decisions while facing a constraint that will become binding in the future. Saltari and Travaglini (2001) investigate the effects of constraints and (the output price) uncertainty on investment. They show that future liquidity constraints affect the equilibrium value of the firm, which becomes a non-monotonic functional form of the fundamental. However, the paper does not consider the consequences of these changes on current investment policy. Then, Saltari and Travaglini (2003) show that future constraints can affect a firm’s investment policy, even when constraints are currently slack. But, the authors illustrate their point with a *parametric* example and do not provide an explicit analytical solution for the dynamic path of the potentially constrained firm: conditions under which the optimal policy leads the firm to anticipate his financing constraint have not been thoroughly explored. All this suggests a need for explicit modeling.

In the present paper we present two main results.

1. We fully characterize the problem of the potentially constrained firm deriving analytical solutions for the dynamic paths of the investment, the q value of the firm, and the Lagrange multiplier.
2. To obtain these optimal trajectories we employ the ‘no arbitrage principle’ even when the constraints are binding. The analytical consequence of this condition is that the optimal investment path for a firm switching between regimes must be continuous and smooth at all times, including the point where the constrained and the unconstrained trajectories meet. This results in two boundary conditions, which allow us to construct the new optimal path.

Of course, this formalization of the constrained optimization problem has also economic implications for empirical papers. First of all, note that in our model the Euler equation provides us with relatively little information. Euler equation test cannot discriminate between the presence and absence of latent constraints: given that anticipated constraints can affect current investment, empirical analysis of investment with constraints is likely to give spurious results. As second point, we derive an explicit expression for the Lagrange multiplier, which can be used to check for misspecification in empirical analyses. This is a step forward in quantitative investigation because often the multiplier associated with financing constraints is parametrized in an ad hoc manner in order to proxy the role of financial resources in the q investment equation.

From a methodological point of view, we study the investment decision problem in the presence of certainty. This means that in this deterministic model rational expectations imply perfect foresight. This makes it possible to separate the effects of constraints from those of uncertainty. We show that the firm ‘overinvests’ at the current time when constraints are still slack. In our framework this initial ‘overinvestment’ is exclusively the result of the optimal investment behavior for a forward-looking firm.

The paper proceeds as follows. In Section 2 we solve the benchmark case of the unconstrained firm. In Section 3, we treat the equivalent problem for the constrained firm. Section 4 studies the behavior of the firm during switching between different financing regimes, and explains how to calculate the accumulation path for a firm, which is free from constraints at the current time, but will meet constraints at some future date. Section 5 uses this same argument to show how future constraints affect the current value of the firm, deriving expressions for the Lagrange multiplier and the firm’s investment policy. Section 6 concludes.

2. The unconstrained firm with an infinite horizon

As a starting point, we briefly analyze the standard case with an infinite horizon and no constraints.

Our set up uses all the standard hypotheses for the q model. We assume that the firm has a constant returns to scale technology with decreasing marginal products. We further assume that labor, output and financial markets are all perfectly competitive, and that the labor supply is perfectly elastic. This allows us to write operating profit as a linear function of capital stock, aK_t , where a is the (constant) marginal profit and K_t is the capital stock at time t .

We assume that the adjustment costs function is quadratic in the rate of investment. Thus, $c(I_t) = I_t + (1/2\omega)I_t^2$, where ω is the reciprocal of the speed at which adjustment costs react to investment, and I_t is the investment rate.

Under these hypotheses the intertemporal problem for the firm can be written as

$$V(K_t) = \max_{I_t} \int_0^{\infty} e^{-rt} \left[aK_t - I_t - \frac{1}{2\omega} I_t^2 \right] dt \quad (1)$$

where r is the interest rate (a constant), $dK_t = (I_t - \delta K_t)dt$ is the accumulation constraint and δ the depreciation rate. In other words, the value of the firm is given by the present value of future net profits. In turn, these are determined by the difference between operating profit aK_t and adjustment costs $I_t + (1/2\omega)I_t^2$.¹

¹ The operating profit aK_t measures the instantaneous profit obtained by the firm optimizing its value with respect to the labor input L_t , with K_t assumed constant. More formally, we can write $\max_{L_t} [L_t^\beta K_t^{1-\beta} - wL_t] = aK_t$ where $a = (1-\beta)(\beta/w)^{\beta/(1-\beta)}$.

