Probing the Standard Model through radiative corrections

Thesis submitted in partial fullfillment of the requirements
for the Ph. D. degree in Physics

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To Lucia and my Parents.

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Chapter 1

Introduction

In this thesis we are going to report on some of the topics investigated during the years at Scuola Normale Superiore, namely weak radiative decays, exemplified by the study of the so called Flavor Changing Neutral Current (FCNC) decays of the b quark, and weak radiative corrections to LEP observables, discussing the computation of the two-loop heavy top effects to the ρ parameter and to the $Z \rightarrow b\bar{b}$ decay width.

1.1   Motivations

These investigations contribute to a general research activity, aimed at testing the Standard Model at the quantum level.

   The Standard Model of electroweak interactions is a very successful theory, in the sense that when a process is considered which can be predicted with a good theoretical accuracy, there is full agreement with the experiment[1]. It appears that every known high energy process can be described at least qualitatively in the context of the SM; the common belief is that where a quantitative analysis lacks, the reason is the inadequacy of our computational techniques, not of the theory in itself.

   Nevertheless there are good reasons of principle to believe that the Standard Model is just an effective theory: for instance there is compelling evidence that φ^4 models, and consequently the very Higgs sector of the Standard Model, is *trivial*. This means that these models cannot be renormalized unless the interaction term is put to zero: even if this result has been rigorously proven only for $d > 4$, the existing partial results speak in favour of an extension to $d = 4$.

   Leaving aside the theoretical motivations, there are aspects of the theory which are still essentially not verified. For instance the symmetry breaking mechanism is not clarified, and the Higgs sector itself lacks completely the experimental confirmation.

   These facts have motivated in the last years a considerable effort aimed at devising extensions of the SM which may cure its theoretical shortcomings, without altering the "low energy" behavior. May be the most promising of these extensions is the Minimal Supersymmetric Standard Model.

   Every extended model predicts the existence of some new particles, but it is quite possible that none of these states will be detected directly in the near future: hence one has to look

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[1]With some possible exceptions, for instance R_b, which is found at LEP two standard deviations higher than the SM expectation, a fact however that is not a serious problem at the present level of statistical accuracy.

for confirmations through indirect effects. The very success of the SM implies that we cannot expect discrepancies with the experiment *at the tree level*, but any SM extension may reflect itself in small corrections to the observables that are already being measured.

It goes without saying that in order to disentangle new physics effects it is first necessary to improve the SM predictions at the level of the present experimental accuracy. In this thesis we shall see in two explicit examples how this program is accomplished in practice.

The examples chosen, the $b \to s\gamma$ decay and the heavy top electroweak corrections, are good representatives of a class of studies which are expected to be particularly apt to improve our understanding of the SM structure: the reason is that perturbative methods appear in these cases well suited to extract accurate theoretical predictions.

1.2 Plan of the Work

The thesis is divided in two parts: in the first part we shall deal with the rare $B \to X_s\gamma$ decay, while in the second part we shall consider the heavy top corrections to LEP parameters.

We shall try to be self contained and to follow a pedagogical approach. The physical framework and the computational techniques are discussed on general grounds, as long as it is possible from first principles.

In Chapter 2 we introduce from a general point of view the study of the rare $B \to X_s\gamma$ decay, which is going to be detailed in the following Chapters. We recall the recent results of the CLEO collaboration, which has measured both the inclusive decay and one of the exclusive modes, $B \to K^\star\gamma$. We show how such a process is described by the Standard Model in the context of the Operator Product Expansion, separating the short distance contributions which can be computed by perturbative techniques and essentially non perturbative quantities. The $b \to s\gamma$ decay is unique in this respect, because the perturbative corrections are very large and enhance the decay rate by a factor ~ 4 ! This is not a sign of failure of perturbation theory: it only reflects the fact that the operator mixing induced by QCD corrections alters the pattern of GIM suppression. We shall see that despite these large corrections the inclusive process can be predicted accurately: terms left out in the perturbative computation are not expected to change the result by more than 20%. We shall briefly discuss why this process is interesting as a "probe for New Physics", by comparing the predictions in the SM, in an extended Higgs model, and in SUSY. We shall see that one cannot expect too much from the comparison with experiment, unless the theoretical predictions are made more accurate.

In Chapter 3 we start the analysis by showing how the techniques of Wilson is used in practice to build an Effective Hamiltonian $\mathcal{H}_{\text{eff}}$ for rare B decays. We compute the coefficients of the Operator Product Expansion in the context of the Standard Model, and we start discussing the effect of QCD corrections on the accuracy of the matching between "full" and effective theory. The use of the effective theory at a mass scale much lower than the scale of the matching results in large perturbative logarithms, which need to be resummed (at least in part) to all orders in perturbation theory: this is equivalent to evolve the effective hamiltonian, as a renormalized operator, from the matching scale to the scale of the decay process in consideration.

The techniques of "renormalization group improved" perturbation theory are briefly reviewed in Chapter 4. We work in the context of dimensionally regularized field theories, and we derive the RG equation using an approach which in our opinion is more transparent than the usual "bare" formulation. In particular, we review some subtleties, like the so called

"evanescent" operators, which need to be taken into account for a consistent treatment of QCD corrections to weak decays. We discuss the resummation of large logarithms and we introduce the analysis of scheme and scale dependence: in particular we discuss how the scale dependence is the reflection of the error in the theoretical computation due to the left-out terms in the perturbative expansion.

In Chapter 5 these tools are applied to the computation of the leading logarithmic corrections to $b \to s\gamma$ decay. We compute the anomalous dimension matrix and use it to evolve the effective hamiltonian from the matching scale $\mu \simeq M_W$ to the scale of the decay $\mu \simeq m_b$. We use a technique which has been developed to simplify the computation and which is expected to be of help in evaluating the next to leading corrections. We argue why one should perform the next to leading computation, and how the accuracy of the existing results would be improved.

In the second part of the thesis we discuss the heavy top corrections to LEP observables. It is known from some time that a large mass splitting in the t, b doublet results in relatively large $O\left(m_t^2\right)$ corrections to "oblique" and "non oblique" LEP observables, that is, related to self energies and vertices. Our contribution has been the evaluation of the $O\left(m_t^4\right)$ corrections. This computation can be regarded as an estimate of the theoretical error implied by the lack of knowledge of the full two loop electroweak corrections.

In Chapter 6 we set up the renormalization framework, recalling on general grounds how it is possible to parameterize the quantum corrections to electroweak parameters. We discuss the basic and derived observables, and we introduce the ρ parameter as a way to compactly take into account the self energy corrections common to all the observables. We argue why an heavy top alters the value of the ρ parameter, and we also discuss the effect of an heavy top through vertex corrections. We briefly recall how such computations have been used to derive bounds on the top mass, which are in agreement with the recent result of the CDF collaboration ($m_t \simeq 170\text{GeV}$).

In Chapter 7 we show how the heavy top corrections can be evaluated by using a reduced model, the so called "gaugeless" limit of the Standard Model. Indeed, in the limit of gauge couplings much smaller than Yukawa couplings, we can forget the propagation of gauge fields, and the model needs only to include the "would be" Goldstone bosons, which are coupled to fermions through the Yukawa term. We show how this limit is well defined starting from the Standard Model formulated in the background field gauge, and we show how the self energy and vertex corrections needed to evaluate the ρ parameter and the $Z \to b\bar{b}$ decay width are related by Ward Identities to correlation functions evaluated in the reduced "Yukawa" model. We show how this Ward Identities reflect the conservation, in the reduced model, of the currents coupled to the Z and $W^{\pm}$ fields.

In Chapter 8 the computation of the heavy top effects is detailed: we define the renormalized model by introducing the subtractions needed to make all the results finite, and we clarify the connection with the basic observables. We compute the one and two loop corrections and we discuss their numerical significance. Then we consider the effect of the corrections on some of the LEP observables. We discuss the validity of the approximation and we argue why one should not consider the heavy top effects as the major source of correction to LEP observables.

The conclusions (Chapter 9) will be devoted to resume the results obtained and to discuss further possibilities, in particular the computation of the Next to Leading Corrections to B decays we are currently involved with.

In the Appendixes we have collected some useful formulas. In particular, in App. A we

fix the notations, relating pseudo-euclidean and minkowskian metrics. In App. B we give definitions and identities for the Dirac algebra in d dimensional space. In App. C we briefly recall the Standard Model formulation, its quantization and the relevant Feynman rules. Finally in App. D we briefly recall the background field gauge method and its application to the Standard Model.

Part I

The $B \to X_s \gamma$ decay

Chapter 2

Overview

2.1 The CLEO results

The CLEO collaboration, at the Cornell Electron Storage Ring (CESR), has reported the first observation of rare B decays connected at the quark level to the $b \to s\gamma$ process [88, 105].

At Cornell B, $\bar{B}$ mesons are produced at the $\Upsilon - 4S$ resonance, a $J = 1$ unstable state of mass $m = 10.5800 \pm 0.0035\text{GeV}$ and total width $\Gamma = 23.8 \pm 2.2\text{MeV}$. Both neutral B^0, $\bar{B}^0$ and charged B^-, B^+ pairs are produced, accompainied by light quark pairs, $u\bar{u}$, $d\bar{d}$, $s\bar{s}$, $c\bar{c}$, whose background can be rejected by subtracting measures taken about 50MeV below the resonance.

The B mesons are produced with a cross section ratio to the continuum roughly of 1 to 2.5, and decay mainly in charmed mesons, through Charged Currents interactions. These modes are used to determine the mass of the B meson and the results show the good resolution

$$
\begin{aligned}
M_{B^0} &= 5280.3 \pm 0.2 \pm 2.0 \\
M_{B^-} &= 5279.9 \pm 0.2 \pm 2.0 \ .
\end{aligned}
\tag{2.1}
$$

The CLEO collaboration has collected [105] 2.15 million B meson pairs at the resonance, and about 6.6 million below resonance. Already in a smaller sample, (1.2 million pairs) the collaboration has reported [88] the observation of 13 events corresponding to the rare $B^0 \to K^\star(892)^0\gamma$, $B^- \to K^\star(892)^-\gamma$, resulting in a branching fraction

$$
BR\left(B \to K^\star\gamma\right) = (4.5 \pm .5 \pm 0.9) \times 10^{-5} \ .
\tag{2.2}
$$

These *exclusive* decay modes were already expected to be dominant among the channels relevant for the $b \to s\gamma$ transitions, since the $K\gamma$ decay was excluded by angular momentum conservation, and the $K^\star(892)$ is the lightest vector meson resonance allowed in the final state.

The two body nature of the decay permits a rather clean reconstruction: the experimental signature is an hard photon (2.6GeV) recoiling against the decay products of the excited $K^\star$ state, a $K\pi$ pair. The $K\pi$ pair mass is reconstructed and required in a range $821 < M_{K\pi} < 971\text{MeV}$, and various cuts are imposed to reject continuum [88].

It has also been possible to measure the *inclusive* decay rate, $BR\left(B \to X_s\gamma\right)$, by looking for an hard photon without reconstructing the accompaining hadronic state but requiring its
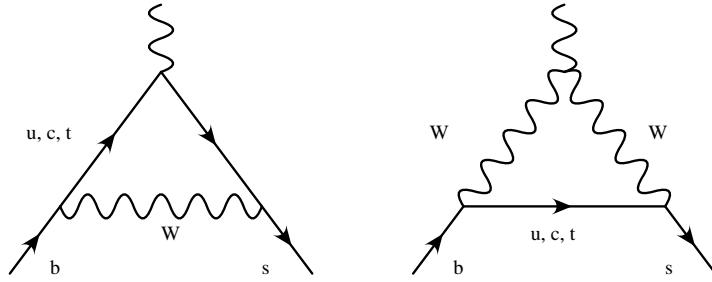
Figure 2.1: One loop diagrams for $b \to s\gamma$ in the SM

mass in the typical strange meson region, $0.5 - 2\,\text{GeV}$. The signature is a photon with energy between 2.2 and 2.7 GeV, this spread resulting from the Fermi motion of the b quark inside B meson. Referring to [105] for a discussion of the identification and background rejection techniques, we can quote the experimental result

$$BR\,(b \to s\gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4}\ ; \tag{2.3}$$

we shall see that this observation is quite consistent with the Standard Model predictions. It is also useful to observe that the following inequalities are true at 95% confidence level

$$1.0 \times 10^{-4} < BR\,(b \to s\gamma) < 4.2 \times 10^{-4}, \tag{2.4}$$

a range of variation important in order to constrain all the non-standard model contributions to the decay.

To size the importance of this rare decays and of the hard experimental effort needed to detect them, let us consider their theoretical interpretation.

2.2 The structure of the $B \to X_s\gamma$ decays

The $b \to s\gamma$ decay, a Flavor Changing Neutral Current process, is not generated in the Standard Model classically, that is, at the tree level. This feature is "protected" at the quantum level by the GIM mechanism: in the limit of equal masses for the up-quarks, the diagrams in Fig. 2.1 cancel exactly, and the flavor diagonal structure of the Neutral Current interactions is preserved from receiving quantum corrections.

As soon as it was recognized that the t quark mass had to be rather large [11], it became apparent the violation of the GIM mechanism, and the generation of FCNC effective interactions, which can be accounted for in an effective low energy theory which incorporates the contributions resulting by the virtual effect of the heavy top.

We shall discuss in the next chapter how the effective theory is generated, for the time being it suffices to observe that thanks to the large mass of the W boson, in analogy to the Fermi theory, decays like $B \to K^\star\gamma$ can be described at short distances, by a local magnetic momentum operator

$$C_{\text{m.m.}}\,(m_b)\,\frac{e}{(4\pi)^2}m_b V_{tb}V_{ts}^\star\bar{s}\sigma_{\mu\nu}b_R F^{\mu\nu}\ , \tag{2.5}$$

with a coefficient C, which depends on the scale of the process and on the short distance physics details. This parameterization is specific to the SM only in the fact that the unitarity of the CKM has been assumed, and the $V_{ub}V_{us}^\star$ have been dropped, but in any effective low-energy theory the essential characteristics do not change.

Note the explicit m_b factor: in the limit of $m_b \to 0$ the b helicity would be conserved and there would be no way to emit a real vector particle. The amplitude for $B \to K^\star \gamma$ is given by the on-shell matrix element [82]

$$A = \frac{eG_F m_b}{2\sqrt{2}\pi^2} C_{\text{m.m.}}(m_b) V_{tb}V_{ts}^\star \eta^\mu \langle K^\star | \bar{s}\sigma_{\mu\nu}q^\nu b_R | B\rangle \tag{2.6}$$

where q, η are momentum and polarization of the outgoing photon. The matrix element contains the long distance physics, the details of the hadronization, and can be parameterized as follows [64]

$$\langle K^\star(k, \varepsilon) | \bar{s}\sigma_{\mu\nu}q^\nu b_R | B(p)\rangle = C_\mu^1 T_1\left(q^2\right) + C_\mu^2 T_2\left(q^2\right) + \text{terms zero on-shell} \tag{2.7}$$

where $q = p - k$ and

$$\begin{aligned} C_\mu^1 &= \epsilon_{\mu\nu\lambda\rho}\varepsilon^\nu p^\lambda k^\rho \\ C_\mu^2 &= \varepsilon_\mu\left(m_B^2 - m_{K^\star}^2\right) - \varepsilon \cdot q\,(p+k)_\mu \ . \end{aligned} \tag{2.8}$$

On shell $T_2\left(q^2 = 0\right) \propto T_1\left(q^2 = 0\right)$, and after proper phase space integration and sum over polarizations one gets

$$\Gamma\left(B \to K^\star \gamma\right) = \frac{\alpha}{32\pi^4} m_b^2 G_F^2 m_B^4 \left(1 - \frac{m_{K^\star}^2}{m_B^2}\right)^3 |V_{tb}V_{ts}^\star|^2\, |C(m_b)|^2 \left|T_1\left(q^2 = 0\right)\right|^2 \ . \tag{2.9}$$

This expression deserves a few comments:

- the rate is proportional to $\alpha_{\text{QED}} G_F^2$, while most other FCNC processes involving leptons (like $b \to X_s e^+ e^-$) or photons are of order $\alpha_{\text{QED}}^2 G_F^2$.

- The CKM coefficients appearing in the expression have not been directly measured, but in the SM they can be deduced from unitarity, and are known to be

$$\begin{aligned} |V_{tb}| &= 0.9987 - 0.9994 \\ |V_{ts}| \simeq |V_{cb}| &= 0.044 \pm 0.006 \ . \end{aligned} \tag{2.10}$$

If one considers a related FCNC process, the $B \to X_d \gamma$ decay, one immediately deduces that

$$\frac{BR(B \to X_d \gamma)}{BR(B \to X_s \gamma)} \simeq \frac{|V_{td}|^2}{|V_{ts}|^2} \simeq 10^{-2} \ , \tag{2.11}$$

which is a consistent suppression.

- The separation between short and long distance physics is made, in the context of OPE, at a scale $m_b \simeq 5\text{GeV}$ which is well within the perturbative QCD range. We shall elaborate on this subject in the following chapters, but the essential is that the large m_b mass makes us confident that the factorization of short and long-distance contributions is reliable.

- The long-distance physics, contained in the T_1 form factor, has been considered for a long time the larger source of uncertainty.

Recently however, as pointed out in [104], a remarkable agreement between Lattice QCD results and QCD sum rules has shown up.

On the lattice two independent groups have computed the form factor $T_1\left(q^2 = 0\right)$; Bernard et al. [64] obtaining $T_1^{\text{Bernard}} = 0.20 \pm 0.02 \pm 0.01$, Bowler et al.(UKQCD collab.) [82] obtaining $T_1^{\text{UKQCD}} = 0.30^{+10}_{-7}$.

From QCD sum rules other results are available: $T_1^{\text{Colangelo et al.}} = 0.35 \pm 0.05$, obtained in [84], $T_1^{\text{Ball}} = 0.37 \pm 0.05$ [83], $T_1^{\text{Narison}} = 0.31 \pm 0.013 \pm 0.06$ [101], $T_1^{\text{Ali et al.}} = 0.32 \pm 0.05$ [97].

This results determine the ratio

$$R_{K^\star} = \frac{BR\left(B \to K^\star\left(892\right)\gamma\right)}{BR\left(B \to X_s\gamma\right)} = \left(\frac{m_B}{m_b}\right)^3 \left(1 - \frac{m_{K^\star}^2}{m_B^2}\right)^3 \left|T_1\left(q^2 = 0\right)\right|^2 \tag{2.12}$$

of the esclusive over inclusive process, and while the lower Lattice results gives a value of the order of 6%, the larger Lattice and the QCD sum rules results give a value larger than 20%: despite the large errors, this result confronts well with the experimental result

$$R_{K^\star}^{\text{exp}} = 0.19 \pm 0.09 \ . \tag{2.13}$$

The agreement is quite encouraging, and calls for further study in order to improve the confidence in the long-distance physics analyses, and eventually reduce the difference in the numerical results.

2.2.1   The inclusive rate

The inclusive rate $\Gamma\left(B \to X_s\gamma\right)$ can be accurately predicted on the basis of perturbative QCD only. The reason is that in a spectator model the *probability* of the decay is determined solely on the basis of the *amplitude* for the quark process $b \to s\gamma$, that is, one assumes that no interferences are present between short and long distance effects.

This is more than an assumption: the Heavy Quark Effective Theory allows to demonstrate that the spectator model is a well definite limit of QCD, the leading term in an expansion in powers of $\frac{\Lambda_{\text{QCD}}}{m_Q}$ (see for instance [100] and references therein). Moreover it can be shown that the expansion starts at the Λ^2/m_Q^2 level, hence for the b quark, $m_b \simeq 5\text{GeV}$, the non spectator corrections are expected to amount to less than 1%.

In the spectator model the inclusive rate is therefore given by

$$\Gamma\left(B \to X_s\gamma\right) = \frac{\alpha_{\text{QED}} G_F^2 m_b^5}{32\pi^4} \left|V_{tb} V_{ts}^\star\right|^2 \left|C_{\text{m.m.}}\left(m_b\right)\right|^2 \ ; \tag{2.14}$$

a more refined discussion of this formula, and of the dependence on the scale used to separate short and long distance contributions shall be given in the following chapters.

The predictivity of this formula is enhanced by reducing the strong dependence on the quark mass m_b, normalizing the rate to the semileptonic b decay, $\Gamma\left(b \to c\,e\,\bar{\nu}_e\right)$. The ratio results

$$R = \frac{\Gamma\left(b \to s\gamma\right)}{\Gamma\left(b \to c\,e\,\bar{\nu}_e\right)} = \frac{|V_{tb}V_{ts}^\star|^2}{|V_{cb}|^2} \frac{6\,\alpha_{\text{QED}}}{\pi\left(1 - f\left(\frac{m_c}{m_b}\right)\right)}\,|C_{\text{m.m.}}\left(m_b\right)|^2 \ , \tag{2.15}$$

where the f function results from the integration over the phase space in the semileptonic b decay

$$f\left(x\right) = x^2\left(8 - 8x^4 + x^6 + x^2 \ln x\right) \ . \tag{2.16}$$

The ratio in Eq. 2.15, or equivalently the ratio of branching fractions

$$R = \frac{BR\left(B \to X_s\gamma\right)}{BR\left(B \to X_c\gamma\right)} \tag{2.17}$$

together with the experimental measure [70]

$$\Gamma\left(B \to X_c\,e\,\bar{\nu}_e\right) = 10.7 \pm 0.5 \tag{2.18}$$

allows to predict the branching fraction, *slowly* dependent on m_t, shown in Fig. 2.2, which includes the short distance QCD corrections at the level of Leading Logs. The error bars accompaining the theoretical values have been computed following [81], and more details will be given in Sec. 5.4.1.

Despite the large errors, the figure shows a remarkable success of the SM and of perturbative QCD. We shall see indeed that the theoretical prediction is strongly influenced by the aforementioned short distance QCD "corrections", and it is enhanced by a factor ranging between 4 and 5 for reasonable values of the t mass. It is worth recalling that even before the measurement of the inclusive decay rate, the knowledge of the $BR\left(B \to K^\star\gamma\right)$ and the SM prediction for the inclusive rate (including the short distance QCD corrections) allowed to obtain a value for the ratio defined in Eq. 2.12

$$R_{K^\star} = 0.15 \pm 0.06 \ . \tag{2.19}$$

In absence of the QCD enhancement this ratio would be of the order of 60%, showing how dramatic is the effect of these corrections. One may wonder if so large corrections can be reliable, and in which sense they are "perturbative".

2.2.2 The QCD enhancement

The substantial short distance QCD enhancement makes the $b \to s\gamma$ process the only known SM process "dominated by two loop effects" [81].

Let us briefly comment on the reasons of this big enhancement.

As already stated, at short distances the Feynman diagrams in Fig. 2.1 generate the decay. This graphs give rise to a coefficient $C_{\text{m.m.}}$ for the magnetic momentum operator in Eq. 2.5, whose value varies between $\simeq 0.17 - 0.21$ for a top mass in the interval $140 - 200\text{GeV}$. However, through QCD corrections also the Fermi type operators, like $(\bar{s}c)_{V-A} \otimes (\bar{c}b)_{V-A}$,
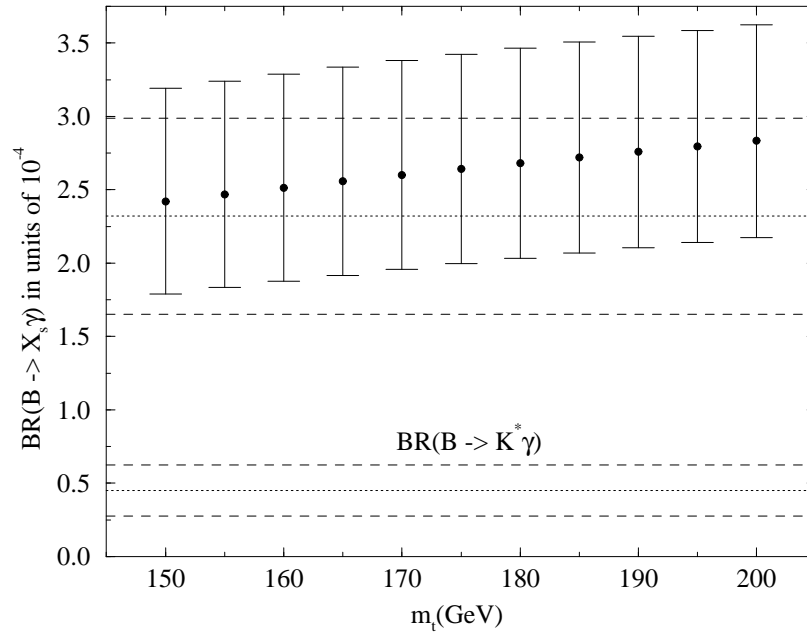
Figure 2.2: Theory and experiment: $B \to X_s\gamma$ in the SM
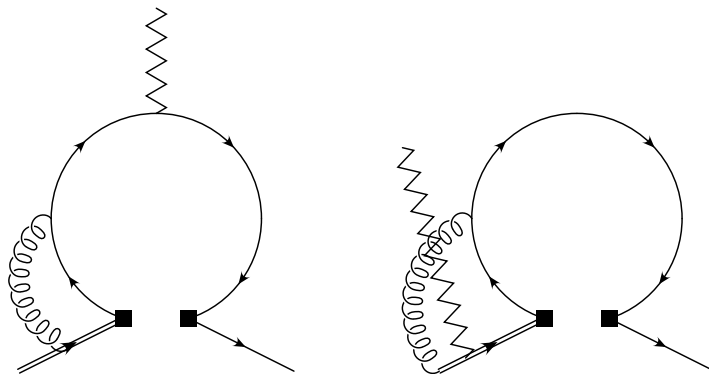
Figure 2.3: Typical two-loop graphs contributing to $b \to s\gamma$

which appear in the Effective Theory below the Electroweak scale, contribute to the process via two-loop Feynman graphs, as in Fig. 2.3.

These operators give rise, order by order in the loop expansion, to contributions proportional to powers of $\alpha_S (M_W) \log \frac{\mu \simeq M_W}{m_b} \simeq 0.33$, where the value of the μ scale has been fixed by matching the effective theory and the underlying complete theory (say, the SM). This corrections can be summed in part to all orders of the perturbative expansion, and can be included in a redefinition of the coefficient of the magnetic momentum operator[1] (schematically)

$$C_{\text{m.m.}} (m) = f_m \left(\frac{\alpha_s (m)}{\alpha_s (M_W)} \right) C_{\text{m.m.}} (M_W) + \ldots + f_F \left(\frac{\alpha_s (m)}{\alpha_s (M_W)} \right) C_{\text{Fermi}} \qquad (2.20)$$

where the m argument of the coefficient means that the renormalization scale appropriate to compute the matrix element in Eq. 2.6 has been "scaled down" to m, and the f functions contain the QCD corrections, which can be expressed in terms of the ratio of the QCD couplings at different scales.

We shall see that the additive renormalization schematically addressed by the "Fermi" coefficient in Eq. 2.20 is responsible for the consistent improvement, while the multiplicative renormalization tends to suppress the rate.

The physical origin can be traced back to the evasion of the GIM mechanism. If the t and c masses where equal, the diagrams in Fig. 2.3 would have been canceled by corresponding graphs with a virtual t quark.

2.3   The $b \to s\gamma$ as a probe of new physics

To be definite, in Sec 2.2 we have briefly addressed the origin of this rare decay in the context of SM. A more complete discussion will be given in the following chapter, but it should be stressed that the $b \to s\gamma$ process is not only a test of the SM but more importantly is a window in the higher energy structure of the theory. The reason is that non standard contributions resulting from the exchange of undiscovered particles are comparable in size to the SM contributions, which are suppressed by the symmetry mechanism preventing large FCNC effects. Hence on general grounds one expects that even if it will be not possible for some time to extend the investigation of the spectrum of elementary particles, the indirect effects will be accessible through the combined study of the different rare decays.

Fixing the attention to $b \to s\gamma$, many detailed analyses exist, in the context of the Minimal Supersymmetric Standard Model (MSSM) [59, 63, 86, 99], the Two Higgs Doublet extension of the scalar sector [79, 80, 81] and in other models, as in the $SU(2)_L \times SU(2)_R \times U(1)$ extension of the electroweak gauge group [89].

The lack of space and direct experience prevents us from adequately reviewing the many interesting contributions to the argument, but a few general considerations are possible.

2.3.1   Two Higgs doublet models

The two Higgs doublet models can be seen as the mildest extension of the electroweak simmetry breaking sector of the SM, and the resulting charged Higgs fields in the spectrum naturally induce FCNC transitions. There are two versions of the model, the so called Type I in which one of the doublets "gives mass" to all the fermions while the other decouples, and

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[1] In the leading log approximation
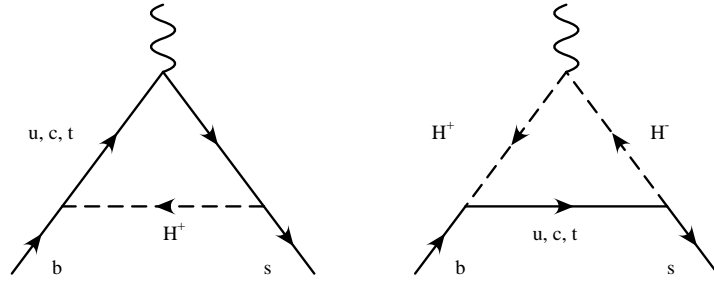
Figure 2.4: Charged Higgs contribution to $b \to s\gamma$.

the Type II model in which one Higgs doublet (φ_2) couples to up-quarks, while the other (φ_1) to down-quarks. The Type II model is more popular because it may possibly give a "natural" explaination of the large mass splitting in the t, b doublets, in terms of a large ratio of the vacuum expectation values, commonly parameterized as

$$\tan\beta = \frac{v_2}{v_1} \; ; \tag{2.21}$$

moreover, this model appears as the minimal extension of the Higgs sector when considering SUSY theories.

In either models there are new contributions to $b \to s\gamma$ through the diagrams in Fig. 2.4, with the general form [80]

$$A_{\text{SM}}\left(\frac{m_t^2}{M_W^2}\right) + \lambda A_H^1\left(\frac{m_t^2}{M_{H_\pm}^2}\right) + \frac{1}{\tan^2\beta}A_H^2\left(\frac{m_t^2}{M_{H_\pm}^2}\right) \tag{2.22}$$

where A_{SM} and $A_H^{1,2}$ are the SM and charged-Higgs contributions respectively; in model I one has $\lambda = -1/\tan^2\beta$, while in model II $\lambda = 1$. For small values of $\tan\beta$ both the models give an enhancement of the amplitude, and in model II the term A_H^1 gives an amplitude always larger than in the SM. However in model II large $\tan\beta$ values are more appealing, as discussed above.

In Fig. 2.5 we show the prediction for two values of $M_{H_\pm}$, already considered in [81]. For a large $M_{H_\pm}$ the SM and the 2HDM become undistinguishable, at the present level of theoretical accuracy.

2.3.2 $b \to s\gamma$ in the MSSM

Good reviews on the subject exist, and we shall limit to a brief account, mainly following [59, 78, 99]. Even restricting the attention to the so called $N = 1$ models, that is, the SUSY models which are derived as low energy theories from a $N = 1$ supergravity model, a large number of parameters exists. The MSSM is obtained with some additional constraints

- a new symmetry, the R parity, is imposed in order to avoid large baryon-lepton number violations.

- Only the minimal number of fields needed to "supersymmetrize" the known spectrum are introduced.
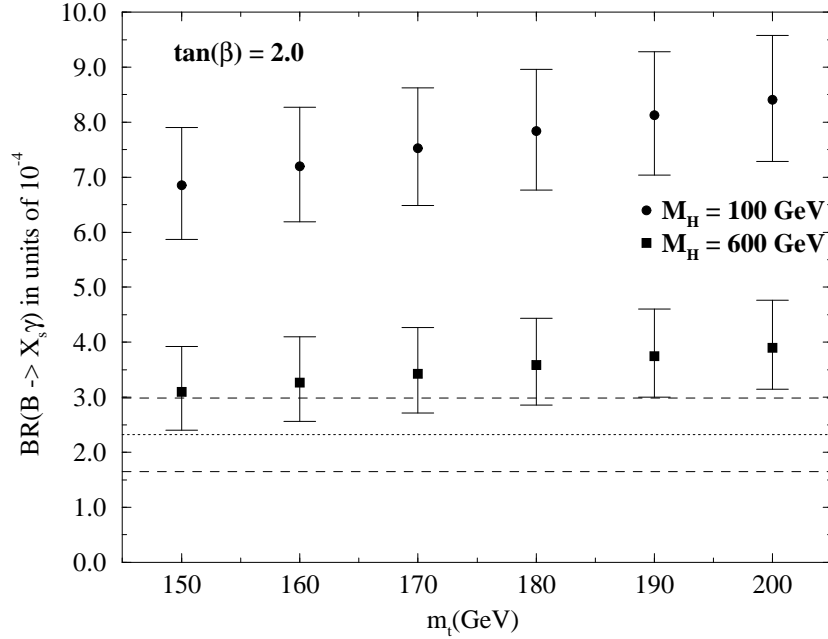
Figure 2.5: $B \to X_s\gamma$ in the 2HDM

- Canonical kinetics terms are chosen for all the scalar fields.

While the particle content is fixed, with these choices there are essentially 4 new parameters with respect to SM (5 if one relaxes the assumption of a flat Kälher metric which underlies the choice of canonical kinetic terms), which determine the low-energy theory. By requiring also that the correct scale of electroweak symmetry breaking is reproduced, and not introducing a third Higgs doublet, the parameters reduce to 3; of these, one is again commonly chosen as $\tan\beta$, while the other two can be for instance chosen as two mass scales characterizing the pattern of SUSY breaking.

The enlargement of the spectrum, with respect to the SM, brings naturally new sources of FCNC contributions: apart the extension of the Higgs sector, discussed in the preceding section, one has an immediate new contribution coming from the supersymmetrization of the W and charged Higgs contributions. The fermion mass eigenstates resulting from $\tilde{W}$ and $\tilde{H}_\pm$ mixing, the so called charginos $\tilde{\chi}_\pm$, are exchanged in loops where the up squarks also circulate.

Other peculiar contributions result from the Flavor Changing vertices connecting quarks and squarks through the exchange of neutralinos $\tilde{\chi}^0$ and gluinos. In summary one has contributions resulting from the following virtual particles

1. W^- and up quarks,

2. charged Higgs fields H^- and up quarks,

3. charginos, $\tilde{\chi}^-$, and up squarks,

4. neutralinos, $\tilde{\chi}^0$ and down squarks,

5. gluinos, $\tilde{g}$, and down squarks

where the last three are peculiar of SUSY. In some of the older analyses the first two contributions were considered dominant, thus giving bounds analogous to the one considered in the SM. However more complete studies [59, 86] show that only the neutralino and gluino contributions can be numerically neglected. On the other hand, while as in the 2HDM the charged Higgs contributions interfere constructively with the SM contribution, there exist large portions of the parameter space in which the chargino amplitude interferes destructively, and it can also become the dominant contribution for a large $\tan\beta$ value.

It is not possible even to account for the exploration of the parameter space needed to make these statements quantitative, so we refer to a recent analysis [99].

We can expect in a near future an improvement of the accuracy of the experimental results, however we see that the possible interference effect among different SUSY contributions makes difficult a clear distinction from the SM. This is even more true in presence of the rather large theoretical uncertainties, mainly originated by the lack of knowledge of higher order QCD corrections.

Chapter 3

Effective theories

3.1 Generalities

In this thesis we consider physical processes which are strongly influenced by quantum corrections. A direct consequence, relevant both for weak decay processes, and the electroweak corrections to LEP observables, is the dependence of the theoretical predictions on widely different mass scales.

The reason is that the SM[1] has the ambition to account, on quantitative grounds, for phenomena occurring at largely different energies, starting from a common fundamental description. For instance, in many weak decays the typical energy scale, the Q value of the reaction, is of the order of 1GeV, while the fundamental interactions result from the exchange of virtual particles with masses of the order of 100GeV. On the other hand, observables determined with experiments at energies of the order of the Z resonance, must be related to low-energy experiments, like the β decay or the νN scattering.

A better understanding of the phenomena, as well as a practical way to properly account for the different scales involved, can be obtained by using the effective field theory formalism [26].

3.1.1 What is an Effective Theory?

A number of useful reviews exist (see for instance [27]), so we shall limit ourselves to a very general discussion, and then proceed through the study of a practical case.

The idea is quite simple: we start from an high energy theory, say the SM, described by a local lagrangian[2]. Suppose it involves *light* fields, ϕ, and *heavy* fields, Φ.

We are going to study a process which can be described in terms of correlations of light fields

$$\langle 0 | \phi(x_1)\phi(x_2)\dots\phi(x_n) | 0 \rangle = \int d\phi \, d\Phi e^{-S_0(\phi,\,\Phi)} \phi(x_1)\phi(x_2)\dots\phi(x_n) \ . \tag{3.1}$$

We can perform the functional integration over the heavy field, and have an equally good description of the light fields interactions. The same Green function will be obtained from a

---

[1]Or any sensible extension of it.

[2]In some sense, this is the *definition* of a fundamental theory, at least for the majority of physicists.

functional integral

$$\int d\phi e^{-S_1(\phi)} \phi(x_1)\phi(x_2)\ldots\phi(x_n) \tag{3.2}$$

involving an *effective* action S_1, which is not, at least in general, local. If we try to express the action in terms of local interactions, we find that an infinite tower of composite operators, involving more and more elementary fields, is required in order to *match* the two descriptions.

Technically this procedure, which allows to go from an effective non local action to a local theory is the Wilson's Operator Product Expansion: in presence of an heavy mass-scale, it allows to organize the infinite tower of non-renormalizable interactions as a Taylor series in inverse powers of the heavy field mass. The short distance physics is then "buried" in the coefficients of the resulting local operators, while the long distance physics remains explicit.

It is worth noting the similarity with the description of critical phenomena and the existence of universality classes: long distance physics is described by an entire class of different theories, differing on the amount of short distance physics which is left explicit.

3.1.2   Matching

In all the practical cases the non-local effective action cannot be determined: instead, one builds a local effective lagrangian, or more generally an effective Hamiltonian, up to some power p of the inverse mass (for instance some order in G_F), which will involve a finite number of operators and coefficients.

For instance, in the case of weak decays, for any initial and final state, which can be written in terms of "light" fields, it must be imposed the indentity

$$\langle f|i\rangle_{\text{full}}(\mu) = \langle f|H_{\text{eff}}|i\rangle(\mu) + O\left(\frac{1}{M^p}\right) \tag{3.3}$$

where H_{eff} is a sum of local *renormalized* operators $N[O]$,

$$H_{\text{eff}} = \sum_j C_j(M,\mu) N[O_j] \tag{3.4}$$

built of light fields, and depending on M through the coefficients C. The bracket on the left hand side addresses an amplitude $i \to f$ evaluated in the "full" theory, while on the right hand side the expectation value of the effective hamiltonian between the same states is evaluated, as specified by the lagrangian of the light theory[3].

A *matching* scale μ is therefore introduced, as the scale where coefficients of the low-energy theory are tuned to reproduce the results of the high energy theory, on the basis of a finite number of *test* processes.

On physical grounds, the matching scale can be regarded as separating short distance from long distance physics. In fact, through the expansion in loops both theories generate logarithmic corrections, depending on the typical momenta and masses involved and an arbitrary renormalization scale. This means, as well known from QCD, that choosing the renormalization scale of the same order as the typical scale of the process, the higher order corrections are minimized.

---

[3]Also the parameters of the light theory undergo finite renormalizations in the matching procedure, but we shall see that they are unobservable, and therefore irrelevant.

In the matching procedure, one is guided by the "complete" theory to set the μ scale in order to have theoretical predictions for the test processes as accurate as possible: so the effective theory is determined at a large scale, while it is to be applied at processes at a lower scale.

3.1.3   Renormalization Group

The coefficients of the effective theory, determined at the matching scale, say $\mu \simeq M$, act as the experiments setting the parameters of a renormalizable field theory. As in a renormalizable field theory, one determines the fundamental parameters, the "couplings" of the theory, at some scale, and then the theory is used at a possibly largely different scale.

Now, at a fixed order in the $\frac{1}{M}$ expansion the effective theory is renormalizable, and the coefficients can be scaled using the renormalization group evolution, in order to improve the convergence of the perturbative expansion.

It is worth then reporting the observation of Georgi [48], "the renormalization group is simply the matching of the theory at the scale μ to the theory at the scale $\mu - d\mu$, without changing the particle content". In other words one integrates out the effect of the energy modes higher than the process in consideration.

The RG evolution results then from the integration of this differential matching procedure, and allows to use in a optimal way the informations coming from perturbation theory.

The Effective Hamiltonian is scaled down from the matching scale $\mu \simeq M$ to the characteristic low energy scale of the physical amplitude

$$\sum_j C\left(\mu \simeq M,\, M\right) \langle f \left| N\left[O_j\right]\right| i \rangle \left(\mu \simeq M\right) \rightarrow \sum_j C\left(\mu \simeq m,\, M\right) \langle f \left| N\left[O_j\right]\right| i \rangle \left(\mu \simeq m\right) \quad (3.5)$$

which means that one determines the coefficients appropriate to the operators renormalized at the scale $\mu \simeq m$.

In this way, the large logarithms that would appear in the matrix elements as a consequence of the QCD corrections in presence of two widely different scales, the renormalization scale of the operators and the typical scale m of the low energy process, are transferred in the coefficients.

In the rest of the Chapter we shall give more details on this procedure, illustrating the matching procedure for the Effective Hamiltonian which describes the rare b decays.

3.2   The FCNC Effective Hamiltonian

In the Standard Model, our "fundamental" theory, the rare b decays are mediated by the graphs in Fig. 3.1, where in the loop circulate the c and t quarks[4], and the virtual dashed lines are the W boson or the charged component of the Higgs multiplet.

A particular gauge is chosen, the so called R_ξ gauge, described in Appendix C: in this gauge the theory is renormalizable by power-counting, and the nonlinear form of the gauge fixing term cancels trilinear vertices A, W, ϕ, reducing the number of graphs to be computed. Most importantly, thanks to the covariant derivative in the gauge fixing term, the electromagnetic Ward-Takahashi identities are unbroken, and there is no need of finite renormalizations to reinforce them.