The firm's objective is to select an admissible trajectory for the control variable I_t that maximizes the value of the firm, as indicated in (1). Admissible paths are defined as trajectories which guarantee the continuity of I_t and K_t , satisfying the initial condition on capital K_0 and the transversality condition.

In the standard case, it is straightforward to show that the value function is linear in K_t . To see this, write the corresponding Bellman equation for problem (1)²:

$$rV(K_t)dt = \max_{I_t} \left[\left(aK_t - I_t - \frac{1}{2\omega} I_t^2 \right) dt + dV \right] \quad (2)$$

The first order conditions for this problem are

$$I_t = \omega(q_t - 1) \quad (3)$$

$$q_t = \frac{a}{r + \delta} + \frac{\dot{q}_t}{r + \delta} \quad (4)$$

where $V_{K_t} \equiv q_t$. Eq. (3) provides the optimal intertemporal condition for the control variable I_t , while the differential Eq. (4) is obtained applying the envelop condition—with respect to the state variable K_t —to the Bellman Eq. (2).

Eq. (4) is the Euler equation and has a simple economic interpretation. It is an arbitrage condition: the value of the firm, q_t , is given by the sum of the present value of future marginal profits, a , discounted at rate $r + \delta$, and the potential capital gain, \dot{q}_t , from reselling the capital on the secondary market. Note that the dynamics of q_t are necessarily continuous. Otherwise, it would be impossible to define \dot{q}_t . It is intuitively clear that this would represent a violation of the 'no arbitrage principle'. In the presence of such a violation the firm would be incorrectly priced and the investment would not be optimal.

The differential Eq. (4) has the solution:

$$q_t = A e^{(r+\delta)t} + \frac{a}{r + \delta}$$

where A is a constant to be determined. Making use of the transversality condition:

$$\lim_{t \rightarrow \infty} e^{-rt} q_t K_t = 0$$

we note that we have to set A equal to zero in order to avoid speculative bubbles and to reach the stable arm of the saddle path, i.e. the one converging to the steady state.

As a consequence, $q = V_{K_t} \equiv \partial V / \partial K$ is a constant ($q = a / (r + \delta)$ does not depend on t). Integrating with respect to capital, we obtain the value function for the firm

$$V(K_t) = \alpha_t + qK_t$$

which is a linear function of capital stock. Substituting this expression in the Bellman equation, one can verify that $\alpha_t = (1/(2\omega))(I_t^2/r)$. That is, the additive term in $V(K_t)$ is due to adjustment costs when—the firm invests more, it has higher adjustment costs, but increases its value—while marginal q is the present value of the future marginal product of capital.

2.1. Finite horizon and perfect capital markets

Let us now assume a finite time horizon T (for an analysis of dynamic optimization problems with finite horizons, see Arrow and Kurz (1970); Leonard and Long (1993)). We will assume that, in addition to the starting level of the capital stock K_0 , we also know the final level. If T is not literally the end of the world, the capital stock left over at the terminal time will have some value in the future. We denote this value by $K(T) = K_T$.

However, economists are often interested in a slightly different formulation of end-of-period conditions. For a firm that intends to continue its existence beyond the planning period $[0, T]$, it may be reasonable to stipulate some minimum acceptable level for the terminal capital instead of a scrap value.

² For Bellman recursive methods in continuous time, see Dixit and Pindyck (1994); Dixit (1993); Kamien and Schwartz (1991); Turnovsky (2000), and Leonard and Long (1993).