---

[4]the u quarks can be completely neglected thanks to the smallness of the corresponding CKM matrix entries
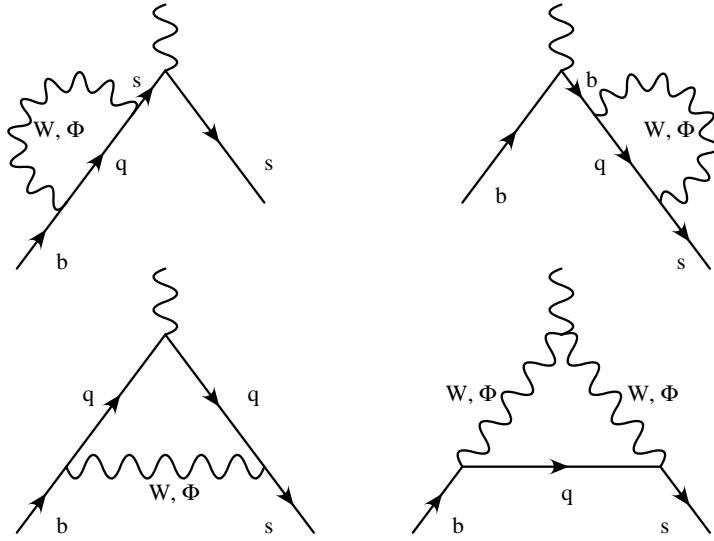
Figure 3.1: $b \to s\gamma$ process in SM
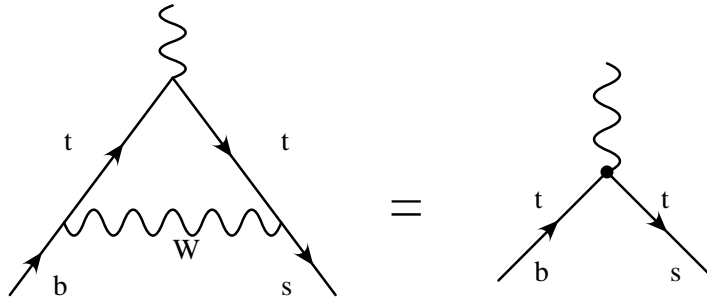
Figure 3.2: Contribution to Effective Hamiltonian from the t quark

The first step in the determination of the effective hamiltonian is the computation of the amplitude for a minimal set of relevant processes in the full theory. In the following we will integrate out the t and $W, (\phi)$ fields: graphs with heavy particles t, W give rise, in the expansion in powers of external momenta, to local interactions which can be reexpressed as a sum over local operators. This is pictorially expressed in Fig. 3.2

By computing the proper vertex part, one finds easily the following effective hamiltonian, resulting from the t exchange only[5]:

$$H_{\text{eff:top}} = \frac{4G_F}{\sqrt{2}} V_{ts}^\star V^{tb} \sum_{i=1}^{8} C_i^t O_i \tag{3.6}$$

where

---

[5]Recall that $\frac{4G_F}{\sqrt{2}} = \left(\frac{g}{\sqrt{2}}\right)^2 \frac{1}{M_W^2}$ is the coupling of two weak currents

$$O_1 = \frac{1}{(4\pi)^2}\bar{s}_L\,\hat{D}\hat{D}\hat{D}\,b_L$$

$$O_2 = \frac{1}{(4\pi)^2}(i\,e\,Q_d)\bar{s}_L\,\left\{\hat{D},\,F_{\mu\nu}\,\sigma_{\mu\nu}\right\}b_L$$

$$O_3 = \frac{1}{(4\pi)^2}(-\,i\,e\,Q_d)\bar{s}_L\,\gamma_\nu\,b_L D_\mu F_{\mu\nu}$$

$$O_4 = -\frac{1}{(4\pi)^2}m_b\,\bar{s}_L\,\hat{D}\hat{D}\,b_R$$

$$O_5 = \frac{1}{(4\pi)^2}(-i\,e\,Q_d)m_b\,\bar{s}_L\,F_{\mu\nu}\sigma_{\mu\nu}\,b_R$$

$$O_6 = \frac{1}{(4\pi)^2}(i\,g_s)\,\bar{s}_L\,\left\{\hat{D},\,T^A G^A_{\mu\nu}\,\sigma_{\mu\nu}\right\}b_L$$

$$O_7 = \frac{1}{(4\pi)^2}(-i\,g_s)\,\bar{s}_L\,\gamma_\nu T^A\,b_L\,(D_\mu G_{\mu\nu})^A$$

$$O_8 = \frac{1}{(4\pi)^2}(-i\,g_s)m_b\,\bar{s}_L T^A G^A_{\mu\nu}\sigma_{\mu\nu}\,b_R \tag{3.7}$$

and the computation gives

$$C_1^t = \frac{(2+x)\,(1-5\,x-2\,x^2)}{6\,(-1+x)^3}+\frac{x^2\,(2+x)\,\log(x)}{(-1+x)^4}$$

$$C_2^t = \frac{46-141\,x+105\,x^2+8\,x^3}{24\,(-1+x)^3}+\frac{(2-5\,x)\,x\,(-2+3\,x)\,\log(x)}{4\,(-1+x)^4}$$

$$C_3^t = \frac{104-312\,x+237\,x^2-47\,x^3}{36\,(-1+x)^3}+\frac{(8-32\,x+54\,x^2-30\,x^3+3\,x^4)\,\log(x)}{6\,(-1+x)^4}$$

$$C_4^t = \frac{x\,(1-x^2+2\,x\,\log(x))}{2\,(1-x)^3}$$

$$C_5^t = \frac{(3-5\,x)\,x}{4\,(-1+x)^2}+\frac{x\,(-2+3\,x)\,\log(x)}{2\,(-1+x)^3}$$

$$C_6^t = \frac{-8+12\,x+15\,x^2-x^3}{24\,(-1+x)^3}+\frac{(2-5\,x)\,x\,\log(x)}{4\,(-1+x)^4}$$

$$C_7^t = \frac{-4-42\,x+21\,x^2+7\,x^3}{36\,(-1+x)^3}+\frac{(-4+16\,x-9\,x^2)\,\log(x)}{6\,(-1+x)^4}$$

$$C_8^t = \frac{(-3+x)\,x}{4\,(-1+x)^2}+\frac{x\,\log(x)}{2\,(-1+x)^3} \tag{3.8}$$

with $x=\frac{m_t^2}{M_W^2}$.

This is not the end of the story, because the graphs with charm exchange cannot be accounted for in the same manner: in fact, one easily recognizes that coefficients C_3^t and C_7^t diverge for $x\to0$, hence the limit $m_c\to0$ in the full theory leaves graphs which are infrared divergent in the limit of zero external momenta and cannot be expanded in series. This is
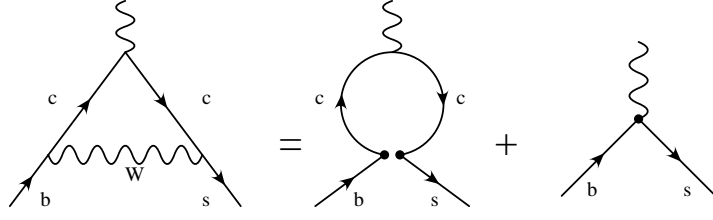
Figure 3.3: Contribution to Effective Hamiltonian from the c quark

what we expect, because of the contribution, in the light theory, of the Fermi-type interaction terms, resulting as the low energy limit of the tree level W-exchange graphs: this interaction is to be included in the effective hamiltonian as

$$\frac{4G_F}{\sqrt{2}} V_{ts}^{\star} V_{tb} O_{10} = -\frac{g^2}{2M_W^2} V_{cs}^{\star} V_{cb} \bar{s}_\alpha \gamma_\mu^L c_\beta \otimes \bar{c}_\beta \gamma_\mu^L b_\alpha \ , \tag{3.9}$$

where the unitarity of the CKM matrix V has been used; α, β are the color indices, and the 9 suffix has been reserved for an operator which will be generated by QCD corrections, having a Fierzed color structure

$$\bar{s}_\alpha \gamma_\mu^L c_\alpha \otimes \bar{c}_\beta \gamma_\mu^L b_\beta \ . \tag{3.10}$$

The 4$-$fermion operator gives rise to the infamous "penguin" graph, in the right hand side in Fig. 3.3: the difference between the graphs in the full theory, and this "penguin" graph is local and can be easily computed. It does not contain IR divergences, because the full and effective theory have the same long distance behaviour, hence it can be expanded in powers of the external momenta. It results the "charm" contribution to the coefficients in Eq. 3.8, again taking into account the unitarity of CKM matrix

$$
\begin{aligned}
C_1^c &= \frac{1}{3} \\
C_2^c &= \frac{23}{12} \\
C_3^c &= \frac{38}{9} - \frac{4}{3}\left(\frac{1}{\varepsilon} - \gamma_E + \log\frac{M_W^2}{4\pi\mu^2}\right) \\
C_6^c &= -\frac{1}{3} \\
C_7^c &= -\frac{7}{9} + \frac{2}{3}\left(\frac{1}{\varepsilon} - \gamma_E + \log\frac{M_W^2}{4\pi\mu^2}\right) \ ;
\end{aligned}
\tag{3.11}
$$

it is important to note that the coefficients C_3, C_7 are explicitly dependent on the subtraction scale μ, as we expected from the general considerations: this dependence is completely canceled by the corresponding dependence in the matrix element of the operator O_{10}. This is true, at this level, by construction, but we shall see that the inclusion of QCD corrections shall make the cancellation only approximate. Moreover, they appear pole parts in the co-efficients: the reason is that the matrix element of operator O_{10} is divergent, while the full

theory amplitude was finite. This divergence is canceled by the coefficients, hence we can redefine the effective hamiltonian as made of renormalized operators, for instance in the $\overline{MS}$ scheme.

Therefore we have the effective hamiltonian

$$H_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{ts}^{\star} V^{tb} \sum_{i=1}^{8} C^i N [O_i] \tag{3.12}$$

with the corrected coefficients

$$
\begin{aligned}
C_1 &= \frac{x\,(1+5\,x)}{2\,(1-x)^3} + \frac{x^2\,(2+x)\,\log(x)}{(-1+x)^4} \ , \\
C_2 &= \frac{x\,(-1-11\,x+18\,x^2)}{8\,(-1+x)^3} + \frac{(2-5\,x)\,x\,(-2+3\,x)\,\log(x)}{4\,(-1+x)^4} \ , \\
C_3 &= \frac{-16+48\,x-73\,x^2+35\,x^3}{12\,(-1+x)^3} + \frac{(8-32\,x+54\,x^2-30\,x^3+3\,x^4)\,\log(x)}{6\,(-1+x)^4} \ , \\
C_4 &= \frac{x\,(1+x)}{2\,(-1+x)^2} + \frac{x^2\,\log(x)}{(1-x)^3} \ , \\
C_5 &= \frac{(3-5\,x)\,x}{4\,(-1+x)^2} + \frac{x\,(-2+3\,x)\,\log(x)}{2\,(-1+x)^3} \ , \\
C_6 &= \frac{(4-x)\,x\,(-1+3\,x)}{8\,(-1+x)^3} + \frac{(2-5\,x)\,x\,\log(x)}{4\,(-1+x)^4} \ , \\
C_7 &= \frac{8-42\,x+35\,x^2-7\,x^3}{12\,(-1+x)^3} + \frac{(-4+16\,x-9\,x^2)\,\log(x)}{6\,(-1+x)^4} \ , \\
C_8 &= \frac{(-3+x)\,x}{4\,(-1+x)^2} + \frac{x\,\log(x)}{2\,(-1+x)^3} \ , \\
C_9 &= 0 \\
C_{10} &= 1 \ .
\end{aligned}
\tag{3.13}
$$

We stress that the results quoted are obtained with operators renormalized in the MS scheme, and the NDR scheme for the treatment of γ_5: when considering QCD corrections we shall see how to compare different schemes and to ensure the scheme independence of physical results.

Chapter 4

Renormalization Group improved perturbation theory

In this somewhat technical Chapter we shall give a brief overview of perturbative renormalization, avoiding the language of "bare" fields and operators; we shall rederive some of the formulas of Renormalization Group, which shall be used in the following Chapters.

4.1 Notations for the Dimensional Regularization

Let us consider a field theory regularized by continuation in the number of dimensions ($d = 4 - 2\varepsilon$), and described in lagrangian formulation by the action

$$S = \int d^d z \left(\mu^2\right)^{-\varepsilon} \sum_i \left(g_i + P_i\right) \mathcal{L}_i\left(z\right) \tag{4.1}$$

where the lagrangian density is intended as a sum over local operators $\mathcal{L}_i$, each multiplied by a charge g_i, and P_i denotes the counterterms needed to render the Green functions finite in the $\varepsilon \mapsto 0$ limit, and also to impose particular renormalization conditions. Note that we suppose that every operator, including the terms in the quadratic part of the lagrangian, carries an explicit coupling, which will be very useful to deduce general identities. We shall also use the abbreviated form

$$S_l = \int d^d z \left(\mu^2\right)^{-\varepsilon} \mathcal{L}_l \ .$$

Further, let us consider a composite operator $G_i = E_i O_i$, where O_i is a product of some of the fields in the lagrangian, and E_i is a product of couplings. The renormalized product is then defined as

$$N\left[G_i\right] = N\left[E_i O_i\right] = \sum_j E_i M_{ij} O_j \ , \tag{4.2}$$

where the sum is carried over a number of operators (a complete basis) sufficient to guarantee that with an appropriate choice of the "counterterms" matrix M every Green function with the insertion of the operator G_i is made finite in the $\varepsilon \mapsto 0$ limit.

In this formulation, the mass parameter appears in the lagrangian explicitly to compensate the change in mass dimensions of the integration measure, and therefore fields and couplings have their canonical (four dimensional) mass dimensions.

We now want to determine the change in the couplings in order to compensate for a change in the scale μ.

4.1.1   The renormalization scale dependence

Let us consider for definiteness a Green function, the correlation of fields $\phi_1, \ldots \phi_n$ and of a local operator O, and take the derivative with respect to μ,

$$
\mu \frac{\partial}{\partial \mu} \langle 0 \, | N\, [G_i] \phi_1 \ldots \phi_n | \, 0 \rangle = \left\langle 0 \left| N\, [G_i] 2\varepsilon \sum_i (g_i + P_i)\, S_i \right| 0 \right\rangle \; , \tag{4.3}
$$

we insert in the Green function the action, multiplied by a factor of ε. Note that derivatives with respect to the couplings g_i can be expressed as renormalized insertions of operators present in the action density [1],

$$
-\frac{\partial}{\partial g_i} \exp(-S) = \int d^d z\, N\, [\mathcal{L}_i(z)] = \int d^d z \sum_j \left(\delta_{ij} + \frac{\partial P_j}{\partial g_i} \right) \mathcal{L}_i(z) \; , \tag{4.4}
$$

and then let us consider also the effect of the derivative

$$
g \frac{\partial}{\partial g} \equiv \sum_i g_i \frac{\partial}{\partial g_i} \; :
$$

we obtain

$$
\begin{aligned}
g \frac{\partial}{\partial g} \langle 0 \, | N\, [G_i] \phi_1 \ldots \phi_n | \, 0 \rangle \;\; = \;\; & \left\langle 0 \left| g \frac{\partial}{\partial g} \sum_j E_i M_{ij} O_j \phi_1 \ldots \phi_n \right| 0 \right\rangle \\
& - \left\langle 0 \left| N\, [G_i] \sum_j \left(g_j + g \frac{\partial P_j}{\partial g} \right) \mathcal{L}_j \phi_1 \ldots \phi_n \right| 0 \right\rangle \; ;
\end{aligned} \tag{4.5}
$$

hence by taking the sum with the appropriate $\simeq \varepsilon$ factor for the second term it is possible to cancel the "classical" contribution, that is the change in the Green function due solely to the measure. One has therefore

$$
\begin{aligned}
\left(\mu \frac{\partial}{\partial \mu} + 2\varepsilon g \frac{\partial}{\partial g} \right) & \langle 0 \, | N\, [G_i] \phi_1 \ldots \phi_n | \, 0 \rangle = \\
= \;\; & \left\langle 0 \left| 2\varepsilon g \frac{\partial}{\partial g} (E_i M_{ij})\, O_j \phi_1 \ldots \phi_n \right| 0 \right\rangle \\
& + \left\langle 0 \left| NG_i \sum_j 2\varepsilon \left(P_j - g \frac{\partial P_j}{\partial g} \right) S_j \phi_1 \ldots \phi_n \right| 0 \right\rangle \; .
\end{aligned} \tag{4.6}
$$

---

[1] *in passim*, observe that the action density is not a renormalized operator!

For simplicity, let us limit to minimal subtraction [2], and have then

$$P_i = \sum_L \sum_{p=1}^{L} \frac{1}{\varepsilon^p} P_i^{p,L} \ ;$$

Eq. 4.6 becomes

$$\left(\mu \frac{\partial}{\partial \mu} + 2\varepsilon g \frac{\partial}{\partial g} \right) \langle 0 \left| N \left[G_i \right] \phi_1 \ldots \phi_n \right| 0 \rangle =$$

$$= \ 2\varepsilon g \frac{\partial \ln E_i}{\partial g} \langle 0 \left| N \left[G_i \right] \phi_1 \ldots \phi_n \right| 0 \rangle$$

$$+ \left\langle 0 \left| 2\varepsilon E_i \sum_j g \frac{\partial M_{ij}}{\partial g} O_j \phi_1 \ldots \phi_n \right| 0 \right\rangle$$

$$+ \left\langle 0 \left| N \left[G_i \right] \sum_j \tilde{\beta}_j S_j \phi_1 \ldots \phi_n \right| 0 \right\rangle$$

$$+ \left\langle 0 \left| N \left[G_i \right] \sum_j 2 \sum_{L,p=2}^{\infty,L} \left(1 - g \frac{\partial}{\partial g} \right) \frac{P_j^{L,p}}{\varepsilon^{p-1}} S_j \phi_1 \ldots \phi_n \right| 0 \right\rangle \ , \tag{4.7}$$

where the "pre" β functions

$$\tilde{\beta}_i \equiv 2 \left(1 - g \frac{\partial}{\partial g} \right) P_j^{L,1} \tag{4.8}$$

are introduced, the usual definitions of β and γ functions will be recovered only when we shall specify the interaction part of the action density.

By comparing with Eq. 4.4 we can modify the term multiplying the $\tilde{\beta}$ in order to interpret it as a derivative with respect to a coupling:

$$\mu \frac{\partial}{\partial \mu} + \left(\tilde{\beta}_i + 2\varepsilon g_i \right) \frac{\partial}{\partial g_i} - 2\varepsilon g \frac{\partial \ln E_i}{\partial g} \langle 0 \left| N \left[G_i \right] \phi_1 \ldots \phi_n \right| 0 \rangle =$$

$$= \ \left\langle 0 \left| 2\varepsilon E_i g \frac{\partial M_{ij}}{\partial g} O_j \phi_1 \ldots \phi_n \right| 0 \right\rangle$$

$$+ \left\langle 0 \left| \tilde{\beta}_l \frac{\partial}{\partial g_l} \left(E_i M_{ij} \right) O_j \phi_1 \ldots \phi_n \right| 0 \right\rangle$$

$$+ \left\langle 0 \left| N \left[G_i \right] \sum_k \left[2 \sum_{L,p=2}^{\infty,L} \left(1 - g \frac{\partial}{\partial g} \right) \frac{P_k^{L,p}}{\varepsilon^{p-1}} \right] - \tilde{\beta}_l \frac{\partial P_k}{g_l} S_k \phi_1 \ldots \phi_n \right| 0 \right\rangle . \tag{4.9}$$

We can now impose that the chosen subtractions make the Green functions finite: as the derivatives with respect to μ and $\{g\}$ do not introduce spurious divergences, we can say that

---

[2]Modified minimal schemes, like the $\overline{MS}$ or the G scheme, can be interpreted as a redefinition of the μ scale and are therefore also minimal, in the sense that the finite subtractions needed to enforce them do not modify our formulas.

the l.h.s. in Eq. 4.9 is finite, and so must be for the r.h.s. First of all, in absence of the operator insertion, we should have then that

$$
\text{r.h.s} = \sum_k \left\langle 0 \left| N\left[G_i\right] \sum_{p=1}^{\infty} \frac{1}{\varepsilon^p} \left[2\left(1 - g\frac{\partial}{\partial g}\right) P_k^{p+1} - \tilde{\beta}_l \frac{\partial P_k^p}{g_l} \right] S_k \phi_1 \dots \phi_n \right| 0 \right\rangle .
$$

(4.10)

is finite, and therefore each term multiplying the different pole parts must be zero. It follows the noteworthy relation

$$
\begin{aligned}
2\hat{L} P_k^{p+1} - \tilde{\beta}_l \frac{\partial P_k^p}{g_l} &= 0 \\
P_k^p &= \sum_L P_k^{L,p} \\
\hat{L} &= 1 - \sum_l g_l \frac{\partial}{\partial g_l} \qquad \text{Loop counter operator .}
\end{aligned}
$$

(4.11)

It will give useful relations between poles at different orders in L. The relation in Eq. 4.11 allows to rewrite Eq. 4.9 as

$$
\begin{aligned}
\left[\mu\frac{\partial}{\partial \mu} + \left(\tilde{\beta}_l + 2\varepsilon g_l\right) \frac{\partial}{\partial g_l} \right] & \langle 0 \left| N\left[G_i\right] \phi_1 \dots \phi_n \right| 0 \rangle = \\
&= \tilde{\beta}_l \frac{\partial \ln E_i}{g_l} \langle 0 \left| N\left[G_i\right] \phi_1 \dots \phi_n \right| 0 \rangle \\
&\quad + \left\langle 0 \left| E_i \left(\tilde{\beta}_l + 2\varepsilon g_l\right) \frac{\partial M_{i,j}}{\partial g_l} O_j \phi_1 \dots \phi_n \right| 0 \right\rangle .
\end{aligned}
$$

(4.12)

Now we can impose that also the Green function with the insertion of the renormalized operator G_i is finite in the $\varepsilon \mapsto 0$ limit. Taking into account that, by definition of minimal subtraction we have

$$
M_{i,j} = \delta_{i,j} + \sum_{L,p} \frac{1}{\varepsilon^p} M_{i,j}^{L,p} ,
$$

(4.13)

we can rewrite with a few algebra Eq. 4.9 as[3]

$$
\begin{aligned}
\left[\mu\frac{\partial}{\partial \mu} + \left(\tilde{\beta}_l + 2\varepsilon g_l\right) \frac{\partial}{\partial g_l} \right] & \langle 0 \left| N\left[G_i\right] \phi_1 \dots \phi_n \right| 0 \rangle = \\
&= \tilde{\beta}_l \frac{\partial \ln E_i}{\partial g_l} \langle 0 \left| N\left[G_i\right] \phi_1 \dots \phi_n \right| 0 \rangle \\
&\quad - \left\langle 0 \left| E_i \sum_L 2L M_{i,j}^{L,1} N\left[G_i\right] \phi_1 \dots \phi_n \right| 0 \right\rangle
\end{aligned}
$$

---

[3]Observe that in presence of the operator insertion, which modifies the number of vertices, the loop counting operator is $L = -g\frac{\partial}{\partial g}$

$$+\left\langle 0\left|E_i\sum_L^\infty 2L\frac{M_{i,j}^{L,1}}{E_j}E_j\sum_{L',p=1}^{\infty,L'}\frac{1}{\varepsilon^p}M_{j\,k}^{L',p}O_k\phi_1\ldots\phi_n\right|0\right\rangle$$

$$+\left\langle 0\left|E_i\sum_{L,p=1}^{\infty,L}\frac{1}{\varepsilon^p}\left(\sum_l\tilde\beta_l\frac{\partial M_{i,j}^{L,p}}{g_l}-2LM_{i,j}^{L,p+1}\right)O_k\phi_1\ldots\phi_n\right|0\right\rangle\ ;$$

$$(4.14)$$

by defining

$$\tilde\gamma_{i,j}=\sum_L 2LM_{i,j}^{L,1}\frac{E_i}{E_j}-\tilde\beta_l\frac{\partial\ln E_i}{\partial g_l}\delta_{i,j}\tag{4.15}$$

we can rewrite the right hand side of Eq. 4.14 as

$$
\begin{aligned}
r.h.s. \quad=\quad & -\tilde\gamma_{i,j}\langle 0\,|N\,[G_i]\phi_1\ldots\phi_n|\,0\rangle\\
& +E_i\left\langle 0\left|\sum_p\frac{1}{\varepsilon^p}\left[\sum_L 2LM_{i,j}^{L,1}\sum_{L'}M_{j,k}^{L',p}\right.\right.\right.\\
& \left.\left.\left.-\sum_L 2LM_{i,k}^{L,p+1}+\sum_{L,l}\tilde\beta_l\frac{\partial M_{i,k}^{L,p}}{g_l}\right]O_k\right|0\right\rangle
\end{aligned}
\tag{4.16}
$$

and the last term has to be zero $\forall p$, hence the recursive relation

$$\sum_L 2LM_{i,k}^{L,p+1}=\sum_{L,L'}2LM_{i,j}^{L,1}M_{j,k}^{L',p}+\sum_{L,l}\tilde\beta_l\frac{\partial M_{i,k}^{L,p}}{\partial g_l}\ .\tag{4.17}$$

In summary we have

$$
\begin{aligned}
&\left\{\left[\mu\frac{\partial}{\partial\mu}+\left(\tilde\beta_l+2\varepsilon g_l\right)\frac{\partial}{\partial g_l}\right]\delta_{i,j}+\tilde\gamma_{i,j}\right\}\langle 0\,|N\,[G_i]\phi_1\ldots\phi_n|\,0\rangle=\\
\tilde\beta_l\quad=\quad & 2\sum_l\left(1-g\frac{\partial}{\partial g}\right)P_l^{L,1}\\
\tilde\gamma_{i,j}\quad=\quad & \sum_L 2LM_{i,j}^{L,1}\frac{E_i}{E_j}-\tilde\beta_l\frac{\partial\ln E_i}{\partial g_l}\delta_{i,j}\ ;
\end{aligned}
\tag{4.18}
$$

this form of the RG equation is written directly in terms of the "poles" of the Feynman graphs (the P and M matrices).

It will be now useful to connect to the common notions of β and γ functions.

4.1.2   The conventional form

Recall that in Eq. 4.1 we have introduced a coupling for each term in the action density, including for instance the kinetic term. This means that if we now want to set to unity the normalization of this term, we need to reinterpret the derivative with respect to this "eliminated" coupling: the usual way is to define a field anomalous dimension.

Following the constructive approach let us introduce the field anomalous dimension and proceed to its definition: for simplicity we limit the discussion to a single field ϕ, as the generalization is trivial. We rewrite Eq. 4.18 as

$$\left[\mu\frac{\partial}{\partial\mu} + \left(\tilde{\beta}_l + 2\varepsilon g_l\right)\frac{\partial}{\partial g_l} + \gamma_\phi\frac{n}{2}\right]\langle 0\,|N\,[G_i]\phi_1\ldots\phi_n|\,0\rangle =$$
$$= \left[\gamma_\phi\frac{n}{2}\delta_{i,j} - \tilde{\gamma}_{i,j}\right]\langle 0\,|N\,[G_j]\phi_1\ldots\phi_n|\,0\rangle\ , \tag{4.19}$$

and note that the n factor can be obtained by using the field counting operator[4]

$$\hat{n} = \int d^d x\,N\left[\phi(x)\frac{\delta S}{\delta\phi(x)}\right]$$

apart a correction due to the presence of the composite operator: one has

$$n\langle 0\,|N\,[O]\phi_1\ldots\phi_n|\,0\rangle = \langle 0\,|N\,[O]\hat{n}\phi_1\ldots\phi_n|\,0\rangle - n\left\langle 0\left|\phi\frac{\partial}{\partial\phi}N\,[O]\phi_1\ldots\phi_n\right|0\right\rangle\ ,$$

hence the r.h.s. of Eq. 4.19 becomes

$$\begin{aligned}
\text{r.h.s} &= \gamma_\phi\frac{1}{2}\left\langle 0\left|N\,[G_i]\sum_l g_l\rho^l S_l\phi_1\ldots\phi_n\right|0\right\rangle \\
&\quad -\gamma_\phi\frac{1}{2}\left\langle 0\left|E_i M_{i,j}\nu^j O_j\phi_1\ldots\phi_n\right|0\right\rangle \\
&\quad -\tilde{\gamma}_{i,j}\langle 0\,|N\,[G_j]\phi_1\ldots\phi_n|\,0\rangle \\
&= \gamma_\phi\frac{1}{2}\left(-\rho_l g_l\frac{\partial}{\partial g_l}\right)\langle 0\,|N\,[G_i]\phi_1\ldots\phi_n|\,0\rangle \\
&\quad \gamma_\phi\frac{1}{2}\left\langle 0\left|\left(\rho^l g_l\frac{\partial}{\partial g_l} - \nu^j\right)E_i M_{i,j}\phi_1\ldots\phi_n\right|0\right\rangle \\
&\quad -\tilde{\gamma}_{i,j}\langle 0\,|N\,[G_j]\phi_1\ldots\phi_n|\,0\rangle\ , \tag{4.20}
\end{aligned}$$

where we label with ν^i the number of fields in the composite operator G_i, and with ρ_i the same quantity for a term $\mathcal{L}^i$ of the lagrangian density. Let it be g_ϕ the coupling associated with the kinetic term $\partial_\mu\phi\partial_\mu\phi$, and define γ_ϕ such that

$$\tilde{\beta}_\phi = -\gamma_\phi g_\phi \qquad \left(\nu^\phi = 2\right)\ ;$$

we obtain from Eq. 4.19, after the rearrangements needed to reassemble the normal products,

---

[4]Note that the derivation is particularly simple in dimensional regularization, thanks to the simple form of the "action principle"
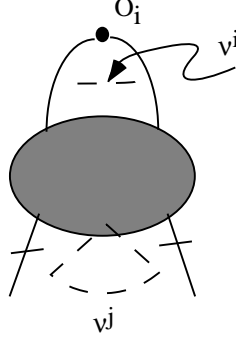
Figure 4.1: A contribution to $M_{i,j}$

$$\left[\mu\frac{\partial}{\partial\mu} + (\beta_l + 2\varepsilon g_l)\frac{\partial}{\partial g_l} + \gamma_\phi\frac{n}{2} + \gamma_{i,j}\right]\langle 0|N\,[G_i]\phi_1\ldots\phi_n|0\rangle =$$

$$= \frac{\gamma_\phi}{2}E_i\left\langle 0\left|\left(\rho_l g_l\frac{\partial}{\partial g_l} + \nu^i - \nu^j\right)M_{i,j}O_j\phi_1\ldots\phi_n\right|0\right\rangle\;;$$

$$\gamma_{i,j} = \tilde{\gamma}_{i,j} + \frac{\gamma_\phi}{2}\left(\nu^i - \rho^l g_l\frac{\partial\ln E_i}{\partial g_l}\right)\delta_{i,j}\;;$$

$$\beta_l = \tilde{\beta}_l + \gamma_\phi\frac{\nu^l}{2}g_l\;. \tag{4.21}$$

The right hand side cannot be written in renormalized form and in fact is identically zero: indeed, let us consider a Feynman diagram with the insertion of the operator O_i contributing to $M_{i,j}$: it has ν^j external lines, as in Fig. 4.1.

Applying the operator $\rho^l g_l\frac{\partial}{\partial g_l}$, it counts with a positive sign the number of lines connected to each vertex of the blob, and with a negative sign and a factor of 2 the internal propagators

$$\rho^l g_l\frac{\partial}{\partial g_l} = -2I + \sum_v \rho^v\;.$$

But one has also

$$\nu^i + \sum_v \rho^v = 2I + \nu^j$$

hence the identity

$$\rho^l g_l\frac{\partial}{\partial g_l} + \nu^i - \nu^j = 0\;. \tag{4.22}$$

results.

4.1.3   Reference Formulas

We have finally written the RG equations for 1PI Green functions (simply changing the sign of the term referring to the field anomalous dimensions) as follows

$$\left\{\left[\mu\frac{\partial}{\partial\mu} + (\beta_l + 2\varepsilon g_l)\frac{\partial}{\partial g_l} - \gamma_\phi\frac{n}{2}\right]\delta_{i,j} + \gamma_{i,j}\right\}\langle 0\,|N\,[G_j]\phi_1\dots\phi_n|\,0\rangle_{1\mathrm{PI}} = 0 \qquad (4.23)$$

and we have found the form of the β and γ functions

$$\gamma_\phi = -\sum_L 2LP_\phi^{L,1}\ ,$$

$$\beta_l = \sum_L 2LP_l^{L,1} - \gamma_\phi\frac{\rho^l}{2}g_l\ ,$$

$$\gamma_{i,j} = \sum_L 2LM_{i,j}^{L,1}\frac{E_i}{E_j} + \left(\frac{\gamma_\phi}{2}\nu^i - \sum_l \beta_l\frac{\partial\ln E_i}{\partial g_l}\right)\delta_{i,j}\ ; \qquad (4.24)$$

moreover, a set of recursive relations for pole parts have been found, and using the topological relation in Eq. 4.22, they can be rewritten as

$$\sum_L 2LM_{i,k}^{L,p+1} = \sum_{L,L'} 2LM_{i,j}^{L,1}M_{j,k}^{L',p}$$

$$+ \sum_{L,l}\beta_l\frac{\partial M_{i,k}^{L,p}}{\partial g_l} + \frac{\gamma_\phi}{2}\sum_L\left(\nu^i - \nu^j\right)M_{i,k}^{L,p}$$

$$\sum_L 2LP_k^{L,p+1} = \sum_{L,l}\left(\beta_l\frac{\partial}{\partial g_l} - \frac{\gamma_\phi}{2}\rho^k\right)P_k^{L,p}\ ; \qquad (4.25)$$

these relations[5] can be expanded in $\hbar$ and give therefore useful checks on the computations: every pole in a diagram at order L in $\hbar$, apart the simple pole, can be predicted in terms of the poles of graphs with $L' < L$ and of the β and γ functions. This is exactly equivalent to say, as we shall see, that order by order in perturbation theory the only new information, in the Renormalization Group Improved perturbation theory, comes from the simple log's.

4.1.4   Finite renormalizations

Let us now consider the possibility of finite renormalizations, that is, we do not simply subtract poles but also finite parts. This can be useful in order to impose renormalization conditions, for instance on the matrix elements of operators. For definiteness, we shall suppose to have already regularized some operator basis $\{N[G]\}$, and we can therefore impose further renormalization conditions with a mixing matrix of finite parts.

---

[5]In proving the second one it can be useful to observe that in a manner similar to the proof of Eq. 4.22 one can show that

$$\sum_l \rho^l g_l\frac{\partial P_k}{\partial g_l} = \rho^k$$

We introduce a matrix F and define the renormalized operators as

$$R\left[G_i\right] = F_{i,j} N\left[G_j\right] ; \tag{4.26}$$

the matrix F is assumed to give finite corrections to counterterms, hence it possesses a loop expansion of the form:

$$F_{i,j} = \delta_{i,j} + \sum_{L=1}^{\infty} F_{i,j}^L . \tag{4.27}$$

It is simple to find the modification to the RG equation Eq. 4.24 induced by finite renormalizations: the RG equation can be written schematically as

$$
\begin{aligned}
\left[\mathcal{D}\hat{\delta} + \hat{\gamma}\right] N\left[\vec{G}_i\right] &= 0 \\
\mathcal{D} &= \mu\frac{\partial}{\partial\mu} + (\beta_l + 2\varepsilon g_l)\frac{\partial}{\partial g_l}
\end{aligned}
\tag{4.28}
$$

and therefore the equation for the renormalized operators is

$$\left[\hat{F}\mathcal{D}\hat{F}^{-1} + \hat{F}\hat{\gamma}\hat{F}^{-1}\right] N\left[\vec{R}_i\right] = 0 . \tag{4.29}$$

It is worth noting that the only source of μ dependence is in the action, Eq. 4.1, hence the $\mu\frac{\partial}{\partial\mu}$ operator commutes with $\hat{F}$. The derivative with respect to the couplings, instead, acts on $\hat{F}$, and as we assume that this matrix is defined in order to give finite renormalization to counterterms, it has the same properties under this derivation as the matrix M: in other words the coupling content is the same, and it follows that (see note 3)

$$g_l\frac{\partial}{\partial g_l}\hat{F} = -L\hat{F} .$$

By using also the identity

$$\hat{F}\mathcal{D}\hat{F}^{-1} = -\left(\mathcal{D}F\right)\hat{F}^{-1}$$

the modified RG equation results

$$
\begin{aligned}
\left[\mathcal{D}\hat{\delta} + \hat{\gamma}'\right] R\left[\vec{G}_i\right] &= 0 \\
\hat{\gamma}' &= \hat{F}\hat{\gamma}\hat{F}^{-1} - \left(\mathcal{D}\hat{F}\right)\hat{F}^{-1} \\
&= \hat{F}\hat{\gamma}\hat{F}^{-1} - \left(\beta_l\frac{\partial\hat{F}}{\partial g_l} - 2\varepsilon\hat{L}\hat{F}\right)\hat{F}^{-1} :
\end{aligned}
\tag{4.30}
$$

the effect of the finite renormalization is a modification of $\hat{\gamma}$ in order to account for the evolution of the finite renormalization matrix, plus a "rotation" of the original γ matrix.

4.1.5 Treatment of evanescent operators

It is well known how in dimensional regularization the arbitrariness in the d dimensional extension of 4 dimensional mathematical objects reflects itself in the appearance of the so-called "evanescent" operators. While we refer to the literature, and to the examples coming from practical computations (Chap. 5), for details on their treatment, in particular on the reduction formulas which are the equivalent of the Zimmermann formulas in the BPHZ formulation, we want here to discuss how evanescent operators modify the $d \to 4$ limit of the RG equation.

Suppose for definiteness to be interested in the evolution of the matrix element of some operator $N[R_1]$, and that in the renormalization process it mixes with a set $\{N[R]\}$ of relevant operators having a non zero limit in 4 dimensions, and a set $\{N[E]\}$ of evanescent operators. The RG equation Eq. 4.23 are coupled, so in general one shall have

$$
\begin{aligned}
\mathcal{D}N\left[\vec{R}\right] + \hat{\gamma}_{R,R}N\left[\vec{R}\right] + \hat{\gamma}_{R,E}N\left[\vec{E}\right] &= 0 \\
\mathcal{D}N\left[\vec{E}\right] + \hat{\gamma}_{E,E}N\left[\vec{E}\right] &= 0 \; ;
\end{aligned}
\tag{4.31}
$$

now the set of evanescent operators is in many cases unbounded, that is, order by order in perturbation theory new operators appear; this is not the ruin of perturbation theory for in the $d \to 4$ limit the matrix elements of evanescent operators can be rewritten as local contributions, which can be therefore accounted for as finite renormalizations of the relevant operators. The well known result is that a reduction matrix exists, defined in perturbation theory, such that

$$
N\left[\vec{E}\right] = \hat{r}_{E,R}N\left[\vec{R}\right] \; ;
\tag{4.32}
$$

the matrix $\hat{r}$ is zero at the classical level.

It is therefore immediate to decouple the evanescent operators in Eq. 4.31, by applying the reduction formula to the first relation in Eq. 4.31:

$$
\begin{aligned}
\left[\mathcal{D}\hat{\delta} + \hat{\gamma}'\right] N\left[\vec{R}\right] &= 0 \\
\hat{\gamma}' &= \hat{\gamma}_{R,R} + \hat{\gamma}_{R,E}\hat{r}_{E,R}
\end{aligned}
\tag{4.33}
$$

the second relation instead gives a consistency check:

$$
\left[(\mathcal{D}\hat{r}_{E,R}) - \hat{r}_{E,R}\left(\hat{\gamma}_{R,R} + \hat{\gamma}_{R,E}\hat{r}_{E,R}\right) + \hat{\gamma}_{E,E}\hat{r}_{E,R}\right]\vec{R} = 0 \; .
\tag{4.34}
$$

It is worth noting that the same results can be obtained with a slightly different procedure, namely by defining a non minimal scheme in which the evanescent operators do not contribute to the matrix elements, as their contribution is canceled by finite renormalizations. In practice this amounts to redefine the normal product

$$
N\left[\vec{E}\right] \to N\left[\vec{E}\right] - \hat{r}_{E,R}N\left[\vec{R}\right]
\tag{4.35}
$$

in such a manner to subtract not only poles proportional to other evanescent operators, but also finite parts proportional to relevant ones.

Let us find the resulting modification in the anomalous dimension matrix, as in section 4.1.4, by defining

$$\hat{F} = \hat{\delta} - \hat{r}_{E,R} \ ,$$

and then using Eq. 4.30 it results

$$\hat{\gamma}' = \left(\hat{\delta} - \hat{r}\right) \hat{\gamma} \left(\hat{\delta} + \hat{r}\right) + (\mathcal{D}\hat{r}) \left(\hat{\delta} + \hat{r}\right) \ ; \tag{4.36}$$

writing Eq. 4.36 in "components", one easily finds that

$$
\begin{aligned}
\hat{\gamma}'_{R,R} &= \hat{\gamma}_{R,R} + \hat{\gamma}_{R,E}\hat{r}_{E,R} \\
\hat{\gamma}'_{R,E} &= \hat{\gamma}_{R,E} \\
\hat{\gamma}'_{E,R} &= -\hat{r}_{E,R}\left(\hat{\gamma}_{R,R} + \hat{\gamma}_{R,E}\hat{r}_{E,R}\right) + \hat{\gamma}_{E,E}\hat{r}_{E,R} \\
\hat{\gamma}'_{E,E} &= \hat{\gamma}_{E,E} - \hat{r}_{E,R}\hat{\gamma}_{R,E} \ .
\end{aligned}
\tag{4.37}
$$

A few comments are in order

- the first relation shows that the "relevant" part of the anomalous dimension matrix is modified exactly as in Eq. 4.33;

- the anomalous dimension mixing of relevant with evanescent is immaterial, as matrix elements of evanescent operators have been set to zero, and is unaltered;

- we have written without comments in Eq. 4.31 that there is no mixing of evanescent operators with relevant ones, and consequently $\hat{\gamma}_{E,R} = 0$. This is true in the minimal subtraction scheme, and if one takes into account the consistency condition in Eq. 4.34, one easily proves that this remains true, $\hat{\gamma}'_{E,R} = 0$; otherwise, the matrix elements of evanescent operators would get non zero contributions from the relevant ones, vanifying our renormalization prescription, Eq. 4.35;

- the last relation shows a modification of the flow of evanescent operators, again immaterial in the $d \to 4$ limit.

In summary, we have shown explicitly how two different ways to deal with evanescent operators, namely reducing them at the level of the RG equation or in the course of the renormalization process leads to the same results for the anomalous dimension matrix.

We shall give explicit examples in Chap. 5.

4.2   The RG improved perturbation theory

In Chapter 3 we have seen how the Effective Hamiltonian resulting from the OPE defines the amplitude for a process $i \to f$ as (we shall drop the 1PI suffix)

$$
\begin{aligned}
A\left(i \to f\right) &= \langle f|i\rangle_{\text{complete}} \\
&\propto G_F \sum_j C_j \left(\frac{\mu}{M_W}, \frac{m_t}{M_W}\right) \langle f\,|N\left[O_j\right]|\,i\rangle_{\text{eff.}}\left(\mu\right) + O\left(1/M_W^4\right) \ .
\end{aligned}
\tag{4.38}
$$

We have been able to compute the coefficients C_j for $\mu \simeq M_W$. Now we want to change the μ scale in order to have $\mu \simeq m_i$, so to avoid large logarithms in the matrix element.

We known that the μ dependence on the l.h.s. is equal to the one on the r.h.s. hence we can write down the identity

$$(\mathcal{D} + \gamma_i + \gamma_f) \langle f | i \rangle_{\text{complete}} = 0 \ . \tag{4.39}$$

On the r.h.s, the same identity holds, dropping unessential constants

$$(\mathcal{D} + \gamma_i + \gamma_f) C_j \left(\frac{\mu}{M_W}, \frac{m_t}{M_W} \right) \langle f | N [O_j] | i \rangle_{\text{eff.}} (\mu) \tag{4.40}$$

but the derivative $\mathcal{D}$ acts both on the coefficients C and on the matrix element: we already know that a change in μ in the matrix element is compensated by a finite renormalization of the couplings and of the operators, that is, it holds the identity

$$\left[(\mathcal{D} + \gamma_i + \gamma_f) \hat{1} + \hat{\gamma} \right] \left\langle f \left| N \left[\vec{O} \right] \right| i \right\rangle \tag{4.41}$$

hence the scale dependence of the coefficients results

$$\mathcal{D} \vec{C} = \hat{\gamma}^T \vec{C} \ . \tag{4.42}$$

It is worth noting that there is no dependence on the external states, otherwise the effective hamiltonian framework would be meaningless. The Eqs. 4.41 and 4.42 are a mere tautology: any redefinition of the "normal product" $N \left[\vec{O} \right]$ can be compensated by a redefinition of the coefficient vector $\vec{C}$, so to leave the physical amplitude invariant.

4.2.1 Leading Logs in QCD

The solution of Eq. 4.42 is completely standard, the derivative $\mathcal{D}$ can be written as a total derivative with respect to μ, by defining the running coupling constants. In the case of QCD, having defined $\alpha_s (\mu)$ such that

$$\mu \frac{\partial}{\partial \mu} \alpha_s (\mu) = \beta_{\text{QCD}} (\alpha (\mu)) \tag{4.43}$$

the Eq. 4.42 can be rewritten as

$$\mu \frac{d}{d\mu} \vec{C} (\mu) = \hat{\gamma}^T (\alpha (\mu)) \vec{C} (\mu) \tag{4.44}$$

whose solution is given by a path-ordered exponential

$$\vec{C} (m) = P \exp \left(\int_M^m \hat{\gamma}^T \frac{d\mu}{\mu} \right) \vec{C} (M) \ ; \tag{4.45}$$

exploiting the property that the anomalous dimension matrix $\hat{\gamma}$ does depend on μ only through the running coupling α one arrives at the well known result

$$\vec{C} (m) = P \exp \left(\int_{\alpha(M)}^{\alpha(m)} \hat{\gamma}^T (a) \frac{1}{\beta (a)} da \right) \vec{C} (M) \ ; \tag{4.46}$$

where one easily recognizes that the β function is the Jacobian of the dimensional transmutation, and the P means "path-ordering".

In the LL approximation, a considerable simplification is possible. The coupling α factorizes and the AD matrix can be diagonalized with a μ independent orthogonal transformation R

$$
\begin{aligned}
\beta_{\text{QCD}}(a) &= -\frac{a^2}{2\pi}\beta_0 \\
\beta_0 &= \frac{11\,C_A - 2\,n_f}{3} \\
\hat{\gamma}^T &= \frac{a}{4\pi}\hat{R}^{-1}\hat{D}\hat{R}
\end{aligned}
\tag{4.47}
$$

hence the path ordering is no longer needed and one immediately obtains

$$
\exp\left[-\int_{\alpha(M)}^{\alpha(m)}\frac{1}{2\beta_0}\frac{d\,a}{a}\hat{R}^T\hat{D}\hat{R}\right] = -\hat{R}^{-1}\frac{1}{2\beta_0}\hat{D}\ln\frac{\alpha(m)}{\alpha(M)}\hat{R}\ .
\tag{4.48}
$$

The evolution of the coefficients results

$$
\vec{C}(m) = \hat{R}^{-1}\left(\frac{\alpha(m)}{\alpha(M)}\right)^{-\frac{\hat{D}}{2\beta_0}}\hat{R}\vec{C}(M)\ .
\tag{4.49}
$$

We can observe that at leading order

$$
\begin{aligned}
\left(\frac{\alpha(m)}{\alpha(M)}\right)^d &= \left[1 + \frac{2\beta_0}{(4\pi)^2}\alpha(M)\ln\frac{M}{m}\right]^{-d} \\
&= 1 + \sum_{i=1}^{\infty}k_i\left[\alpha(M)\ln\frac{M}{m}\right]^i\ :
\end{aligned}
\tag{4.50}
$$

all the terms of the form $\alpha\ln$ have been summed, as expected.