In such a case the transversality condition can be written as³:

$$q(T) \geq 0 \quad \text{and} \quad [K(T) - K_T]q(T) = 0$$

If $K(T) > K_T$ then the restriction does not bind, and the outcome is the same as if there were no restriction and condition $q(T) = 0$ is met. But if the restriction is binding, the amount of terminal capital actually left will correspond exactly to the minimum required level $K(T) = K_T$, and the terminal shadow value will be positive $q(T) > 0$.

As before, the problem is to choose the investment trajectory that maximizes the value of the firm:

$$V(K_t) = \max_{I_t} \int_0^T e^{-rt} \left[aK_t - I_t - \frac{1}{2\omega} I_t^2 \right] dt \quad \text{with} \quad K(0) = K_0 \quad \text{and} \quad K(T) = K_T \quad (5)$$

With a finite horizon, the general form of the solution is the same as before

$$q_t = Ae^{(r+\delta)t} + \frac{a}{r+\delta} \quad (6)$$

but in this case the value of the constant A depends on T , K_0 and K_T . To see this, substitute the value of the firm (6) in the investment Eq. (3) to obtain

$$I_t = \omega \left(Ae^{(r+\delta)t} + \frac{a}{r+\delta} - 1 \right)$$

or:

$$\dot{K}_t = \omega \left(Ae^{(r+\delta)t} + \frac{a}{r+\delta} - 1 \right) - \delta K_t$$

The solution of this differential equation is:

$$K_t = K_0 e^{-\delta t} + \frac{\omega A}{r+2\delta} (e^{(r+\delta)t} - e^{-\delta t}) + \frac{\omega}{\delta} \left(\frac{a}{r+\delta} - 1 \right) (1 - e^{-\delta t}) \quad (7)$$

To determine the value of the constant A , we use the terminal value of the capital stock:

$$K_T = K_0 e^{-\delta T} + \frac{\omega A}{r+2\delta} (e^{(r+\delta)T} - e^{-\delta T}) + \frac{\omega}{\delta} \left(\frac{a}{r+\delta} - 1 \right) (1 - e^{-\delta T})$$

Solving this equation for A , we obtain:

$$A = \frac{r+2\delta}{\omega} \frac{K_T - \left(K_0 e^{-\delta T} + \frac{\omega}{\delta} \left(\frac{a}{r+\delta} - 1 \right) (1 - e^{-\delta T}) \right)}{e^{(r+\delta)T} - e^{-\delta T}} \quad (8)$$

Thus, marginal q_t depends on K_T and on T . Note that, in this case, constant A is positive, implying that the q value of the firm is higher than in the steady state.

3. Finite horizon with a financing constraint

Assume now that investment decisions are subject to a financing constraint. For example, if capital markets are imperfect, investment policy could be restricted by a ceiling on available credit. We write the constraint as

$$\dot{K}_t \leq mK_t \quad (9)$$

where m is an exogenous parameter.⁴ Eq. (9) says that the constraint affects the maximum rate $\dot{K}_t/K_t \leq m$ at which the firm can enlarge its initial capital endowment. Of course, the constraint on \dot{K}_t also impinges on gross investment. In

³ See Chang (1992), chapter 8, for a more thorough exposition of the transversality condition in finite time.

⁴ One can endogenize the parameter m , for instance studying the optimal relationship between financial resources and their uses. This clearly complicates the formal analysis without substantially changing the results here obtained.

the constrained regime, this is given by the expression:

$$I_t = (m + \delta)K_t$$

It is important to note that the assumption of a finite time horizon is essential for the constrained problem to be relevant. Expression (9) is an effective constraint only when time is a *scarce* resource. This is because the constraint affects the velocity with which the system (I_t, K_t) converges to its ultimate values.

Simulation runs of our model show that when T is very large, the starting level for investment and the trajectories followed by the constrained and unconstrained system tend to coincide. That is, as T increases, the constrained and the unconstrained paths are very similar. Hence, the solution for investment has the following turnpike property: for T sufficiently large, the system spends most of the time in the neighborhood of the long-run equilibrium, changing its direction only in proximity to the boundary time T , following either the constrained or unconstrained trajectory so as to obtain K_T at T . Finally, note that the solution for large T implies that the present value of capital, $e^{-rT}q_T K_T$, tends to zero. This means that, for T sufficiently large, the standard transversality condition applies.

In this set up the first order condition (3) becomes

$$I_t = \omega(q_t - 1 - \lambda_t) \quad (10)$$

where λ_t is the Lagrange multiplier, the shadow value of the constraint (9), and the corresponding Euler equation is:

$$\dot{q}_t = (r + \delta)q_t - a - \lambda_t(m + \delta) \quad (11)$$

$$\text{with } \lambda_t \geq 0, \quad mK_t - \dot{K}_t \geq 0, \quad \lambda_t(mK_t - \dot{K}_t) = 0 \quad (12)$$

4. Switching between regimes: constraints and absence of arbitrage

The following analysis draws attention to the possibility of regime switching. We have two cases. It may happen that a firm, which is financing constrained at the current time will become unconstrained and remain so. As said above this problem has already been studied (Chatelain (1998)) and it has no implication for the value of the firm because it finally falls in the standard neoclassical investment regime. The second and more interesting case is when the firm is initially unconstrained but it will find itself facing a constraint at some later date. In this scenario, a forward looking firm anticipates this outcome and its investment policy changes at the outset. It is with this effect that we are concerned.

Solving this problem poses a methodological question: the new optimal investment path cannot be obtained by simply pasting together the unconstrained and the constrained parts of the trajectory. It is likely that, along the constrained trajectory, the accumulation rate m , over the time interval $[0, T]$ will be too low for the firm to reach K_T . As we will see in a moment, pasting the two paths together does not yield the desirable features required of an optimal solution. This means that to calculate the optimal trajectory we have to compute not only a new value for A —the starting point for the new trajectory—but also t^* , the time at which the unconstrained and the constrained trajectories should meet. In other words, t^* is an *endogenous* variable: when the firm chooses how much to accumulate now, it determines its future capital stock, and the time at which the constraint will become binding. These considerations are reflected in the analytical method used to solve the optimal control problem, when the firm switches from the unconstrained to the constrained regime.

To deal with this problem, note that, as Arrow and Kurz (1970) have shown, in the unconstrained model the investment policy has to ensure that the accumulation paths for K_t and q_t are continuous and smooth for any period t . The first derivative \dot{q}_t exists and it is continuous only when K_t and q_t are continuously differentiable. One interpretation of this condition, which involves the Euler equation, is that the *no arbitrage condition* is satisfied.

But, for the same reason, the optimal paths of K_t and q_t in the *constrained* scenario must also be continuously differentiable. Along the optimal path the investment dynamics are driven by the marginal value of the firm. In our set up, the value of the firm summarizes the effects of all factors relevant to the investment decision. It follows that a future financing constraint will cause an upward jump in the current value of q_t because future restrictions lead to an immediate increase in the marginal value of the firm. In these conditions, the optimal decision for the owners of the firm will be to start off with a higher level of investment.

The main analytical problem is how to characterize this optimal behavior. We impose two boundary conditions, ensuring the continuity and smoothness of the new trajectories describing the changes of K_t and q_t for the switch from the unconstrained to the constrained status. These allow us to draw a new path which is optimal in the intermediate phase when the constraint is slack. It should be observed that, if these two conditions were violated, q_t would be mispriced and investors could obtain arbitrage profits at the current time by buying the shares of the firm at a price lower than their true value, determined by q_t .

4.1. The boundary conditions

To solve our problem, we impose the following two boundary conditions

$$K_{t^*}^{\text{NC}} = K_{t^*}^{\text{C}} \quad (13)$$

$$\dot{K}_{t^*}^{\text{NC}} = \dot{K}_{t^*}^{\text{C}} \quad (14)$$

where K_t^{NC} and K_t^{C} indicate the capital stock, respectively, in the unconstrained and in the constrained states. These conditions require that K_t should be continuous and smooth along the optimal trajectory at the optimal switching time.⁵ Note that this is coherent with Arrow and Kurz's theorem of continuity of the state variables (1970, p.57, Proposition 12).