At the next to leading order, logarithms of the form $\alpha^n\ln^{n-1}$ are summed, and so on. Detailed discussions of the NLL approximation can be found in a number of works (see for instance [22, 46, 77], and references therein). It is worth reminding that this successive approximations are not directly connected with the loop expansion. For instance we shall see that the two-loop determination of the ADM in the $b \to s\gamma$ process allows only the resummation of the leading logarithms. This is due to a cancellation, at one loop one has no contribution to the amplitude of the form in Eq. 4.50.

4.2.2 Scheme and scale dependence

Physics is determined by the amplitude and therefore schematically by the combination $\vec{C}\cdot N\left[\vec{O}\right]$, therefore any scheme dependence implied by a different regularization *and* renormalization scheme for the composite operators must be compensated by an appropriate redefinition of the coefficients.

The scale (μ) dependence itself is the reflection of the existence of a family of subtraction schemes, at different renormalization points, connected by the RG equation, whose coefficients

we are able to determine only approximately. In other words a scale dependence is introduced when we relate two subtraction schemes belonging to the same family, as the one used to compute the coefficients of the effective theory at the large mass scale M to the one appropriate to evaluate the matrix elements at the lower scale m.

In fact there are three sources of approximation: the initial conditions of the evolution, at the higher scale M, the coefficients of the differential equation, that is the β function(s) and the anomalous dimension matrix, and finally the matrix elements at the lower scale m.

This scale dependence has nothing to do with the non-perturbative effects[6] but it is intrinsic of the perturbative expansion and of the reorganization of the perturbative series implied by the RG resummation. Assume for simplicity to have only one operator $N\left[O\right]$ contributing the some process $i \to f$, and to be able to sum the series

$$\left\{\alpha^n \ln^n\right\}, \ \left\{\alpha^n \ln^{n-1}\right\}, \ \ldots \left\{\alpha^n \ln^{n-p}\right\} \ , \tag{4.51}$$

having computed the ADM up to the $p+1$ order in the α_s expansion.

Then a *consistent* determination of the amplitude

$$A\left(i \to f\right) = C\left(m\right)\left\langle f\left|N\left[O\right]\right|i\right\rangle\left(m\right) \tag{4.52}$$

at the lower scale requires the knowledge of the initial conditions of the differential Eq. 4.44, $C\left(M, \alpha\right)$ up to p order, and analogously for the matrix element $\left\langle f\left|N\left[O\right]\right|i\right\rangle\left(m\right)$. For instance, terms α^p from the matrix element, when multiplied by the terms of the series $\left\{\alpha^n \ln^n\right\}$ in the coefficient, generate contributions of the form $\alpha^{n+p} \ln^n$.

On the other hand terms of the form $\alpha^n \ln^{n-p-1}$ are not summed, in this approximation: this left-out logarithms are the source of the residual scale dependence. This dependence can be sized varying the lower scale around the typical scale of the $i \to f$ process, for instance in the $b \to s\gamma$ case around the scale m_b, and can be reduced, but never eliminated, only through higher order computations.

4.2.3   Sizing the scale dependence

In order to estimate the scale dependence it is necessary to consider the amplitude at the next to leading accuracy. For simplicity we shall limit ourselves to the case of a single operator in QCD (see for instance [46], and [73] for the general case), therefore the amplitude is

$$C\left(M, \mu, \alpha\right)\left\langle f\left|N\left[O\right]\right|i\right\rangle\left(m, \mu, \alpha\right) \tag{4.53}$$

where we have made explicit the dependence on the coupling α, the two scales M, m and the renormalization point μ. If all the intermediate computations were exact, the μ dependence of the matrix element would be canceled by the one of the coefficient.

Let us assume that having $\mu \simeq m$ guarantees that no large logarithms appear in the expression for the matrix element: it means that it is possible to compute the μ dependence of the matrix element accurately, even not using the resummation techniques. The μ dependence is a purely short distance effect[7], which means that a perturbative evaluation of the matrix element should suffice in checking the extent of cancellation of the μ dependence.

---

[6]A further source of approximation, especially at the lower scale

[7]Barring infrared complications, which in any case would be coped with including soft bremsstrahlung.

The strategy is as follows: we can write down the explicit expression for the Wilson coefficient, in dependence on M and μ, using the Renormalization Group, and check the cancellation of the μ dependence against the matrix element, computed at the same accuracy in perturbation theory. This will give an *idea* of the implied accuracy.

Given the NLL expression for the β function and anomalous dimension,

$$
\begin{aligned}
\beta_{QCD} &= -\frac{\beta_0}{2\pi}\alpha^2 - \frac{\beta_1}{8\pi^2}\alpha^3 \\
\gamma &= \gamma_0\frac{\alpha}{4\pi} + \gamma_1\frac{\alpha^2}{16\pi^2}
\end{aligned}
\tag{4.54}
$$

we easily write down the expression for the RG evolution operator

$$
\begin{aligned}
U(\mu_1,\,\mu_2) &= \exp\left[\int_{\alpha(\mu_2)}^{\alpha(\mu_1)} d\alpha \frac{\gamma(\alpha)}{\beta(\alpha)}\right] \\
&= \left(\frac{\alpha(\mu_2)}{\alpha(\mu_1)}\right)^{\frac{\gamma_0}{2\beta_0}}\left[1 + R\left(\frac{\alpha(\mu_2)}{4\pi} - \frac{\alpha(\mu_1)}{4\pi}\right)\right] \\
R &= \frac{\gamma_1}{2\beta_0} - \frac{\gamma_0\beta_1}{2\beta_0^2} \,.
\end{aligned}
\tag{4.55}
$$

We have

$$
C(M,\,\mu,\,\alpha) = U(\mu,\,M)\,C(M,\,M,\,\alpha)
\tag{4.56}
$$

and at the same accuracy we need to specify the Wilson coefficient leading α dependence:

$$
C(M,\,M,\,\alpha) = \left(1 + \frac{\alpha(M)}{4\pi}B\right)\,.
\tag{4.57}
$$

Now the μ dependence of the matrix element is easily obtained by expanding the RG equation to the appropriate order

$$
\begin{aligned}
\langle f\,|N\,[O]|\,i\rangle(\mu) &= U(m,\,\mu)\,\langle f\,|N\,[O]|\,i\rangle(m) \\
U(m,\,\mu) &= \left[1 + \frac{\alpha}{4\pi}\gamma_0\ln\left(\frac{m}{\mu}\right)\right]\,.
\end{aligned}
\tag{4.58}
$$

Finally it is possible to write down the full expression for the amplitude, at NLL accuracy[8]

$$
\begin{aligned}
A(i \to f) = \left(\frac{\alpha(M)}{\alpha(\mu)}\right)^{\frac{\gamma_0}{2\beta_0}} &\left[1 + R\left(\frac{\alpha(M)}{4\pi} - \frac{\alpha(\mu)}{4\pi}\right)\right]\left[1 + \frac{\alpha(M)}{4\pi}B\right] \\
&\left[1 + \frac{\alpha(m)}{4\pi}\gamma_0\ln\left(\frac{m}{\mu}\right)\right]\langle f\,|N\,[O]|\,i\rangle(m)\,.
\end{aligned}
\tag{4.59}
$$

---

[8]The expression for the running coupling at NLL accuracy is

$$
\frac{\alpha(m)}{4\pi} = \frac{1}{\beta_0\ln\left(\frac{m^2}{\Lambda^2}\right)} - \frac{\beta_1}{\beta_0^3}\frac{\ln\ln\left(\frac{m^2}{\Lambda^2}\right)}{\ln^2\left(\frac{m^2}{\Lambda^2}\right)}
$$

One easily recognizes that the dominant μ dependence in the leading term is canceled by the logarithmic term coming from the evolution of the matrix element, as it should be.

It is also important to note that in general it may be possible, given a NLL computation, to find an appropriate range in the scale μ which minimizes the size of the NLL corrections with respect to the LL ones: this is what one usally refers to as "setting the scale" of the leading computation by going to NL order. This is equivalent to say that for reasonable values of the NL coefficients in the γ function, corresponding μ values can be found in order to reproduce the NL results with a LL expression. This somewhat "empirical" result helps to deduce the error implicit in the LL approximation by varying the μ scale in a reasonable range. The range itself however can be sized only explicitly checking the correspondence with reasonable values of γ_1.

Chapter 5

QCD corrections to the $b \to s\gamma$ process

We are now in position to discuss in full details the short distance QCD corrections to the $b \to s\gamma$ decay. This process has been studied by many authors and the LO corrections are by now well established. Nevertheless it is instructive to make a short review of the researches in this field.

5.1 Review of the existing results

The great interest of the $b \to s\gamma$ decay has been stimulated by the growing limits on the top mass. Indeed, the initial analyses based on the assumption of a large m_t had found a substantial enhancement of the decay rate due to QCD perturbative corrections; however different results had been obtained by groups working with different regularization schemes for γ_5.

The group of Grinstein et al. [37, 47], using the so called Naive Dimensional Regularization scheme (NDR), based on a fully anticommuting γ_5 even in d dimensions, found an enhancement of about a factor of 4, while the enhancement was limited to a factor of about 2 in the work of Grigjanis et al. [38], based on the so called Dimensional REDuction scheme (DRED) [15].

We stress that both the NDR and DRED schemes are known to be internally unconsistent, that is, to give wrong results in definite computations. For instance the NDR scheme does not reproduce the Adler anomaly, while the DRED scheme is known to fail at three loops (in 4 dimensions) [18, 19].

It is quite easy to show the internal unconsistency of the NDR scheme: consider the following trace

$$T_{\mu\nu\rho\sigma} = \text{Tr}\left[\gamma_5 \gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\alpha\right] \tag{5.1}$$

and assume to first contract the α indices without exploiting the cyclicity of the trace and the assumed anticommuting property of γ_5 with the other γ matrices: the result is

$$T_{\mu\nu\rho\sigma} = (d-8)\,\text{Tr}\left[\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma\right] \tag{5.2}$$

on the other hand, bringing the γ_α matrices close, one obtains

$$T_{\mu\nu\rho\sigma} = -d \operatorname{Tr}\left[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right] \ . \tag{5.3}$$

Clearly the two results show that it is not possible to have a fully anticommuting γ_5 and consistently define its trace with 4 γ matrices. A possible way out is to maintain γ_5 a 4 dimensional object, as in the t'Hooft-Veltman scheme (HV).

This result does not mean that NDR (and DRED) schemes cannot be applied in many practical cases: as long as there is no need to give an explicit definition of the trace in Eq. 5.3, the NDR regularization will give unambiguous results. A caveat, all the subtleties of evanescent operators will still be present, see for instance [57, 76]. In fact various authors [22, 73, 77] have been able to apply NDR, DRED, HV schemes successfully to the QCD corrections to weak hamiltonians, obtaining consistent results.

As the NDR scheme does not appear to cause unconsistencies when applied to $b \to s\gamma$ decay at the leading log approximation, we confirmed in [57] the results of Grinstein et al.

All these works were performed in a reduced basis, not including the mixing with a certain class of 4 fermion operators. The first author to extend the computation to include all the mixings relevant in the LO approximation has been Misiak [58], using the NDR scheme. The resulting corrections to the previous NDR computations are small, showing that the approximation used in [37] was good.

Nevertheless the use of a complete basis has led to understand the origin of the disagreement between the pioneering works. Ciuchini et al. [76] have shown that the operators completing the basis contribute to the $b \to s\gamma$ amplitude, through one-loop graphs, at zeroth order in QCD. This contribution is scheme dependent (for instance, in the HV scheme is absent), and compensates the scheme dependence of the AD matrix.

We shall fully describe the re-evaluation of the LO QCD corrections [102, 103], based on the scheme recently used by Curci et al. [74], a method we shall call from now on the "symmetrized" scheme. In our opinion this technique greatly simplifies the calculation and will be useful in the NLL three loop computation.

5.2   The symmetrized scheme

In the Chapter 3 we have derived the Effective Hamiltonian for the rare B decays: here we add the operators required by the QCD corrections

$$
\begin{aligned}
\mathcal{H}_{\text{eff}} &= \frac{4G_F}{\sqrt{2}} V_{ts}^\star V^{tb} \sum_{i=1}^{14} C_i O_i \\
O_1 &= \frac{1}{(4\pi)^2} \bar{s}_L \, \hat{D}\hat{D}\hat{D}\, b_L \\
O_2 &= \frac{(i\,e\,Q_d)}{(4\pi)^2} \bar{s}_L \left\{ \hat{D},\, F_{\mu\nu}\,\sigma_{\mu\nu} \right\} b_L \\
O_3 &= \frac{(-i\,e\,Q_d)}{(4\pi)^2} \bar{s}_L \, \gamma_\nu \, b_L D_\mu F_{\mu\nu} \\
O_4 &= -\frac{1}{(4\pi)^2} m_b \, \bar{s}_L \, \hat{D}\hat{D}\, b_R
\end{aligned}
$$

$$
\begin{aligned}
O_5 &= \frac{(-i\,e\,Q_d)}{(4\pi)^2} m_b\,\bar{s}_L\,F_{\mu\nu}\sigma_{\mu\nu}\,b_R \\[6pt]
O_6 &= \frac{(i\,g_s)}{(4\pi)^2}\,\bar{s}_L\left\{\hat{D},\,T^A G_{\mu\nu}^A\,\sigma_{\mu\nu}\right\}b_L \\[6pt]
O_7 &= \frac{(-i\,g_s)}{(4\pi)^2}\,\bar{s}_L\,\gamma_\nu T^A\,b_L\,(D_\mu G_{\mu\nu})^A \\[6pt]
O_8 &= \frac{(-i\,g_s)}{(4\pi)^2} m_b\,\bar{s}_L\,T^A G_{\mu\nu}^A\,\sigma_{\mu\nu}\,b_R \\[6pt]
O_9 &= \left(\bar{s}_\alpha\gamma_\mu^L c_\beta\right)\otimes\left(\bar{c}_\beta\gamma_\mu^L b_\alpha\right) \\[6pt]
O_{10} &= \left(\bar{s}_\alpha\gamma_\mu^L c_\alpha\right)\otimes\left(\bar{c}_\beta\gamma_\mu^L b_\beta\right) \\[6pt]
O_{11/13} &= \left(\bar{s}_\alpha\gamma_\mu^L b_\alpha\right)\otimes\sum_{q=u,\ldots b}\left(\bar{q}_\beta\gamma_\mu^{L/R}q_\beta\right) \\[6pt]
O_{12/14} &= \left(\bar{s}_\alpha\gamma_\mu^L b_\beta\right)\otimes\sum_{q=u,\ldots b}\left(\bar{q}_\beta\gamma_\mu^{L/R}q_\alpha\right)\ .
\end{aligned}
\tag{5.4}
$$

In principle working off-shell they also appear non gauge invariant operators, which do not contribute to physics as they are proportional to motion equations. This has been avoided by using a background gauge fixing term for the gluon field, thus ensuring that the gauge invariant basis in Eq. 5.18 closes under renormalization.

The coefficients different from zero have already been given in Eq. 3.13, and it is to be noted that they are evaluated in the NDR scheme: in the LO approximation the scheme dependence of the coefficients at the $\mu \simeq M_W$ scale is irrelevant, at least for the $b \to s\gamma$ process.

Two comments about the basis in Eq. 5.4 are needed: first of all this off-shell basis can be reduced, using the equations of motion, and we shall elaborate on this in the following. Secondly, we have seen that in the d dimensional formulation of the RG equation they are present also evanescent operators. In what follows we assume that a subtraction scheme is chosen in order to decouple them from the coefficient evolution: this will be explicitly shown where appropriate.

Let us consider the general structure of the Effective Hamiltonian, leaving aside the field content

$$
\begin{aligned}
\mathcal{H}_{\text{eff}} &= \sum_i C_i^{R,HV} N\left[R_i\right] + \sum_i C_i^{L,HV} N\left[L_i\right] \\
&\quad + \sum_i C_i^{LL,HV} N\left[(L\otimes L)_i\right] + \sum_i C_i^{LR,HV} N\left[(L\otimes R)_i\right]\ ;
\end{aligned}
\tag{5.5}
$$

$N\left[L, R\right]$ stand for operators bilinear in fermion fields, like the magnetic momentum operator, and $N\left[L\otimes L_i\right]$, $N\left[L\otimes R_i\right]$ stand for current-current operators.

The symbol R and L reminds the presence of the chiral projectors $P_{\frac{L}{R}} = \frac{1}{2}(1\pm\gamma_5)$. This effective hamiltonian is obtained from the "complete" theory, say the $SU(2)\times U(1)$ model: if we make in the complete theory the substitutions

$$
\begin{aligned}
\gamma_5 &\to \bar{\gamma}_5 = -\gamma_5 \\
\epsilon_{\mu\nu\rho\sigma} &\to \bar{\epsilon}_{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}
\end{aligned}
\tag{5.6}
$$

the resulting effective hamiltonian (in the same regularization and renormalization scheme) is identical to the one in Eq. 5.4, except for the same substitutions in Eq. 5.6.

But the QCD corrections are independent on the convention used for defining γ_5, hence the RG evolution equation is invariant under the trasformation (5.6)

$$\left[\mu\frac{d}{d\mu}+\hat{\gamma}\right]\begin{pmatrix} N\left[L\otimes L\right] \\ N\left[L\otimes R\right] \\ N\left[R\right] \\ N\left[L\right] \end{pmatrix}=0 \quad \mapsto \quad \left[\mu\frac{d}{d\mu}+\hat{\gamma}\right]\begin{pmatrix} N\left[R\otimes R\right] \\ N\left[R\otimes L\right] \\ N\left[L\right] \\ N\left[R\right] \end{pmatrix}=0 \ . \tag{5.7}$$

In other words the anomalous dimension matrix $\hat{\gamma}$ is the same for the two set of operator in Eq. 5.7 and for the purpose of the calculation of the AD matrix any linear combinations of the two sets is equivalent. It is convenient in particular to consider the RG evolution of the symmetric combination,

$$\left[\mu\frac{d}{d\mu}+\hat{\gamma}\right]\begin{pmatrix} \frac{1}{2}\left(N\left[L\otimes L\right]+N\left[R\otimes R\right]\right) \\ \frac{1}{2}\left(N\left[L\otimes R\right]+N\left[R\otimes L\right]\right) \\ \frac{1}{2}\left(N\left[R+L\right]\right) \end{pmatrix}=0 \ , \tag{5.8}$$

because it is possible to redefine these operators in order to make the γ_5 matrix disappear. This is trivial for the symmetrized magnetic momentum operator, while for the 4-fermion operators a change of the d dimensional extension is needed.

A complete (infinite) basis for the Clifford algebra in d dimensions is given by the completely antisymmetric products of γ matrices [20, 30, 56, 74],

$$\gamma^{(n)} \equiv \gamma_{\mu_1,\mu_2,\ldots\mu_n} = \frac{1}{n!}\sum_{p\in\Pi_n}(-1)^p\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_n} \ . \tag{5.9}$$

In 4 dimensions the structures $\gamma^{(n)}$ survive only for $n \leq 4$, and one can write

$$\begin{aligned} \frac{1}{2}\left[(L\otimes L)+(R\otimes R)\right] &= (\gamma^{(1)}\otimes\gamma^{(1)})+\frac{1}{3!}(\gamma^{(3)}\otimes\gamma^{(3)}) \\ \frac{1}{2}\left[(L\otimes R)+(R\otimes L)\right] &= (\gamma^{(1)}\otimes\gamma^{(1)})-\frac{1}{3!}(\gamma^{(3)}\otimes\gamma^{(3)}) \ . \end{aligned} \tag{5.10}$$

These equations can be taken as *definitions* for the tensor products in d dimensions; in practice we change the d-dimensional extension of the "symmetric" operators, with the aim to simplify the evaluation of the AD matrix.

The key advantage over the HV scheme is that there is no split between 4 dimensional and $d-4$ dimensional objects (the indices in Eq. 5.9 are d dimensional). For example, working in the well definite basis of $\left(\gamma^{(n)}\otimes\gamma^{(n)}\right)$ structures, for arbitrary integer n, relevant operators, having $n \leq 4$, and evanescent ones having $n \geq 5$ are treated in a unified manner.

We stress that the definitions in Eq. 5.9 render this scheme different from the HV one.

In fact, working in the HV scheme, the effective hamiltonian in Eq. 5.4 can be cast in the form

$$\mathcal{H}_{eff} = \sum_i C_i^{HV}(\mu)N[O_i^{(+)}](\mu) + C_i^{HV}(\mu)N[O_i^{(-)}](\mu), \tag{5.11}$$

where $O_i^{(+)}$ and $O_i^{(-)}$ are the symmetrized and antisymmetrized operators. The two classes of operators are not mixed by QCD corrections, and so the two terms of $\mathcal{H}_{eff}$ are separately RG invariant

$$\mu \frac{d}{d\mu} \sum_i C_i^{HV}(\mu) N[O_i^{(+)}](\mu) = \mu \frac{d}{d\mu} \sum_i C_i^{HV}(\mu) N[O_i^{(-)}](\mu) = 0 \; . \tag{5.12}$$

The redefinition of the symmetric part amounts to add of evanescent operators $E_i^{(+)}$ to the $O_i^{(+)}$, defining the "symmetric" and extended operators

$$\bar{O}_i^{(+)} = O_i^{(+)} + E_i^{(+)} \; . \tag{5.13}$$

Using the $\bar{O}_i^{(+)}$ operators one obtains an AD matrix different from the one obtained with the $O_i^{(+)}$ operators, which are defined as in the HV scheme. The mismatch must be readsorbed in finite renormalizations, which ensure that we can obtain the AD matrix in the HV scheme, and therefore also the evolution of the antisymmetric part.

This is possible as a renormalized evanescent operator can be expanded on a complete basis of relevant ones [27] with finite reduction coefficients r_{ij},

$$\left\langle f \left| N \left[E_i^{(+)} \right] \right| i \right\rangle = r_{ij} \left\langle f \left| N \left[\bar{O}_j^{(+)} \right] \right| i \right\rangle \; . \tag{5.14}$$

For instance imposing the condition

$$\sum_i C_i^{HV}(\mu) N \left[O_i^{(+)} \right](\mu) = \sum_i C_i^{(+)}(\mu) N \left[\bar{O}_i^{(+)} \right](\mu) \tag{5.15}$$

one has

$$C_i^{(+)}(\mu) = \left(\delta_{ij} - r_{ji} \right) C_i^{HV}(\mu) \; . \tag{5.16}$$

The RG evolution of $C_i^{(+)}(\mu)$ coefficients is guided by the anomalous dimension matrix governing the evolution of the symmetrized basis, and using the Eq. 5.16 it is simple to find the evolution of $C_i^{HV}(\mu)$.

Alternatively the Eq. 5.16 can be interpreted as a redefinition of the normal product $N[O]$ as a non-minimal subtraction procedure: we shall use the notation $N'[O]$ for this prescription.

The advantage of the method is that the r_{ij} coefficients are $O(\hbar)$; hence the most cumbersome part of the computation, the determination of the AD matrix, can be done in the symmetrized scheme, while the matching needed to write down the amplitude requires a computation at one loop order less.

In the following we shall use the following symmetrized and extended basis

$$\mathcal{H}_{\text{eff}}^{(+)} \equiv \frac{G_F}{\sqrt{2}} V_{ts}^\star V_{tb} \sum_i C_i^{(+)} O_i^{(+)} \tag{5.17}$$

where the operators are defined

$$\bar{O}_1^{(+)} = \frac{1}{(4\pi)^2}\bar{s}\,\slashed{D}\slashed{D}\slashed{D}\,b \qquad\qquad \bar{O}_2^{(+)} = \frac{(i\,e\,Q_d)}{(4\pi)^2}\,\bar{s}\,\{\slashed{D},\,F_{\mu\nu}\,\sigma_{\mu\nu}\}\,b$$

$$\bar{O}_3^{(+)} = \frac{(-i\,e\,Q_d)}{(4\pi)^2}\,\bar{s}\,\gamma_\nu\,b\,D_\mu F_{\mu\nu} \qquad\qquad \bar{O}_4^{(+)} = -\frac{1}{(4\pi)^2}m_b\,\bar{s}\,\slashed{D}\slashed{D}\,b$$

$$\bar{O}_5^{(+)} = \frac{(-i\,e\,Q_d)}{(4\pi)^2}\,m_b\,\bar{s}\,F_{\mu\nu}\sigma_{\mu\nu}\,b \qquad\qquad \bar{O}_6^{(+)} = \frac{(i\,g_s)}{(4\pi)^2}\,\bar{s}\,\{\slashed{D},\,T^A G_{\mu\nu}^A\,\sigma_{\mu\nu}\}\,b$$

$$\bar{O}_7^{(+)} = \frac{(-i\,g_s)}{(4\pi)^2}\,\bar{s}\,\gamma_\nu T^A\,b\,(D_\mu G_{\mu\nu})^A \qquad\qquad \bar{O}_8^{(+)} = \frac{(-i\,g_s)}{(4\pi)^2}\,m_b\,\bar{s}T^A G_{\mu\nu}^A \sigma_{\mu\nu}\,b \tag{5.18}$$

$$\bar{O}_{9,n}^{(+)} = \frac{1}{n!}(\bar{s}_\alpha\gamma^{(n)}c_\beta)\otimes(\bar{c}_\beta\gamma^{(n)}b_\alpha) \qquad\qquad \bar{O}_{10,n}^{(+)} = \frac{1}{n!}(\bar{s}_\alpha\gamma^{(n)}c_\beta)\otimes(\bar{c}_\beta\gamma^{(n)}b_\alpha)$$

$$\bar{O}_{11,n}^{(+)} = \frac{1}{n!}(\bar{s}_\alpha\gamma^{(n)}b_\alpha)\otimes\sum_q(\bar{q}_\beta\gamma^{(n)}q_\beta) \qquad \bar{O}_{12,n}^{(+)} = \frac{1}{n!}(\bar{s}_\alpha\gamma^{(n)}b_\beta)\otimes\sum_q(\bar{q}_\beta\gamma^{(n)}q_\alpha)$$

and the non-zero coefficients are normalized as follows, with respect to the ones in Eq. 3.13:

$$C_i^{(+)} = 2\,C_i \qquad i = 1,\ldots 8 \quad C_{10,1}^{(+)} = C_{10,3}^{(+)} = C_{10} \quad . \tag{5.19}$$

5.2.1 On-shell and off-shell formulation

To make contact with the work of other authors [76, 58], we shall also use an on-shell basis, obtained from Eq. 5.4 by exploiting the motion equations. The reason for computing QCD corrections off-shell is that the redundancy of the computation provides further checks, and the intermediate results can be saved in view of the planned NLO computation: nevertheless, given the AD matrix, it will be convenient to reduce the results to the following on-shell basis.

$$\begin{aligned}
\mathcal{H}_1 &= \frac{G_F}{\sqrt{2}}V_{ts}^\star V_{tb}\sum_i C_i^{\text{on-shell}}Q_i \\
Q_1 &= (\bar{s}c)_{V-A}\,(\bar{c}b)_{V-A}\,(1\otimes 1)_F \\
Q_2 &= (\bar{s}c)_{V-A}\,(\bar{c}b)_{V-A}\,(1\otimes 1) \\
Q_{3,5} &= (\bar{s}b)_{V-A}\sum_{q=u,d,s,c,b}(\bar{q}q)_{V\pm A}\,(1\otimes 1) \\
Q_{4,6} &= (\bar{s}b)_{V-A}\sum_{q=u,d,s,c,b}(\bar{q}q)_{V\pm A}\,(1\otimes 1)_F \\
Q_7 &= \frac{(-ieQ_d)}{(4\pi)^2}m_b\bar{s}\sigma_{\mu\nu\,(V+A)}F_{\mu\nu}b \\
Q_8 &= \frac{(-ig_s)}{(4\pi)^2}m_b\bar{s}\sigma_{\mu\nu\,(V+A)}\hat{G}_{\mu\nu}b \ . \tag{5.20}
\end{aligned}$$

It is worth recalling the general properties of an anomalous dimension matrix $\hat{\gamma}$ for an operator basis not reduced by motion equations: following [5, 6, 7], it has the block form

$$\begin{array}{cc} & \text{relevant} \quad \text{motion equations} \\ \begin{array}{c}\text{relevant} \\ \text{motion equations}\end{array} & \begin{pmatrix} \hat{X} & \hat{Y} \\ 0 & \hat{Z} \end{pmatrix} , \end{array}$$

and we shall explicitly check this result in the computation. Please note that QCD corrections background gauge fixing term has been used for the gluon field, thus ensuring that even off-shell there is no need to introduce non gauge invariant operators, and that the basis in Eq. 5.18 closes under renormalization.

The symmetrized and extended version of the basis in Eq. 5.20 will be given by the following combinations of the operators appearing in Eq. 5.18

$$
\begin{aligned}
\bar{Q}_1^{(+)} &= \bar{O}_{(9,1)}^{(+)} + \bar{O}_{(9,3)}^{(+)} \\
\bar{Q}_2^{(+)} &= \bar{O}_{(10,1)}^{(+)} + \bar{O}_{(10,3)}^{(+)} \\
\bar{Q}_{3/5}^{(+)} &= \bar{O}_{(11,1)}^{(+)} \pm \bar{O}_{(11,3)}^{(+)} \\
\bar{Q}_{4/6}^{(+)} &= \bar{O}_{(12,1)}^{(+)} \pm \bar{O}_{(12,3)}^{(+)} \\
\bar{Q}_7^{(+)} &= \bar{O}_5^{(+)} \\
\bar{Q}_8^{(+)} &= \bar{O}_8^{(+)} \\
\bar{Q}_9^{(+)} &= \bar{O}_{(9,1)}^{(+)} - \bar{O}_{(9,3)}^{(+)} \\
\bar{Q}_{10}^{(+)} &= \bar{O}_{(10,1)}^{(+)} - \bar{O}_{(10,3)}^{(+)} \ ,
\end{aligned}
\tag{5.21}
$$

where the last two operators are introduced only in order to have and invertible relation, and as can be expected decouple.

5.3 The computation

5.3.1 One loop results

We list in Fig. 5.1 and in Fig. 5.2 the general structure of the graphs needed to renormalize off-shell the operators $\bar{O}_{1\ldots 8}^{+}$. Here and in the following figures the external zigzag line addresses a gluon or a photon. Note that the graphs in Fig. 5.2 are proportional to α_s^2, but as pointed by Misiak [58] they are required even at leading order since they give rise to the mixing of the operator $\bar{O}^7$ with the operators $\bar{O}_{(11,n),(12,n)}^{+}$.

The computation of the Feynman graphs in Fig. 5.1 results in the $\{1\ldots 8\} \times \{1\ldots 8\}$ sector of the one-loop anomalous dimension matrix $\hat{\gamma}^{(1)}$,

$$
\frac{\alpha_s}{4\pi}
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\left(
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
-2\,C_F\xi & 0 & 0 & 6\,C_F & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8\,C_F & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6\,C_F-2\,C_F\xi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8\,C_F & 0 & 0 & 0 \\
-12\,C_F & 4\,C_F & 0 & 0 & 4\,C_F & 4\,C_F & 0 & 4(3\,C_F-C_A) \\
2\,C_F & 0 & \frac{8\,C_F}{3} & 0 & 0 & 0 & \frac{8}{3}\,C_F+3\,C_A & 0 \\
0 & 0 & 0 & -12\,C_F & 8\,C_F & 0 & 0 & 4(4\,C_F-C_A)
\end{array}
\right) ,
\tag{5.22}
$$
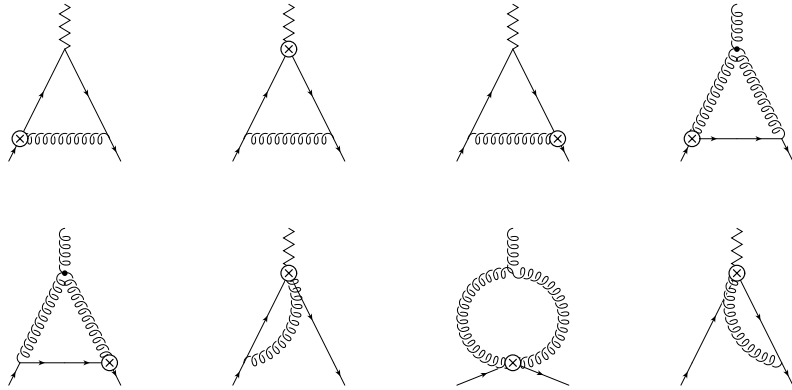
Figure 5.1: Mixing among operators $\bar{O}^+_{1\ldots 8}$.
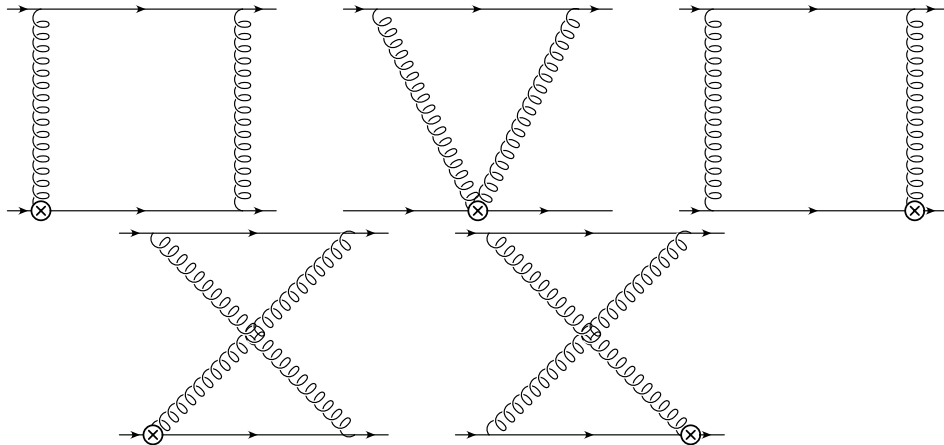
Figure 5.2: Mixing of operators $\bar{O}^+_{1\ldots 8}$ with $\bar{O}^+_{(11-12,\,n)}$.
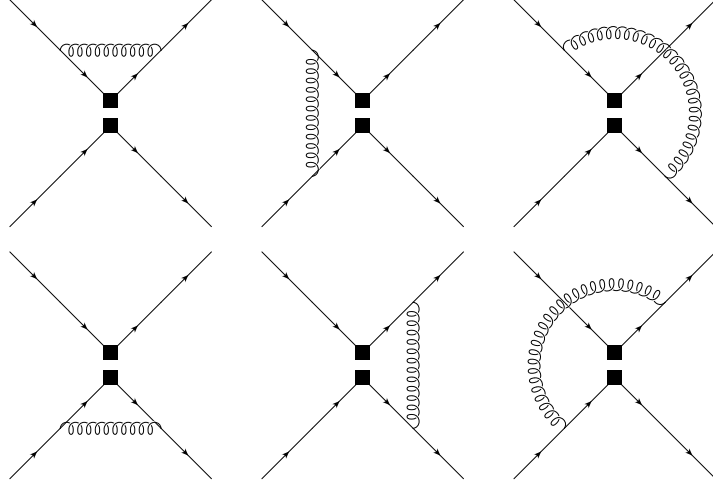
Figure 5.3: Mixing among 4-fermion operators.
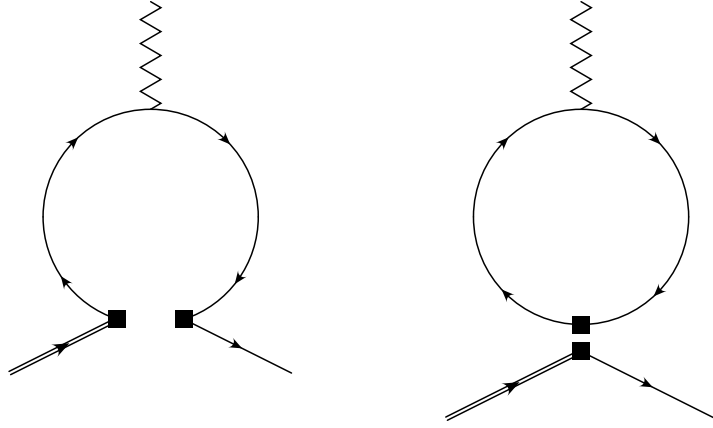
Figure 5.4: Mixing of 4-fermion operators with $\bar{O}^+_{1\ldots8}$ operators.

while the graphs in Fig. 5.2 connect the operator $\bar{O}^+_7$ to four fermion operators, resulting in

$$
\left(\frac{\alpha_s}{4\pi}\right)^2 7 \left(\begin{array}{cccc} (11,1) & (11,3) & (12,1) & (12,3) \\ \frac{-5+5\,C_A^2-10\,C_A\,C_F}{2} & \frac{9\,C_A^2-30\,C_A\,C_F+24\,C_F^2}{2} & 0 & \frac{24\,C_F-9\,C_A}{2} \end{array}\right) . \tag{5.23}
$$

The renormalization of four fermion operators, off-shell, results from the graphs in Fig. 5.3 and in Fig. 5.4.

The self mixing of four fermion operators at one-loop, in Fig. 5.3, connects Dirac structures with $\Delta n = +2,\, 0,\, -2$

$$
\frac{\alpha_s}{4\pi} \begin{array}{c} (9,n) \\ (10,n) \end{array} \left(\begin{array}{ccc} (9,n-2) & (9,n) & (9,n+2) \\ \frac{(6-n)\,(n-5)}{C_A} & 2\,C_F\,((1-n)\,(n-3)-\xi) & \frac{-((1+n)\,(2+n))}{C_A} \\ \frac{(n-6)\,(n-5)}{2} & 3\left(-2+4\,n-n^2\right) & \frac{(1+n)\,(2+n)}{2} \end{array}\right) , \tag{5.24}
$$

$$\frac{\alpha_s}{4\pi} \begin{array}{c} (9,n) \\ (10,n) \end{array} \begin{array}{ccc} (10,n-2) & (10,n) & (10,n+2) \\ \left(\begin{array}{ccc} \frac{(n-6)\,(n-5)}{2} & 0 & \frac{(1+n)\,(2+n)}{2} \\ \frac{(4\,C_F - C_A)\,(n-6)\,(n-5)}{2} & (3\,C_A - 4\,C_F)\,(2 - 4\,n + n^2) - 4\,C_F\,(1+\xi) & \frac{(4\,C_F - C_A)\,(1+n)\,(2+n)}{2} \end{array} \right) \end{array} .$$

The same mixings occur in the $\bar{O}^+_{(11,n),(12,n)}$ sector.

The penguin graphs in Fig. 5.4 give rise to the mixing of the 4-fermion operators with $\bar{O}^+_{3,7}$ at the α_s^0 order

$$\begin{array}{c} (9,n) \\ (10,n) \\ (11,n) \\ (12,n) \end{array} \begin{array}{cc} 3 & 7 \\ \left(\begin{array}{cc} -\frac{2(n-2)}{3n!}(-1)^n c_n(0) & \frac{(n-2)}{3n!}(-1)^n c_n(0) \\ -C_A\frac{2(n-2)}{3n!}(-1)^n c_n(0) & 0 \\ C_A\frac{8}{3}\bar{n}_f\delta_{n,1} + \frac{2(n-2)}{3n!}(-1)^n c_n(0) & \frac{2(n-2)}{3n!}(-1)^n c_n(0) \\ \frac{8}{3}\bar{n}_f\delta_{n,1} + C_A\frac{2(n-2)}{3n!}(-1)^n c_n(0) & \frac{8}{3}n_f\delta_{n,1} \end{array} \right) \end{array} . \qquad (5.25)$$

The meaning of the symbol c_n is explained in App. B together with the other definitions and useful formulas. We use the symbols $n_f = u + d$, $\bar{n}_f = d - 2u$, with u and d being the number of up and down quark species active.

It is to be noted that at leading order only the $n = \{1, 3, 5\}$ values are needed, because the effective hamiltonian starts with $n = \{1, 3\}$ and at one loop only the $n = 5$ evanescent arises. We find convenient to give also the results restricted to this set of n values

$$\frac{\alpha_s}{4\pi} \begin{array}{c} (9,1) \\ (9,3) \\ (10,1) \\ (10,3) \end{array} \begin{array}{cccccc} (9,1) & (9,3) & (9,5) & (10,1) & (10,3) & (10,5) \\ \left(\begin{array}{cccccc} 0 & -\frac{6}{C_A} & 0 & 0 & 6 & 0 \\ -\frac{6}{C_A} & 0 & -\frac{20}{C_A} & 6 & 0 & 20 \\ 3 & 3 & 0 & -\frac{3}{C_A}-6\,C_F & -\frac{3}{C_A}+6\,C_F & 0 \\ 3 & 3 & 10 & -\frac{3}{C_A}+6\,C_F & -\frac{3}{C_A}-6\,C_F & 20\,C_F - \frac{10}{C_A} \end{array} \right) \end{array} , \qquad (5.26)$$

$$\begin{array}{c} (9,\,1) \\ (9,\,3) \\ (10,\,1) \\ (10,\,3) \\ (11,\,1) \\ (11,\,3) \\ (12,\,1) \\ (12,\,3) \end{array} \begin{array}{cc} 3 & 7 \\ \left(\begin{array}{cc} -\frac{8}{3} & \frac{4}{3} \\ -\frac{8}{3} & \frac{4}{3} \\ -\frac{8\,C_A}{3} & 0 \\ -\frac{8\,C_A}{3} & 0 \\ \frac{8}{3}\left(1 + C_A\,\bar{n}_f\right) & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3}\left(C_A + \bar{n}_f\right) & \frac{8}{3}n_f \\ \frac{8}{3}\,C_A & 0 \end{array} \right) \end{array} . \qquad (5.27)$$

It is well known that renormalized operators proportional to motion equations mix only between themselves and that their anomalous dimension matrix is not gauge independent. In fact, we can observe that operator $N\left[\bar{O}^+_1\right]$, proportional to the s motion equation, mixes only with itself and operator $N\left[\bar{O}^s_4\right]$: this one mixes with itself only, and both operators have anomalous dimension matrix depending on ξ [5]. The difference $N\left[\bar{O}^+_2 - \bar{O}^+_5\right]$ is proportional

to a combination of the s and b motion equations and does not evolve at one-loop. Analogously the operator $N\left[\bar{O}_6^+ - \bar{O}_8^+\right]$ mixes with itself and with $N\left[\bar{O}_1^+\right]$, $N\left[\bar{O}_2^+ - \bar{O}_5^+\right]$, $N\left[\bar{O}_4^+\right]$. Finally the combination

$$N\left[\bar{O}_7^+ + \frac{\alpha_s}{4\pi}\left(\frac{1}{2\,C_A}\bar{O}_{(11,\,1)}^+ - \frac{1}{2}\bar{O}_{(12,\,1)}^+\right)\right]$$

is proportional to the equation of motion of the gluon. It is worth noting that the elimination of the operator $N\left[\bar{O}_7^+\right]$ in favor of the four fermion operators introduces a factor $\frac{1}{\alpha_s}$ which combines with the α_s^2 in Eq. 5.23 to give a result relevant for the leading order computation [58].

5.3.2   Two loop results

The two-loop mixing of the four fermion operators with operators $\bar{O}_{1,...8}^+$ can be obtained from the computation of the Feynman diagrams in Figs. 5.5, 5.6 and in Figs. 5.7, 5.8: the "closed" loop graphs are relevant only for the renormalization of operators $\bar{O}_{((11-12),\,n)}^+$.

The massive b quark is represented by the heavy-faced lines. The b propagator is expanded in series of m_b up to the first order and the resulting massless graphs are regularized in the infrared region by the flow of the external momenta: the diagrams have to be evaluated with zero and one mass insertions. The loop integrals have been computed with the help of the algorithms developed by Chetyrkin et al. [21], implemented in the *Mathematica* [62] symbolic manipulation language.   As in the case of the one loop computation, the results can be given for arbitrary n and are listed in Tabb. 5.1, 5.2, 5.3. We refer to App. B for symbols and definitions; let us just note that the symbol f_{nj} results from traces involving elements of the $\{\gamma^{(n)}\}$ basis.

We stress that having the results for arbitrary n will be useful for the NLO computation.