These conditions identify new admissible paths for the intermediate phase, when constraints are still slack, based on the consideration that investment dynamics along the optimal path investment do not simply reflect profits but also anticipate the discounted value of future constraints.

As we will see shortly, firms facing future constraints have a higher q_t since future restrictions increase their current marginal value. This implies that the optimal policy for the owners will be to start off with a higher level of investment.

4.1.1. The continuity condition

Condition (13) states that at the (endogenous) time t^* , K_t must have the same value on both the constrained and the unconstrained trajectories. Condition (14) assures us that, to avoid arbitrage opportunities, both trajectories have to be smooth at the time they meet.

If there is no constraint, Eq. (7) defines the optimal trajectory for K_t :

$$K_t^{\text{NC}} = K_0 e^{-\delta t} + \frac{\omega A}{r + 2\delta} (e^{(r+\delta)t} - e^{-\delta t}) + \frac{\omega}{\delta} \left(\frac{a}{r + \delta} - 1 \right) (1 - e^{-\delta t}) \quad (15)$$

On the other hand, the presence of the constraint (9) forces K_t to follow the restricted trajectory

$$\dot{K}_t = mK_t \rightarrow K_t^{\text{C}} = Ce^{mt}$$

where C is a constant whose value depends on the boundary value K_T . We use this piece of information to determine the value of C :

$$K_T = Ce^{mT} \rightarrow C = K_T e^{-mT}$$

⁵ Condition (13) can be seen as a 'value matching condition' because it matches the values of K_t^{NC} to that of K_t^{C} . Note that we have two unknowns, the switching time t^* and the corresponding value of capital at that time, K_{t^*} . This explains why this kind of problem is dubbed a 'free boundary' problem, and why we need a second condition in addition to (13). The general mathematical theory is of little help in such a case. The conditions applicable to free boundaries must come from economic considerations (see Dixit and Pindyck (1994), p. 108–10). In our context, the no arbitrage principle defines the criterion. We require that at t^* the values of K_t^{NC} and K_t^{C} should meet tangentially at the boundary, that is $\dot{K}_{t^*}^{\text{NC}} = \dot{K}_{t^*}^{\text{C}}$, which is condition (14). This is generally called 'high order contact' or 'smooth pasting' condition because it requires not just the values but also the derivatives of the two functions to match at the boundary. The intuition is the following. As investment approaches the boundary, the firm realizes that future investment plans will be limited by the constraint. The firm will anticipate this, so that future constraints will be reflected in the firm's current decisions. Thus, it is not surprising that K_t^{NC} smoothly to K_t^{C} , becoming tangent at the trigger value K_t^* in t^* , in such a way as to satisfy the no-arbitrage condition.

By virtue of the continuity condition (13), at the optimal endogenous time t^* the unconstrained and constrained capital stocks must be equal:

$$K_{t^*}^{\text{NC}} = Ce^{mt^*} \quad (16)$$

Using this equation together with (15), we obtain an expression for $A(t)$ (where we made explicit the dependence on time):

$$A(t) = \frac{r + 2\delta}{w} \frac{Ce^{mt} - [K_0 e^{-\delta t} + \frac{\omega}{\delta} (\frac{a}{r+\delta} - 1)(1 - e^{-\delta t})]}{e^{(r+\delta)t} - e^{-\delta t}} \quad (17)$$

Note that when $t=T$, then $Ce^{mT} = K_T$ while the remaining terms in the function have the same value as in the unconstrained case. In other words, the value of $A(T)$ is the same as in the unconstrained scenario.

$A(t)$ is maximum at t^* . This implies that $A(t)$ has a lower value in the unconstrained scenario than in the constrained case, $A(T) < A(t^*)$. Since, $A(t)$ determines the optimal starting level for I_t , the initial level of investment in the potentially constrained scenario is higher than the one in the scenario without constraints. Hence, the optimal behavior for the potentially constrained firm is to ‘overinvest’ at the outset in order to achieve K_T at time T .

4.1.2.. The smoothness condition

Let us now consider the second condition. For a firm facing future constraints, t^* is the optimal endogenous time for the unconstrained trajectory to meet the constrained accumulation path. Eq. (14) requires that at t^* net investment should be the same on both the constrained and the unconstrained trajectories. This implies that K_t should be smooth at time t^* : at t^* both trajectories should have the same accumulation rate.

Using the first order condition, $I = \omega(q_t - 1)$, net investment on the unconstrained trajectory can be written as:

$$\dot{K}_t^{\text{NC}} = \omega \left(Ae^{(r+\delta)t} \frac{a}{r+\delta} - 1 \right) - \delta K_t$$

On the other hand, the constraint on the accumulation rate is:

$$\dot{K}_t^{\text{C}} = mK_t$$

Hence, the smoothness condition (14) can be expressed as:

$$\omega \left(Ae^{(r+\delta)t^*} + \frac{a}{r+\delta} - 1 \right) - \delta K_{t^*} = mK_{t^*} \quad (18)$$

where $K_{t^*} = Ce^{mt^*} = K_T e^{-m(T-t^*)}$. Solving Eq. (18) for $A(t)$, we get:

$$A(t) = \left[\frac{m+\delta}{\omega} Ce^{mt} - \left(\frac{a}{r+\delta} - 1 \right) \right] - e^{(r+\delta)t} \quad (19)$$

Finally, putting together (17) and (19), we obtain the values of the ‘constant’ and of the optimal switching time, $A(t^*)$ and t^* . Fig. 1 illustrates the relationship between the two conditions. As is clear from the figure, t^* is optimal when the value of $A(t)$ is maximum. This is a consequence of the maximization principle underlying the Bellman equation.

5. The value of the firm

As we said above the Euler equation must be valid even when the constraint binds. To focus on this crucial implication note that the continuity of \dot{K}_t implies the continuity of I_t . Hence, given the relationship

$$I_t = \omega(q_t - 1 - \lambda_t)$$

both q_t and λ_t must be continuous. Consequently, from (11)—the Euler equation in the presence of a constraint— \dot{q}_t is also continuous. In other words, q_t must be continuously differentiable. Therefore, the following boundary conditions must hold at t^* :

$$q_{t^*}^{\text{NC}} = q_{t^*}^{\text{C}} \quad (20)$$

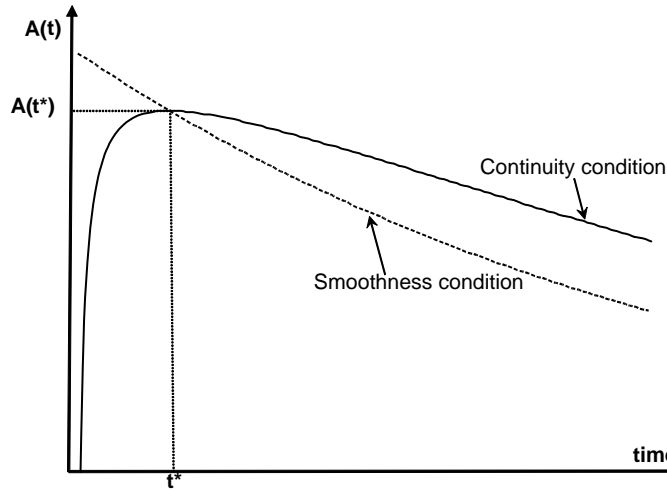


Fig. 1. The smoothness and continuity conditions.

$$\dot{q}_t^{\text{NC}} = \dot{q}_t^{\text{C}} \quad (21)$$

As before, these two conditions assure the absence of arbitrage opportunities in the constrained scenario: the switching between regimes takes place without discontinuity. Eq. (21) allows us to determine the dynamics of the value of the firm along the constrained trajectory.⁶