5.3.3   Reduction of evanescent operators

Before giving the results for $n = 1, 3$, that are needed in this leading logarithmic computation, we perform the reduction of the evanescent operators with $n = 5$. We follow the scheme detailed in Sec. 4.1.5, computing the graphs in Fig. 5.9, which allow to express the insertion of the evanescent operators in the Green functions in terms of the insertion of relevant operators, with coefficients addressed by the matrix $\hat{r}$

$$\hat{r} = \begin{array}{c} \\ (9,5) \\ (10,5) \\ (11,5) \\ (12,5) \end{array} \begin{array}{cccc} 3 & 5 & 7 & 8 \end{array} \left(\begin{array}{cccc} \frac{2}{5} & 0 & -\frac{1}{5} & 0 \\ \frac{2}{5}C_A & 0 & 0 & 0 \\ -\frac{2}{5} & \frac{8}{15} & -\frac{2}{5} & \frac{8}{15} \\ -\frac{2}{5}C_A & \frac{8}{15}C_A & 0 & 0 \end{array}\right). \tag{5.28}$$

For the present computation only columns 5, 8 are relevant, while the columns 3, 7 contribute through equations of motion to four fermion operators,

$$\hat{r}_{\text{on-shell}} = \frac{\alpha_s}{4\pi} \begin{array}{c} \\ (9,5) \\ (10,5) \\ (11,5) \\ (12,5) \end{array} \begin{array}{cc} (11,1) & (12,1) \end{array} \left(\begin{array}{cc} \frac{1}{10\,C_A} & -\frac{1}{10} \\ 0 & 0 \\ \frac{1}{5\,C_A} & -\frac{1}{5} \\ 0 & 0 \end{array}\right), \tag{5.29}$$

Figure 5.5: "Open" graphs contributing to the renormalization of $\bar{O}^{+}_{(9\ldots 12,\, n)}$.
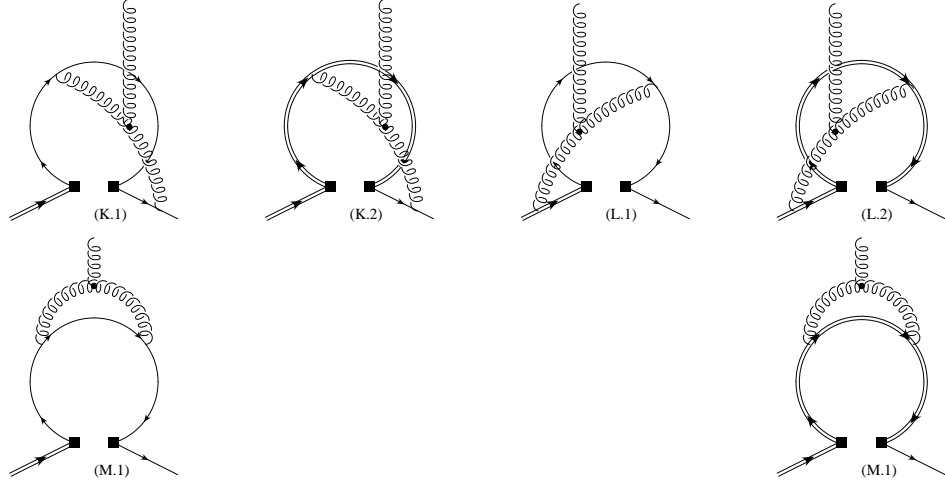
Figure 5.6: "Open" graphs for the renormalization of $\bar{O}^+_{(9\dots12,\,n)}$. Non abelian couplings.

and are relevant only for the NLO computation.

By combining the reduction coefficients in Eq. 5.28, the one loop anomalous dimension matrix in Eq. 5.26 and the results of the two loop computation listed in Tabb. 5.1, 5.2, 5.3, we are able to give the anomalous dimension matrix relevant at leading order, where a factor $\frac{\alpha_s}{4\pi}$ is understood.

$$
\begin{array}{c}
\begin{array}{cccc}
\quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad
\end{array}\\[4pt]
\begin{array}{c}
(9,1)\\
(9,3)\\
(10,1)\\
(10,3)\\
(11,1)\\
(11,3)\\
(12,1)\\
(12,3)
\end{array}
\left(
\begin{array}{cccc}
\frac{2\,C_F}{3} & \frac{-44\,C_F}{9} & \frac{-88\,C_F}{27} & -2\,C_F \\[4pt]
\frac{2\,C_F}{9} & \frac{-44\,C_F}{9} & 8\,C_A-\frac{1688\,C_F}{27} & -2\,C_F \\[4pt]
0 & 0 & \frac{-40\,C_A\,C_F}{3} & 0 \\[4pt]
0 & 0 & \frac{4\,C_A\,(3-14\,C_F)}{3} & 0 \\[4pt]
\frac{4\,C_F}{3} & \frac{20\,C_F}{9} & \frac{-176\,C_F}{27}+\frac{8\,C_A\,C_F\,\bar{n}_f}{3} & -16\,C_F \\[4pt]
\frac{4\,C_F}{9} & \frac{20\,C_F}{9} & \frac{944\,C_F}{27} & -24\,C_F \\[4pt]
\frac{4\,C_F\,(2\,\bar{n}_f-3\,n_f)}{9} & \frac{16\,C_F\,(2\,\bar{n}_f-3\,n_f)}{27} & \frac{40\,C_A\,C_F}{3}+\frac{8\,C_F\,(95\,\bar{n}_f-102\,n_f)}{81} & \frac{4\,C_F\,(2\,\bar{n}_f-3\,n_f)}{3} \\[4pt]
0 & \frac{4\,C_F\,\bar{n}_f}{3} & \frac{32\,C_A\,C_F}{3}+\frac{88\,C_F\,\bar{n}_f}{9} & 0
\end{array}
\right.
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{cc}
\qquad 5 \qquad & \qquad 6 \qquad
\end{array}\\[4pt]
\begin{array}{c}
(9,1)\\
(9,3)\\
(10,1)\\
(10,3)\\
(11,1)\\
(11,3)\\
(12,1)\\
(12,3)
\end{array}
\left(
\begin{array}{cc}
-8\,C_F & \frac{C_A}{2}+\frac{10\,C_F}{9} \\[4pt]
-8\,C_F & \frac{C_A}{2}+\frac{10\,C_F}{9} \\[4pt]
0 & 1 \\[4pt]
0 & 1 \\[4pt]
-4\,C_F & C_A+\frac{20\,C_F}{9} \\[4pt]
\frac{-32}{3\,C_A}+\frac{32\,C_A}{3}+\frac{28\,C_F}{3} & C_A+\frac{20\,C_F}{9}+2\left(n_f-\frac{2}{3}\bar{n}_f\right) \\[4pt]
4\,C_A\,C_F & 2+\frac{(16\,C_F-18\,C_A)\left(2\,\bar{n}_f-3\,n_f\right)}{27} \\[4pt]
\frac{52\,C_A\,C_F}{3}+\frac{8\,C_F\,\bar{n}_f}{3} & 2+\frac{(4\,C_F-C_A)\left(3\,n_f-2\,\bar{n}_f\right)}{3}
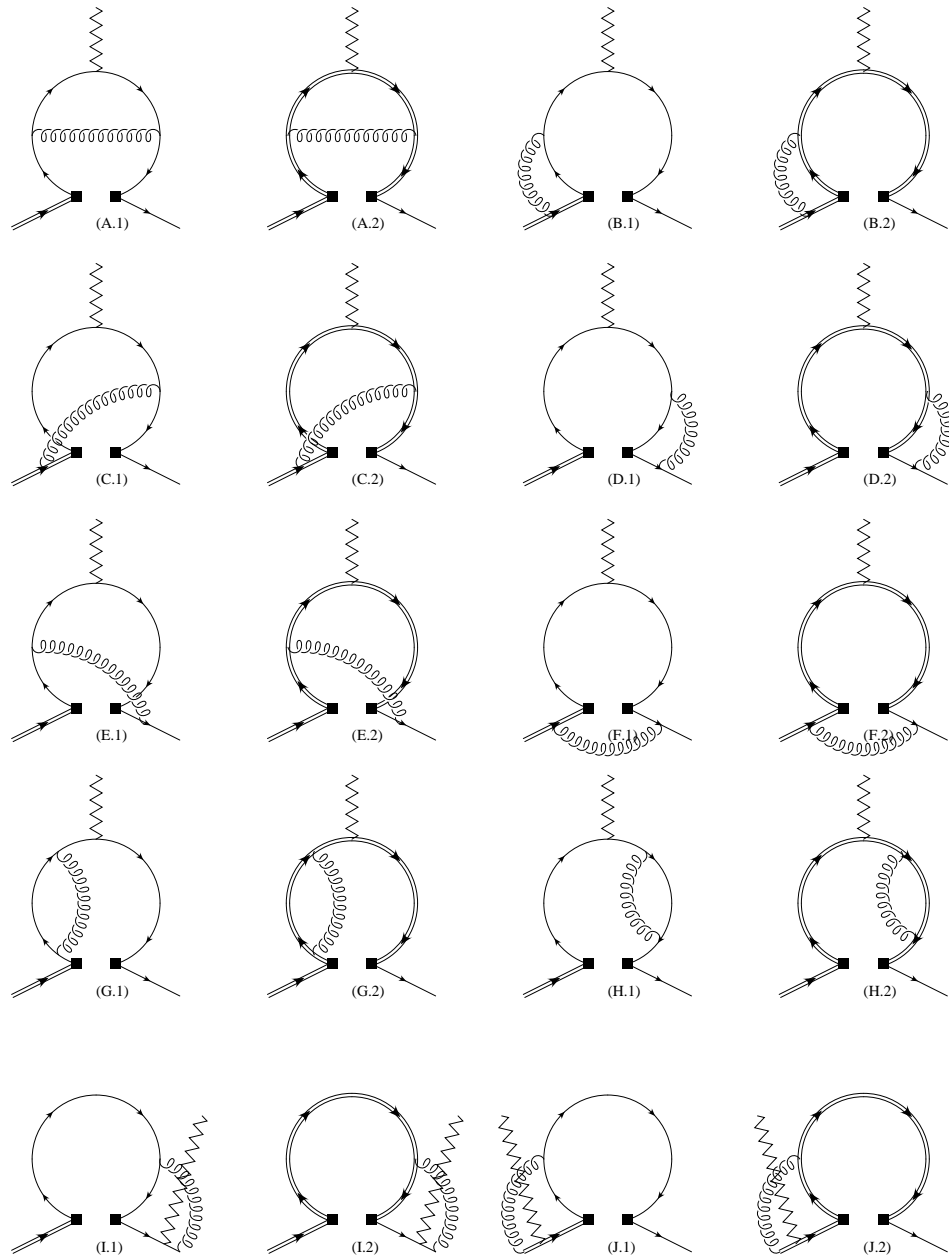\end{array}
\right)
\end{array}
$$
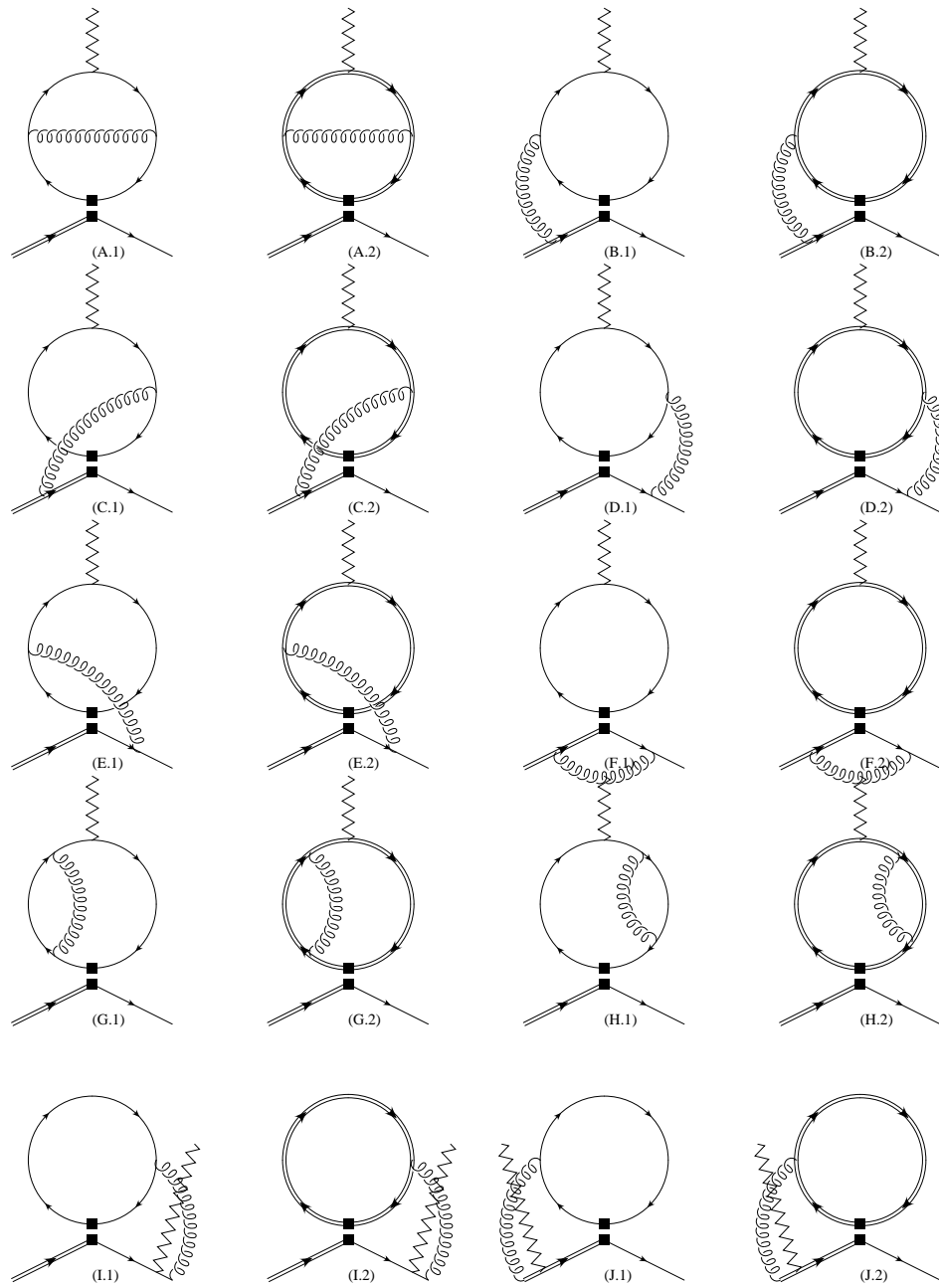
Figure 5.7: 'Closed" graphs contributing to the renormalization of $\bar{O}^+_{(11-12,\,n)}$.
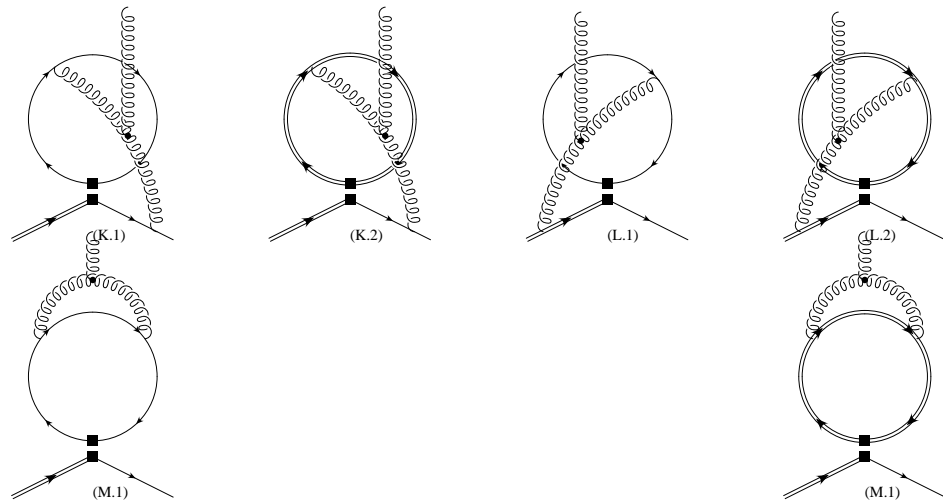
Figure 5.8: "Closed" graphs for the renormalization of $\bar{O}^+_{(11-12, n)}$. Non abelian couplings.
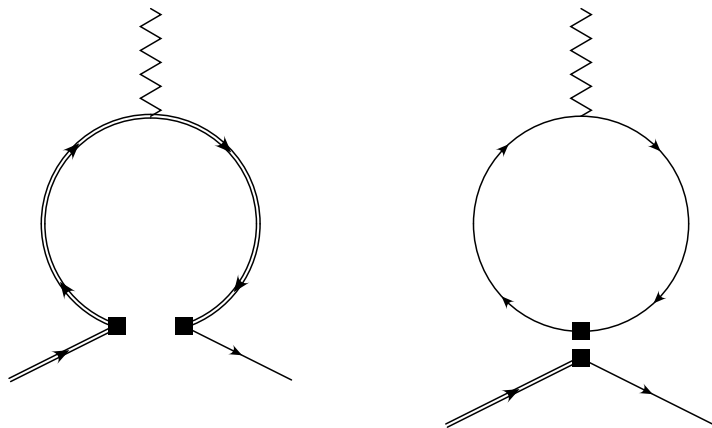
Figure 5.9: Diagrams relevant at LO for the reduction of evanescent operators.

Table 5.1: Two loop AD matrix entries for $\bar{O}^+_{(9,n)}$, $\bar{O}^+_{(10,n)}$, resulting from the mixing with $\bar{O}^+_{1...8}$

γ entry	
$(9,n),1$	$(-1)^n \frac{C_F}{3n!} ((1-n)\, c_n(0) + (2-n)\, c_n(1))$
$(9,n),2$	$\frac{2\,C_F}{n!} \left[-f_{n,3}(0) + (-1)^n \frac{(2-n)}{9} c_n(0) \right]$
$(9,n),3$	$\frac{4\,C_F}{n!} \left[-f_{n,3}(0) + \frac{n(n-1)(-1)^n}{18} \left(1908 - 1124n + 213n^2 - 13n^3 \right) c_{n-2}(0) \right.$
	$\left. + \frac{(-1)^n}{54} \left(160 - 290n + 177n^2 - 33n^3 \right) c_n(0) \right.$
	$\left. + \frac{2(n-2)(-1)^n}{9} \left(4 - 12n + 3n^2 \right) c_n(1) \right]$
$(9,n),4$	$\frac{C_F\,(2-n)(-1)^n}{2n!} c_n(0)$
$(9,n),5$	$\frac{2\,C_F\,(n-2)}{3n!} \left(2n^2 - 8n + 3 \right)(-1)^n c_n(0)$
$(9,n),6$	$\frac{4\,C_F - C_A}{4n!} f_{n,3}(0) + (-1)^n \frac{(8\,C_F - 9\,C_A)(2-n)}{36n!} c_n(0)$
$(9,n),7$	$\frac{(4\,C_F - C_A)\,f_{n,3}(0)}{2n!}$
	$+ \frac{(-1)^n(C_A - 2\,C_F)\,n\,(n-1)}{18n!} \left(1908 - 1124\,n + 213\,n^2 - 13\,n^3 \right) c_{n-2}(0)$
	$+ \frac{(-1)^n}{108n!} \left(3\,C_A\,(4-n)\,(3 - 46\,n + 24\,n^2) \right.$
	$\left. - 4\,C_F\,(58 - 230\,n + 177\,n^2 - 33\,n^3) \right) c_n(0)$
	$+ \frac{(-1)^n}{18n!}\,(n-2)\left(4\,C_F\,(-11 + 24\,n - 6\,n^2) + 3\,C_A\,(-3 - 8\,n + 2\,n^2) \right) c_n(1)$
$(9,n),8$	$\frac{(-1)^n(n-2)}{24n!} \left(8\,C_F\,(-3 + 8\,n - 2\,n^2) + C_A\,(1 - 16\,n + 4\,n^2) \right) c_n(0)$
$(10,n),1\,(2)$	0
$(10,n),3$	$\frac{C_A\,C_F(-1)^n}{9n!} \left[n\,(n-1)\,(1908 - 1124\,n + 213\,n^2 - 13\,n^3)\, c_{n-2}(0) \right.$
	$\left. + (108 - 148\,n + 83\,n^2 - 13\,n^3)\, c_n(0) \right]$
$(10,n),4\,(5)$	0
$(10,n),6$	$\frac{1}{2n!} f_{n,3}(0)$
$(10,n),7$	$\frac{1}{n!} f_{n,3}(0) + \frac{(-1)^n\,n\,(n-1)}{36n!} \left(-1908 + 1124\,n - 213\,n^2 + 13\,n^3 \right) c_{n-2}(0)$
	$+ \frac{(-1)^n}{36n!} \left(24 + 32\,n - 35\,n^2 + 9\,n^3 \right) c_n(0) + \frac{(-1)^n\,(n-2)}{3n!} \left(-3 + 8\,n - 2\,n^2 \right) c_n(1)$
$(10,n),8$	$\frac{(-1)^n(n-2)}{6n!} \left(-3 + 8\,n - 2\,n^2 \right) c_n(0)$

$$
\begin{array}{c}
\begin{array}{cc} \quad\quad\quad 7 \quad\quad\quad & \quad\quad\quad 8 \quad\quad\quad \end{array} \\
\begin{array}{c}
(9,1) \\ (9,3) \\ (10,1) \\ (10,3) \\ (11,1) \\ (11,3) \\ (12,1) \\ (12,3)
\end{array}
\left(
\begin{array}{cc}
\frac{25\,C_A}{3} - \frac{88\,C_F}{27} & \frac{-11\,C_A}{6} + 4\,C_F \\[4pt]
\frac{-5\,C_A}{3} + \frac{688\,C_F}{27} & \frac{-11\,C_A}{6} + 4\,C_F \\[4pt]
-\frac{10}{3} & 2 \\[4pt]
10 & 2 \\[4pt]
\frac{50\,C_A}{3} - \frac{176\,C_F}{27} & \frac{13\,C_A}{3} - 4\,C_F \\[4pt]
\frac{944\,C_F + 126\,C_A}{27} + \frac{44}{3}\left(n_f - \frac{2}{3}\bar{n}_f \right) & \frac{92\,C_F - 19\,C_A}{3} + 4\left(n_f - \frac{2}{3}\bar{n}_f \right) \\[4pt]
-\frac{20}{3} + \frac{(56\,C_F - 252\,C_A)\left(2\,\bar{n}_f - 3\,n_f\right)}{81} & -8 + \frac{5\,C_A\left(2\,\bar{n}_f - 3\,n_f\right)}{9} \\[4pt]
16 + \frac{22\,(4\,C_F - C_A)\left(3\,n_f - 2\,\bar{n}_f\right)}{9} & \frac{2\,(4\,C_F - C_A)\left(3\,n_f - 2\,\bar{n}_f\right)}{3}
\end{array}
\right) .
\end{array}
$$

5.3.4 On-shell results

By applying the motion equations, it is now possible to give the results in the following on-shell basis which closely corresponds to the symmetrization of the one used by Ciuchini et

Table 5.2: Two loop AD matrix entries for $\bar{O}^+_{(11,n)}$, resulting from the mixing with $\bar{O}^+_{1\dots 8}$

γ entry	
$(11,n),1$	$\frac{2\,C_F\,(-1)^n}{3\,n!}\left[(1-n)\,c_n(0)+(2-n)\,c_n(1)\right]$
$(11,n),2$	$\frac{2\,C_F}{n!}\left[f_{n,3}(0)+\frac{2}{9}\,(2-n)\,(-1)^n\,c_n(0)\right]$
$(11,n),3$	$\frac{C_F}{n!}\left[8\,C_A\,\bar{n}_f\delta_{n,1}+4\,f_{n,3}(0)\right.$
	$\qquad +\frac{2}{9}\,(-1)^n\,(n-1)\,n\,\left(-1908+1124\,n-213\,n^2+13\,n^3\right)\,c_{n-1}(0)$
	$\qquad +\frac{2}{27}\,(-1)^n\,\left(-58+230\,n-177\,n^2+33\,n^3\right)\,c_n(0)$
	$\qquad \left.+\frac{4}{9}\,(-1)^n\,(n-2)\,\left(-11+24\,n-6\,n^2\right)\,c_n(1)\right]$
$(11,n),4$	$\frac{C_F}{n!}\left[\frac{1}{3}\left(39-68\,n+20\,n^2+3\,(2-n)\,(-1)^n\right)\,c_n(0)+2\,(n-3)\,(n-1)\,c_n(1)\right]$
$(11,n),5$	$\frac{C_F}{3n!}\left[2\,(n-6)\,(n-5)^2\,(n-3)\,(n-1)\,n\,c_{n-2}(1)\right.$
	$\qquad -\frac{1}{3}\left(45+100\,n-170\,n^2+56\,n^3-4\,n^4\right.$
	$\qquad\quad \left.+2\,(n-2)\,\left(2\,n^2-8\,n+3\right)\,(-1)^n\right)\,c_n(0)$
	$\qquad \left.+2\,(n-3)\,(n-1)\,\left(-5+3\,n-n^2\right)\,c_n(1)\right]$
$(11,n),6$	$\frac{1}{n!}\left[12\,n_f\delta_{n,3}+\frac{1}{2}\,(4\,C_F-C_A)\,f_{n,3}(0)\right.$
	$\qquad \left.+\frac{(-1)^n}{18}\,(9\,C_A-8\,C_F)\,(n-2)\,c_n(0)\right]$
$(11,n),7$	$\frac{1}{n!}\left[88\,n_f\delta_{n,3}+(4\,C_F-C_A)\,f_{n,3}(0)\right.$
	$\qquad +\frac{1}{9}\,(-1)^n\,(2\,C_F-C_A)\,(n-1)\,n\,\left(-1908+1124\,n-213\,n^2+13\,n^3\right)\,c_{n-2}(0)$
	$\qquad +\frac{1}{54}\,(-1)^n\,\left(3\,C_A\,(n-4)\,\left(-3+46\,n-24\,n^2\right)+\right.$
	$\qquad +\,4\,C_F\,\left.\left(-58+230\,n-177\,n^2+33\,n^3\right)\right)\,c_n(0)$
	$\qquad \left.+\frac{1}{9}\,(-1)^n\,(2-n)\,\left(3\,C_A\,\left(3+8\,n-2\,n^2\right)+4\,C_F\,\left(11-24\,n+6\,n^2\right)\right)\,c_n(1)\right]$
$(11,n),8$	$\frac{1}{n!}\left[\frac{1}{3}(-C_A+2\,C_F)\,(n-6)\,(n-5)^2\,(n-3)\,(n-1)\,n\,c_{n-2}(1)\right.$
	$\qquad +\frac{1}{18}\left(C_A\,\left(99+24\,n-125\,n^2+40\,n^3-2\,n^4\right)+\right.$
	$\qquad +\,2\,C_F\,\left.\left(-45-100\,n+170\,n^2-56\,n^3+4\,n^4\right)\right)\,c_n(0)$
	$\qquad +\frac{1}{3}(n-3)\,(n-1)\,\left(C_A\,n+2\,C_F\,\left(-5+3\,n-n^2\right)\right)\,c_n(1)+24\,n_f\,\delta_{n,3}$
	$\qquad \left.+\frac{1}{12}\,(-1)^n\,(n-2)\,\left(8\,C_F\,\left(-3+8\,n-2\,n^2\right)+C_A\,\left(1-16\,n+4\,n^2\right)\right)\,c_n(0)\right]$

al. [76, 77] in the HV case,

$$
\begin{aligned}
Q_1^+ &= \bar{Q}_1^+ - E_1^+ = \bar{O}^+_{(9,\,1)} + \bar{O}^+_{(9,\,3)} - E_1^+ \\
Q_2^+ &= \bar{Q}_2^+ - E_2^+ = \bar{O}^+_{(10,\,1)} + \bar{O}^+_{(10,\,3)} - E_2^+ \\
Q_{3/5}^+ &= \bar{Q}_{3/5}^+ - E_{3/5}^+ = \bar{O}^+_{(11,\,1)} \pm \bar{O}^+_{(11,\,3)} - E_{3/5}^+ \\
Q_{4/6}^+ &= \bar{Q}_{4/6}^+ - E_{4/6}^+ = \bar{O}^+_{(12,\,1)} \pm \bar{O}^+_{(12,\,3)} - E_{4/6}^+ \\
Q_7^+ &= \bar{Q}_7^+ = \bar{O}_5^+ \\
Q_8^+ &= \bar{Q}_8^+ = \bar{O}_8^+ \\
Q_9^+ &= \bar{Q}_9^+ - E_9^+ = \bar{O}^+_{(9,\,1)} - \bar{O}^+_{(9,\,3)} - E_9^+ \\
Q_{10}^+ &= \bar{Q}_{10}^+ - E_{10}^+ = \bar{O}^+_{(10,\,1)} - \bar{O}^+_{(10,\,3)} - E_{10}^+ \,.
\end{aligned}
\tag{5.30}
$$

The last two operators have been introduced to have an invertible relation between the bases $\bar{O}_i^+$ and Q_i^+; as expected they decouple from the others in the RG evolution. The

Table 5.3: Two loop AD matrix entries for $\bar{O}^+_{(12,n)}$, resulting from the mixing with $\bar{O}^+_{1...8}$

γ entry	
$(12,n),1$	$-\frac{4}{3}C_F n_f \delta_{n,1}$
$(12,n),2$	$-\frac{16}{9}C_F n_f \delta_{n,1} + 4C_F \bar{n}_f \delta_{n,3}$
$(12,n),3$	$-\frac{8}{81}C_F\left(102 n_f - 81\bar{n}_f\right)\delta_{n,1} + \frac{88}{3}C_F \bar{n}_f \delta_{n,3}$
	$\quad + \frac{C_A C_F (-1)^n}{9\,n!}\left[(n-1)\,n\left(-1908 + 1124\,n - 213\,n^2 + 13\,n^3\right)c_{n-2}(0)\right.$
	$\quad\quad \left. + \left(-108 + 148\,n - 83\,n^2 + 13\,n^3\right)c_n(0)\right]$
$(12,n),4$	$-4C_F n_f \delta_{n,1}$
$(12,n),5$	$\frac{C_A C_F}{3\,n!}\left[(n-6)\,(n-5)^2\,(n-3)\,(n-1)\,n\,c_{n-2}(1)\right.$
	$\quad + \frac{1}{3}\left(45 - 80\,n + 56\,n^2 - 12\,n^3\right)c_n(0)$
	$\quad \left. + (n-3)\,(n-1)\left(-2 - 5\,n + n^2\right)c_n(1)\right] + 8C_F\,\bar{n}_f\,\delta(n,3)$
$(12,n),6$	$\frac{1}{n!}f_{n,3}(0) + \frac{2}{9}\left(9C_A - 8C_F\right)n_f \delta_{n,1} + \left(4C_F - C_A\right)n_f \delta_{n,3}$
$(12,n),7$	$\frac{28}{27}\left(9C_A - 2C_F\right)n_f \delta_{n,1} + \frac{22}{3}\left(4C_F - C_A\right)n_f \delta_{n,3} + \frac{2}{n!}f_{n,3}(0)$
	$\frac{(-1)^n}{3n!}\left[\frac{1}{6}(n-1)\,n\left(-1908 + 1124\,n - 213\,n^2 + 13\,n^3\right)c_{n-2}(0)\right.$
	$\quad + \frac{1}{6}\left(24 + 32\,n - 35\,n^2 + 9\,n^3\right)c_n(0)$
	$\quad \left. + 2\,(2-n)\,\left(3 - 8\,n + 2\,n^2\right)c_n(1)\right.$
$(12,n),8$	$\frac{-5C_A}{3}n_f \delta_{n,1} + 2\left(4C_F - C_A\right)n_f \delta_{n,3}$
	$\quad + \frac{1}{3n!}\left[\frac{1}{2}(n-6)\,(n-5)^2\,(n-3)\,(n-1)\,n\,c_{n-2}(1)\right.$
	$\quad + \frac{1}{3}\left(73\,n^2 - 18 - 62\,n - 22\,n^3 + 2\,n^4\,(-1)^n\left(18 - 57\,n + 36\,n^2 - 6\,n^3\right)\right)c_n(0)$
	$\quad \left. + \frac{1}{2}(n-3)\,(n-1)\left(-4 + 11\,n - 3\,n^2\right)c_n(1)\right]$

presence of the evanescent operators takes into account the difference of the d-dimensional extensions. In the basis $\bar{Q}^+_i$ we give the final results for the AD matrix in the following block form:

$$\hat{\gamma} = \frac{\alpha_s}{4\pi}\left(\begin{array}{cc} \hat{\gamma}_{ff} & \hat{\gamma}_{fm} \\ \hat{0} & \hat{\gamma}_{mm} \end{array}\right) . \tag{5.31}$$

The matrices $\hat{\gamma}_{ff}$, $\hat{\gamma}_{mm}$ result from the one loop computation and are scheme independent; they guide the RG evolution respectively in the four fermion and in the magnetic momentum sectors,

$$\hat{\gamma}_{ff} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} -\frac{6}{C_A} & 6 & 0 & 0 & 0 & 0 \\ 6 & -\frac{6}{C_A} & -\frac{2}{3C_A} & \frac{2}{3} & -\frac{2}{3C_A} & \frac{2}{3} \\ 0 & 0 & -\frac{22}{3C_A} & \frac{22}{3} & -\frac{4}{3C_A} & \frac{4}{3} \\ 0 & 0 & \frac{2(9C_A - n_f)}{3C_A} & -\frac{2(9 - n_f C_A)}{3C_A} & -\frac{2n_f}{3C_A} & \frac{2n_f}{3} \\ 0 & 0 & 0 & 0 & \frac{6}{C_A} & -6 \\ 0 & 0 & -\frac{2n_f}{3C_A} & \frac{2n_f}{3} & -\frac{2n_f}{3C_A} & \frac{2(n_f - 18C_F)}{3} \end{array}\right) \end{array} ,$$

$$\hat{\gamma}_{mm} = \begin{array}{c} \\ 7 \\ 8 \end{array}\begin{array}{c} \begin{array}{cc} 7 & 8 \end{array} \\ \left(\begin{array}{cc} 8C_F & 0 \\ 8C_F & 16C_F - 4C_A \end{array}\right) \end{array} . \tag{5.32}$$

The matrix γ_{fm} connecting the four fermion and the magnetic momentum sectors results from the two-loop computation and it is scheme dependent, even at leading order, as pointed out by Ciuchini et al. [76, 77]

$$
\hat{\gamma}_{fm} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \left(\begin{array}{cc} 7 & 8 \\ 0 & 6 \\ -\frac{232\,C_F}{9} & -\frac{8\,C_A}{3} + \frac{92\,C_F}{9} \\ \frac{280\,C_F}{9} & \frac{280\,C_F}{9} + 6\,n_f \\ \frac{64\,C_A\,C_F}{3} + \frac{4\,C_F}{9}\left(27\,\bar{n}_f - 4\,n_f\right) & -4 - \frac{8\,C_A}{3}\,n_f + \frac{92\,C_F}{9}\,n_f \\ -\frac{104\,C_F}{3} & \frac{32\,C_A}{3} - \frac{104\,C_F}{3} - 6\,n_f \\ -\frac{40\,C_A\,C_F}{3} - \frac{4\,C_F}{9}\left(4\,n_f + 27\,\bar{n}_f\right) & -8 + \frac{10\,C_A}{3}\,n_f - \frac{124\,C_F}{9}\,n_f \end{array} \right) . \tag{5.33}
$$

5.3.5 Matrix elements and scheme independence

The scheme dependence of the matrix $\hat{\gamma}_{fm}$ is compensated in the physical amplitude by the matrix elements of the four fermion operators.

The "even" contribution to the amplitude for the $b \to s\gamma$ decay has the form

$$
< s\gamma|\mathcal{H}_{eff}^+|b> = \bar{C}_7^s \left\langle s\gamma \left| N\left[Q_7^+\right] \right| b \right\rangle_0 + \sum_{n=3}^{6} \bar{C}_n^s \left\langle s\gamma \left| N\left[Q_n^+\right] \right| b \right\rangle_1 \tag{5.34}
$$

where the subscripts 0, 1 address the order in $\hbar$ of the matrix element, and an analogous formula holds for the $b \to sg$ amplitude [1].

The one loop matrix elements of the operators $Q_{3,4,5,6}^+$ result from Feynman diagrams analogous to the ones listed in Fig. 5.9 and with a mass insertion. If we define our scheme preserving the naive four dimensional Fierz simmetry they give a zero result. In fact in this case the "open" diagram in Fig. 5.9 with a mass insertion is proportional to the "closed" one, which is set to zero by the trace. The symmetry can be preserved by using the HV scheme for γ_5 and maintaining the four dimensional definition for the current-current products [77].

In other regularization schemes this is no longer true and in general one obtains a local, finite contribution. This difference can be reinterpreted as a finite correction[2] to the coefficient C_7^s or alternatively to the operators. In our scheme the following relations hold

$$
\left\langle s\gamma \left| \left(\begin{array}{c} \bar{O}_{11,n}^+ \\ \bar{O}_{12,n}^+ \end{array} \right) \right| b \right\rangle_1 =
$$
$$
\left(\frac{\left(-36 + 64\,n - 19\,n^2\right)}{36}\frac{c_n(0)}{n!} + \frac{(1-n)\,(-3+n)}{6}\frac{c_n(1)}{n!} \right) \left(\begin{array}{c} 1 \\ C_A \end{array} \right) \left\langle s\gamma \left| \bar{O}_5^+ \right| b \right\rangle_0
$$
$$
\left\langle sg \left| \left(\begin{array}{c} \bar{O}_{11,n}^+ \\ \bar{O}_{12,n}^+ \end{array} \right) \right| b \right\rangle_1 = \tag{5.35}
$$
$$
\left(\frac{\left(-36 + 64\,n - 19\,n^2\right)}{36}\frac{c_n(0)}{n!} + \frac{(1-n)\,(-3+n)}{6}\frac{c_n(1)}{n!} \right) \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \left\langle sg \left| \bar{O}_8^+ \right| b \right\rangle_0 .
$$

---

[1] The bar over the coefficients reminds us that they are obtained after applying motion equations: for instance $\bar{C}_7^s = C_2^s + C_5^s$, $\bar{C}_8^s = C_6^s + C_8^s$.

[2] see, f.i. in [81], the so called "effective coefficients" formalism

Using the definitions in Eq. 5.30 we can write

$$
\begin{aligned}
\left\langle s\,\gamma \left| \bar{Q}_i^+ \right| b \right\rangle &= Z_{7i} \left\langle s\,\gamma \left| \bar{Q}_7^+ \right| b \right\rangle & i = \{1 \ldots 6\} \\
\left\langle s\,g \left| \bar{Q}_i^+ \right| b \right\rangle &= Z_{8i} \left\langle s\,g \left| \bar{Q}_8^+ \right| b \right\rangle & i = \{1 \ldots 6\}
\end{aligned}
\tag{5.36}
$$

where following [76] we define the vectors

$$
\begin{aligned}
\vec{Z}_7 &= \left(0, 0, \frac{8}{3}, \frac{8\,C_A}{3}, -\frac{2}{3}, -\frac{2\,C_A}{3}, 0, 0 \right) \\
\vec{Z}_8 &= \left(0, 0, \frac{8}{3}, 0, -\frac{2}{3}, 0, 0, 0 \right) .
\end{aligned}
\tag{5.37}
$$

We can perform the finite renormalizations needed to compare the results with the ones obtained in the HV scheme. We match the renormalization schemes by defining a non-minimal subtraction

$$
\begin{aligned}
N'\left[\bar{Q}_i^+ \right] &= \left(\hat{1} + \delta\hat{F} \right)_{ij} N\left[\bar{Q}_j^+ \right] \\
\delta\hat{F}_{ij} &= \begin{cases} -Z_{ji} & j = \{7, 8\} \\ 0 & j = \{1, \ldots 6\} \end{cases}
\end{aligned}
\tag{5.38}
$$

which sets to zero the matrix elements in Eq. 5.36, as they are in the HV scheme. According to the formula in Eq. 4.30 the AD matrix is modified as follows

$$
\hat{\gamma}' = \left(\hat{1} + \delta\hat{F} \right) \hat{\gamma} \left(\hat{1} + \delta\hat{F} \right)^{-1} .
\tag{5.39}
$$

Using the results in Eq. 5.37, one easily obtains the AD matrix in the HV scheme

$$
\hat{\gamma}'_{mf} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}
\begin{pmatrix}
& 7 & 8 \\
& 0 & 6 \\
& -\frac{208\,C_F}{9} & \frac{116\,C_F}{9} - 4\,C_A \\
& \frac{232\,C_F}{9} & \frac{232\,C_F}{9} - 8\,C_A + 6 n_f \\
\frac{8\,C_F}{9} n_f + 12\,C_F\,\bar{n}_f & 12 + \left(\frac{116\,C_F}{9} - 4\,C_A \right) n_f \\
-16\,C_F & 4\,C_A - 16\,C_F - 6\,n_f \\
\frac{8\,C_F}{9} n_f - 12\,C_F\,\bar{n}_f & -8 + \left(2\,C_A - \frac{100\,C_F}{9} \right) n_f
\end{pmatrix}
\tag{5.40}
$$

which coincides with the result in [76].

5.4 Effective Hamiltonian at $\mu \simeq m_b$

We have obtained the expression for the AD matrix in the HV scheme:

$$
\hat{\gamma}_{\mathrm{HV}} = \frac{\alpha_s}{4\pi}
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\begin{pmatrix}
\overset{1}{-2} & \overset{2}{6} & \overset{3}{0} & \overset{4}{0} & \overset{5}{0} & \overset{6}{0} & \overset{7}{0} & \overset{8}{6} \\
6 & -2 & -\frac{2}{9} & \frac{2}{3} & -\frac{2}{9} & \frac{2}{3} & -\frac{832}{27} & \frac{140}{27} \\
0 & 0 & -\frac{22}{9} & \frac{22}{3} & -\frac{4}{9} & \frac{4}{3} & \frac{928}{27} & \frac{1090}{27} \\
0 & 0 & \frac{44}{9} & \frac{4}{3} & -\frac{10}{9} & \frac{10}{3} & -\frac{272}{27} & \frac{1024}{27} \\
0 & 0 & 0 & 0 & 2 & -6 & -\frac{64}{3} & -\frac{118}{3} \\
0 & 0 & -\frac{10}{9} & \frac{10}{3} & -\frac{10}{9} & -\frac{38}{3} & \frac{592}{27} & -\frac{1406}{27} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{32}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{32}{3} & \frac{28}{3}
\end{pmatrix} ,
\tag{5.41}
$$

where the values $n_f = 5$, $\bar{n}_f = -1$ have been inserted, as appropriate to the 5 quark theory below M_W.

With the help of *Mathematica*[62] it is easy to find numerically eigenvectors and eigenvalues of the transpose of the matrix in Eq. 5.41

$$
\hat{\gamma}_{\mathrm{HV}}^{T} = \frac{\alpha_s}{4\pi} \hat{R}^{-1} \cdot \hat{D} \cdot R
\tag{5.42}
$$

$$
\hat{D} =
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\begin{pmatrix}
\overset{1}{10.6667} & \overset{2}{0} & \overset{3}{0} & \overset{4}{0} & \overset{5}{0} & \overset{6}{0} & \overset{7}{0} & \overset{8}{0} \\
0 & 9.33333 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6.26549 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -6.48583 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2.23328 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -13.7907 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -8
\end{pmatrix}
\tag{5.43}
$$

$$
\hat{R} \simeq
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\begin{pmatrix}
\overset{1}{6.882} & \overset{2}{6.528} & \overset{3}{64.18} & \overset{4}{65.69} & \overset{5}{-32.51} & \overset{6}{-9.038} & \overset{7}{1} & \overset{8}{8} \\
1.442 & 1.725 & 10.66 & 11.51 & -4.603 & -0.9293 & 0 & 1 \\
-0.08882 & -0.1224 & -0.6888 & -0.7894 & 0.1511 & -0.1074 & 0 & 0 \\
-0.2444 & 0.1827 & -0.8566 & 0.4248 & 0.2403 & 0.3399 & 0 & 0 \\
-0.04516 & -0.03186 & -0.2895 & -0.2496 & -0.9597 & 0.03731 & 0 & 0 \\
-0.02363 & 0.04644 & -0.02528 & 0.1849 & -0.3489 & -0.9184 & 0 & 0 \\
0.7071 & 0.7071 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.7071 & 0.7071 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} .
\tag{5.44}
$$

Then we can determine the evolution of the coefficients at leading logarithmic order, on the basis of Eq. 4.49

$$
\vec{C}(m_b) = \hat{R}^{-1} \cdot \left(\frac{\alpha_s(m_b)}{\alpha_s(M_W)} \right)^{\frac{-\hat{D}}{2\beta_0}} \cdot \hat{R} \cdot \vec{C}(M_W) .
\tag{5.45}
$$

The numerical result for the coefficient $C_7^{\text{on-shell}}$ of the magnetic momentum operator, is at $\mu = m_b$

$$r \;=\; \frac{\alpha_s\,(m_b)}{\alpha_s\,(M_W)}$$

$$C_7^{\text{on-shell}}\,(m_b) \;=\; r^{-0.69565}\,C_7^{\text{on-shell}}\,(M_W) + 8\left(r^{-0.69565} - r^{-0.6087}\right)C_8^{\text{on-shell}}\,(M_W)$$

$$+ \;\left(6.52786\,r^{-0.69565} - 13.797\,r^{-0.608696} + 3.89614\,r^{-0.40862}\right.$$

$$+ \;2.57143\,r^{-0.26087} + 0.034332\,r^{-0.14565} + 0.22793\,r^{0.422989}$$

$$\left.+ \;0.428571\,r^{0.521739} + 0.111299\,r^{0.899395}\right)C_2^{\text{on-shell}}\,(M_W)\ . \quad (5.46)$$

The values[3] of the on-shell coefficients can be obtained from Eq. 3.13, taking into account the change in the normalization of the operators

$$C_2^{\text{on-shell}}\,(M_W) \;=\; 1$$

$$C_7^{\text{on-shell}}\,(M_W) \;=\; 2\left(C_2\,(M_W) + C_5\,(M_W)\right) = \frac{8x^3 + 5x^2 - 7x}{4\,(x-1)^3} - \frac{3\,x^2\,(3\,x - 2)}{2\,(x-1)^4}\,\ln x$$

$$C_8^{\text{on-shell}}\,(M_W) \;=\; 2\left(C_6\,(M_W) + C_8\,(M_W)\right) = \frac{-x^3 + 5x^2 + 2x}{4\,(x-1)^3} - \frac{3x^2}{2\,(x-1)^4}\,\ln x\ .$$

Following Buras et al. [81] we shall normalize the QCD coupling to the value determined at LEP,

$$\alpha_s\,(M_Z = 91.175\text{GeV}) \simeq 0.12 \tag{5.47}$$

using consistently the leading order approximation for the running coupling,

$$\alpha_s\,(\mu) = \frac{\alpha_s\,(M_Z)}{1 + \frac{\alpha_s(M_Z)}{2\pi}\frac{11\,C_A - 2n_f}{3}\ln\left(\frac{\mu}{M_Z}\right)}\ . \tag{5.48}$$

With this choice the ratio r has, still for $\mu = m_b$, the value

$$r \simeq 1.7 \tag{5.49}$$

with the choices $m_b = 5\text{GeV}$, $M_W = 80.14$, and the formula for the coefficient $C_7^{\text{on-shell}}$ becomes

$$C_7^{\text{on-shell}}\,(m_b) \simeq 0.97\,C_2^{\text{on-shell}} + 0.69\,C_7^{\text{on-shell}}\,(M_W) - 0.26\,C_8^{\text{on-shell}}\,(M_W)\ ; \tag{5.50}$$

as anticipated the multiplicative renormalization tends to suppress the ratio.