To solve the system formed by Eqs. (20) and (21), write the Euler equation for the unconstrained part of the trajectory:

$$\dot{q}_t^{\text{NC}} = (r + \delta)q_t^{\text{NC}} - a$$

The Euler equation for the constrained part of the trajectory is:

$$\dot{q}_t^{\text{C}} = (r + \delta)q_t - a - \lambda_t(m + \delta) \quad (22)$$

To solve this equation, we have to provide an explicit expression for the Lagrange multiplier λ_t . Investment along the constrained trajectory is given by:

$$I_t = w(q_t^{\text{C}} - 1 - \lambda_t)$$

and $I_t = (m + \delta)K_t$, it follows that:

$$(m + \delta)K_t = \omega(q_t^{\text{C}} - 1 - \lambda_t)$$

The Lagrange multiplier can thus be expressed as a function of q_t^{C} :

$$\lambda_t = q_t^{\text{C}} - 1 - \frac{(m + \delta)K_t}{\omega}$$

Substituting this expression in (22), we obtain the Euler equation for the constrained part of the trajectory

$$\dot{q}_t^{\text{C}} = (r + \delta)q_t^{\text{C}} - a - (m + \delta) \left[q_t^{\text{C}} - 1 - \frac{(m + \delta)K_t}{\omega} \right]$$

⁶ If the firm knows that the future investment will be constrained, it anticipates at the current time that its fundamental value will not grow beyond an endogenous maximum upper value. This information will affect the investment decisions at the outset changing the firm's value over time as the barrier draws closer. As a result future constraint will be reflected in the firm's current value. In these circumstances, it is not surprising that q_t^{NC} must converge smoothly to q_t^{C} , becoming tangent at t^* in such a way as to satisfy the no-arbitrage condition.

that is, using the fact that on the constrained path $K_t = Ce^{mt}$:

$$\dot{q}_t^C = (r-m)q_t^C - [a - (m + \delta)] + \frac{(m + \delta)^2}{\omega} Ce^{mt} \quad (23)$$

The dynamics of q_t^C is the solution to (23), that is

$$q_t^C = De^{(r-m)t} + E + Fe^{mt}$$

where

$$E = \frac{a - (m + \delta)}{r - m}, \quad F = \frac{1}{r - 2m} \frac{(m + \delta)^2}{\omega} C$$

We now examine the continuity condition for q_t . Along the unconstrained trajectory we know that:

$$q_t^{NC} = A(t^*)e^{(r+\delta)t} + \frac{a}{r + \delta}$$

Applying the continuity condition (20), we obtain:

$$A(t^*)e^{(r+\delta)t^*} + \frac{a}{r + \delta} = De^{(r-m)t^*} + E + Fe^{mt^*} \quad (24)$$

Given the optimal values of $A(t^*)$ and t^* , this equation determines the constant D and the corresponding optimal value for q_t along the switching trajectory.

Summing up, the value of the firm is:

$$q_t = \begin{cases} q_t^{NC} = A(t^*)e^{(r+\delta)t} + \frac{a}{r + \delta}, & \text{for } t \leq t^* \\ q_t^C = De^{(r-m)t} + E + Fe^{mt}, & \text{for } t > t^* \end{cases} \quad (25)$$

This last equation neatly shows that the effects of future constraints are included in the market value of the firm, captured by q_t . This is clearly true for the constrained component in the q_t value of the firm since, as can be seen from Eq. (24), constant D depends directly on rate m and optimal switching time t^* . But it is also true for the unconstrained part of the q_t value since t^* and $A(t^*)$ both reflect the value of future constraint m (see Eqs. (17) and (19)). Of course, this is a consequence of the no arbitrage condition, which works analytically through the continuity and the smoothness conditions. The economic implication is that future constraints immediately affect a firm's investment decisions, by increasing its current value. A further consequence is that the existence of financial constraints is per se not sufficient to establish cash flow as a significant regressor in the standard investment equation. Conversely, future constraints are not necessary to obtain cash flow effects.