Considering the range of variation of the coefficients for m_t in the $130 - 180\text{GeV}$ interval,

$$C_7^{\text{on-shell}}\,(M_W) \;=\; 0.95\ -\ 1.20$$
$$C_8^{\text{on-shell}}\,(M_W) \;=\; (-0.17)\ -\ (-0.20) \tag{5.51}$$

one can make a few comments:

---

[3]in the Standard Model

- the mixing with the gluon magnetic momentum operator is not much relevant.

- The contribution from the 4 fermion operator is of the same order of magnitude, and with the same sign, as the one from photon magnetic momentum operator.

- As the C_7 coefficient increases with m_t, the relative importance of the QCD enhancement decreases.

5.4.1 Error analysis

Let us go back to the formula in Eq. 2.15, which expresses the ratio of the $b \to s\gamma$ to the $b \to ce\bar{\nu}_e$ decay rate.

$$R = \frac{\Gamma\left(b \to s\gamma\right)}{\Gamma\left(b \to c\,e\,\bar{\nu}_e\right)} = \frac{|V_{tb}V_{ts}^\star|^2}{|V_{cb}|^2} \frac{6\,\alpha_{\mathrm{QED}}}{\pi\left(1 - f\left(\frac{m_c}{m_b}\right)\right)} |C_{\mathrm{m.m.}}(m_b)|^2 \ , \tag{5.52}$$

We want to briefly explain how they have been derived the theoretical error bars shown in Figg. 2.2, 2.5. This topic has been addressed at length by the authors of [81], and we shall be schematic.

In the above mentioned papers 6 sources of uncertainty are discussed: the resulting errors can be added in quadrature, as there is no reason to assume they are correlated.

(i) The use of a spectator model is now know to correspond to the leading order in the $1/m_b$ expansion, and the existing literature [49, 65, 66] suggests that at most a 10% error can be ascribed to this origin.

(ii) The cancellation of the rapidly varying factor m_b^5, using the normalization to the semileptonic decay rate, introduces a dependence on the ratio $z = m_c/m_b$, which is known [43] to be $z = 0.316 \pm 0.013$, and originates a 6% error.

(iii) The combination of CKM parameters occurring in Eq. 2.15, assuming unitarity and taking into account limits from CP violation in K physics, is

$$\frac{|V_{tb}V_{ts}^\star|^2}{|V_{cb}|^2} = 0.95 \pm 0.04 \ . \tag{5.53}$$

(iv) It is common to use α_s coming from the Z physics: $\alpha_s = 0.12 \pm 0.01$. This determination however results by using NLL expressions for the observables at the Z peak, therefore it is in general unconsistent to use this determination at the present level of theoretical accuracy of the $b \to s\gamma$ process. The authors of [81] have checked that using two or one loop expressions for α_s in the coefficient evolution gives an error 4 times smaller than the one resulting by the inaccuracy in the determination of the initial value. It appears therefore reasonable to use consistently a leading logarithm expression for α_s and maintain the range of variation of the initial condition, as it comes from LEP.

(v) In Sec. 4.2.3 we have discussed how the expression for the coefficient evolution is altered by going from the leading to the next-to-leading expression: we have also seen how the dependence on the arbitrary renormalization scale is reduced. Also in the present case it is possible to check [81] the explicit reduction of μ dependence, however the central value

cannot be predicted without the NLL terms in the anomalous dimension matrix. On the other hand, as the variation of the admissible *perturbative* values of these parameters can be put in correspondence with a variation of the μ scale of the LL computation, the resulting error can be reasonably estimated by varying the scale between, say, $1/2m_b$ and $2m_b$. It is worth noting that even $1/2m_b \simeq 2.5\text{GeV}$ is a scale sufficiently high to trust perturbative QCD, so the assumption that NL coefficients remain "reasonable" is justified. The resulting error is about 25%

(vi) The branching ratio $BR\left[B \to X_s\gamma\right]$ is further affected by the experimental error on $BR\left[B \to X_c e\bar{\nu}_e\right] = (10.4 \pm 0.4)\,\%$.

The lack of the NLL analysis is the main source of error: a NLL computation could reduce the uncertainty under 10%, while it is currently larger than 25%. We have already seen how difficult is at the moment to discriminate among the different models, in view of these large uncertainty. We shall come back to this point after discussing the second topic of this thesis, the heavy top effects at LEP, because other processes sensible to the short distance structure of the Standard Model, like the $Z \to b\bar{b}$ decay, put complementary limits to the New Physics parameters and a comparative discussion is mandatory.

Part II

Heavy Top effects at LEP

Chapter 6

Overview

It is known from a long time that an heavy top quark results in relatively large quantum corrections to LEP observables. This heavy top dependence allowed to derive, together with low energy data, bounds on the m_t value. At the present time the top quark appears to have been discovered at CDF, and the same calculations which helped to set the bounds are used to "subtract" the top effect from precision electroweak experiments, in order to uncover possible New Physics effects.

In this Chapter we would like to provide an introduction to these topics and to set up a framework for the detailed discussion of a part of the radiative corrections, the $O\left(G_\mu^2 m_t^4\right)$ effects.

6.1 The renormalization scheme

It is useful to recall the methods commonly used to analyze the radiative corrections to the SM, limiting to the aspects more relevant for the study of the heavy top corrections at LEP.

The Standard Model is a renormalizable theory, which means that a finite number of experiments is sufficient to its complete determination, allowing to make predictions for all the other processes.

Given the set of "bare" parameters of the SM lagrangian, say $\{a_o^i\}$, an equal number of experiments $\{e^i\}$ has to be performed. The theoretical relations

$$e^i\left(a_o\right) \tag{6.1}$$

can then be inverted to obtain

$$a_o^i\left(e\right) \tag{6.2}$$

and specify any other experiment in terms of the basic ones.

The choice of the defining experiments is largely arbitrary, with the only requirement of independence, in order to allow the inversion of the relations in Eq. 6.1. In practice, the accuracy in the determination of the a_o^i parameters is maximized choosing basic observables which are on one hand experimentally clean and on the other hand computable (for instance within perturbation theory) to an high degree of precision.

6.1.1   Basic observables

The specification of the SM gauge content $SU(2) \times U(1)$, together with the electroweak symmetry breaking scale, given by the vacuum expectation value (*vev*) for the Higgs fields, require the determination of three basic quantities, which are the two gauge couplings

$$g_0, \, g_0'$$

(6.3)

and the vacuum expectation value v_0. Other than these quantities, the theory is specified by the fermion Yukawa couplings and the quartic Higgs self coupling.

For what concerns the three basic parameters, an optimal set of experiments is provided by [17]

- the determination of the fine structure constant α, coming from the Josephson effect or by the $(g-2)$ experiment

$$\alpha^{-1} = 137.0359895(61) \ .$$

(6.4)

- The Fermi constant

$$G_\mu = 1.16639(2) \times 10^{-5} \, \mathrm{GeV}^{-2}$$

(6.5)

as determined from the lifetime of the muon and by the theoretical formula

$$\tau_\mu^{-1} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} \left(1 - 8\frac{m_e^2}{m_\mu^2} \right) \left[1 + \frac{\alpha}{\pi} \left(25 - \pi^2 \right) \left(1 + \frac{\alpha}{3\,\pi} \ln \frac{m_\mu^2}{m_e^2} \right) \right]$$

(6.6)

including the QED corrections to the 4−fermion interaction. [1]

- The Z−mass, as determined by the LEP experiment [90]

$$M_Z = 91.187 \pm 0.007 \, \mathrm{GeV}$$

(6.9)

---

[1] One easily recognizes that the log appearing in the right hand side is the first contribution of the leading log series contributing to the running QED constant, as given by

$$\alpha \left(M \right)^{-1} = \alpha^{-1} + \frac{1}{3\pi} \sum_f Q_f^2 \ln \frac{M^2}{m_f^2}$$

(6.7)

where the sum runs over all the fermionic degrees of freedom which are "active" at the m_μ scale, hence one could have also written

$$\tau_\mu^{-1} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} \left(1 - 8\frac{m_e^2}{m_\mu^2} \right) \left[1 + \frac{\alpha \left(m_\mu \right)}{\pi} \left(25 - \pi^2 \right) \right] \ ,$$

(6.8)

which is rather an affectation at this level, but will be important when considering experiments at the scale of the $SU(2) \times U(1)$ breaking.

For what concerns the Yukawa couplings, they are eliminated in favor of the fermion masses: we just recall that while lepton masses are unambiguously defined because these particles are observed in *isolation*, the masses of the light u, d, s quarks (or better the ratios m_u/m_d, m_s/m_d) are extracted from pion and kaon masses using chiral symmetry, while c, b masses come either from the study of charmonium spectra, and by the D and B meson masses. In dealing with LEP observables it is possible to avoid the resulting uncertitude, for instance in the expression for the running QED coupling the hadronic contribution can be related to the $e^+ e^- \to \gamma \to$ hadrons using a dispersive representation.

A different role is played by the masses of the top quark and of the Higgs field (which is a substitute for the quartic Higgs coupling): in the analysis of experiments up to the electroweak breaking scale, and in the language of effective theories, heavy fields can be removed at the price of renormalizing the effective lagrangian [25]: but the decoupling theorem, as we shall see, is "evaded", and the corrections grow with the top mass.

6.1.2 Derived observables

At the tree level, the W mass is related to the basic observables by the relation

$$M_W^2 = M_Z^2 \cos^2 \theta_W \tag{6.10}$$

where

$$\sin^2 \theta_W \cos^2 \theta_W = \frac{\pi \alpha}{\sqrt{2} G_\mu M_Z^2} \ . \tag{6.11}$$

At the quantum level, if we maintain the definition of the Weinberg angle as fixing the γ and Z fields as mass eigenstates, the relation with the vector boson masses is altered as follows

$$\sin^2 \theta_W \to s_\theta^2 = 1 - \frac{M_W^2}{\rho \, M_Z^2} \ ; \tag{6.12}$$

the ρ parameter has a tree value of 1 for a minimal SM or extensions based on doublets of scalar fields. However, it is also altered by quantum corrections. It is customary to define

$$s_\theta^2 = \left(1 - \frac{M_W^2}{M_Z^2} \right) + \frac{M_W^2}{M_Z^2} \Delta \rho \equiv s_W^2 + c_W^2 \Delta \rho \tag{6.13}$$

where the parameter $s_W^2 = 1 - M_W^2/M_Z^2$ is just a substitute for the W mass, while s_θ^2 describes the coupling constants in the neutral current sector and enters neutrino scattering and observables at the Z peak.

From the relation of the Charged Current coupling to the effective Fermi constant G_μ

$$G_\mu = \frac{e^2}{8 \, s_\theta^2 M_W^2} = \frac{\pi \, \alpha}{2 s_\theta^2 c_\theta^2 \rho M_Z^2} \tag{6.14}$$

one obtains the definition of s_θ in terms of the fundamental observables, and the calculable ρ parameter

$$s_\theta^2 \, c_\theta^2 = \frac{\pi \, \alpha}{\sqrt{2} G_\mu M_Z^2 \rho}, \tag{6.15}$$

in other words the ρ parameter enters the overall normalization of the Neutral Current vertex

$$\frac{e}{2s_\theta c_\theta} = \left(\sqrt{2}G_\mu M_Z^2 \rho\right)^{1/2} \tag{6.16}$$

and can be compactly defined as the ratio of the neutral current to charged current interaction at low energy

$$\rho = \frac{G_{NC}(0)}{G_{CC}(0)} \ . \tag{6.17}$$

It must be stressed that s_θ^2 is not a fundamental observable, it is a derived quantity and can be tested in the appropriate experiments.

The fact that the ρ parameter is nearly equal to one, that is, the corrections are "naturally" small, is the reflection of a underlying symmetry of the SM (see for instance [26]). Indeed, in the limit in which the $U(1)$ gauge coupling g' goes to zero and also the Yukawa couplings go to zero, the SM possesses a gauged $SU(2)_L$ symmetry and also a global $SU(2)_R$ symmetry. When the Higgs field acquires a *vev*, this symmetry is broken down to diagonal $SU(2)$, the so called "custodial" symmetry. This means that the neutral currents and the charged currents, being part of an $SU(2)$ multiplet must have the same coefficients, hence $\rho = 1$ at the tree level.

6.2   Heavy top effects

The heavy top effects are a manifestation of the mass-generation mechanism in the SM, that is, of the Yukawa coupling to the Higgs field [10, 12]. It is sometimes stated that these effects, growing with m_t, are an "evasion" of the Applequist-Carazzone theorem [4]. This can be misleading: simply the theorem does not apply. Recall that the theorem asserts that sending a *dimensionful* parameter, say the mass of one of the fields to infinity, the low energy theory results by considering the lagrangian with the heavy field removed, including finite, unobservable redefinition of the couplings, plus corrections which are depressed by powers of the large mass. But the $m_t \to \infty$ limit results from a *dimensionless* parameter, the Yukawa coupling g_t, going to ∞. Diagrammatically, we shall see that powers of the Yukawa coupling from the vertices can over-compensate the m_t terms from the denominators.

6.2.1   Heavy top effects in the ρ parameter

We have seen that the ρ parameter is protected by the "custodial" symmetry, which is broken by the Yukawa couplings. What really counts is not the mass of the individual quarks, but the mass splitting within a multiplet: for instance, if the b and t quarks were both heavy and degenerate, the $SU(2)$ custodial would remain unbroken.

The language of effective field theories is well suited to analyze the effect of the mass splitting [25]. Suppose we are given the t, b doublet, with $m_b \ll M_W$, $m_t \gg M_W$; we can divide the range of scales in three regions: below M_W the theory can be described with an effective Fermi theory, that is, with W, Z, t fields integrated out. At a scale around M_W a theory with the t field integrated out is used, while finally at energies of the order of m_t the full structure of the theory is recovered. The matching between these descriptions, that is, the
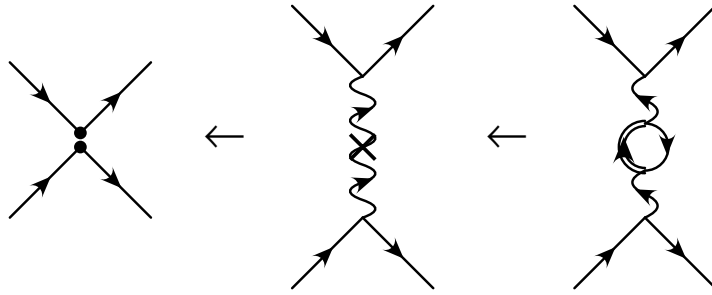
Figure 6.1: From the effective theory to the fundamental theory

procedure needed to derive the coefficients of the effective lagrangian, at lower energy scales, from the fundamental description at higher energy scales, originates the residual dependence of the low energy theory from the parameters of the "fundamental" theory.

The coefficients of the four fermion operators, considered at a matching scale of about $\mu \equiv M_W$, are connected to the gauge boson mass terms of the intermediate scale theory, while on his turn the gauge boson mass terms are affected by the heavy t quark present in the higher scale theory (Fig. 6.1). In practice then the ρ parameter results from the different effect of the heavy quark on the $W^{\pm}$ and the W^3 mass terms. In this language it is particularly clear why the effect is so relevant, and depends on the square of the top mass: the mass terms in the effective lagrangian of the intermediate scale theory are dimension two operators, hence nothing prevents them to depend on m_t^2.

We anticipate the result at one-loop order,

$$\rho = 1 + N_c \frac{G_\mu m_t^2}{8\pi^2\sqrt{2}} \tag{6.18}$$

where N_c is the number of colors, which shows that to this order the effect comes entirely from fermion loops.

### 6.2.2 Heavy top effects in vertices

The effect on the ρ parameter can be regarded as a self-energy effect; in the scheme which sets among the fundamental parameters the Z mass and the Fermi coupling G_μ, it results from the $W^{\pm}$ self-energy.

Other heavy top effects stem from the vertex corrections, and the most relevant, from a numerical point of view, are the corrections to the $Z \to f\bar{f}$ vertex [33, 39, 40, 41]. We shall see how in the limit of a very heavy top only the exchange of the would-be-Goldstone fields is relevant to the computation of the m_t^2 corrections, as in Fig. 6.2.

The vertex is proportional to $|V_{qf}|^2$, hence due to the smallness of V_{td}, V_{ts} coefficients [2] the heavy top effects are relevant numerically only for the $Z \to b\bar{b}$ vertex. In the following we shall always set $V_{tb} = 1$.

Due to the $SU(2)_L$ symmetry, a purely left handed correction results to the lowest order $Z \to b\bar{b}$ vertex: we again anticipate the result for the amplitude, at leading order

---

[2]$|V_{td}| = 0.004 \div 0.015$, $|V_{ts}| = 0.030 \div 0.048$[70]
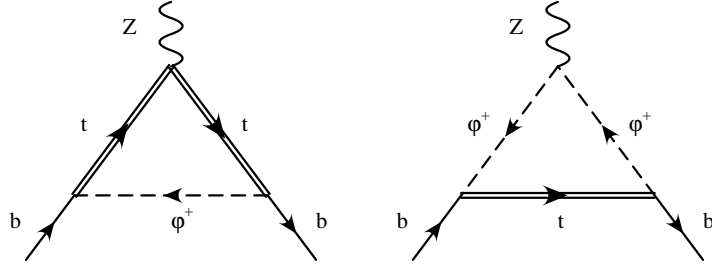
Figure 6.2: Leading corrections to $Z \to b\bar{b}$ in the $m_t \to \infty$ limit.

$$
\begin{aligned}
A_\mu \left(Z \to b\bar{b}\right) &= -i\frac{e}{2\, s_\theta\, c_\theta} \left[\left(1 - \frac{2}{3}s_\theta^2 + \tau\right)\gamma_\mu^L - \frac{2}{3}s_\theta^2\gamma_\mu^R\right] \\
\tau &= -\frac{G_\mu m_t^2}{4\pi^2\sqrt{2}} \; .
\end{aligned}
\tag{6.19}
$$

Apart the overall factor, the corrections can be included in a redefinition of the vector and axial electroweak couplings,

$$
g_{bV} = 1 - \frac{4}{3}s_\theta^2 + \tau \qquad\qquad g_{bA} = 1 + \tau
\tag{6.20}
$$

thus separating the vertex corrections to the amplitude, included in τ, from the corrections induced by the very connection to the fundamental observables, included in s_θ.

6.2.3 Bounds on top mass

We have to remind, before discussing the bounds, that a comparison with experiment is possible only introducing the main sources of radiative corrections to electroweak observables, coming from the "pure" QED effects[3] and by the QCD corrections.

Their inclusion is mandatory and straightforward: in this thesis it is impossible to account for the whole set of radiative corrections, but it goes without saying that in graphs and tables all the relevant effects other than the heavy top ones are included whenever needed.

Low energy data

The most precise low energy informations come from the determination of NC/CC neutrino-nucleon cross section ratios [68], by the CDHS[35, 45] and CHARM[34] experiments: the accuracy is about 1%. At tree level, this ratios are given by [23]

$$
\begin{aligned}
R_\nu &= \left(\frac{M_W}{M_Z}\right)^4 \frac{1 - 2s_W^2 + \frac{10}{9}\left(1 + r\right)s_W^4}{2\left(1 - s_W^2\right)^2} \\
R_{\bar{\nu}} &= \left(\frac{M_W}{M_Z}\right)^4 \frac{1 - 2s_W^2 + \frac{10}{9}\left(1 + \frac{1}{r}\right)s_W^4}{2\left(1 - s_W^2\right)^2}
\end{aligned}
\tag{6.21}
$$

---

[3]As shown by Okun and collaborators [91, 108], for some time QED effects have been sufficient to account for all the radiative corrections.

where $r = \sigma^\nu_{CC}/\sigma^{\bar\nu}_{CC} \simeq 0.4$. Leaving aside the QED, box and vertex corrections, the large m_t effects can be incorporated in this formulas by replacing $s^2_W \to s^2_W + \Delta\rho c^2_W$. But in the expression for R_ν the dependence on s^2_W is weak:

$$R_\nu = \frac{1}{2}\left(\frac{M_W}{M_Z}\right)^4 \left(1 + 0.56 s^2_W + O\left(s^4_W\right)\right) . \tag{6.22}$$

Hence the ratio M_W/M_Z, when extracted from R_ν, depends very weakly on m_t; this stability is the result of an accidental cancellation, and does not appear neither in the $R_{\bar\nu}$ ratio nor in the νe cross sections. Combining this M_W/M_Z measurement and the other low energy νe, νq, $e q$ data one can obtain an upper limit $m_t < 168\text{GeV}$ at $68\%C.L.$, for $M_H \simeq M_Z$[51], but including also the direct measurement at CDF of the ratio of vector-boson masses, together with the theoretical prediction from the input parameters α, G_μ, M_Z, one can obtain a global fit (see for instance [68])

$$m_t = 122^{+46}_{-32}\text{GeV} \qquad (68\%\text{C.L.}) \tag{6.23}$$

for $40\text{GeV} < M_H < 1\text{TeV}$.

LEP data

We refer to a number of good reviews [44, 53, 60, 67, 92, 95] for a complete discussion of the observables studied at LEP around the Z resonance. We shall here be brief because we will come back to this subject in the Chapter 8, when also the two-loop corrections will be considered.

It suffices here for the purpose of illustration, to focus on one of the most precisely measured observables, the total width Γ_Z. In Fig. 6.3 it is shown the comparison between theory and experiment. The various curves, plotted in dependence of m_t, for different values of the Higgs mass, are superimposed on the band of experimental values.

One can note a few features common also to similar plots for the various partial widths, like $\Gamma\left(Z \to \mu^+\mu^-\right)$ or the forward-backward asymmetries: the M_H dependence is rather slow, as a consequence of the Veltman "screening" theorem [9]: leading quantum corrections depend logarithmically on M_H.

Combining limits extracted from plots like the one in Fig. 6.3 with low energy informations, and the determination of M_W from CDF, different [68] analyses agree on the following limits on m_t, at 1σ level

$$m_t = 144^{+23}_{-26}\,^{+19}_{-23}\text{GeV} \tag{6.24}$$

where the second error is desumed by varying M_H around a central value of 300GeV[4].

The very significance of these bounds relies on the stability of the perturbative expansion: predicting an heavy top mass starting from an expansion in powers of m_t calls for a check of the next perturbative order. In the following we shall describe the calculation of g^4_t effects to LEP parameters, which has been useful, at the time it was performed, to improve the confidence in the existing results. Presently the g^4_t effects are included in the analyses of experimental

---

[4]A more recent analysis [111] finds a result in remarkable agreement with the recent CDF results, namely

$$m_t = 175 \pm 11^{+17}_{-19}\text{GeV} \qquad \text{for } M_H = 300^{+700}_{-240}\text{GeV} .$$
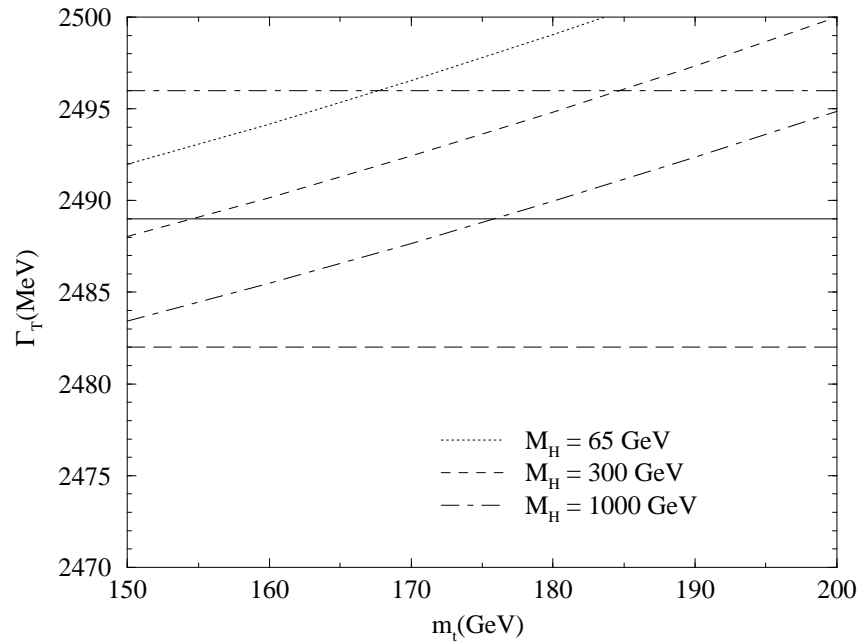
Figure 6.3: The total Z width from LEP

data, since some observables are being measured at a comparable level of accuracy: however we shall see that, in view of the present probable value of the top mass, the g_t^4 effect cannot be considered dominant, and eventually a full two loop calculation shall be mandatory.

Chapter 7

Strategy to compute heavy top corrections

Our aim is to evaluate all the corrections growing like $G_\mu\, m_t^2$, $G_\mu^2\, m_t^4$ to the Standard Model parameters [1]; we have seen that it is known how these effects stem from the Yukawa sector of the theory [52, 50]. The idea is to define an appropriate limit such that for $g_t \gg g, g'$ the theory reduces to a pure Yukawa model, that is, to a scalar theory, simplifying the computation.

The easiest way to define this limit is to work in a renormalizable gauge: in this way only charged and pseudoscalar Higgs fields, not "eaten up" by the gauge fields, have couplings with the $(t,\ b)$ doublet proportional to g_t. As a consequence only Feynman diagrams resulting by the exchange of these scalar particles are to be considered, while the quantum gauge fields are irrelevant, and the gauge fields appear only on the external lines and can be considered of classical nature.

These considerations are made rigorous by using the background field formulation of the Standard Model, which allows to relate different Green functions by classical Ward Identities; in particular it results that quantum corrections to the ρ parameter and to the $Z \to b\bar{b}$ vertex are connected to correlations of scalar fields. The advantage over a standard formulation, which would lead to Slavnov-Taylor Identities, is that it is made manifest how these large g_t effects are completely decoupled from the gauge structure of the theory. The gauge structure almost completely disappears, in the sense that gauge fields do not propagate, and they couple as external sources to the classical $SU(2) \otimes U(1)$ currents of the theory.

It can be useful to give a pictorial resume of the strategy

<div align="center">

Full Model

Classical $SU(2) \otimes U(1)$ gauge symmetry $\to$ BRS symmetry

</div>

Fields

$$(A,\ Z,\ W) \cup (H,\ \chi,\ \varphi^{\pm}) \cup (t,\ b)$$

Inputs

$$\alpha_{e.m.} \quad G_\mu \quad M_Z \quad m_t \quad (m_H)$$

---

[1]Times functions of the ratio m_t/m_H, and disregarding QCD effects

$$\Downarrow$$
$$\text{Background Field Gauge}$$
$$\Downarrow$$

- Fields doubling $\Phi \rightarrow \Phi^q + \Phi^{\text{cl}}$

- Ward Identities on external lines

- Slavnov-Taylor Identities, ghosts, ... in internal lines

$$\Downarrow$$

$$g \ll g_t \qquad \longrightarrow \qquad M_{W,Z} \ll m_t$$

$$\Downarrow$$

ghosts and gauge fixing related vertices irrelevant; gauge bosons coupled as external currents

$$\Updownarrow$$

Gaugeless limit of the Standard Model

7.1   The Ward Identities

In App. C and D the formulation of the quantum Standard Model in the renormalizable non-linear gauge and then in the background field gauge is discussed.

The formulas given there are rather involved, in the attempt to write down results as general as possible, but the philosophy is quite elementary, and the results are self explanatory.

Having chosen to preserve the classical symmetries of the theory, by using a background field formulation, we are allowed to impose on the renormalized effective action Γ the same Ward identities[2] we would obtain in the "tree-level" theory (see Appendix D).

As discussed in the previous chapter, for the purpose of evaluating the large m_t effects we can reduce the matter content of the Standard Model only to the third quark doublet (t, b).

By exploiting the invariance under the so called type II transformations, which are pure rotations on the quantum fields, we can obtain 4 generating equations for the classical Ward Identities relating $1PI$ Green functions, corresponding to the 4 generators of $SU(2) \otimes U(1)$

$$\sum_{\text{f}} \delta_i^{II} f^{\text{cl}}(x) \frac{\delta\Gamma}{\delta f^{\text{cl}}(x)} = 0 \qquad\qquad i = 1 \ldots 4 \ . \tag{7.1}$$

Taking into account the formulas C.39, C.40, C.41 we can easily write down their detailed form: in the following the $\mp, \pm$ symbols whould be understood as summations over the contributions from oppositely charged fields.

A-current Ward Identity

$$
\begin{aligned}
\partial_\mu^x \frac{\delta\Gamma}{\delta A_\mu(x)} \;=\;& \pm ie W_\mu^\pm \frac{\delta\Gamma}{\delta W_\mu^\pm} \pm ie\phi^\pm \frac{\delta\Gamma}{\delta\phi^\pm} \\
& - \sum_{q=t,b} ie Q_q \left[\frac{\delta\Gamma}{\delta q(x)} q(x) + \bar{q}(x) \frac{\delta\Gamma}{\delta\bar{q}(x)} \right]
\end{aligned}
\tag{7.2}
$$

---

[2]in absence of external quantum legs or ghosts

Z-current Ward Identity

$$\partial_\mu^x \frac{\delta\Gamma}{\delta Z_\mu(x)} =$$

$$\pm ig\cos_W W_\mu^\pm \frac{\delta\Gamma}{\delta W_\mu^\pm} \pm \frac{ie}{\tan 2\theta_W}\phi^\pm \frac{\delta\Gamma}{\delta\phi^\pm} + \frac{e}{\sin 2\theta_W}\chi\frac{\delta\Gamma}{\delta H} - (H+v)\frac{e}{\sin 2\theta_W}\frac{\delta\Gamma}{\delta\chi}$$

$$-\sum_q \frac{ie}{\sin_W\cos_W}\left[\frac{\delta\Gamma}{\delta q(x)}\left(T^3 P_L - \sin_W^2 Q\right)q(x) + \bar{q}(x)\left(T^3 P_R - \sin_W^2 Q\right)\frac{\delta\Gamma}{\delta\bar{q}(x)}\right] ,$$

$$(7.3)$$

here the projector P_L on left states has been explicitly introduced.

Charged current Ward Identities

Here *no* summation on signs $\pm, \mp$ is understood

$$\left(\partial_\mu \pm igW_\mu^3\right)\frac{\delta\Gamma}{\delta W_\mu^\pm} =$$

$$\pm iW_\mu^\mp\left(e\frac{\delta\Gamma}{\delta A_\mu} + g\frac{\delta\Gamma}{\delta Z_\mu}\right) + \frac{g}{2}\phi^\mp\left(\mp i\frac{\delta\Gamma}{\delta H} + \frac{\delta\Gamma}{\delta\chi}\right)$$

$$+\frac{g}{2}\left(\pm i\left(H+v\right) - \chi\right)\frac{\delta\Gamma}{\delta\phi^\pm} - \frac{ig}{\sqrt{2}}\sum_q\left(\frac{\delta\Gamma}{\delta q_L}\sigma^\pm q_L + \bar{q}_L\sigma^\pm\frac{\delta\Gamma}{\delta\bar{q}_L}\right) \qquad (7.4)$$

Having fully described the symmetry content of the theory, we can specialize to the processes under investigation.

7.2 Identities relevant for $Z \to b\bar{b}$ and the ρ parameter

By differentiating Eq. 7.3 with respect to b fields one easily obtains

$$\partial_\mu^x \frac{\delta\Gamma}{\delta Z_\mu(x)b(y)\bar{b}(z)} =$$

$$-\frac{ev}{\sin 2\theta_W}\frac{\delta\Gamma}{\delta\chi(x)\delta b(y)\delta\bar{b}(z)}$$

$$-\frac{ie}{\sin_W\cos_W}\left[-\frac{\delta\Gamma}{\delta b(x)\delta\bar{b}(z)}\left(-\frac{1}{2}P_L + \frac{1}{3}\sin_W^2\right)\delta(x-y)+\right.$$

$$\left.+\delta(x-z)\left(-\frac{1}{2}P_R + \frac{1}{3}\sin_W^2\right)\frac{\delta\Gamma}{\delta b(y)\delta\bar{b}(x)}\right] \qquad (7.5)$$

which allows to express the irreducible $Z \to b\bar{b}$ vertex in terms of the corresponding vertex for $\chi \to b\bar{b}$[3].

---

[3]Note a useful identity relating the $W^+ \to t\bar{b}$ vertex with the corresponding one for ϕ^+

The identities relevant for the ρ parameter can be analogously obtained by considering the wave function renormalization of fields Z, χ and $W^{\pm}$, $\phi^{\pm}$ [4] and the tadpole term for H:

$$\partial_{\mu}^x \frac{\delta\Gamma}{\delta Z_{\mu}(x)\delta Z_{\nu}(y)} = -\frac{g\,v}{2\,c} \frac{\delta\Gamma}{\delta\chi(x)\delta Z_{\nu}(y)}$$

$$\partial_{\mu}^x \frac{\delta\Gamma}{\delta Z_{\mu}(x)\delta\chi(y)} = \frac{g}{2\,c} \left[\delta(x-y)\frac{\delta\Gamma}{\delta H(x)} - v\frac{\delta\Gamma}{\delta\chi(x)\delta\chi(y)} \right] \ . \tag{7.7}$$

As a renormalization condition we shall set to zero the tadpole term, thus imposing that the vacuum expectation value v does not renormalize: consequently the two relations in Eq. 7.7 combine to give

$$\partial_{\mu}^x \partial_{\mu}^y \frac{\delta\Gamma}{\delta Z_{\mu}(x)\delta Z_{\nu}(y)} = \frac{g^2\,v^2}{4\,c^2} \frac{\delta\Gamma}{\delta\chi(x)\delta\chi(y)} \ . \tag{7.8}$$

We can also relate the renormalization of $W^{\pm}, \phi^{\pm}$ fields by writing

$$\partial_{\mu}^x \frac{\delta\Gamma}{\delta W_{\mu}^+(x)\delta\phi^-(y)} = \frac{g}{2}\delta(x-y) \left(-i\frac{\delta\Gamma}{\delta H(x)} + \frac{\delta\Gamma}{\delta\chi(x)} \right) + \frac{igv}{2} \frac{\delta\Gamma}{\delta\phi^+(x)\phi^-(y)}$$

$$\partial_{\mu}^x \frac{\delta\Gamma}{\delta W_{\mu}^+(x)\delta W_{\nu}^-(y)} = i\delta(x-y) \left[e\frac{\delta\Gamma}{\delta A_{\nu}(y)} + g\frac{\delta\Gamma}{\delta Z_{\nu}(y)} \right] + \frac{igv}{2} \frac{\delta\Gamma}{\delta\phi^+(x)\delta W_{\nu}^-(y)}$$

$$\tag{7.9}$$

and then, again setting to zero the tadpoles,

$$\partial_{\mu}^x \partial_{\nu}^y \frac{\delta\Gamma}{\delta W_{\mu}^+(x)\delta W_{\nu}^-(y)} = \left(\frac{g\,v}{2} \right)^2 \frac{\delta\Gamma}{\delta\phi^+(x)\phi^-(y)} \ . \tag{7.10}$$

7.2.1 Form factors

In momentum space, the identities in Eqs. 7.8, 7.10 can be rewritten as follows:

$$p_{\mu}p_{\nu}\Pi_{\mu\nu}^Z(p) = \left(\frac{gv}{2c} \right)^2 \Pi_{\chi}(p)$$

$$p_{\mu}p_{\nu}\Pi_{\mu\nu}^W(p) = \left(\frac{gv}{2} \right)^2 \Pi_{\phi}(p) \ . \tag{7.11}$$

The self energy at zero momentum gives the correction to the mass,

$$\partial_{\mu}^x \frac{\delta\Gamma}{\delta W_{\mu}^+(x)\delta b(y)\delta\bar{t}(z)} = \frac{igv}{2} \frac{\delta\Gamma}{\delta\phi^+(x)\delta b(y)\delta\bar{t}(z)}$$

$$- \frac{ig}{\sqrt{2}} \left[-\frac{\delta\Gamma}{\delta t(x)\delta\bar{t}(z)}P_L\delta(x-y) + \delta(x-z)P_R\frac{\delta\Gamma}{\delta b(y)\delta\bar{b}(x)} \right] \tag{7.6}$$

[4]Note that the background gauge fixing sets to zero the mixing Z, χ only in the quantum quadratic part of the action

$$\Pi^Z_{\mu\nu}\Big|_{p=0} = (-i)\,\eta_{\mu\nu}\delta m^2_Z$$
$$2\eta_{\mu\nu}\delta m^2_Z = im^2_Z\frac{\partial\Pi_\chi}{\partial p_\mu\partial p_\nu} \tag{7.12}$$

where the mass m_Z appearing in the formula is the tree level one.

We shall use the notation, for the self energy of the scalar fields,

$$\Pi_f = ip^2\Sigma_f + O\,(1)$$

and obtain the mass renormalizations

$$\frac{\delta m^2_Z}{m^2_Z} = i\frac{\partial\Pi^\chi}{\partial p^2}\Big|_{p=0} = -\Sigma_\chi$$
$$\frac{\delta m^2_W}{m^2_W} = i\frac{\partial\Pi^\phi}{\partial p^2}\Big|_{p=0} = -\Sigma_{\phi\pm}\ . \tag{7.13}$$

The unrenormalized ratio $\bar{\rho}$ is then given by

$$\bar{\rho} = \frac{m^2_W + \delta m^2_W}{c\,(m^2_Z + \delta m^2_Z)}$$
$$= 1 + \hbar\left(\Sigma^{(1)}_\chi - \Sigma^{(1)}_{\phi\pm}\right) + \hbar^2\left[\left(\Sigma^{(2)}_\chi - \Sigma^{(2)}_{\phi\pm}\right) + \Sigma^{(1)}_\chi\left(\Sigma^{(1)}_\chi - \Sigma^{(1)}_{\phi\pm}\right)\right] \tag{7.14}$$

where the orders of $\hbar$ have been put in evidence. We shall see in the following that a term $\left(1 + \Sigma_{\phi\pm}\right)$ will be readsorbed by the renormalization procedure in the definition of the Fermi coupling G_μ. Consequently the renormalized ρ parameter will be obtained by the formula

$$\rho = 1 + \hbar\left(\Sigma^{(1)}_\chi - \Sigma^{(1)}_{\phi\pm}\right) + \hbar^2\left[\left(\Sigma^{(1)}_\chi - \Sigma^{(1)}_{\phi\pm}\right)^2 + \left(\Sigma^{(2)}_\chi - \Sigma^{(2)}_{\phi\pm}\right)\right] + O\left(\hbar^3\right)\ , \tag{7.15}$$

which deserves a few comments: first of all, it is made evident that only the difference of the self energies for charged and pseudoscalar unphysical Higgs fields has to be computed. Next we note that at $\hbar^2$ order there are two contributions, one resulting from reducible diagrams and proportional to the square of the one-loop coontributions, the other coming from two-loop diagrams. It is therefore common to present the results of a perturbative evaluation of the ρ parameter in a different form,

$$\frac{1}{\rho} - 1 = -\hbar\left(\Sigma^{(1)}_\chi - \Sigma^{(1)}_{\phi\pm}\right) - \hbar^2\left(\Sigma^{(2)}_\chi - \Sigma^{(2)}_{\phi\pm}\right) + O\left(\hbar^3\right)\ . \tag{7.16}$$

A similar analysis for the Ward Identity in Eq. 7.5 (relevant for the $Z \to b\bar{b}$ vertex) is slightly simplified if we choose as renormalization condition for the b propagator a unit residue at the mass pole (at $m_b = 0$ in the present approximation): the tree level terms cancel in both sides of the equation, and it is immediate to show that the quantum corrections of the vertices at zero exchanged momentum are related by

$$\langle Z_\mu b\bar{b}\rangle_{\text{quantum}} = -\frac{i\,g\,v}{2\,c}\frac{\partial}{\partial k_\mu}\langle\chi b\bar{b}\rangle\,\|_{k=0} \tag{7.17}$$

where k is the momentum entering the χ, Z line.

7.3 The Yukawa model

The computation of the ρ parameter and the $Z \to b\bar{b}$ vertex have been reduced, through Eq. 7.15 and 7.17, to the evaluation of correlations of scalar fields. Now it comes the final step, in the heavy top limit these fields are described by the following Yukawa lagrangian, resulting by dropping the gauge couplings and fields from the SM lagrangian, in the renormalizable, background field gauge fixed form:

$$
\begin{aligned}
\mathcal{L} &= -\bar{Q}_L \slashed{\partial} Q_L - \bar{t}_R \slashed{\partial} t_R - \bar{b}_R \slashed{\partial} b_R - (\partial_\mu \Phi)^\dagger (\partial_\mu \Phi) \\
&- V\left(\Phi^\dagger \Phi\right) - g_t \left\{ \bar{Q}_L \tilde{\Phi} t_R + \bar{t} \tilde{\Phi}^\star Q_L \right\} \\
Q_L &= \begin{pmatrix} t_L \\ b_L \end{pmatrix} \qquad \Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(H' + i\chi) \end{pmatrix}
\end{aligned}
\tag{7.18}
$$

with the usual form for the potential V

$$
V(x) = \mu^2 x + \lambda x^2 .
\tag{7.19}
$$

The Ward Identities we have discussed before are now derived by supplementing this lagrangian with the coupling of the external W, Z fields to the conserved classical currents,

$$
\Delta \mathcal{L} = -\frac{i\,g}{c}\left(\frac{1}{2} J_\mu^3 - s^2 J_\mu^{\text{em}}\right) - \frac{i\,g}{\sqrt{2}}\left(J_\mu^+ W_\mu^+ + J_\mu^- W_\mu^-\right) ,
\tag{7.20}
$$

resulting from the different symmetry generators

- baryon $U(1)$

$$
\hat{J}_\mu^B = \bar{t}\gamma_\mu t + \bar{b}\gamma_\mu b
\tag{7.21}
$$

- weak hypercharge $U(1)$

$$
\hat{J}_\mu^Y = \frac{1}{3}\bar{t}_L \gamma_\mu t_L + \frac{4}{3}\bar{t}_R \gamma_\mu t_R + \frac{1}{3}\bar{b}_L \gamma_\mu b_L - \frac{2}{3}\bar{b}_R \gamma_\mu b_R - \Phi^\star \partial_\mu \Phi + \partial_\mu \Phi^\star \Phi
\tag{7.22}
$$

- T^3 generator of $SU(2)$

$$
\hat{J}_\mu^3 = \bar{t}_L \gamma_\mu t_L - \bar{b}_L \gamma_\mu b_L + (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + (\Phi_0^\star \partial_\mu \Phi_0 - \Phi_0 \partial_\mu \Phi_0^\star)
\tag{7.23}
$$

- Combining in the usual way the currents J^3 and J^Y one gets the e.m. current $\hat{J}^{em}$

$$
\hat{J}_\mu^{em} = \frac{2}{3}\bar{t}\gamma_\mu t - \frac{1}{3}\bar{b}\gamma_\mu b + (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+)
\tag{7.24}
$$

- $T^\pm$ generators of $SU(2)$

$$
\begin{aligned}
\hat{J}_\mu^- &= \bar{t}_L \gamma_\mu b_L + \frac{1}{\sqrt{2}}\left(H \partial_\mu \phi^- - \phi^- \partial \mu H\right) + \frac{i}{\sqrt{2}}\left(\chi \partial_\mu \phi^- - \phi^- \partial_\mu \chi\right) \\
\hat{J}_\mu^+ &= -\bar{b}_L \gamma_\mu t_L + \frac{1}{\sqrt{2}}\left(H \partial_\mu \phi^+ - \phi^+ \partial \mu H\right) - \frac{i}{\sqrt{2}}\left(\chi \partial_\mu \phi^+ - \phi^+ \partial_\mu \chi\right) .
\end{aligned}
\tag{7.25}
$$

In other words, the classical Ward Identities reduce to the conservation of these Noether currents; in particular, as the electromagnetic current is exactly conserved, we have that only the renormalization of current J^3 is important.

Note that Eq. 7.18 is written without specifying the phase of the theory (this is the meaning of the H' notation): when considering the symmetry breaking a subtlety arises and the conclusions of the previous section, that is, the Ward Identities relating the effective Z and χ vertices, are to be translated in this language by stating that the current coupled to the Z field is still

$$\hat{J}^3_\mu = \bar{t}_L \gamma_\mu t_L - \bar{b}_L \gamma_\mu b_L + (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + i \left(H \partial_\mu \chi - \chi \partial_\mu H \right) \tag{7.26}$$

but the conserved current is

$$J^3_\mu = \hat{J}^3_\mu + iv \partial_\mu \chi \; , \tag{7.27}$$

meaning that, as operators

$$\partial_\mu \hat{J}_\mu = -iv \Box \chi + \text{motion equations} \tag{7.28}$$

and the insertion of the current $\hat{J}_\mu$ is put in relation with the insertion of operator $\partial_\mu \chi$. Analogously one has that

$$J^\pm_\mu = \hat{J}^\pm_\mu + \frac{v}{\sqrt{2}} \partial \phi^\pm \tag{7.29}$$

are the charged conserved currents.

We stress that this is an information coming from the full model, in the non-linear background field formulation, which allowed to derive the Ward Identities.

In summary, by considering the SM quantized in the background field gauge, we have shown that the same Ward Identities, in the complete theory and in the so-called gaugeless theory, relate the quantities most sensitive to m_t to correlations of scalar fields, which can be studied in a pure Yukawa model.

Chapter 8

Heavy Top effects in the Gaugeless Limit

We are now in position to describe in detail the computations needed to obtain the $O\left(G_\mu^2 m_t^4\right)$ corrections. We start by defining the renormalized model.

8.1 Renormalized model

We work in dimensional regularization using the NDR scheme for the γ_5 matrix. We have seen in considering the computation of QCD corrections to the $b \to s\gamma$ process that up to two loops it is possible to avoid any resulting regularization ambiguity.

The following renormalization constants are introduced, to cancel poles and perform additional subtractions in order to impose suitable conditions, without breaking the classical symmetries

$$
\begin{aligned}
Q_L &\to Z_L^{1/2} Q_L & \mu^2 &\to \left(\mu^2 - \delta\mu^2\right) Z_\phi^{-1} \\
t_R &\to Z_R^{1/2} t_R & \lambda &\to \lambda Z_\lambda Z_\phi^{-2} \\
g_t &\to Z_g g_t & v &\to Z_\phi^{1/2}\left(v - \delta v\right) \\
\Phi &\to Z_\phi^{1/2} \Phi
\end{aligned}
\tag{8.1}
$$

where the redefinitions of v, Z_λ, μ^2 contain also Z_ϕ for reasons of opportunity. Indeed, one has

$$
\begin{pmatrix} \phi^+ \\ \frac{H+i\chi}{\sqrt{2}} + \frac{v}{\sqrt{2}} \end{pmatrix} \to Z_\phi^{1/2} \begin{pmatrix} \phi^+ \\ \frac{H+i\chi}{\sqrt{2}} + \frac{v-\delta v}{\sqrt{2}} \end{pmatrix} \equiv Z_\phi^{1/2} \Phi \;.
\tag{8.2}
$$

The renormalized lagrangian for the Yukawa model is therefore

$$
\begin{aligned}
\mathcal{L} =\ & -Z_L \bar{Q}_L \slashed{D} Q_L - Z_R \bar{t}_R \slashed{\partial} t_R - \bar{b}_R \slashed{\partial} b_R - Z_\phi \left(\partial_\mu \Phi\right)^\dagger \left(\partial_\mu \Phi\right) \\
& - g_t Z_g \left(Z_L Z_R Z_\phi\right)^{1/2} \left\{\bar{Q}_L \tilde{\Phi} t_R + \bar{t} \tilde{\Phi}^\star Q_L\right\} \\
& - \left(\mu^2 - \delta\mu^2\right) \Phi^\dagger \Phi - \lambda Z_\lambda \left(\Phi^\dagger \Phi\right)^2
\end{aligned}
\tag{8.3}
$$

where for each Z constant we mean $Z = 1 + \hbar \delta Z$. No wave function renormalization is needed for the b_R, which completely decouples in this reduced model.

Apart the subtraction of poles, the counterterms arising from these redefinitions are set according to the following renormalization conditions:

Tadpole cancellation $\qquad T + iv^3 \left[\dfrac{\delta\mu^2}{v^2} + 2\lambda\dfrac{\delta v}{v} - \lambda Z_\lambda \right] = 0$.

Higgs mass at its classical value $m_H = 2\lambda v^2$

$$\Pi_H \bigg|_{p^2=-m_H^2} + i \left[\delta\mu^2 - 3v^2\lambda\delta Z_\lambda + 6\lambda v\delta v \right] = 0 \tag{8.5}$$

Higgs Propagator residue set to 1. In order to preserve the explicit invariance of the action only one of the residues of the Higgs fields can be fixed; choosing the physical one, the condition is

$$\frac{\partial}{\partial p^2} \Pi_H \bigg|_{p^2=-m_H^2} - i\delta Z_\phi = 0 \ . \tag{8.6}$$

Masses of χ, $\phi^\pm$. In this restricted model we forget masses proportional to the gauge couplings, when compared to the m_t, m_H, so we set to zero the masses of the unphysical degrees of freedom.

$$\Pi_{\chi|\phi^\pm} \bigg|_{p^2=0} + i \left[\delta\mu^2 - v^2\lambda\delta Z_\lambda + 2\lambda v\delta v \right] = 0 \ ; \tag{8.7}$$

it is crucial that the two self energies are equal at $p^2 = 0$. As one may expect this condition coalesces with the one on the tadpole and no new constraint arises.

Bottom propagator . In this model it is easy to see that no mass is generated for the b quark, so the only condition we impose is on the residue

$$\begin{aligned} \Pi_b &= \Sigma_b^L \slashed{p}^L \\ \Sigma_b^L + \delta Z_L &= 0 \end{aligned} \tag{8.8}$$

Top Mass . The perturbative expansion generates a non zero axial term for the top self energy, which can be set to zero at the physical pole: given the expression

$$\Pi_t = \Sigma_V \slashed{p} + \Sigma_A \slashed{p}\gamma_5 + im_t\Sigma_s \tag{8.9}$$

one can impose

$$\Sigma_A \bigg|_{p^2=-m_t^2} + \frac{1}{2}\left(\delta Z_L - \delta Z_R \right) = 0 \ . \tag{8.10}$$

We fix the mass at the tree level value with the condition

$$\left\{\not{p}\left[\Sigma_V + \frac{1}{2}\left(\delta Z_L + \delta Z_R\right)\right] + im_t\left[\Sigma_S - \frac{\delta m_t}{m_t}\right]\right\}\bigg|_{\not{p}=im_t} = 0 \ , \tag{8.11}$$

where δm_t is only a shortcut for

$$\delta m_t = \delta Z_g + \frac{1}{2}\left(\delta Z_L + \delta Z_R + \delta Z_\phi\right) - \frac{\delta v}{v} \ ; \tag{8.12}$$

it follows that

$$\left(\Sigma_V + \Sigma_S\right)\bigg|_{p^2=-m_t^2} - \left(\delta Z_g + \frac{1}{2}\delta Z_\phi\right) + \frac{\delta v}{v} = 0 \ . \tag{8.13}$$

Decay constant G_μ . The value of the μ decay constant cannot be of course obtained by this model: we may use an indirect argument, noting that contributions to the process $\mu \to e\,\bar{\nu}_e\,\nu_\mu$ arise from corrections on the line of the $\phi^\pm$ exchanged: these corrections, together with the mass and wave-functions renormalizations, give rise to a non zero residue which can be incorporated in the definition of the physical coupling.

In short, if

$$\Pi_{\phi^\pm}\left(p^2\right) - \Pi_{\phi^\pm}\left(0\right) = ip^2\Sigma_{\phi^\pm} \tag{8.14}$$

is the $\phi^\pm$ self energy after mass renormalization, then the propagator has the residue

$$R_{\phi^\pm} = \frac{1}{Z_\phi}\left(1 + \Sigma_{\phi^\pm}\bigg|_{p^2=0}\right) \ ; \tag{8.15}$$

again terms proportional to the gauge coupling g have been omitted.

The resulting correction on G_μ is given by

$$G_\mu^{1\ \text{loop}} = G_\mu^{\text{tree}}\left(1 + \Sigma_{\phi^\pm}\bigg|_{p^2=0} - \delta Z_\phi\right) \tag{8.16}$$

Now the tree level relation between G_μ and the other couplings is

$$\frac{G_\mu}{\sqrt{2}} = \frac{1}{2v^2} \tag{8.17}$$

so we can proceed by fixing also the residue of the $\phi^\pm$ propagator to 1 and setting to zero the correction to v, thus preserving the relation 8.17.

It is to be stressed that the only physical inputs are m_t, G_μ, m_H. The other conditions like the ones on the propagator residues are just conventional, to simplify the expression of amplitudes.
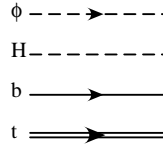
Figure 8.1: Legenda for fields in Feynman Diagrams
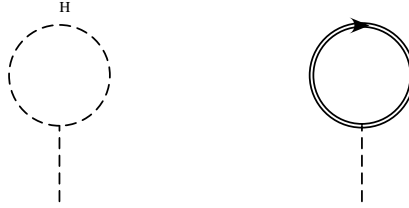

Figure 8.2: Higgs "tadpole" diagrams

8.1.1 The renormalization constants

Here we list the one-loop computations needed to set up the renormalization framework up to two-loops.
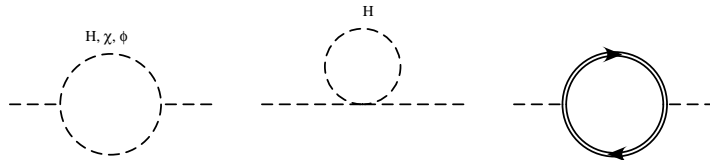
The abbreviation

$$\Delta = \frac{1}{\varepsilon} - \gamma_E + \log 4\pi \tag{8.18}$$

is used, where $\varepsilon = (4 - d)/2$, and the fields are addressed according to the legenda in Fig. 8.1.

Tadpole (Fig. 8.2)

$$\frac{T}{v} = i \left\{ 3m_H^2 \frac{\lambda}{(4\pi)^2} \left[\Delta + 1 - \log \frac{m_H^2}{\mu^2} \right] - 2m_t^2 \frac{g_t^2}{(4\pi)^2} N_c \left[\Delta + 1 - \log \frac{m_t^2}{\mu^2} \right] \right\} \tag{8.19}$$

where N_c is the number of colors.
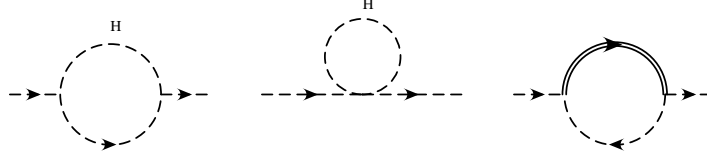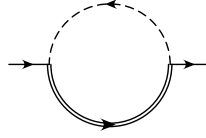

Figure 8.3: Higgs or χ self energy

Figure 8.4: ϕ self energy

Figure 8.5: Bottom self energy

Higgs self energy (Fig. 8.3)

$$
\begin{aligned}
\Pi_H\left(p^2\right) &= i\left(p^2+m_H^2\right)\left[\frac{\lambda}{(4\pi)^2}\left(12-2\pi\sqrt{3}\right)-\frac{g_t^2}{(4\pi)^2}N_c\left(\Delta+1-\log\frac{m_H^2}{\mu^2}\right)\right]+ \\
&+ im_H^2\left[\frac{\lambda}{(4\pi)^2}\left(\Delta+27-3\pi\sqrt{3}-15\log\frac{m_H^2}{\mu^2}\right)\right]+O\left(p^2+m_H^2\right)^2
\end{aligned}
\tag{8.20}
$$

χ, $\phi^\pm$ **self energy** (Fig. 8.3, 8.4)

$$
\begin{aligned}
\Pi_\chi\left(p^2\right) &= ip^2\left[-\frac{\lambda}{(4\pi)^2}-\frac{g_t^2}{(4\pi)^2}N_c\left(\Delta-\log\frac{m_H^2}{\mu^2}\right)\right]+ \\
&+ im_H^2\left[3\frac{\lambda}{(4\pi)^2}\left(\Delta+1-\log\frac{m_H^2}{\mu^2}\right)+O\left(\frac{m_t^2}{m_H^2}\right)\right]+O\left(p^2\right)^2\ , \\
\Pi_{\phi^\pm} &= ip^2\left[-\frac{\lambda}{(4\pi)^2}-\frac{g_t^2}{(4\pi)^2}N_c\left(\Delta+\frac{1}{2}-\log\frac{m_t^2}{\mu^2}\right)\right]+ \\
&+ im_H^2\left[3\frac{\lambda}{(4\pi)^2}\left(\Delta+1-\log\frac{m_H^2}{\mu^2}\right)+O\left(\frac{m_t^2}{m_H^2}\right)\right]+O\left(p^2\right)^2
\end{aligned}
\tag{8.21}
$$

where terms proportional to T/v have bee omitted.
b **self energy** (Fig. 8.5)

$$
\begin{aligned}
\Pi_b &= \Sigma_L^b\slashed{p}P_L\ , \\
\left.\Sigma_L^b\right|_{p^2\simeq0} &= \frac{g_t^2}{(4\pi)^2}\left[\frac{1}{2}\Delta+\frac{3}{4}-\frac{1}{2}\log\frac{m_t^2}{\mu^2}\right]
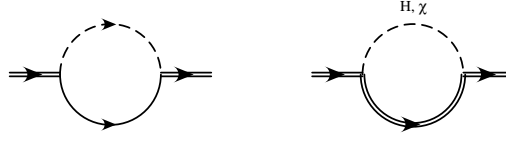\end{aligned}
\tag{8.22}
$$

Figure 8.6: Top self energy

t **self energy** (Fig. 8.6)

$$
\begin{aligned}
\Pi_t &= \Sigma_V \slashed{p} + \Sigma_A \slashed{p}\gamma_5 + im_t\Sigma_S \ , \\
\Sigma_V\Big|_{p^2=-m_t^2} &= \frac{g_t^2}{(4\pi)^2}\left[\frac{3}{4}\Delta + \frac{11}{8} + \frac{1}{4}\log r - \frac{3}{4}\log\frac{m_t^2}{\mu^2}\right] \ , \\
\Sigma_A\Big|_{p^2=-m_t^2} &= \frac{g_t^2}{(4\pi)^2}\left[-\frac{1}{4}\Delta - \frac{1}{2} + \frac{1}{4}\log\frac{m_t^2}{\mu^2}\right] \ , \\
\Sigma_S\Big|_{p^2=-m_t^2} &= \frac{g_t^2}{(4\pi)^2}\left[-\frac{1}{2} + \frac{1}{2}\log\frac{m_t^2}{m_H^2}\right] \ .
\end{aligned}
\tag{8.23}
$$

The renormalization constants are then

$$
\begin{aligned}
\delta Z_L &= \frac{g_t^2}{(4\pi)^2}\left[-\frac{3}{4} - \frac{1}{2}\Delta + \frac{1}{2}\log\frac{m_t^2}{\mu^2}\right] \ , \\
\delta Z_R &= \frac{g_t^2}{(4\pi)^2}\left[-\frac{7}{4} - \Delta + \log\frac{m_t^2}{\mu^2}\right] \ , \\
\frac{\delta m_t}{m_t} &= \frac{g_t^2}{(4\pi)^2}\left[-\frac{3}{8} + \frac{3}{4}\log r\right] \ .
\end{aligned}
\tag{8.24}
$$

The δv constant is set equal to zero: it follows that the counterterm $\delta m_t/m_t$ defines also the finite vertex counterterms, in accordance with Eq. 8.3 and Eq. 8.12.

8.2   One loop order

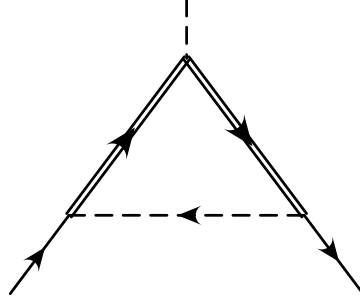The expression for the $\phi^{\pm}$, χ self energies in Eq. 8.21 together with Eq. 7.15 allows to obtain the ρ parameter at one loop order

$$
\rho \equiv 1 + \delta\rho_1 = 1 + \frac{N_c}{2}\frac{g_t^2}{(4\pi)^2}
\tag{8.25}
$$

where $N_c = 3$ is the number of colors. The tree level relations

$$
g_t = \frac{\sqrt{2}\,m_t}{v} \qquad \frac{G_\mu}{\sqrt{2}} = \frac{1}{2\,v^2}
\tag{8.26}
$$

Figure 8.7: One loop $\chi b\bar{b}$ vertex

can be used to write the result in terms of the physical inputs,

$$\rho = 1 + N_c \frac{G_\mu \, m_t^2}{8\pi^2 \sqrt{2}} \ . \tag{8.27}$$

The computation of the single Feynman diagram contributing to the $\chi\bar{b}b$ vertex, in Fig. 8.7 allows to obtain

$$\langle \chi b\bar{b} \rangle = \frac{1}{v} \left(-i\slashed{k}^L \right) \frac{g_t^2}{(4\pi)^2} \tag{8.28}$$

and using Eq. 7.17 one obtains the quantum contribution to the amplitude

$$\langle Z_\mu b\bar{b} \rangle_{\text{quantum}} = -\frac{g}{2\,c} \frac{g_t^2}{(4\pi)^2} \gamma_\mu^L \ . \tag{8.29}$$

Before expressing also this result in terms of physical inputs, let us consider the two-loop contributions.

8.3 Two-loop order: $Z \to b\bar{b}$

The Feynman diagrams contributing to the $\langle \chi b\bar{b} \rangle$ are listed in Fig. 8.8. The results are normalized to the one loop calculation, and an overall factor

$$g_0 = -\frac{i\slashed{q}^L}{v} \frac{g_t^2}{(4\pi)^2} \tag{8.30}$$

is understood, where q is the momentum entering the χ vertex.

Only two mass scales are present, m_t and m_H, and apart an overall scale factor and the renormalization logarithms they appear only in the combination

$$r = \frac{m_t^2}{m_H^2} \ . \tag{8.31}$$

The "technology" for the computation of this diagrams is well known, and we have obtained analytical results for arbitrary values of the ratio r; for compactness, the results graph by graph will be reported in the asymptotic small r limit, and when appropriate in the large r limit, but the full analytical results will be also discussed in the following.
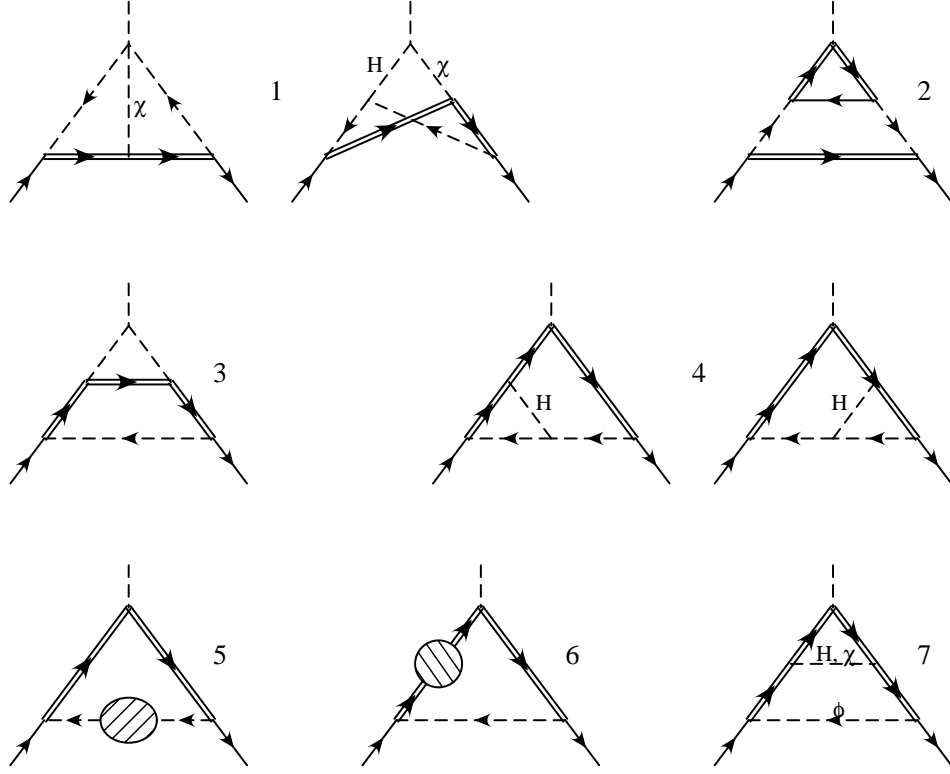
Figure 8.8: Two loop contributions to the $\chi b\bar{b}$ vertex

Non divergent diagrams

We recall that the results are normalized to g_0:

Diagrams $1 - 4, 7$

$$
\begin{aligned}
g_1 &= \frac{g_t^2}{(4\pi)^2} \left[\frac{1}{16} + \frac{3}{8} \log r \right] \\
g_2 &= \frac{g_t^2}{(4\pi)^2} N_c \left[-\frac{1}{2} \right] \\
g_3 &= \frac{g_t^2}{(4\pi)^2} \left[\frac{27}{32} - \frac{\pi^2}{12} - \frac{11}{16} \log r + \frac{1}{16} \log^2 r + O\left(r \log r\right) \right] \\
g_4 &= \frac{g_t^2}{(4\pi)^2} \left[-\frac{1}{16} + \frac{\pi^2}{12} - \frac{7}{8} \log r + \frac{1}{4} \log^2 r + O\left(r \log r\right) \right] \\
g_7 &= \frac{g_t^2}{(4\pi)^2} \left[\frac{1}{4} + \frac{\pi^2}{24} + \frac{1}{2} \log r + O\left(r \log r\right) \right]
\end{aligned}
\tag{8.32}
$$

Divergent diagrams

It is convenient to impose some of the renormalization conditions diagram by diagram: in particular, the mass renormalization of the $\phi^\pm$ fields entering the diagram 5. We list also the

counterterm for the t self energy entering graph 6.

Diagram 5 The result after mass subtraction only is

$$
g_5^{\text{mass subtracted}} = \frac{g_t^2}{(4\pi)^2} \left[\frac{N_c}{2} - \frac{1}{72} - \frac{1}{12} \log r + O\left(r \log r\right) \right] +
$$
$$
+ \left[-\frac{\lambda}{(4\pi)^2} - \frac{g_t^2}{(4\pi)^2} N_c \left(\Delta + \frac{1}{2} - \log \frac{m_t^2}{\mu^2} \right) - \delta Z_\phi \right] \tag{8.33}
$$

where as usual $\Delta = \frac{1}{\varepsilon} - \gamma_E + \log 4\pi$. The wave function renormalization δZ_ϕ coming from the condition on the Higgs propagator residue does not cancel the second line in Eq. 8.33 completely: it leaves a finite result which will cancel against the redefinition of G_μ.

Diagram 6 Here we show separately the graph and the counterterm induced by mass and wave function renormalization.

$$
g_6^{\text{unsubtracted}} = \frac{g_t^2}{(4\pi)^2} \left[-\frac{1}{\varepsilon} - \frac{1}{2} + 2\gamma_E + \frac{\pi^2}{24} + \frac{1}{4} \log r + 2 \log \frac{m_t^2}{4\pi\mu^2} \right] . \tag{8.34}
$$

The counterterm results from the insertion of the following self energy correction

$$
\text{ct} = \not{p}^L \left[-\frac{3}{4} - \frac{1}{2}\Delta + \frac{1}{2} \log \frac{m_t^2}{\mu^2} \right]
$$
$$
+ \not{p}^R \left[-\frac{7}{4} - \Delta + \log \frac{m_t^2}{\mu^2} \right]
$$
$$
+ i m_t \left[\frac{3}{8} - \frac{3}{4} \log r \right] ; \tag{8.35}
$$

when inserted in the one loop graph the counterterm results

$$
g_6^{\text{counterterm}} = \frac{g_t^2}{(4\pi)^2} \left[\frac{1}{\varepsilon} - 2\gamma_E + \frac{13}{8} - \frac{3}{4} \log r - 2 \log \frac{m_t^2}{\mu^2} \right] . \tag{8.36}
$$

The total result is therefore

$$
g_6 = \frac{g_t^2}{(4\pi)^2} \left[\frac{9}{8} + \frac{\pi^2}{24} - \frac{1}{2} \log r \right] . \tag{8.37}
$$

Other finite counterterms

The counterterms for the t and $\phi^\pm$ self energies have been sufficient to obtain a finite result: it only remains to include finite counterterms resulting from the Yukawa couplings. Having fixed δv to zero, as a consequence of the definition of G_μ, the counterterms for the vertices $\langle \chi t\bar{t} \rangle, \langle \phi^+ b\bar{t} \rangle, \dots$ are connected to the mass counterterm of the t: each of these vertices is multiplied by the factor

$$
\left[1 + \delta Z_g + \frac{1}{2} \left(\delta Z_L + \delta Z_R + \delta Z_\phi \right) \right] = \left[1 + \frac{\delta m_t}{m_t} \right] ; \tag{8.38}
$$

hence one has a contribution, proportional to the one loop result, of the form

$$g_c = 3 \frac{\delta m_t}{m_t} = \frac{g_t^2}{(4\pi)^2} \left[-\frac{9}{8} + \frac{9}{4} \log(r) \right] \ . \tag{8.39}$$

8.3.1 Cumulative result

It is convenient to define the dimensionless expansion parameter

$$x = \frac{G_\mu m_t^2}{8\pi^2 \sqrt{2}} \tag{8.40}$$

and write the $\langle \chi \, b\bar{b} \rangle$ in the form

$$\langle \chi b\bar{b} \rangle = \frac{i \slashed{q}^L}{v} \tau \tag{8.41}$$

where

$$\tau = -2x \left(1 + x \, \tau^{(2)} \right) \ . \tag{8.42}$$

In the small r, $(m_H \gg m_t)$ limit we have

$$\tau^{(2)} = \frac{311 + 24\pi^2 + 282 \log r + 90 \log^2 r}{144} \ , \tag{8.43}$$

while in the large r limit

$$\tau^{(2)} = \frac{27 - \pi^2}{3} \ . \tag{8.44}$$

To understand the meaning of the τ parameter, note that the $Z \to b\bar{b}$ amplitude can be written as follows

$$
\begin{aligned}
A \left(Z_\mu \to b\bar{b} \right) &= \frac{e}{4\,s\,c} \left(g_v \gamma_\mu + g_a \gamma_\mu \gamma_5 \right) \\
g_v &= 1 - \frac{4}{3} s^2 + \tau \\
g_a &= 1 + \tau \ .
\end{aligned}
\tag{8.45}
$$

We recall that $s = \sin \theta_W$ is not in our renormalization scheme a physical input and it is eliminated through the relation[1] in Eq. 6.11

$$\frac{\hat{\alpha}_{\text{em}}}{s^2 \, c^2} = \frac{1}{\pi} \left(\sqrt{2} G_\mu M_Z^2 \rho \right) \ . \tag{8.47}$$

---

[1] s^2 should not be confused with the definition of

$$\sin_W^2 = 1 - \frac{M_W^2}{M_Z^2} \tag{8.46}$$

The decay width is therefore given by

$$\Gamma\left(Z \to b\bar{b}\right) = \frac{N_c}{24\pi}\frac{G_\mu}{\sqrt{2}}M_Z^3\rho\left(|g_v|^2 + |g_a|^2\right) \ ,$$

(8.48)

where contributions proportional to m_b/M_W have been dropped. They will be reinserted when dealing with numerical results and physical observables. Here it was interesting to show the dependence on the ρ parameter, coming from the renormalization conditions. This dependence is conventionally called *oblique*, as it is common to all the physical observable as a consequence of the theoretical framework, in opposition to the *non oblique* corrections, like τ itself.

We require now the full two-loop heavy t expression for the ρ parameter.

8.4 Two loop order: ρ

Let us recall that defining the renormalized model in Sec. 7.2.1 we have fixed G_μ by adsorbing in the physical value the self energy contribution from $\phi^\pm$: we have used the relation

$$G_\mu^{\text{phys}} = G_\mu^{\text{bare}}\left(1 + \Sigma_{\phi^\pm}^{(1)}\right) \ .$$

(8.49)

Explicitly, in the formula in Eq. 7.14

$$\delta\rho = (1 + \Sigma_{\phi^\pm}^{(1)})\left(\Sigma_\chi^{(1)} - \Sigma_{\phi^\pm}^{(1)}\right) + \left(\Sigma_\chi^{(1)} - \Sigma_{\phi^\pm}^{(1)}\right)^2 + \left(\Sigma_\chi^{(2)} - \Sigma_{\phi^\pm}^{(2)}\right)$$

(8.50)

the prefactor in the first term was adsorbed in the definition of G_μ: and $\delta\rho$ results calculable in terms of finite measurable quantities[2].

The two loop graphs needed to compute the difference of self energies of the unphysical Higgs fields at two loop level are listed in Fig. 8.9, and the individual contributions are addressed as follows

$$\delta\rho_\# = \Sigma_\chi - \Sigma_\phi$$

(8.51)

where $\#$ numbers the graph, starting from 2 because the number 1 was used for the one-loop result. Again the detailed results are given in the small r limit, but the full analytical formulas have been used in the numerical evaluation.

Diagram 2 There is no contribution to ϕ self energy, while the χ self energy gives

$$\delta\rho_2 = \left(\frac{g_t^2}{(4\pi)^2}\right)^2 N_c\left[-\frac{1}{2} + \frac{\pi^2}{6} + \frac{1}{2}\left(\log r + \log^2 r\right)\right]$$

(8.52)

---

[2]We include in the definition of Σ_f the subtraction of counterterms, both infinite and finite parts, as in the case of the $Z \to b\bar{b}$ process
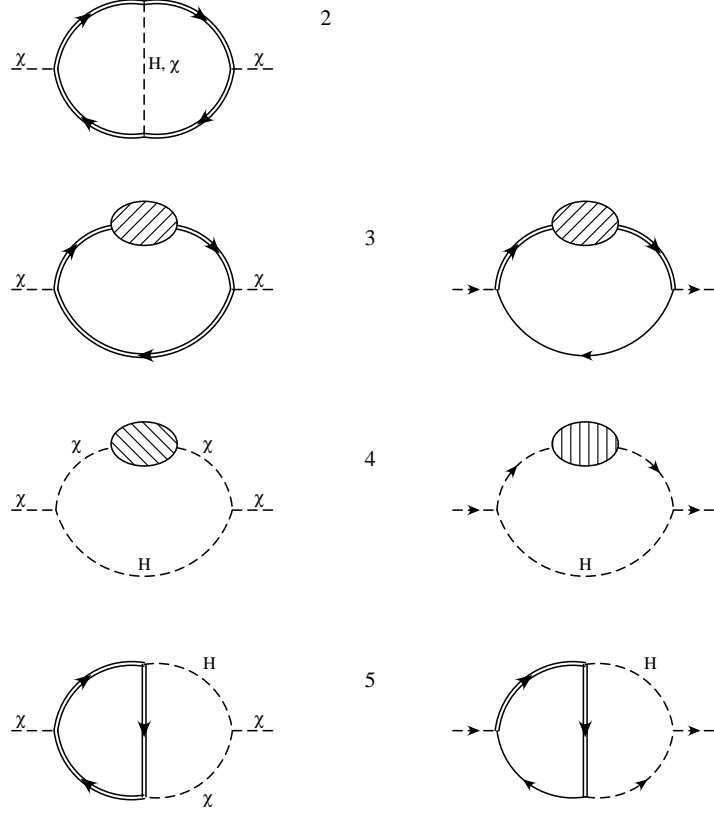
Figure 8.9: Two loop graphs for the computation of ρ parameter

Diagram 3 In the difference of the self energies the square pole cancels

$$\delta\rho_3^{\text{bare}} =$$
$$\left(\frac{g_t^2}{(4\pi)^2}\right)^2 N_c \left[\frac{25}{16} - \frac{\pi^2}{4} + \frac{1}{8}\log r - \frac{1}{8}\log^2 r - \frac{3}{2}\left(\frac{1}{2\varepsilon} - \gamma_E - \log\frac{m_t^2}{4\pi\mu^2}\right)\right] \ .$$
(8.53)

The t self energy counterterm gives the contribution

$$\delta\rho_3^{\text{count.}} = \left(\frac{g_t^2}{(4\pi)^2}\right)^2 N_c \left[\frac{7}{8} + \frac{3}{2}\left(\frac{1}{2\varepsilon} - \gamma_E - \log\frac{m_t^2}{4\pi\mu^2}\right)\right]$$
(8.54)

and the sum is finite as expected

$$\delta\rho_3 = \left(\frac{g_t^2}{(4\pi)^2}\right)^2 N_c \left[\frac{39}{16} - \frac{\pi^2}{4} + \frac{1}{8}\log r - \frac{1}{8}\log^2 r\right]$$
(8.55)

Diagram 4

$$\delta\rho_4 \ = \ \left(\frac{g_t^2}{(4\pi)^2}\right)^2 N_c \left(\frac{3}{2} + \frac{\pi^2}{6} + \log r\right)$$
(8.56)

Diagram 5

$$\delta\rho_5 = \left(\frac{g_t^2}{(4\pi)^2}\right)^2 N_c \left(\frac{\pi^2}{6} + \log r\right)$$ (8.57)

Finite counterterms As in the case of the $Z \to b\bar{b}$ process, we have to include finite vertex counterterms, in analogy with Eq. 8.39

$$
\begin{aligned}
\delta\rho_6 &= \frac{2\delta m_t}{m_t}\delta\rho_1 \\
&= 2\frac{g_t^2}{(4\pi)^2}\left(-\frac{3}{8} + \frac{3}{4}\log r\right)\delta\rho_1 \\
&= \left(\frac{g_t^2}{(4\pi)^2}\right)^2\left(-\frac{3}{8} + \frac{3}{4}\log r\right) \ .
\end{aligned}
$$ (8.58)

The two loop contribution is therefore

$$
\begin{aligned}
\Sigma_\chi^{(2)} - \Sigma_{\phi^\pm}^{(2)} &= \sum_{n=2}^{6}\delta\rho_n \\
&= \left(\frac{g_t^2}{(4\pi)^2}\right)^2 N_c\frac{49 + 4\pi^2 + 54\log r + 6\log^2 r}{16} \ .
\end{aligned}
$$ (8.59)

We already know that

$$\Sigma_\chi^{(1)} - \Sigma_{\phi^\pm}^{(1)} = \delta\rho_1 = \frac{N_c}{2}\frac{g_t^2}{(4\pi)^2} \ ,$$ (8.60)

and combining the results we obtain

$$
\begin{aligned}
\delta\rho &= \hbar\delta\rho_1 + \hbar^2\left[\delta\rho_1^2 + \sum_{n=2}^{6}\delta\rho_n\right] \\
&= \frac{N_c}{2}\frac{g_t^2}{(4\pi)^2}\left[1 + \frac{g_t^2}{8(4\pi)^2}\left(49 + 4N_c + 4\pi^2 + 54\log r + 6\log^2 r\right)\right] \\
&= N_c x\left[1 + x\frac{49 + 4N_c + 4\pi^2 + 54\log r + 6\log^2 r}{4}\right] \ .
\end{aligned}
$$ (8.61)

This result confirms a preceding computation [36], performed in the $m_H = 0$ limit.

8.5   Summary of the results

We have already discussed in the previous chapter that it is common to present the ρ parameter as

$$\frac{1}{\rho} - 1 = -N_c x \left(1 + x\rho^{(2)}\right) \qquad\qquad x = \frac{G_\mu m_t^2}{8\pi^2\sqrt{2}} \; ; \tag{8.62}$$

recalling the definition of τ

$$\tau = -2x \left(1 + x\tau^{(2)}\right) \;, \tag{8.63}$$

the asymptotic expansions of the two loop contributions $\rho^{(2)}$, $\tau^{(2)}$ are

(i) for $r = (m_t/m_H)^2 \ll 1$

$$\begin{aligned}
\rho^{(2)} &= \frac{49}{4} + \pi^2 + \frac{27}{2}\log r + \frac{3}{2}\log^2 r + \\
&\quad + \frac{r}{3}\left(2 - 12\pi^2 + 12\log r - 27\log^2 r\right) + \\
&\quad + \frac{r^2}{48}\left(1613 - 240\pi^2 - 1500\log r - 720\log^2 r\right) + O\left(r^3\right)
\end{aligned} \tag{8.64}$$

$$\begin{aligned}
\tau^{(2)} &= \frac{1}{144}\Big[311 + 24\pi^2 + 282\log r + 90\log^2 r + \\
&\quad - 4r\left(40 + 6\pi^2 + 15\log r + 18\log^2 r\right) + \\
&\quad + \frac{3r^2}{100}\left(24209 - 6000\pi^2 - 45420\log r - 18000\log^2 r\right)\Big] + O\left(r^3\right)
\end{aligned}$$

where some sub-asymptotic contributions are included in order to improve the convergence to the exact formula.

(ii) for $r \gg 1$

$$\begin{aligned}
\rho^{(2)} &= 19 - 2\pi^2 - \frac{4\pi}{\sqrt{r}} + O\left(\frac{\log r}{r}\right) \\
\tau^{(2)} &= \frac{27 - \pi^2}{3} + \frac{4\pi}{\sqrt{r}} + O\left(\frac{\log r}{r}\right) \;.
\end{aligned} \tag{8.65}$$

The $r \to \infty$ limit of eq. (8.66) confirms the result given in ref. [36].

    These asymptotic expressions are not sufficient for an accurate evaluation of the second-order coefficients $\rho^{(2)}$ and $\tau^{(2)}$ for $m_t \simeq m_H$.

    In Tab. 8.5 the numerical values of $\rho^{(2)}$ and $\tau^{(2)}$ as functions of m_H/m_t are therefore given.

    The exact numerical results are compared with the asymptotic expansions in Fig. 8.10: one can see that already for $m_H \geq 2\, m_t$ the asymptotic expansion is fairly accurate.

m_H/m_t	$\rho^{(2)}$	$\tau^{(2)}$	m_H/m_t	$\rho^{(2)}$	$\tau^{(2)}$	m_H/m_t	$\rho^{(2)}$	$\tau^{(2)}$
0.1	-1.82	4.67	1.1	-7.30	1.38	2.1	-9.66	1.37
0.2	-2.70	3.90	1.2	-7.62	1.32	2.2	-9.81	1.42
0.3	-3.46	3.30	1.3	-7.91	1.27	2.3	-9.96	1.47
0.4	-4.13	2.83	1.4	-8.19	1.24	2.4	-10.1	1.53
0.5	-4.72	2.46	1.5	-8.44	1.23	2.5	-10.2	1.60
0.6	-5.25	2.16	1.6	-8.68	1.23	2.6	-10.4	1.66
0.7	-5.74	1.92	1.7	-8.90	1.24	2.7	-10.5	1.73
0.8	-6.18	1.73	1.8	-9.11	1.26	2.8	-10.6	1.80
0.9	-6.58	1.59	1.9	-9.30	1.29	2.9	-10.7	1.87
1.0	-6.95	1.47	2.0	-9.48	1.33	3.0	-10.8	1.95

Table 8.1: Values of $\rho^{(2)}$ and $\tau^{(2)}$ as functions of m_H/m_t

These results have been confirmed by other authors [96, 106]; they have also been able to fully exploit algebraic identities of the Li$_2$ functions in order to obtain compact analytical results[3].

All in all, the corrections to the one-loop computation are small, a fact that justifies a posteriori the perturbative expansion, even for rather large values of the g_t coupling. Indeed, for $m_t \simeq 180\,\text{GeV}$ one has $g_t \simeq 1.0$, and the typical expansion parameter

$$\alpha_t = \frac{g_t^2}{4\pi} = \frac{G_\mu m_t^2}{\pi\sqrt{2}} \tag{8.66}$$

has the value of $\alpha_t \simeq 0.08$.

The good convergence of the perturbative expansion is shown in Fig. (8.11), where the first and second order expressions for $\frac{1}{\rho} - 1$ and τ are compared, for different values of m_H in the $m_t = 150$–$250\,\text{GeV}$ interval: the most affected quantity is the ρ parameter, and even for $m_t = 250\,\text{GeV}$ one has less than a 10% correction of the one-loop contributions.

8.5.1 Physical observables

Let us consider the influence of these heavy top effects on the various LEP observables.

Observe first that we could also have defined the ρ and τ parameter from the $Z \to \mu^+\mu^-$ and $Z \to b\bar{b}$ width: reinserting the phase space factors, one has

$$\Gamma\left(Z \to \mu^+\mu^-\right) = \rho\frac{G_\mu M_Z^3}{8\pi\sqrt{2}}\left(g_{\mu V}^2 + g_{\mu A}^2\right)$$

$$\Gamma\left(Z \to b\bar{b}\right) = \rho\frac{G_\mu M_Z^3}{8\pi\sqrt{2}}\sqrt{1 - \frac{4m_b^2}{M_Z^2}}\left[\left(g_{bB}^2 + g_{bA}^2\right)\left(1 + 2\frac{m_b^2}{M_Z^2}\right) - 6g_{bA}^2\frac{m_b^2}{M_Z^2}\right]$$

$$(8.67)$$

$$g_{\mu V} = 1 - 4s^2 \quad , \qquad g_{\mu A} = 1$$

$$(8.68)$$

$$g_{bV} = 1 - \frac{4}{3}s^2 + \tau \quad , \qquad g_{bA} = 1 + \tau$$

---

[3]In [106] an explicit check of the Ward-Takahashi identities used by us has been given

where

$$s^2 = \frac{1}{2} \left(1 - \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}G_\mu M_Z^2 \rho}} \right) \ . \tag{8.69}$$

Formulas are greatly simplified by the possibility of writing the heavy top corrections as a redefinition of the Weinberg angle, through the relation in Eq. 8.69, and of the axial and vector couplings of the Z field with the b field. The results can then be included in the so called "improved Born" approximation for the LEP observables, together with part of the QCD and "pure QED" effects.

For instance, considering the effect on the $e^+e^- \to \mu^+\mu^-$ forward backward asymmetry

$$A_{FB}^\mu = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} \tag{8.70}$$

where

$$\sigma_{F(B)} = 2\pi \int_{0(-1)}^{1(0)} d\left(\cos\theta\right) \frac{d\sigma}{d\Omega}, \tag{8.71}$$

in this limit it can be simply written as

$$A_{FB}^\mu = \frac{3g_{\mu V}^2 g_{\mu A}^2}{\left(g_{\mu V}^2 + g_{\mu A}^2\right)^2} \tag{8.72}$$

and depends on ρ through the effective Weinberg angle s.

But to illustrate the effect of this two-loop computation on the electroweak precision observables currently measured at LEP, it is necessary to include also the other relevant corrections: in Figs. 8.12, 8.13, 8.14 we report the Standard Model predictions for the Z leptonic width, $\Gamma\left(Z \to e^+e^-\right)$, the forward-backward $e^+e^- \to \mu^+\mu^-$ asymmetry at the Z pole, A_{FB}^μ (both affected by ρ only) and the total Z width, Γ_Z (affected by ρ and τ).

The graphs are obtained using the ZFITTER code [71]; the dashed lines include the m_t^4 effects *irreducible*, that is, coming from the two-loop computations, for different values of the Higgs mass.

Superimposed is the band of experimental values:

$$
\begin{aligned}
\Gamma_Z^{\text{Tot}} &= 2497.4 \pm 3.8 \text{MeV} \\
\Gamma_Z^l &= 83.96 \pm 0.18 \text{MeV} \\
A_{FB}^\mu &= 0.0170 \pm 0.0016 \ .
\end{aligned}
\tag{8.73}
$$

To judge the importance of the two-loop effect, the dot-dashed curve at $m_H = 2\text{TeV}$ can be compared with the one-loop result, at the same value of the Higgs mass, given by the full line and resulting by setting to zero the second-order coefficients $\rho^{(2)}$ and $\tau^{(2)}$. The $m_H = 2\text{TeV}$ has been chosen because the effect is larger, but unless the top mass is pushed at unreasonably large values, the effect is at most of per mille.

8.5.2   Validity of the gaugeless approximation

Is the gaugeless approximation sensible, for the real world? We have seen that for m_t in the $150 - 200$GeV range, the corresponding range of $g_t \simeq 5.74 \times 10^{-3}\, m_t/\text{GeV}$ is $0.86 - 1.15$, while the gauge couplings are $g = e/\sin\theta_W \simeq 0.64$, $g' = e/\cos\theta_W \simeq 0.34$; alternatively, recall that in the two-loop computations we have systematically neglected M_W, M_Z when compared with m_t, a procedure which cannot be justified on numerical grounds.

It is useful to consider one of the original computations at one loop of the heavy top corrections to the $Z \to b\bar{b}$, performed in [41] without the $M_W \ll m_t$ approximation, in the renormalizable t'Hooft-Feynman gauge. It is possible to separate the contribution of the graphs with the exchange of $W^\pm$ and the $\phi^\pm$ fields and the result is that even for $m_t \geq 2M_W$ the "leading" m_t^2 contribution is only a factor of two greater than the subleading one.

This means that while the large m_t limit is sufficient for an order of magnitude estimate, the actual accuracy in LEP experiments *requires* the inclusion of the one-loop subleading contributions.