5.1. The Lagrange multiplier

Another important feature of this model is that it makes it possible to derive an explicit expression for the Lagrange multiplier λ_t —which as usual measures the additional cost of external resources when capital markets are imperfect. Thus, the cost of an additional unit of capital is equal to $1 + (1/\omega)I_t + \lambda_t$.

To derive the expression for the Lagrange multiplier, start with the investment equation for the constrained part of the trajectory, $I_t = \omega(q_t - 1 - \lambda_t)$. Solving for q_t gives:

$$q_t = 1 + \lambda_t + \frac{1}{\omega}(m + \delta)K_t$$

Next, differentiate this expression with respect to time to get:

$$\dot{q}_t = \dot{\lambda}_t + \frac{1}{\omega}(m + \delta)mK_t$$

Substituting these equations in the Euler equation corresponding to the constrained status, Eq. (23), we obtain the differential equation for λ_t :

$$\dot{\lambda}_t + \frac{1}{\omega}(m + \delta)mK_t = (r + \delta) \left[1 + \lambda_t + \frac{1}{\omega}(m + \delta)K_t \right] - (m + \delta)\lambda_t - a$$

Finally, use the fact that $K_t = Ce^{mt}$ to find the following solution for the Lagrange multiplier

$$\lambda_t = Qe^{(r-m)t} + P + Le^{mt}$$

where

$$P = \frac{a - (r + \delta)}{r - m}, \quad L = C \frac{m + \delta}{\omega(2m - r)}(r + \delta - m), \quad Q = -(P + Le^{mt^*})e^{(m-r)t^*}$$

As already stated, λ_t is continuous. This does not mean, however, that λ_t has to be smooth. In fact, λ_t has a ‘kink’ at the point where the constraint becomes binding. Before t^* λ_t has a value of zero.

5.2. Optimal Investment

Having obtained the path for q_t and λ_t , it is easy to determine optimal investment policy. We already know that in the absence of constraints the investment trajectory is given by:

$$I_t = \omega(q_t^{\text{NC}} - 1)$$

We also know that in the constrained regime investment is:

$$I_t = \omega(q_t^{\text{C}} - 1 - \lambda_t)$$

Using the expressions previously calculated for λ_t and q_t , we are able to determine the investment path:

$$I_t = \begin{cases} \omega(q_t^{\text{NC}} - 1), & \text{for } t \leq t^* \\ \omega(q_t^{\text{C}} - 1 - \lambda_t), & \text{for } t > t^* \end{cases}$$

Note that, given the continuity of K_t and \dot{K}_t , the only condition we can impose on investment is continuity. We cannot impose smoothness. Investment, like the Lagrange multiplier, has a kink at t^* .

6. Conclusions

In this paper, we have shown that the presence of financing constraints affects firms’ behavior even when current investment is far below the level where the constraint binds: for forward looking firms, marginal q_t incorporates the effects of future financing constraints, and the Euler equation is always valid even when the investment path switches from the unconstrained to the constrained regime.

To show this result we have employed the no arbitrage principle in the constrained scenario, in order to determine the optimal value of a firm switching between regimes. By excluding ‘jumps’ in the accumulation rate \dot{K}_t and in capital gains \dot{q}_t when constraints bind, the principle implies that the paths for capital stock K_t and the q_t value must be continuous and smooth.

Of course, these analytical results depend on the model’s assumptions. It is clear, for instance, that starting levels for investment and investment trajectories depend on the functional form used to express adjustment costs, the form in which we express constraints and the absence of uncertainty. A key task for future work is thus to generalize the model to include uncertainty.

Nonetheless, we would argue that the ‘no arbitrage principle’ is valid in all financing scenarios. Only by respecting this condition is it possible to ensure correct pricing of firms and optimal investment decisions.

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