At the two loop level no complete computation exists, but other authors [107] have attempted to estimate the subleading contributions by working in a reduced SM model, limiting themselves to an $SU(2)$ gauge group[4]. They find that the leading $O\left(G_\mu^2 m_t^2\right)$ and the subleading $O\left(G_\mu^2 m_t^2 M_Z^2\right)$ corrections are of the same order of magnitude, and of the same sign. Similar results have been obtained in [112], estimating in a particular process (neutrino scattering on a leptonic target) the subleading terms, which are again found to be even larger than the "leading" ones. This results are a clear indication that at the moment the two-loop heavy top effects can be considered as a partial estimation of the theoretical error of the one-loop approximation, and that whenever the two loop heavy-top effects become observable, they will require a much greater theoretical effort to be compared with experiments.

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[4]in other words, QED effects are not present
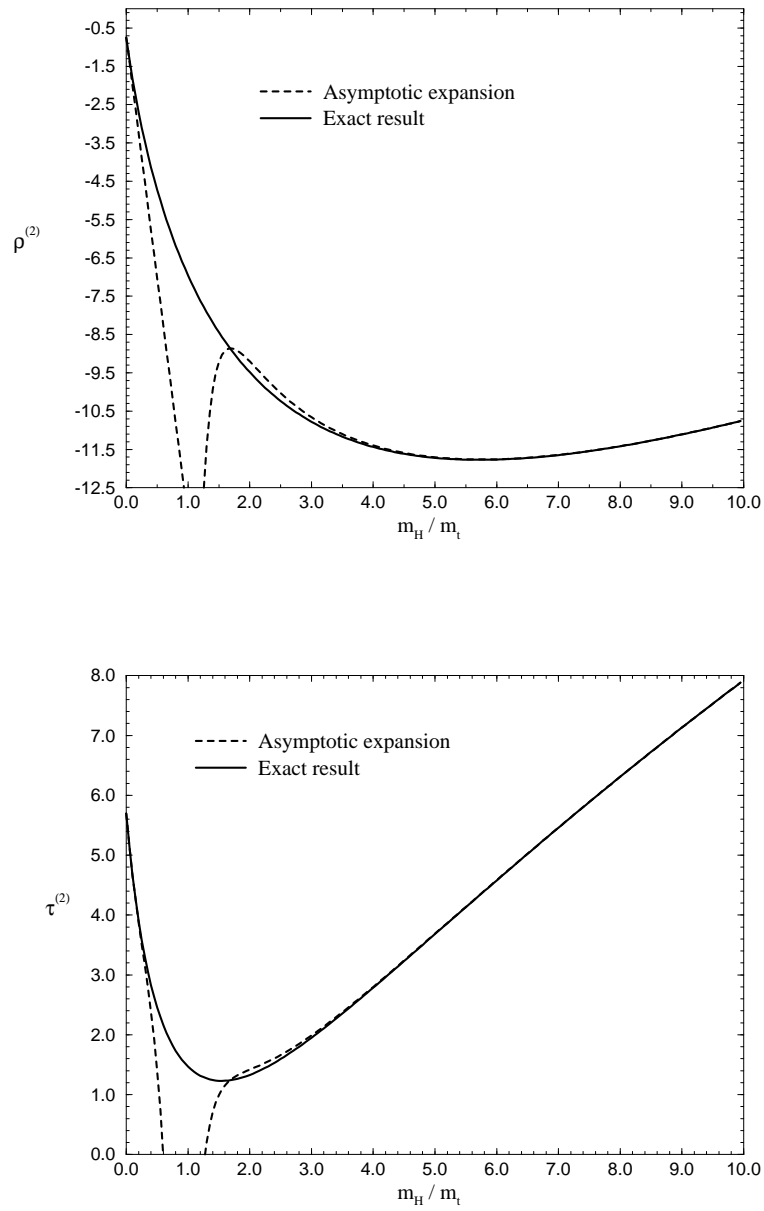
Figure 8.10: Asymptotic expansions and exact results for $\rho^{(2)}$ and $\tau^{(2)}$
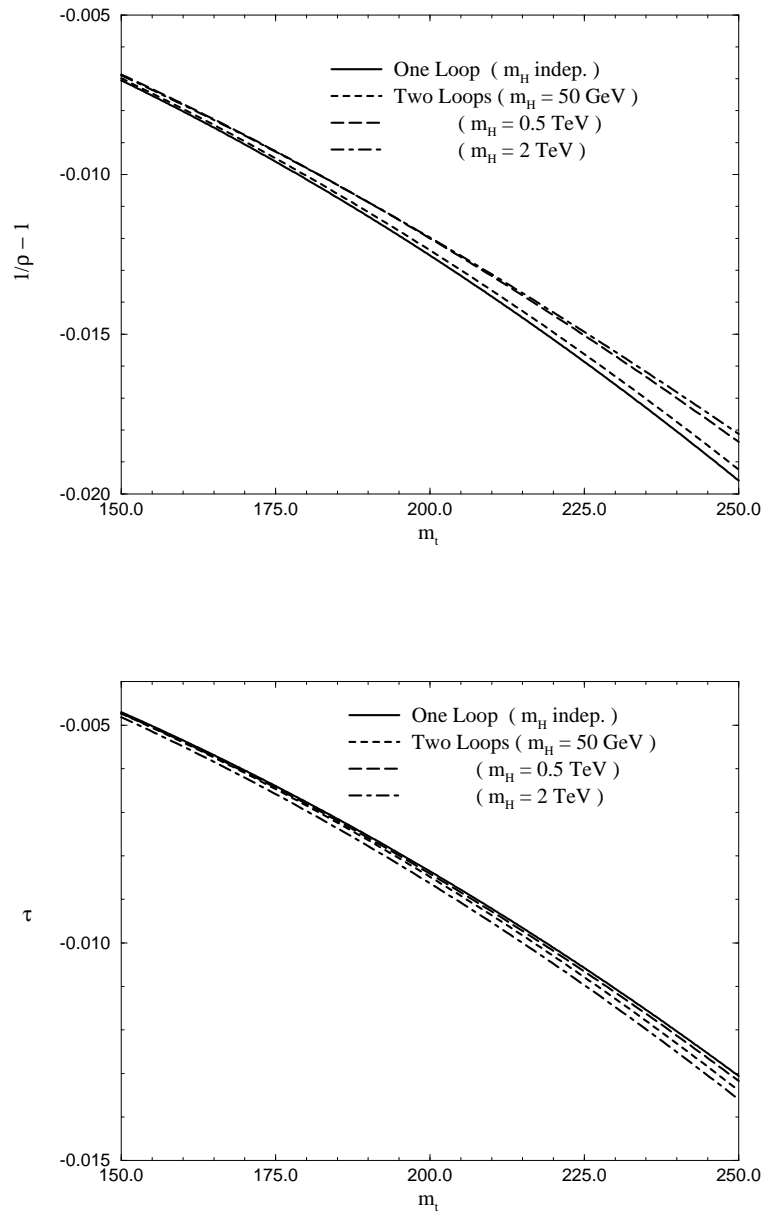
Figure 8.11: Convergence of the perturbative expansion: $1/\rho - 1$ and τ
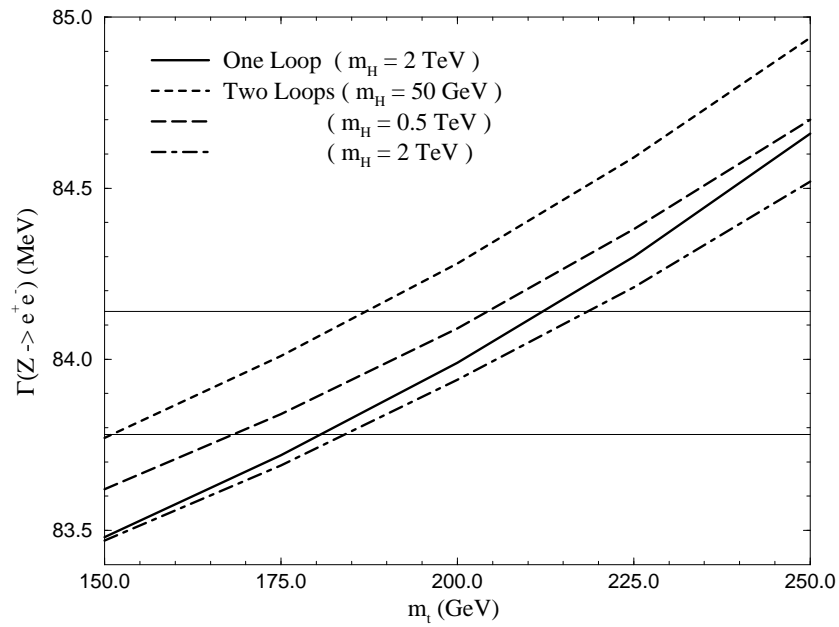
Figure 8.12: Partial width $\Gamma\left(Z \to e^+ e^-\right)$ (ZFITTER)
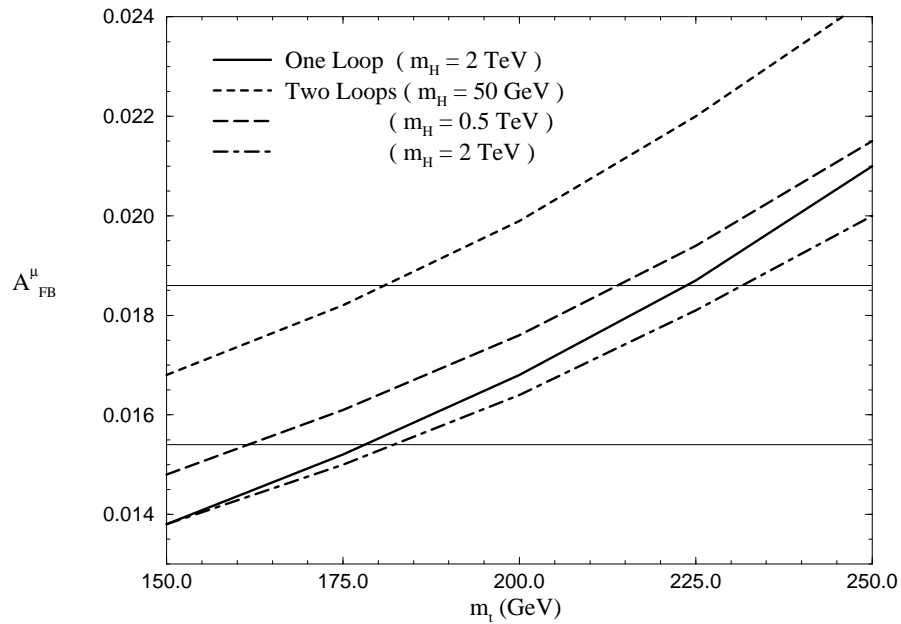
Figure 8.13: Forward-backward asymmetry A_{FB}^{μ} (ZFITTER)
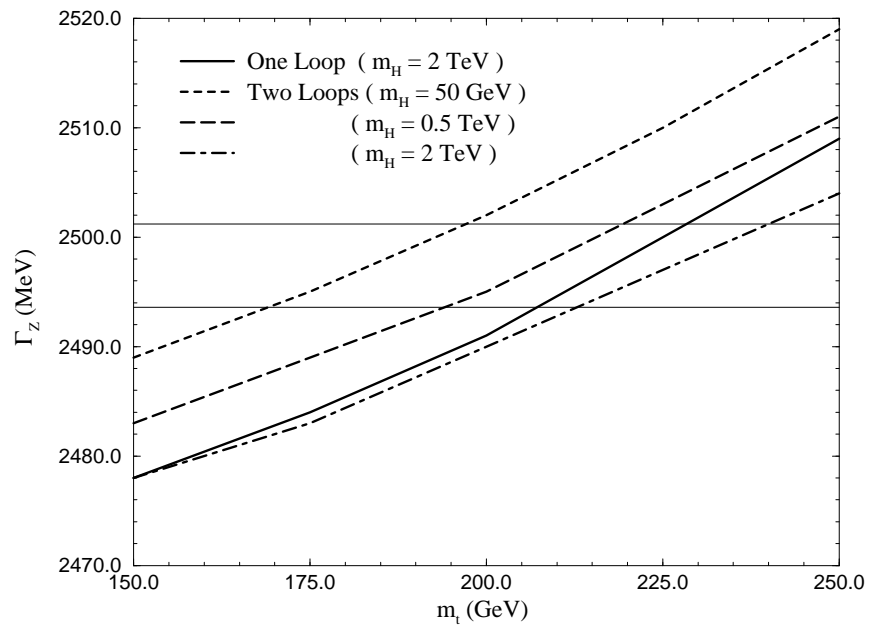
Figure 8.14: Total Z width Γ_Z^{tot} (ZFITTER)

Chapter 9

Conclusions and perspectives

We have presented in this thesis the computation of some radiative corrections to weak decays in the context of the Standard Model. We believe that the processes we have studied, the $b \to s\gamma$ and the $Z \to b\bar{b}$ decays, shall receive further attention in the future, thanks to their sensitivity to New Physics.

For what concerns the $b \to s\gamma$ decay, we have seen that the prediction of the inclusive decay is in good agreement with the experiment, and that this may be regarded not only as a success of the Standard Model but also of perturbative QCD itself: the dramatic enhancement of the rate resulting from the resummation of QCD logarithms is something which is difficult to miss.

The study of the $O\left(m_t^4\right)$ corrections to the $Z \to b\bar{b}$ decay did not result, on the contrary, in a dramatic effect. We have seen that the corrections to the one loop expressions are relatively small, improving our confidence in the phenomenological analysis of the LEP data. Our opinion is that these corrections are not sufficient to account for the bulk of the two loop effects: presently they can be regarded as an estimate of the error implied by the perturbative expansion.

To conclude the work it seems interesting to us to give an example of a phenomenological analysis based on these two independent processes.

9.1   An example: bounds on two Higgs doublets models

We have seen in Chapter 2 how the presence of an extended Higgs sector may affect the prediction for the $b \to s\gamma$ decay rate, in dependence of the mass of the extra charged Higgs particle, $M_{H^\pm}$. It can be easily grasped, on the basis of the discussion in Chapter 6, that the same extended Higgs sector contributes also to the $Z \to b\bar{b}$ process.

We have already discussed how this contributions are parameterized by two other quantities, $M_{H^\pm}$ and $\tan\beta$: a comparison between theory and experiments allows to exclude portions of these parameter space.

The limits coming from the $b \to s\gamma$ process can be defined observing that in Model II the contribution from the extra doublet enhances the rate, for smaller values of $M_{H^\pm}$. We define excluded values of $M_{H^\pm}$, $\tan\beta$ the ones for which the inequality

$$BR^{\mathrm{th}}\left(B \to X_s\gamma\right) - 2 \times \left(\begin{array}{c} \text{theo.} \\ \text{error} \end{array}\right) > BR^{\mathrm{exp}}\left(B \to X_s\gamma\right) + 2 \times \left(\begin{array}{c} \text{exp.} \\ \text{error} \end{array}\right) \qquad (9.1)$$
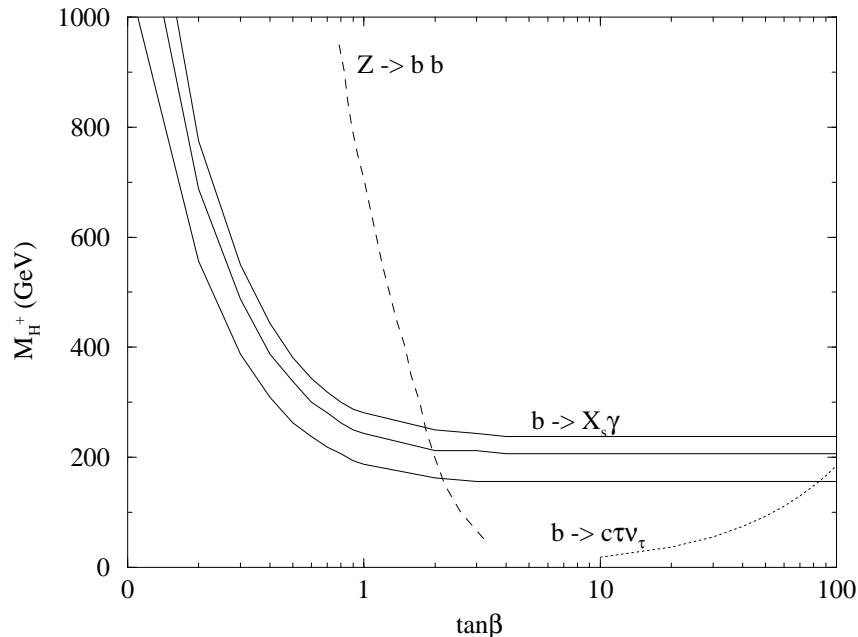
Figure 9.1: Lower limits for the mass of the charged Higgs boson

holds. This relation gives a lower limit on $M_{H^\pm}$, displayed by solid lines in Fig. 9.1, varying m_t around the central CDF value.

We refer to [109] (and references therein) for detailed formulas on the bound from the $Z \to b\bar{b}$ process: in the same figure the dashed lines display the resulting bound for the central CDF m_t mass.

For completeness the lower bound coming from the $b \to c\tau\bar{\nu}_\tau$ [42, 98] is displayed with a dotted line. It is obtained by the simple formula

$$\tan \beta < 0.54 \frac{M_{H^\pm}}{\text{GeV}} \ . \tag{9.2}$$

We can appreciate that in the analysis of this extended model the two processes are complementary, and they already exclude large portions of the plane. As larger values of $\tan \beta$ are preferred, in order to explain the large splitting in the $(t,\ b)$ multiplet, one can see that the $b \to s\gamma$ process gives the most important constraint.

9.2   Future work

In the study of the $b \to s\gamma$ process, presently theoretical and experimental accuracies are comparable. We can expect in the near future an improvement of the statistics of the CLEO experiment, and therefore the $b \to s\gamma$ decay shall require a complete next-to-leading computation. We have seen that such a computation is mandatory if one is willing to obtain sensible answers to the question whether the effect comes in its entirety from the Standard Model or

not. The techniques we have applied to this computation are in our opinion well suited to simplify the required three loop computations.

Once a complete three loop evaluation of the QCD corrections will be available, it will be necessary also to re-consider the determination of the Effective Hamiltonian, obtained by the matching procedure at the higher evolution scale, as described in Chapter 3. In order to exploit the higher precision in the evolution coefficients, it will be necessary to extend up to two loops the matching, not only for the Standard Model but for all the extensions one is interested to compare, including the Minimal Supersymmetric Standard Model.

For what concerns the $Z \to b\bar{b}$ process, and more generally the corrections to the observables measured at LEP, we have mentioned some works where different parts of the two loop effects have been computed. A complete evaluation of the electroweak corrections does not exist, and we cannot say, at the present level of experimental accuracy, that such an effort is mandatory.

We can just say that a complete analysis of LEP observables at the "per mille" level of accuracy shall require the full inclusion of two loop electroweak corrections. We hope that the techniques discussed in this thesis may result of help in this task.

9.3   Acknowledgments

Appendix A

Notations

In this thesis both the pseudo-Euclidean (E) notation and the Minkowsky (M) notations have been used, so it is useful to give here a translation table.

A.0.1   Metrics

The metric tensor is given by

$$\text{Minkowsky:} \quad a \cdot b = \eta^{\mu\nu} a_\mu b_\nu \qquad\qquad \text{Euclidean:} \quad a \cdot b = \delta_{\mu\nu} a_\mu b_\nu \tag{A.1}$$

where $\eta^{\mu\nu} = (-1, +1, +1, +1)$., so that $\quad a \cdot b = \eta^{\mu\nu} a_\mu b_\nu = \vec{a} \cdot \vec{b} - a_0 b_0$. The transcription in the PE notation is given in the following table

$$
\begin{aligned}
x^{M0} &= -i x_4^E & \partial_0^M &= i \partial_4^E \\
x^{Mi} &= x_i^E & \partial_i^M &= \partial_i^E
\end{aligned}
\tag{A.2}
$$

For a vector field in E notation the fourth component is purely imaginary

$$
\begin{aligned}
A^{M0} &= -i A_4^E & A^{Mi} &= A_i^E \\
A_0^M &= i A_4^E & A_i^M &= A_i^E \\
G_{0i}^M &= i G_{4i}^E & G_{ij}^M &= G_{ij}^E
\end{aligned}
\tag{A.3}
$$

hence for instance

$$G_{\mu\nu}^M G^{M\mu\nu} = G_{\mu\nu}^E G_{\mu\nu}^E \ . \tag{A.4}$$

Fourier Transform

We adopt the following conventions for Fourier transforms of fields

$$\phi(x) = \int \frac{d^4 k}{(2\pi)^4} e^{i p_\mu x^\mu} \tilde{\phi}(p) \tag{A.5}$$

as an example

$$\bar{u}(p_-) \gamma_\mu v(p_+) A^\mu(k) \tag{A.6}$$

with $k = p_+ + p_-$ incoming in the vertex is the amplitude for pair creation in electrodynamics, written in momentum space. Using this convention the expression for the four momentum is

$$
\begin{aligned}
p^\mu &= (E, \vec{p}) = \left(i\frac{\partial}{\partial t}, -i\vec{\nabla} \right) \\
\partial^\mu &= \frac{\partial}{\partial x_\mu} = \left(-\frac{\partial}{\partial t}, \vec{\nabla} \right)
\end{aligned}
\tag{A.7}
$$

hence one has the correspondence

$$
p^\mu = -i\partial^\mu \qquad \partial^\mu = ip^\mu
\tag{A.8}
$$

Spinors and Gamma Matrices

The correspondence for spinors in the two notations is as follows

$$
\psi^M = \psi^E \qquad\qquad\qquad \bar{\psi}^M = i\psi^\dagger \gamma^0 = \bar{\psi}^E = \psi^\dagger \gamma_4
\tag{A.9}
$$

where gamma matrices, defined by

$$
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}
\tag{A.10}
$$

$$
\gamma_5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = +i\gamma_0 \gamma_1 \gamma_2 \gamma_3 = \frac{i}{4!} \varepsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma
\tag{A.11}
$$

[1] obey the following correspondences

$$
\begin{aligned}
\gamma^{M0} &= -i\gamma_4^E & \gamma^{Mi} &= \gamma_i^E \\
\gamma_0^M &= +i\gamma_4^E & \gamma_i^M &= \gamma_i^E \\
\gamma_5^M &= \gamma_5^E = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \ .
\end{aligned}
\tag{A.12}
$$

For the computation of squared amplitudes it will be useful also

$$
\gamma_i^\dagger = \gamma_i \quad \gamma_0^\dagger = -\gamma_0 = \gamma^0 \quad \gamma_5^\dagger = \gamma_5
\tag{A.13}
$$

$$
\gamma_0 \gamma_\mu^\dagger \gamma_0 = \gamma_\mu
\tag{A.14}
$$

while for chiral theories the expressions of left and right projectors are

$$
P_{L/R} = \frac{1}{2} \left(1 \mp \gamma_5 \right)
\tag{A.15}
$$

so that one has two component Weyl spinors

$$
\begin{aligned}
\gamma_5 R &= R & \bar{R}\gamma_5 &= -\bar{R} \\
\gamma_5 L &= -L & \bar{L}\gamma_5 &= +\bar{L}
\end{aligned}
\tag{A.16}
$$

obtained from a 4-component spinor via projection

$$
R = P_R \psi \quad L = P_L \psi
\tag{A.17}
$$

note that one has

$$
\bar{\psi}\psi = \bar{L}R + \bar{R}L \quad \bar{\psi}\gamma_\mu \psi = \bar{L}\gamma_\mu L + \bar{R}\gamma_\mu R \ .
\tag{A.18}
$$

Unpolarized density matrices, and Dirac equations are given by

$$
\begin{aligned}
\sum_{spin} u_\alpha(P) \bar{u}_\beta(P) &= (-i\slashed{P} + m) & (i\slashed{P} + m)\, u(P) &= 0 \\
\sum_{spin} v_\alpha(P) \bar{v}_\beta(P) &= (-i\slashed{P} - m) & (i\slashed{P} - m)\, v(P) &= 0 \ .
\end{aligned}
\tag{A.19}
$$

---

[1] $\varepsilon_{0123} = -1 \qquad \varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\alpha\beta} = -2 \left(\eta_\rho^\alpha \eta_\sigma^\beta - \eta_\rho^\beta \eta_\sigma^\alpha \right)$

A.0.2   Lagrangian and Generating functional

The measure in the Generating Functional for Green functions is different in the two metrics: the correspondence is

$$
\begin{aligned}
e^{-S^E} &= e^{iS^M} \\
-\int \left(d^4x\right)^E \mathcal{L}^E &= i \int \left(d^4x\right)^M \mathcal{L}^M \\
\mathcal{L}^E &= -\mathcal{L}^M
\end{aligned}
\tag{A.20}
$$

The generating functional for Green functions is given by

$$
Z\left[J\right]^E = \exp\left[-S^E + \int dx J \cdot \phi\right] \qquad Z\left[J\right]^M = \exp\left[iS^M + i \int dx J \cdot \phi\right] .
\tag{A.21}
$$

The generating functional of connected functions is given by

$$
W\left[J\right]^E = \ln Z\left[J\right]^E \qquad W\left[J\right]^M = \tfrac{1}{i} \ln Z\left[J\right]^E .
\tag{A.22}
$$

Defining the "classical field" ϕ_c functional of J, as

$$
\phi_c\left[J\right]^E = \frac{\delta W[J]^E}{\delta J} \qquad \phi_c\left[J\right]^M = \frac{\delta i W[J]^M}{i\delta J} \quad ,
\tag{A.23}
$$

one obtains by Legendre transform the generators of $1PI$ Green functions

$$
\Gamma\left[\phi_c\right]^E = -W\left[J\right]^E + \int dx J \cdot \phi_c \qquad \Gamma\left[\phi_c\right]^M = W\left[J\right]^M - \int dx J \cdot \phi_c \quad .
\tag{A.24}
$$

A.0.3   QED in Minkowsky notation

The lagrangian is given by

$$
\mathcal{L}_{QED}^M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}\left(\gamma^\mu \mathcal{D}_\mu + m\right)\psi
\tag{A.25}
$$

where

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \mathcal{D}_\mu = \partial_\mu - i\,e\,Q\,A_\mu
\tag{A.26}
$$

note that the electric charge Q of the electron is -1.

Charge conjugation

The Dirac equation is given by

$$
\left[\left(\partial_\mu - ieQA_\mu\right)\gamma^\mu + m\right]\psi = 0
\tag{A.27}
$$

taking the adjoint and transposing one has, equivalently

$$
\left[-\gamma_\mu^{\mathrm{T}}\left(\partial^\mu + ieQA^\mu\right) + m\right]\bar{\psi}^{\mathrm{T}} = 0
\tag{A.28}
$$

the charge conjugation operator C, equipped with the property

$$
C^{-1}\gamma_\mu^{\mathrm{T}} C = -\gamma_\mu
\tag{A.29}
$$

allows to define the charge conjugated spinor $\psi^{\mathrm{c}} = C^{-1}\bar{\psi}^{\mathrm{T}}$ obeying the Dirac equation with the reversed sign for the charge

$$
\left[\left(\partial_\mu + ieQA_\mu\right)\gamma^\mu + m\right]\psi^{\mathrm{c}} = 0 .
\tag{A.30}
$$

It is possible to choose C with the following properties:

$$
\begin{aligned}
C^{\mathrm{T}} &= C^{\dagger} = -C = C^{-1} \\
CC^{\dagger} &= C^{\dagger}C = 1 \\
C^2 &= -1 \ .
\end{aligned}
$$
(A.31)

Some useful relations are

$$
\begin{aligned}
C^{-1}\gamma_5^{\mathrm{T}}C &= \gamma_5 \\
C\gamma_5 C^{-1} &= \gamma_5^{\mathrm{T}} \\
C\gamma_\mu^{\mathrm{L}}C^{-1} &= -\left(\gamma_\mu^{\mathrm{R}}\right)^{\mathrm{T}} \\
C\gamma_\mu^{\mathrm{R}}C^{-1} &= -\left(\gamma_\mu^{\mathrm{L}}\right)^{\mathrm{T}} \ .
\end{aligned}
$$
(A.32)

A.0.4 $SU(N)$ gauge theory in Minkowsky notation

The pure gauge lagrangian is given by

$$
\mathcal{L}_{gauge}^M = -\frac{1}{4}\,G_{\mu\nu}^a G^{a\mu\nu}
$$
(A.33)

where

$$
G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^a A_\nu^b
$$
(A.34)

and $a = 1, \cdots, N^2 - 1$. labels the generators, while the fermionic part is given by:

$$
\mathcal{L}_{fermion} = -\sum_{i=1}^{n}\left(\bar{\psi}_i\gamma^\mu \mathcal{D}_\mu \psi_i + m_i \bar{\psi}_i \psi_i\right)
$$
(A.35)

where $\mathcal{D}_\mu$ è is the covariant derivative.

$$
\mathcal{D}_\mu = \partial_\mu - igT^a A_\mu^a
$$
(A.36)

and matrices T^a satisfy the algebra:

$$
\left[T^a, T^b\right] = if^{abc}T^c \ .
$$
(A.37)

The field tensor $G_{\mu\nu}$ can be expressed in terms of the commutator of covariant derivatives

$$
[\mathcal{D}_\mu, \mathcal{D}_\nu] = -igT_a G_{\mu\nu}^a
$$
(A.38)

fixing so the sign of the charge in A.34 on the basis of A.36.

gauge transformations

Some basical properties are

$$
\begin{array}{ll}
\delta_\omega A_\mu^a = \partial_\mu \omega^a - g f_{abc}\omega^b A_\mu^c & \delta_\omega G_{\mu\nu}^a = -g f^{abc}\omega^b G_{\mu\nu}^c \\
\delta_\omega \psi = ig\omega^a T_a \psi & \delta_\omega \bar{\psi} = -ig\omega^a \bar{\psi}T_a
\end{array}
$$
(A.39)

Generators of the algebra

Fundamental representation:

$$T^a T^a = C_F I \qquad \text{where} \quad C_F = \frac{N^2 - 1}{2N} \tag{A.40}$$

Adjoint representation:

$$T^a_{bc} = i f_{abc} \tag{A.41}$$

$$T^a T^a = C_A I \qquad \text{where} \quad C_A = N \tag{A.42}$$

Minkowsky-Euclidean correspondence

$$i S^M = i \int (dx)^M \left(-\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \bar{\psi} \left(\gamma^\mu \mathcal{D}_\mu + m \right) \psi \right) = -S^E \tag{A.43}$$

where

$$S^E = \int (dx)^E \left(\frac{1}{4} G_{\mu\nu} G_{\mu\nu} + \bar{\psi} \left(\gamma_\mu \mathcal{D}_\mu + m \right) \psi \right) \tag{A.44}$$

Appendix B

Dirac algebra in d dimensions

Some useful dirac algebra definitions and identities valid for a d dimensional space, mostly taken from [1, 20, 30].

B.1 Clifford algebra

A complete basis of dirac algebra in d dimensions is provided by the completely antisymmetric products of γ matrices:

$$\gamma_{\mu_1 \mu_2 \dots \mu_n} = \frac{1}{n!} \sum_p (-1)^p \, \gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_n} \equiv g_{\mu\nu}^n \gamma_{\nu_1} \gamma_{\nu_2} \dots \gamma_{\nu_n} \ , \tag{B.1}$$

where it has been introduced, following Avdeev [30] the antisymmetrization operator

$$g_{\mu\nu}^m = \frac{1}{m!} \sum_{\pi \in s_m} (-1)^{p(\pi)} \, \delta_{\mu_1 \nu_{\pi(1)}} \dots \delta_{\mu_m \nu_{\pi(m)}} \tag{B.2}$$

which is a projector

$$g_{\mu\nu}^m g_{\nu\rho}^m = g_{\mu\rho}^m \tag{B.3}$$

with the trace given by

$$g_{\mu\mu}^m = \begin{pmatrix} d \\ m \end{pmatrix} \equiv \frac{1}{m!} \prod_{i=1}^m (d - m + i) \ . \tag{B.4}$$

In the limit $d \to 4$ the transcription identities are as follows:

$$
\begin{aligned}
\gamma_{\mu_1 \mu_2 \dots \mu_n} &= 0 \qquad n > 4 \\
\gamma_{\mu_1 \mu_2 \mu_3 \mu_4} &= -\gamma_5 \\
\gamma_{\mu_1 \mu_2 \mu_3} &= -\varepsilon_{\mu_1 \mu_2 \mu_3 \rho} \gamma_5 \gamma_\rho \\
\frac{1}{3!} \gamma_{\mu\nu\rho} \otimes \gamma_{\mu\nu\rho} &= \gamma_\mu \gamma^5 \otimes \gamma_\mu \gamma^5 \ .
\end{aligned}
\tag{B.5}
$$

In d dimension some useful relations are then

$$
\begin{aligned}
\gamma_\mu \gamma_\nu \gamma_\rho \otimes \gamma_\mu \gamma_\nu \gamma_\rho &= \gamma_{\mu\nu\rho} \otimes \gamma_{\mu\nu\rho} + (3d - 2) \, \gamma_\mu \otimes \gamma_\mu \\
\gamma_\mu \gamma_\nu \gamma_\rho \otimes \gamma_\rho \gamma_\nu \gamma_\mu &= -\gamma_{\mu\nu\rho} \otimes \gamma_{\mu\nu\rho} + (3d - 2) \, \gamma_\mu \otimes \gamma_\mu \ .
\end{aligned}
\tag{B.6}
$$

The projection on the canonical basis of any product of gamma matrices can be obtained by using recursive relations

$$\gamma_\nu \gamma_{\mu_1,\mu_2...,\mu_n} = \gamma_{\nu,\mu_1...,\mu_n} - \sum_i (-1)^i \delta_{\nu\mu_i} \gamma_{\mu_1...\mu_{i-1},\mu_{i+1}...,\mu_n} \tag{B.7}$$

$$\gamma_{\mu_1,\mu_2...,\mu_n} \gamma_\nu = \gamma_{\mu_1...,\mu_n,\nu} + \sum_i (-1)^{n-i} \delta_{\nu\mu_i} \gamma_{\mu_1...\mu_{i-1},\mu_{i+1}...,\mu_n} \tag{B.8}$$

. The following trace identity holds:

$$\mathrm{Tr}\left(\gamma^{(m)}\gamma^{(n)}\right) = n! \delta^{m,n} s_m g^m_{\mu\nu} \mathrm{Tr}(1)$$
$$s_n = (-1)^{\frac{n(n-1)}{2}} . \tag{B.9}$$

Some contraction identities:

$$\gamma_{\mu_1\mu_2...\mu_n} \gamma_{\mu_1} = (-1)^{n-1} (d-n+1) \gamma_{\mu_2...\mu_n}$$
$$\gamma_{\mu_1\mu_2...\mu_n} \gamma_{\mu_1\mu_2...\mu_n} = c_n$$
$$c_n = s_n \prod_{i=0}^{n-1} (d-i) ; \tag{B.10}$$

the c_n coefficient can been expressed as a series in ε defining

$$c_n = c_n(0) + \varepsilon\, c_n(1) + \varepsilon^2 c_n(2) + O(\varepsilon^3) .$$

They are also useful the following commutation identities:

$$\{\gamma_{\mu_1,\mu_2,...,\mu_n}, \ \gamma_\nu\} = 2\gamma_{\mu_1,\mu_2,...,\mu_n,\nu} \qquad n \text{ even}$$
$$[\gamma_{\mu_1,\mu_2,...,\mu_n}, \ \gamma_\nu] = 2\gamma_{\mu_1,\mu_2,...,\mu_n,\nu} \qquad n \text{ odd}$$
$$\tag{B.11}$$

and the reduction identities:

$$\gamma_\mu \gamma^{(n)} \gamma_\mu = (-1)^n (d-2n) \gamma^{(n)}$$
$$\gamma^{(n)} \gamma_\mu \gamma^{(n)} = s_n (2n-d) \prod_{i=1}^{n-1} (i-d) \gamma_\mu \qquad n \geq 2$$
$$= (-1)^n \frac{d-2n}{d} c_n \tag{B.12}$$
$$\gamma^{(n)} \gamma_\mu \gamma_\nu \gamma^{(n)} = \frac{1}{d(d-1)} \left[\left(d^2 - (d-2n)^2\right)\delta_{\mu\nu} + \left((d-2n)^2 - d\right)\gamma_\mu\gamma_\nu\right]\gamma^{(n)}\gamma^{(n)}$$
$$\gamma^{(m)} \gamma^{(n)} \gamma^{(m)} = f_{m,n} \gamma^{(n)} \tag{B.13}$$
$$f_{m,n} = m! s_n s_{m+n} \sum_{l=0}^{\min(m,n)} (-1)^l \binom{n}{l} \binom{d-n}{m-l} . \tag{B.14}$$

In Eq. B.12 summation over repeated indices is understood.

Again the f coefficients can be expanded in series of ε

$$f_{m,n} = f_{m,n}(0) + \varepsilon f_{m,n}(1) + O(\varepsilon^2) ,$$

and obey the consistency condition

$$\delta_{m,k}\left(\mathrm{Tr}\left(1\right)\right)^2 = \sum_{n=0}^{\infty} \frac{s_n\,s_k}{m!\,n!} f_{m,n} f_{n,k} \; . \tag{B.15}$$

Let us report a note of Avdeev about the use of Fierz transformations: first of all, the following relation holds

$$\gamma^{(m)} \otimes \gamma^{(m)} = \frac{1}{\mathrm{Tr}\left(1\right)} \sum_{n=0}^{\infty} \frac{s_n}{n!} f_d^{m\,n} \left(\gamma^{(n)} \otimes \gamma^{(n)}\right)_F \tag{B.16}$$

which, together with Eq. B.15 and the consequences

$$f_{0\,n} = 1 \qquad \text{and} \qquad f_{n\,0} = n!\,s_n \left(\begin{array}{c} d \\ n \end{array}\right) \tag{B.17}$$

imposes the condition

$$\left(\mathrm{Tr}\left(1\right)\right)^2 = \sum_{l=0}^{\infty} \left(\begin{array}{c} d \\ l \end{array}\right) = 2^d \; . \tag{B.18}$$

It follows that Fierz transformations (B.16) are inconsistent with the usual choice $\mathrm{Tr}\left(1\right) = 4$!

On the other hand, as the values of $f_{m,n}$ follow from purely combinatorial relations in Eq. B.14, contraction identities remain valid.

We have used in the computation reduction identities of the following form

$$\gamma_\mu \gamma^{(n)} \otimes \gamma_\mu \gamma^{(n)} = \gamma^{(n+1)} \otimes \gamma^{(n+1)} + n\left(d-n+1\right)\gamma^{(n-1)} \otimes \gamma^{(n-1)} \tag{B.19}$$

$$\gamma^{(n)} \gamma_\mu \otimes \gamma^{(n)} \gamma_\mu = \gamma^{(n+1)} \otimes \gamma^{(n+1)} + n\left(d-n+1\right)\gamma^{(n-1)} \otimes \gamma^{(n-1)} \tag{B.20}$$

$$\gamma_\mu \gamma^{(n)} \otimes \gamma^{(n)} \gamma_\mu = \left(-1\right)^n \left(\gamma^{(n+1)} \otimes \gamma^{(n+1)} - n\left(d-n+1\right)\gamma^{(n-1)} \otimes \gamma^{(n-1)}\right) \tag{B.21}$$

The relations in Eq. B.19 are fundamental and allow to reduce recursively any product of the form

$$\gamma_{\mu_1} \cdots \gamma_{\mu_k} \gamma^{(n)} \gamma_{\mu_{k+1}} \cdots \gamma_{\mu_m} \otimes \gamma_{\mu_{\pi(1)}} \cdots \gamma_{\mu_{\pi(h)}} \gamma^{(n)} \gamma_{\mu_{\pi(h+1)}} \cdots \gamma_{\mu_{\pi(m)}} \tag{B.22}$$

where $\pi \in S_m$ is a permutation of indexes.

Appendix C

Standard Model

In this appendix the notations for the Standard Model are set, in Minkowskian space, using $\eta_{\mu\nu}$ notation (see App. A).

C.1   Generalities

The electro-weak model is based on a $SU(2)_{left} \times U(1)_Y$ symmetry: the conserved hypercharge Y is associated with the $U(1)$ sector.

C.1.1   Matter fields

The matter fields are organized in $SU(2)$ doublets with left chirality, and right singlets: we resume in the following the quantum numbers and the composition of doublets.

Leptons

$$L_l = \tfrac{1}{2}\,(1+\gamma_5) \begin{pmatrix} \nu_l \\ l^- \end{pmatrix} \qquad\qquad R_l = \tfrac{1}{2}\,(1-\gamma_5)\,e^- \qquad l \in \{e,\,\mu,\,\tau\} \qquad (C.1)$$

Assignments of weak hypercharge:

$$Q = I_3 + \frac{1}{2}Y \qquad Y L_l = -L_l Y R_l = -2R_l \tag{C.2}$$

Quarks

$$L_{q_\uparrow} = \frac{1}{2}\,(1+\gamma_5) \begin{pmatrix} q_\uparrow \\ q_\downarrow \end{pmatrix} \qquad\qquad \begin{cases} R_{q_\uparrow} = \tfrac{1}{2}\,(1-\gamma_5)\,q_\uparrow \\[2mm] R_{q_\downarrow} = \tfrac{1}{2}\,(1-\gamma_5)\,q_\downarrow 4b \end{cases} \tag{C.3}$$

where

$$q_\uparrow \in u,\,c,\,t \qquad\qquad q_\downarrow \in d,\,s,\,b\,. \tag{C.4}$$

Weak hypercharge assignments:

$$Y\,L_{q_\uparrow} = \frac{1}{3}\,L_{q_\uparrow} \qquad Y\,R_{q_\uparrow} = \frac{4}{3}\,R_{q_\uparrow} \qquad Y\,R_{q_\downarrow} = -\frac{2}{3}\,R_{q_\downarrow}\,. \tag{C.5}$$

The mass eigenstates do not coincide with the states that diagonalize the weak interaction matrix: the relation is given by the well known Cabibbo-Kobayashi-Maskawa matrix

$$\begin{cases} L'_u = \frac{1}{2}\left(1+\gamma_5\right)\begin{pmatrix} u \\ d' \end{pmatrix} & d' = V_{ud}d + V_{us}s + V_{ub}b \\[2em] L'_c = \frac{1}{2}\left(1+\gamma_5\right)\begin{pmatrix} c \\ s' \end{pmatrix} & s' = V_{cd}d + V_{cs}s + V_{cb}b \quad , \\[2em] L'_t = \frac{1}{2}\left(1+\gamma_5\right)\begin{pmatrix} t \\ b' \end{pmatrix} & b' = V_{td}d + V_{ts}s + V_{tb}b \end{cases} \tag{C.6}$$

where the primed fields are eigenstates of W.I., while the non primed are mass eigenstates.

The notation $V_{q\uparrow q\downarrow}$ will be used for the elements of V, and $V^\star_{q\downarrow q\uparrow}$ for the ones of $V^\dagger$.

C.1.2 Gauge field content

Four gauge fields are associated to the local version of the $SU(2)\times U(1)$ symmetry: three $W^{1\,2\,3}_\mu$ associated with $SU(2)$ rotations, one B_μ with the remaining $U(1)$ (hypercharge) symmetry. The respective charges are g and g', and the covariant derivative is given by

$$\mathcal{D}_\mu = I\partial_\mu - ig\vec{W}_\mu \cdot \vec{T} - ig'B_\mu \frac{Y}{2} \tag{C.7}$$

while the *classical* lagrangian is

$$\mathcal{L}_{SU(2)\times U(1)} = \mathcal{L}_{gauge} + \mathcal{L}_{leptons} + \mathcal{L}_{quarks} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} \tag{C.8}$$

and the Higgs and Yukawa terms will provide the mean to introduce masses in an invariant manner.

The pure gauge lagrangian is then

$$\mathcal{L}_{gauge} = -\frac{1}{4}\vec{W}_{\mu\nu} \cdot \vec{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{C.9}$$

where

$$\vec{W}_{\mu\nu} = \partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu + g\vec{W}_\mu \times \vec{W}_\nu \tag{C.10}$$

with $\left(\vec{W}_\mu \times \vec{W}_\nu\right)_i = \epsilon_{ijk}W^j_\mu W^k_\nu$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{C.11}$$

C.1.3 Matter fields lagrangian

The lepton lagrangian is

$$\mathcal{L}_{lepton} = -\sum_{l=e,\mu,\tau}\bar{L}_l\gamma \cdot \mathcal{D}L_l - \sum_{l=e,\mu,\tau}\bar{R}_l\gamma \cdot \mathcal{D}R_l \tag{C.12}$$

while hadrons are incorporated in

$$\mathcal{L}_{quark} = -\sum_{q\uparrow=u,c,t}\bar{L}'_{q\uparrow}\gamma \cdot \mathcal{D}L'_{q\uparrow} - \sum_{q\uparrow=u,c,t}\bar{R}_{q\uparrow}\gamma \cdot \mathcal{D}R_{q\uparrow} - \sum_{q\downarrow=d,s,b}\bar{R}_{q\downarrow}\gamma \cdot \mathcal{D}R_{q\downarrow} \tag{C.13}$$

C.1.4   Higgs fields

The Higgs field is defined as a doublet of complex fields transforming under the fundamental representation of $SU(2)$ with assigned hypercharge +1.

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi_\circ \end{pmatrix} \qquad Y\Phi = \Phi \tag{C.14}$$

The form of the Higgs lagrangian is fixed by the gauge symmetry, while the couplings in the potential part are chosen in order to allow a spontaneous symmetry breaking.

$$\mathcal{L}_{Higgs} = -\mathcal{D}\Phi^\dagger \mathcal{D}\Phi - \mu^2 \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi \right)^2 \ . \tag{C.15}$$

The interaction of the Higgs field with matter fields is then of Yukawa type:

$$\mathcal{L}_{Yukawa} = \mathcal{L}_{Yukawa-quarks} + \mathcal{L}_{Yukawa-leptons} \tag{C.16}$$

$$
\begin{aligned}
\mathcal{L}_{Yukawa-leptons} = & \ - \ G_l \bar{L}_l \cdot \Phi R_l + h.c. \\
\mathcal{L}_{Yukawa-quarks} = & \ - \ G_\downarrow^{q_\uparrow \, q_\downarrow} \bar{L}'_{q_\uparrow} \cdot \Phi R_{q_\downarrow} + h.c. \\
& \ - \ G_\uparrow^{q_\uparrow \, q_\downarrow} \bar{L}'_{q_\uparrow} \cdot \tilde{\Phi} R_{q_\uparrow} + h.c.
\end{aligned}
\tag{C.17}
$$

where the conjugated doublet $\tilde{\Phi}$ is given by:

$$\tilde{\Phi} = \begin{pmatrix} \phi_\circ^* \\ -\phi^- \end{pmatrix} \qquad Y\tilde{\Phi} = -\tilde{\Phi} \tag{C.18}$$

and the Yukawa couplings are proportional to the ratio of the quark mass to the massive gauge field mass

$$
\begin{aligned}
G_l &= \tfrac{g}{\sqrt{2}M_W} m^l \\
G_\downarrow^{ij} &= \tfrac{g}{\sqrt{2}M_W} m^j V_{ij} \qquad\qquad\qquad G_\uparrow^{ij} = \tfrac{g}{\sqrt{2}M_W} m^i \delta_{ij}
\end{aligned}
\ . \tag{C.19}
$$

C.1.5   Symmetry breaking

Choosing a negative sign for the mass term, $\mu^2 < 0$, the Higgs field acquires an expectation value: we may choose the phases in order to have it in the form:

$$\Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}} \left(H + i\chi \right) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v \end{pmatrix} \tag{C.20}$$

$$\tilde{\Phi} = \begin{pmatrix} \frac{1}{\sqrt{2}} \left(H - i\chi \right) \\ -\phi^- \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} v \\ 0 \end{pmatrix} \tag{C.21}$$

where

$$v = \sqrt{\frac{-\mu^2}{\lambda}} = \frac{2M_W}{g} \ . \tag{C.22}$$

Thus we obtain for the lepton lagrangian

$$
\begin{aligned}
\mathcal{L}^l_{Yukawa} = & -\ m_l\left(1+\frac{g}{2m_W}H\right)\bar{l}l + i\frac{g}{2m_W}m_l\chi\bar{l}\gamma_5 l \\
& -\ \frac{g}{\sqrt{2}m_W}m_l\left(\bar{\nu}_l\phi^+ Rl + \bar{l}\phi^- L\nu_l\right)
\end{aligned}
\tag{C.23}
$$

Analogously for the mass lagrangian of the quarks and their coupling to $H, \Phi^{(+,-)}, \chi$:

$$
\mathcal{L}_{Yukawa-quarks} = \mathcal{L}_{mass,H} + \mathcal{L}_\Phi + \mathcal{L}_\chi\ ;
\tag{C.24}
$$

H – quarks lagrangian:

$$
\mathcal{L}_{mass,H} = -\left(1+\frac{g}{2m_W}H\right)\sum_{quarks} m_q\bar{q}q\ ;
\tag{C.25}
$$

$\Phi^{(+,-)}$ – quarks lagrangian:

$$
\begin{aligned}
\mathcal{L}_\Phi = & -\ \frac{g}{\sqrt{2}}\frac{m_d^j}{m_W}\left(V_{ij}\bar{u}_L^i\phi^+ d_R^j + V_{ji}^\dagger\bar{d}_R^j\phi^- u_L^i\right) \\
& +\ \frac{g}{\sqrt{2}}\frac{m_u^i}{m_W}\left(V_{ji}^\dagger\bar{d}_L^j\phi^- u_R^i + V_{ij}\bar{u}_R^i\phi^+ d_L^j\right)
\end{aligned}
\tag{C.26}
$$

χ – quarks lagrangian:

$$
\mathcal{L}_\chi = i\frac{g}{2M_W}\chi\sum_{quarks}\left(m_{q_\downarrow}\bar{q}_\downarrow\gamma_5 q_\downarrow - m_{q_\uparrow}\bar{q}_\uparrow\gamma_5 q_\uparrow\right)\ ;
\tag{C.27}
$$

Weinberg angle

The gauge covariant interaction of Higgs and gauge fields gives rise to mass terms for the latter: one gauge field remains massless, corresponding to the unbroken $U(1)$ electromagnetic symmetry. To put in evidence this symmetry breaking pattern it is convenient to define "rotated" fields

$$
\begin{aligned}
W_\mu^3 &= Z_\mu\cos\theta_W + A_\mu\sin\theta_W \\
B_\mu &= A_\mu\cos\theta_W - Z_\mu\sin\theta_W
\end{aligned}
\tag{C.28}
$$

$$
\begin{aligned}
A_\mu &= B_\mu\cos\theta_W + W_\mu^3\sin\theta_W \\
Z_\mu &= W_\mu^3\cos\theta_W - B_\mu\sin\theta_W
\end{aligned}
\tag{C.29}
$$

The covariant derivative turns into the following

$$
\begin{aligned}
\mathcal{D}_\mu = I\partial_\mu\ & -\ ig\left(W_\mu^1 T^1 + W_\mu^2 T^2\right) \\
& -\ i\left(g'\frac{Y}{2}\cos\theta_W + gT^3\sin\theta_W\right)A_\mu \\
& -\ i\left(gT^3\cos\theta_W - g'\frac{Y}{2}\sin\theta_W\right)Z_\mu
\end{aligned}
\tag{C.30}
$$

The condition that the field A_μ is associated with the photon imposes that

$$
g'\frac{Y}{2}\cos\theta_W + gT^3\sin\theta_W = e\left(T^3+\frac{Y}{2}\right) = eQ
\tag{C.31}
$$

$$e = g' \cos \theta_W = g \sin \theta_W \tag{C.32}$$

hence the relations

$$\frac{g'}{g} = \tan \theta_W \qquad\qquad e = \frac{g' g}{\sqrt{g'^2 + g^2}} \; . \tag{C.33}$$

We may then define charge lowering and raising operators of the weak isospin (and hence of the electric charge, according to Eq. C.31

$$T^{\pm} = T^1 \pm i T^2 \; . \tag{C.34}$$

We may individuate the charged components of the gauge fields:

$$W_{\mu}^{\pm} = \frac{1}{\sqrt{2}} \left(W_{\mu}^1 \mp i W_{\mu}^2 \right) \tag{C.35}$$

and the covariant derivative can be simply written as:

$$\mathcal{D}_{\mu} = I \partial_{\mu} - \frac{ig}{\sqrt{2}} \left(W_{\mu}^+ T^+ + W_{\mu}^- T^- \right) - ieQA_{\mu} - ieQ'Z_{\mu} \tag{C.36}$$

where the neutral charge matrix Q' is introduced

$$Q' = T^3 \cot \theta_W - \frac{Y}{2} \tan \theta_W = \frac{1}{\sin \theta_W \cos \theta_W} \left(T^3 - \sin \theta_W^2 Q \right) \; . \tag{C.37}$$

C.1.6   Field transformations

It is useful to reformulate transformation laws in terms of rotated fields: to this end let us set

$$\begin{aligned} \omega^{\pm} &= \tfrac{1}{\sqrt{2}} \left(\omega^1 \mp i\omega^2 \right) \\ \omega_A &= \omega' \cos \theta_W + \omega^3 \sin \theta_W \qquad \omega_Z = \omega^3 \cos \theta_W - \omega' \sin \theta_W \end{aligned} \; . \tag{C.38}$$

It follows for the transformation of gauge fields

$$\begin{aligned} \delta A_{\mu} &= \partial_{\mu} \omega_A - ie \left(\omega^- W_{\mu}^+ - \omega^+ W_{\mu}^- \right) \\ \delta Z_{\mu} &= \partial_{\mu} \omega_Z - ig \cos \theta_W \left(\omega^- W_{\mu}^+ - \omega^+ W_{\mu}^- \right) \\ \delta W_{\mu}^{\pm} &= \partial_{\mu} \omega^{\pm} \mp ie \left(\omega^{\pm} A_{\mu} - \omega_A W_{\mu}^{\pm} \right) \mp ig \cos \theta_W \left(\omega^{\pm} Z_{\mu} - \omega_Z W_{\mu}^{\pm} \right) \end{aligned} \tag{C.39}$$

while for the Higgs it results

$$\begin{aligned} \delta H &= \frac{ig}{2} \left(\omega_- \phi_+ - \omega_+ \phi_- \right) + \omega_Z \chi \frac{e}{\sin 2\theta_W} \\ \delta \chi &= \frac{g}{2} \left(\omega_- \phi_+ + \omega_+ \phi_- \right) - \omega_Z (H + v) \frac{e}{\sin 2\theta_W} \\ \delta \phi_{\pm} &= \pm \frac{ig}{2} \omega_{\pm} (H + v) - \frac{g}{2} \omega_{\pm} \chi \pm ie \left(\omega_A + \frac{1}{\tan 2\theta_W} \omega_Z \right) \phi_{\pm} \end{aligned} \tag{C.40}$$

and finally for any doublet or singlet of matter fields

$$\delta \Psi = \left[\frac{ig}{\sqrt{2}} \left(\omega^+ T^+ + \omega^- T^- \right) + ieQ\omega_A + ieQ'\omega_Z \right] \Psi \tag{C.41}$$

C.1.7   The complete Lagrangian

The gauge lagrangian may be expressed as follows

$$\mathcal{L}_{gauge} = -\frac{1}{2}W_{\mu\nu}^+ W^{-\mu\nu} - \frac{1}{4}W_{\mu\nu}^3 W^{3\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{C.42}$$

where

$$
\begin{aligned}
W_{\mu\nu}^\pm &= \left(\partial_\mu \mp igW_\mu^3\right)W_\nu^\pm - \left(\partial_\nu \mp igW_\nu^3\right)W_\mu^\pm \\
W_{\mu\nu}^3 &= \partial_\mu W_\nu^3 - \partial\nu W_\mu^3 - ig\left(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^-\right)
\end{aligned} \tag{C.43}
$$

that is, evidencing the subgroup $U(1)$ of electromagnetism

$$
\begin{aligned}
\mathcal{L}_{gauge} = &- \frac{1}{4}F_{\mu\nu}F_{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z_{\mu\nu} \\
&+ ig\left(\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3\right)W_\mu^+ W_\nu^- \\
&+ \frac{1}{2}g^2\left(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+\right)W_\mu^+ W_\nu^- \\
&- \frac{1}{2}\left|\left(\partial_\mu - igW_\mu^3\right)W_\nu^+ - \left(\partial_\nu - igW_\nu^3\right)W_\mu^+\right|^2
\end{aligned} \tag{C.44}
$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad\qquad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \; . \tag{C.45}$$

The hadronic lagrangian is, in full form

$$
\begin{aligned}
\mathcal{L}_{quark} = &- \sum_{quarks} \bar{q}\gamma^\mu\left(\partial_\mu - ieQA_\mu - ieQ'Z_\mu\right)q \\
&+ \frac{ig}{\sqrt{2}}\left(V_{ji}\bar{u}_L^j\gamma^\mu W_\mu^+ d_L^i + V_{ij}^\dagger \bar{d}_L^i\gamma^\mu W_\mu^- .u_L^j\right) \; .
\end{aligned} \tag{C.46}
$$

Separating the Higgs lagrangian in kinetic and potential parts, $\mathcal{L}_{Higgs} = \mathcal{T}_{Higgs} - \mathcal{V}_{Higgs}$ we have

$$
\begin{aligned}
\mathcal{T}_{Higgs} = &- \left(\partial_\mu\phi^- + i\frac{g}{2}W_\mu^3\phi^- + i\frac{g'}{2}B_\mu\phi^- + i\frac{g}{\sqrt{2}}W_\mu^-\phi_\circ^*\right) \\
&\times \left(\partial^\mu\phi^+ - i\frac{g}{2}W^{3\mu}\phi^+ - i\frac{g'}{2}B^\mu\phi^+ - i\frac{g}{\sqrt{2}}W^{+\mu}\phi_\circ\right) \\
&- \left(\partial_\mu\phi_\circ^* - i\frac{g}{2}W_\mu^3\phi_\circ^* + i\frac{g'}{2}B_\mu\phi_\circ^* + i\frac{g}{\sqrt{2}}W_\mu^+\phi^-\right) \\
&\times \left(\partial^\mu\phi_\circ + i\frac{g}{2}W^{3\mu}\phi_\circ - i\frac{g'}{2}B^\mu\phi_\circ - i\frac{g}{\sqrt{2}}W^{-\mu}\phi^+\right)
\end{aligned} \tag{C.47}
$$

that is, putting in evidence the charged gauge fields

$$
\begin{aligned}
\mathcal{T}_{Higgs} = &- \left(\partial_\mu\phi^- + i\frac{g}{2}W_\mu^3\phi^- + i\frac{g'}{2}B_\mu\phi^-\right) \\
&\times \left(\partial^\mu\phi^+ - i\frac{g}{2}W^{3\mu}\phi^+ - i\frac{g'}{2}B^\mu\phi^+\right)
\end{aligned}
$$

$$
\begin{aligned}
- \quad & \left(\partial_\mu \phi_\circ^* - i\frac{g}{2} W_\mu^3 \phi_\circ^* + i\frac{g'}{2} B_\mu \phi_\circ^* \right) \\
\times \quad & \left(\partial^\mu \phi_\circ + i\frac{g}{2} W^{3\mu} \phi_\circ - i\frac{g'}{2} B^\mu \phi_\circ \right) \\
+ \quad & i\frac{g}{\sqrt{2}} W^{+\mu} \left(\phi_\circ \partial_\mu \phi^- - \phi^- \partial_\mu \phi_\circ + ig' B_\mu \phi_\circ \phi^- \right) + \text{h.c.} \\
- \quad & \frac{1}{2} g^2 W^{+\mu} W_\mu^- \left(\phi^+ \phi^- + \phi_\circ^* \phi_\circ \right) \ .
\end{aligned} \tag{C.48}
$$

It is convenient to define the covariant derivative in the subgroup $U(1) \times U(1)$:

$$
\begin{aligned}
\hat{\mathcal{D}}_\mu &= I\partial_\mu - igW_\mu^3 T^3 - ig'\frac{Y}{2} B_\mu \\
&= \partial_\mu - ieQ A_\mu - ieQ' Z_\mu \\
Q' &= \frac{1}{\sin_W \cos_W} \left(T^3 - \sin_W^2 Q \right) \ .
\end{aligned} \tag{C.49}
$$

An useful relation is the following

$$
\frac{v}{2} \left(gW_\mu^3 - g' B_\mu \right) = \frac{v}{2}\frac{gg'}{e} Z_\mu = m_Z Z_\mu \tag{C.50}
$$

Setting

$$
\phi_\circ = S + \frac{v}{\sqrt{2}} \equiv \frac{H + i\chi}{\sqrt{2}} + \frac{v}{\sqrt{2}} \tag{C.51}
$$

$$
\hat{\mathcal{D}}_\mu S - \left(\hat{\mathcal{D}}_\mu S \right)^* = i\sqrt{2} \left(\partial_\mu \chi + \frac{1}{2}\frac{gg'}{e} Z_\mu H \right) \tag{C.52}
$$

one is able to rewrite the kinetic Higgs lagrangian in the form

$$
\begin{aligned}
\mathcal{T}_{Higgs} = \quad - \quad & \left(\hat{\mathcal{D}}_\mu \phi^+ \right)^* \hat{\mathcal{D}}^\mu \phi^+ - \left(\hat{\mathcal{D}}_\mu S \right)^* \hat{\mathcal{D}}^\mu S \\
- \quad & m_Z Z_\mu \left(\partial^\mu \chi + \frac{1}{2}\frac{gg'}{e} Z^\mu H \right) \\
+ \quad & im_W W_\mu^+ \left(\partial^\mu + ig' B^\mu \right) \phi^- + \text{h.c.} \\
+ \quad & i\frac{g}{\sqrt{2}} W_\mu^+ \left(S\partial^\mu \phi^- - \phi^- \partial^\mu S + ig' B^\mu S\phi^- \right) + \text{h.c.} \\
- \quad & \frac{1}{2} g^2 W_\mu^+ W^{-\mu} \left(\phi^+ \phi^- + S^* S \right) - g m_W W_\mu^+ W^{-\mu} H \\
- \quad & m_W^2 W_\mu^+ W^{-\mu} - \frac{1}{2} m_Z^2 Z_\mu Z^\mu \ .
\end{aligned} \tag{C.53}
$$

Noting that

$$
\lambda = \tfrac{1}{8} g^2 \left(\tfrac{m_H}{m_W} \right)^2 \qquad \mu^2 = -\tfrac{1}{2} m_H^2 \tag{C.54}
$$

it is possible to rewrite the potential part as follows

$$
\begin{aligned}
-\mathcal{V}_{Higgs} = \quad - \quad & \frac{1}{2} m_H^2 H^2 - \frac{1}{4} g m_W \left(\frac{m_H}{m_W} \right)^2 H \left(H^2 + \chi^2 \right) \\
- \quad & \frac{1}{32} g^2 \left(\frac{m_H}{m_W} \right)^2 \left(H^2 + \chi^2 \right)^2 - \frac{1}{2} g m_W \left(\frac{m_H}{m_W} \right)^2 H\phi^+ \phi^- \\
- \quad & \frac{1}{8} g^2 \phi^+ \phi^- \left(H^2 + \chi^2 \right) - \frac{1}{8} g^2 \left(\phi^+ \phi^- \right)^2
\end{aligned} \tag{C.55}
$$

C.2   Quantization

A perturbative evaluation of matrix elements requires to break the gauge invariance and introduce a gauge fixing. In this section we want to discuss a few technical points that result of great practical advantage.

C.2.1   Non linear gauge fixing

After the symmetry breaking, the interaction between gauge fields and non-physical Higgs fields gives rise to spurious terms in the quadratic part of the lagrangian: it is possible a gauge choice which explicitly eliminates this coupling, avoiding so mixed propagators between gauge and Higgs fields. The gauge-fixing is given by:

$$\mathcal{L}_{g.f.} = -\frac{1}{2\alpha}\,(\partial_\mu A^\mu)^2 - \frac{1}{2\eta}\,(\partial_\mu Z^\mu + \eta m_Z \chi)^2 - \frac{1}{\xi}\left|(\partial_\mu - igW_\mu^3)W^{+\mu} + i\xi m_W \phi^+\right|^2 \tag{C.56}$$

Putting together gauge fixing and Higgs lagrangian one has

$$
\begin{aligned}
\mathcal{L}_{Higgs} + \mathcal{L}_{g.f.} = \quad & - \quad \frac{1}{2\alpha}\,(\partial_\mu A^\mu)^2 - \frac{1}{2\eta}\,(\partial_\mu Z^\mu)^2 - \frac{1}{\xi}\left|(\partial_\mu - igW_\mu^3)W^{+\mu}\right|^2 \\
& - \quad \left(\hat{\mathcal{D}}_\mu \phi^+\right)^* \hat{\mathcal{D}}^\mu \phi^+ - \left(\hat{\mathcal{D}}_\mu S\right)^* \hat{\mathcal{D}}^\mu S - \frac{\eta}{2}m_Z^2 \chi^2 - \xi m_W^2 \phi^+ \phi^- \\
& - \quad \frac{1}{2}\frac{gg'}{e}m_Z Z_\mu Z^\mu H \\
& + \quad \left(\frac{gg'}{e}m_W W_\mu^+ Z^\mu \phi^- + \text{h.c.}\right) \\
& + \quad i\frac{g}{\sqrt{2}}W_\mu^+ \left(S\partial^\mu \phi^- - \phi^- \partial^\mu S + ig'B^\mu S\phi^-\right) + \text{h.c.} \\
& - \quad \frac{1}{2}g^2 W_\mu^+ W^{-\mu}\left(\phi^+ \phi^- + S^* S\right) - gm_W W_\mu^+ W^{-\mu} H \\
& - \quad m_W^2 W_\mu^+ W^{-\mu} - \frac{1}{2}m_Z^2 Z_\mu Z^\mu \tag{C.57}
\end{aligned}
$$

We may collect terms in the total lagrangian containing only gauge fields

$$
\begin{aligned}
\mathcal{L}_{gauge\ total} = \quad & - \quad \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\,(\partial_\mu A^\mu)^2 \\
& - \quad \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{2\eta}\,(\partial_\mu Z^\mu)^2 - \frac{1}{2}m_Z^2 Z_\mu Z^\mu \\
& + \quad ig\left(\partial_\mu W_\nu^3 - \partial_\nu W_\mu^3\right)W^{+\mu}W^{-\nu} \\
& + \quad \frac{1}{2}g^2\left(W_\mu^+ W_\nu^- - W_\mu^- W_\nu^+\right)W^{+\mu}W^{-\nu} \\
& - \quad \frac{1}{2}\left|\left(\partial_\mu - igW_\mu^3\right)W_\nu^+ - \left(\partial_\nu - igW_\nu^3\right)W_\mu^+\right|^2 \\
& - \quad \frac{1}{\xi}\left|(\partial_\mu - igW_\mu^3)W^{+\mu}\right|^2 - m_W^2 W_\mu^+ W^{-\mu} \tag{C.58}
\end{aligned}
$$

and terms coupling gauge and Higgs fields

$$\mathcal{L}_{gauge-Higgs} = \quad - \quad \left(\hat{\mathcal{D}}_\mu \phi^+\right)^* \hat{\mathcal{D}}^\mu \phi^+ - \left(\hat{\mathcal{D}}_\mu S\right)^* \hat{\mathcal{D}}^\mu S - \frac{\eta}{2}m_Z^2 \chi^2 - \xi m_W^2 \phi^+ \phi^-$$

$$- \frac{1}{2} \frac{gg'}{e} m_Z Z_\mu Z^\mu H - g m_W W_\mu^+ W^{-\mu} H$$

$$+ \left(\frac{gg'}{e} m_W W_\mu^+ Z^\mu \phi^- + \text{h.c.} \right)$$

$$+ i \frac{g}{\sqrt{2}} W_\mu^+ \left(S \partial^\mu \phi^- - \phi^- \partial^\mu S + i g' B^\mu S \phi^- \right) + \text{h.c.}$$

$$- \frac{1}{2} g^2 W_\mu^+ W^{-\mu} \left(\phi^+ \phi^- + S^* S \right) \tag{C.59}$$

of course the unitarity is reinforced introducing anti-commuting scalar fields: the ghosts.

C.2.2   Ghosts and Slavnov Invariance

It is convenient to rewrite the gauge fixing lagrangian in the form

$$
\begin{aligned}
\mathcal{L}_{g.f.} &= -\frac{1}{2\alpha} F_A^2 - \frac{1}{2\eta} F_Z^2 - \frac{1}{\xi} F_+ F_- \\
F_A &= \partial_\mu A^\mu \\
F_Z &= \partial_\mu Z^\mu + \eta m_Z \chi \\
F_\pm &= \left(\partial_\mu \mp i g W_\mu^3 \right) W^{\pm \mu} \pm i \xi m_W \Phi^\pm
\end{aligned}
\tag{C.60}
$$

The properties of transformation under $SU(2) \times U(1)$ are

$$
\begin{aligned}
\delta_\omega F_A &= \Box \omega_A - i e \partial^\mu \left(\omega^- W_\mu^+ - \omega^+ W_\mu^- \right) \\
\delta_\omega F_Z &= \Box \omega_Z - i g \cos \theta_W \partial^\mu \left(\omega^- W_\mu^+ - \omega^+ W_\mu^- \right) \\
&\quad - \eta m_Z \left[\omega_Z \left(H + v \right) \frac{e}{\sin 2\theta_W} - \frac{g}{2} \left(\omega^- \Phi^+ + \omega^+ \Phi^- \right) \right] \\
\delta_\omega F_\pm &= \partial^\mu \left(\partial_\mu \mp i g W_\mu^3 \right) \omega^\pm \mp i g W_\mu^3 \left(\partial^\mu \mp i g W_\mu^3 \right) \omega^\pm \\
&\quad \pm i g \omega_3 \left(\partial^\mu \mp i g W^{3\mu} \right) W_\mu^\pm \mp g^2 W_\mu^\pm \left(\omega^- W^{\mu+} - \omega^+ W^{\mu-} \right) \\
&\quad - \xi m_W \left[\frac{g}{2} \omega^\pm \left(H + v \pm i\chi \right) + e \left(\omega_A + \frac{1}{\tan 2\theta_W} \omega_Z \right) \Phi^\pm \right]
\end{aligned}
\tag{C.61}
$$

Following Becchi-Rouet-Stora we may define the gauge transformation with parameters $\omega^i = X^i \zeta, \quad \omega' = X' \zeta$ where the compensating fields X^i, X' anticommute, so as the $x-$independent parameter ζ. The BRS transform s is defined as the *right* derivative with respect to ζ of a gauge transformation parameterized in terms of fields X. In the basis defined by the same rotation which diagonalizes the mass matrix of gauge fields one has

$$
\begin{aligned}
X^\pm &= \tfrac{1}{\sqrt{2}} \left(X^1 \mp i X^2 \right) \\
X_A &= X' \cos \theta_W + X^3 \sin \theta_W \quad X_Z = X^3 \cos \theta_W - X' \sin \theta_W
\end{aligned}
\tag{C.62}
$$

and this fields are introduced in the action with the following lagrangian

$$
\begin{aligned}
L_{g.c.} &= + \bar{X}_A s F_A + \bar{X}_Z s F_Z + \bar{X}^+ s F_+ + \bar{X}^- s F_- \\
\bar{X}_A s F_A &= - \partial^\mu \bar{X}_A \partial_\mu X_A + i e \partial^\mu \bar{X}_A \left(X^- W_\mu^+ - X^+ W_\mu^- \right) \\
\bar{X}_Z s F_Z &= - \partial^\mu \bar{X}_Z \partial_\mu X_Z - \eta m_Z^2 \bar{X}_Z X_Z + i g \cos \theta_W \partial^\mu \bar{X}_Z \left(X^- W_\mu^+ - X^+ W_\mu^- \right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\quad \eta m_Z \frac{gg'}{2e}\bar{X}_Z X_Z H + \eta m_Z \frac{g}{2}\left(\bar{X}_Z X^-\Phi^+ + \bar{X}_Z X^+\Phi^-\right) \\
\bar{X}^{\pm}sF_{\pm} = &\quad -\quad \left(\partial^\mu \pm igW^{\mu 3}\right)\bar{X}^{\pm}\left(\partial_\mu \mp igW^3_\mu\right)X^{\pm} - \xi m_W^2 \bar{X}^{\pm}X^{\pm} \\
&\quad \pm\quad ig\bar{X}^{\pm}X^3\left(\partial^\mu \mp igW^{\mu 3}\right)W^{\pm}_\mu - g^2\left(\bar{X}^{\pm}X^{\mp}W^{\pm}_\mu - \bar{X}^{\pm}X^{\pm}W^{\mp}_\mu\right)W^{\mu\pm} \\
&\quad -\quad \xi m_W \frac{g}{2}\bar{X}^{\pm}X^{\pm}\left(H \pm i\chi\right) - \xi m_W e\left(\bar{X}^{\pm}X_A + \frac{1}{\tan 2\theta_W}\bar{X}^{\pm}X_Z\right)\Phi^{\pm}
\end{aligned}
$$

$$\text{(C.63)}$$

where s is the BRS transformation.

In the following for completeness we report the effect of BRS transformation.

The explicit expression for gauge fields is

$$
\begin{aligned}
sA_\mu &= \partial_\mu X_A - ie\left(X^-W^+_\mu - X^+W^-_\mu\right) \\
sZ_\mu &= \partial_\mu X_Z - ig\cos\theta_W\left(X^-W^+_\mu - X^+W^-_\mu\right) \\
sW^{\pm}_\mu &= \partial_\mu X^{\pm} \mp ie\left(X^{\pm}A_\mu - X_A W^{\pm}_\mu\right) \mp ig\cos\theta_W\left(X^{\pm}Z_\mu - X_Z W^{\pm}_\mu\right)
\end{aligned}
$$

$$\text{(C.64)}$$

while for Higgs fields

$$
\begin{aligned}
sH &= \frac{ig}{2}\left(X^-\Phi^+ - X^+\Phi^-\right) + \frac{e}{\sin 2\theta_W}X_Z\chi \\
s\chi &= \frac{g}{2}\left(X^-\Phi^+ + X^+\Phi^-\right) - \frac{e}{\sin 2\theta_W}X_Z\left(H + v\right) \\
s\Phi^{\pm} &= \pm\frac{ig}{2}X^{\pm}\left(H + v\right) - \frac{g}{2}\chi X^{\pm} \pm ie\left(X_A + \frac{1}{\tan 2\theta_W}X_Z\right)\Phi^{\pm}
\end{aligned}
$$

$$\text{(C.65)}$$

and lastly for the compensating (ghost) fields

$$
\begin{array}{ll}
sX^i = \frac{g}{2}\epsilon^{ijk}X^j X^k & sX^B = 0 \\
s\bar{X}^i = \frac{1}{\xi}F_i & s\bar{X}^B = \frac{1}{\xi}F_B
\end{array}
$$

$$\text{(C.66)}$$

that is,

$$
\begin{array}{ll}
sX^{\pm} = \mp ie\left(X_A + \frac{1}{\tan\theta_W}X_Z\right)X^{\pm} & s\bar{X}^{\pm} = \frac{1}{\xi}F_{\mp} \\
sX_A = -ieX^+X^- & s\bar{X}_A = \frac{1}{\alpha}F_A \\
sX_Z = -\frac{ie}{\tan\theta_W}X^+X^- & s\bar{X}_Z = \frac{1}{\eta}F_Z \; .
\end{array}
$$

$$\text{(C.67)}$$

It is easy to verify, thanks to the nilpotence of the BRS transformation ($s^2 \equiv 0$), that the variation of the Gauge Fixing Lagrangian is compensated by the variation of the Ghost terms. The original gauge invariant lagrangian is also BRS invariant, hence at a classical level the gauge fixing breaks the classical invariance but leaves a BRS invariance. In App. D we shall see how it is possible, and in many cases convenient, a different choice of the gauge fixing which enables to preserve also the classical invariance.

C.2.3 Feynman rules

It is now useful to report the Feynman rules for the electro-weak sector of the Standard Model, for the given choice of gauge fixing. The modifications required by the Background Field Gauge will be discussed in App. D.

The convention chosen in the following is that momenta and charges enter the vertex.

Masses

$$
\begin{array}{rcl}
W^{\pm}\left(Z_{\mu}\right) & : & m_W\left(m_Z\right) \\
\Phi^{\pm} & : & \sqrt{\xi}m_W \\
\chi & : & \sqrt{\eta}m_Z \\
H & : & m_H
\end{array}
\tag{C.68}
$$

Propagators

$$
\text{scalar field} \qquad \left\langle \Phi\left(-p\right)\Phi\left(p\right)\right\rangle = \frac{(-i)}{p^2+m^2}
$$

$$
\text{fermionic field} \qquad \left\langle \bar{\psi}\psi \right\rangle = \frac{(-i)}{i/p+m}
$$

$$
\text{vector field } W^{\pm} \qquad \left\langle W_{\mu}^{\pm}\left(-p\right)W_{\nu}^{\mp}\left(p\right)\right\rangle = \frac{(-i)}{p^2+m_w^2}\left[\eta_{\mu\nu}-\left(1-\xi\right)\frac{p_{\mu}p_{\nu}}{p^2+\xi m_w^2}\right]
$$

$$
\text{vector field } Z \qquad \left\langle Z_{\mu}\left(-p\right)Z_{\nu}\left(p\right)\right\rangle = \frac{(-i)}{p^2+m_z^2}\left[\eta_{\mu\nu}-\left(1-\eta\right)\frac{p_{\mu}p_{\nu}}{p^2+\eta m_z^2}\right]
$$

$$
\text{vector field } A \qquad \left\langle A_{\mu}\left(-p\right)A_{\nu}\left(p\right)\right\rangle = \frac{(-i)}{p^2+\lambda^2}\left[\eta_{\mu\nu}-\left(1-\alpha\right)\frac{p_{\mu}p_{\nu}}{p^2+\alpha\lambda^2}\right]
\tag{C.69}
$$

Matter fields-gauge fields couplings

$$
\begin{array}{rcl}
\left\langle \bar{u}_i W_{\mu}^+ d_j \right\rangle & = & -\dfrac{g}{\sqrt{2}}V_{ij}\gamma_{\mu}^L \\[2mm]
\left\langle \bar{d}_i W_{\mu}^- u_j \right\rangle & = & -\dfrac{g}{\sqrt{2}}V_{ij}^{\dagger}\gamma_{\mu}^L \\[2mm]
\left\langle \bar{\nu}_l W_{\mu}^+ e \right\rangle & = & -\dfrac{g}{\sqrt{2}}\gamma_{\mu}^L \\[2mm]
\left\langle \bar{\psi}_q A_{\mu}\psi \right\rangle & = & -eQ_q\gamma_{\mu} \\[2mm]
\left\langle \bar{u}Z_{\mu}u \right\rangle & = & -\dfrac{e}{\sin_W \cos_W}\left[\dfrac{1}{2}\gamma_{\mu}^L - \dfrac{2}{3}\sin_W^2 \gamma_{\mu}\right] \\[3mm]
\left\langle \bar{d}Z_{\mu}d \right\rangle & = & -\dfrac{e}{\sin_W \cos_W}\left[-\dfrac{1}{2}\gamma_{\mu}^L + \dfrac{1}{3}\sin_W^2 \gamma_{\mu}\right] \\[3mm]
\left\langle \bar{l}Z_{\mu}l \right\rangle & = & e\dfrac{1}{2}\left[\tan_W\left(2\gamma_{\mu}^R + \gamma_{\mu}^L\right) - \cot_W \gamma_{\mu}^L\right] \\[3mm]
\left\langle \bar{\nu}_l Z_{\mu}\nu_l \right\rangle & = & e\dfrac{1}{2}\left[\tan_W + \cot_W\right]\gamma_{\mu}^L
\end{array}
\tag{C.70}
$$

Matter fields-Higgs fields coupling

$$
\left\langle \bar{u}_i \Phi^+ d_j \right\rangle = \frac{ig}{\sqrt{2}}V_{ij}\left(\frac{m^i}{m_W}P_L - \frac{m^j}{m_W}P_R\right)
$$

$$\langle \bar{d}_i \Phi^- u_j \rangle = \frac{-ig}{\sqrt{2}} V_{ij}^\dagger \left(\frac{m^i}{m_W} P_L - \frac{m^j}{m_W} P_R \right)$$

$$\langle \bar{\nu}_l \Phi^+ l \rangle = \frac{-ig}{\sqrt{2}} \frac{m_l}{m_W} P_R$$

$$\langle \bar{l} \Phi^- \nu_l \rangle = \frac{-ig}{\sqrt{2}} \frac{m_l}{m_W} P_L$$

$$\langle \bar{f} H f \rangle = -\frac{ig}{2} \frac{m_f}{m_W}$$

$$\langle \bar{l} \chi l \rangle = -\frac{g}{2} \frac{m_l}{m_W} \gamma_5$$

$$\langle \bar{u} \chi u \rangle = +\frac{g}{2} \frac{m_u}{m_W} \gamma_5$$

$$\langle \bar{d} \chi d \rangle = -\frac{g}{2} \frac{m_d}{m_W} \gamma_5 \tag{C.71}$$

Gauge fields self interactions

$$\left\langle \left(\begin{array}{c} A_\mu \\ Z_\mu \end{array} \right) W_\rho^+ (p_+) W_\sigma^- (p_-) \right\rangle =$$

$$i \left(\begin{array}{c} e \\ g \cos_W \end{array} \right) \left[(p_+ - p_-)_\mu \eta_{\rho\sigma} + 2\eta_{\mu\sigma} (p_+ + p_-)_\rho - 2\eta_{\mu\rho} (p_+ + p_-)_\sigma \right.$$

$$\left. + \left(1 - \frac{1}{\xi} \right) \left(\eta_{\mu\rho} p_\sigma^- - \eta_{\mu\sigma} p_\rho^+ \right) \right]$$

$$\langle W_\mu^+ W_\nu^- W_{\mu'}^+ W_{\nu'}^- \rangle = ig^2 \left[2\eta_{\mu\mu'} \eta_{\nu\nu'} - \eta_{\mu\nu} \eta_{\mu'\nu'} - \eta_{\mu\nu'} \eta_{\mu'\nu} \right]$$

$$\langle W_\mu^3 W_{\mu'}^3 W_{\nu+}^+ W_{\nu-}^- \rangle = i \left(-g^2 \right) \left[2\eta_{\mu\mu'} \eta_{\nu+\nu-} - \eta_{\mu\nu+} \eta_{\mu'\nu-} - \eta_{\mu\nu-} \eta_{\mu'\nu+} \right]$$

$$\langle A_\mu A_{\mu'} W_{\nu+}^+ W_{\nu-}^- \rangle = i \left(-e^2 \right) \left[2\eta_{\mu\mu'} \eta_{\nu+\nu-} - \eta_{\mu\nu+} \eta_{\mu'\nu-} - \eta_{\mu\nu-} \eta_{\mu'\nu+} \right]$$

$$\langle Z_\mu Z_{\mu'} W_{\nu+}^+ W_{\nu-}^- \rangle = i \left(-g^2 \cos_W^2 \right) \left[2\eta_{\mu\mu'} \eta_{\nu+\nu-} - \eta_{\mu\nu+} \eta_{\mu'\nu-} - \eta_{\mu\nu-} \eta_{\mu'\nu+} \right]$$

$$\langle A_\mu Z_{\mu'} W_{\nu+}^+ W_{\nu-}^- \rangle = i \left(-eg \cos_W \right) \left[2\eta_{\mu\mu'} \eta_{\nu+\nu-} - \eta_{\mu\nu+} \eta_{\mu'\nu-} - \eta_{\mu\nu-} \eta_{\mu'\nu+} \right] \tag{C.72}$$

Gauge fields-Higgs interactions

$$\langle \Phi (p_+) A_\mu \Phi (p_-) \rangle = ie (p_- - p_+)_\mu$$

$$\langle \Phi (p_+) Z_\mu \Phi (p_-) \rangle = \frac{i}{2} \left(g \cos_W - g' \sin_W \right) (p_- p_+)$$

$$\langle W_\nu^+ Z_\mu \Phi^- \rangle = i \frac{gg'}{e} m_W \eta_{\mu\nu}$$

$$\langle Z_\mu Z_{\mu'} H \rangle = -\frac{i}{2} \frac{gg'}{e} m_Z \eta_{\mu\mu'}$$

$$\langle W_\mu^+ W_{\mu'}^- H \rangle = -ig m_W \eta_{\mu\mu'}$$

$$\langle W_\mu^\pm \Phi^\mp (p^\mp) H (p^H) \rangle = \pm \frac{ig}{2} \left(p_\mu^H - p_\mu^\mp \right)$$

$$\left\langle W_\mu^\pm \Phi^\mp \left(p^\mp \right) \chi \left(p^\chi \right) \right\rangle \;\; = \;\; \frac{g}{2} \left(p_\mu^\mp - p_\mu^\chi \right)$$

$$\left\langle Z_\mu \left(p^Z \right) H\chi \right\rangle \;\; = \;\; \frac{1}{2} \left(g \cos_W + g' \sin_W \right) p_\mu^Z$$

$$\left\langle Z_\mu Z_\nu \left(\begin{array}{c} HH \\ \chi\chi \end{array} \right) \right\rangle \;\; = \;\; -i\frac{1}{4} \left(g \cos_W + g' \sin_W \right)^2 \eta_{\mu\nu}$$

$$\left\langle W_\mu^+ W_\nu \left(\begin{array}{c} HH \\ \chi\chi \end{array} \right) \right\rangle \;\; = \;\; -i\frac{1}{2} g^2 \eta_{\mu\nu}$$

$$\left\langle A_\mu A_{\mu'} \Phi^+ \Phi^- \right\rangle \;\; = \;\; -i2e^2 \eta_{\mu\mu'}$$

$$\left\langle A_\mu Z_{\mu'} \Phi^+ \Phi^- \right\rangle \;\; = \;\; -ie^2 \left(\cot_W - tan_W \right) \eta_{\mu\mu'}$$

$$\left\langle Z_\mu Z_{\mu'} \Phi^+ \Phi^- \right\rangle \;\; = \;\; -ie^2 \frac{1}{2} \left(\cot_W - tan_W \right)^2 \eta_{\mu\mu'}$$

$$\left\langle W_\mu^+ W_{\mu'}^- \Phi^+ \Phi^- \right\rangle \;\; = \;\; -i\frac{g^2}{2} \eta_{\mu\mu'}$$

$$\left\langle W_\mu^\pm A_\nu H \Phi^\mp \right\rangle \;\; = \;\; -\frac{eg}{2} \eta_{\mu\nu}$$

$$\left\langle W_\mu^\pm A_\nu \chi \Phi^\mp \right\rangle \;\; = \;\; \mp i\frac{eg}{2} \eta_{\mu\nu}$$

$$\left\langle W_\mu^\pm Z_\nu H \Phi^\mp \right\rangle \;\; = \;\; \frac{eg'}{2} \eta_{\mu\nu}$$

$$\left\langle W_\mu^\pm Z_\nu \chi \Phi^\mp \right\rangle \;\; = \;\; \pm i\frac{eg'}{2} \eta_{\mu\nu}$$

$$(\text{C.73})$$

Higgs fields self interactions

$$r \;\; = \;\; \frac{m_H}{m_W}$$

$$\left\langle HHH \right\rangle \;\; = \;\; -i\frac{3}{2} g m_W r^2$$

$$\left\langle H \left(\begin{array}{c} \chi\chi \\ \Phi^+ \Phi^- \end{array} \right) \right\rangle \;\; = \;\; -i\frac{1}{2} g m_W r^2$$

$$\left\langle HHHH \right\rangle \;\; = \;\; -i\frac{3}{4} g^2 r^2$$

$$\left\langle HH\chi\chi \right\rangle \;\; = \;\; -i\frac{1}{4} g^2 r^2$$

$$\left\langle \chi\chi\chi\chi \right\rangle \;\; = \;\; -i\frac{3}{4} g^2 r^2$$

$$\left\langle \left(\begin{array}{c} HH \\ \chi\chi \end{array} \right) \Phi^+ \Phi^- \right\rangle \;\; = \;\; -i\frac{1}{4} g^2$$

$$\left\langle \Phi^+ \Phi^- \Phi^+ \Phi^- \right\rangle \;\; = \;\; -i\frac{1}{2} g^2$$

$$(\text{C.74})$$

Ghost masses

$$X_A \; : \qquad 0$$

$$
\begin{aligned}
X_Z &: & \sqrt{\eta}\, m_Z \\
X^\pm &: & \sqrt{\xi}\, m_W
\end{aligned}
\tag{C.75}
$$

Ghost-Gauge interactions

$$
\begin{aligned}
\left\langle \bar{X}_A \left(p^A \right) X^\mp W_\mu^\pm \right\rangle &= \mp i e p_\mu^A \\
\left\langle \bar{X}_Z \left(p^Z \right) X^\mp W_\mu^\pm \right\rangle &= \mp i g \cos_W p_\mu^Z \\
\left\langle \bar{X}^\pm \left(\bar{p} \right) W_\mu^3 X^\pm \left(p \right) \right\rangle &= \pm i g \left(p - \bar{p} \right)_\mu \\
\left\langle \bar{X}^\pm \left(\bar{p} \right) W_\mu^3 W_\nu^3 X^\pm \left(p \right) \right\rangle &= -i g^2 \eta_{\mu\nu} \\
\left\langle \bar{X}^\pm X^3 W_\mu^\pm \left(p^\pm \right) \right\rangle &= \mp i g p_\mu^\pm \\
\left\langle \bar{X}^\pm X^3 W_\mu^3 W_\nu^\pm \right\rangle &= i g^2 \\
\left\langle \bar{X}^\pm X^\pm W_\mu^\pm W_\nu^\mp \right\rangle &= i g^2 \\
\left\langle \bar{X}^\pm X^\mp W_\mu^\pm W_\nu^\pm \right\rangle &= -i g^2
\end{aligned}
\tag{C.76}
$$

Ghost-Higgs interactions

$$
\begin{aligned}
\left\langle \bar{X}_Z X_Z H \right\rangle &= -i \eta m_Z \frac{g g'}{e} \\
\left\langle \bar{X}_Z \Phi^\pm X^\mp \right\rangle &= i \eta m_Z \frac{g}{2} \\
\left\langle \bar{X}^\pm \begin{pmatrix} H \\ \chi \end{pmatrix} X^\pm \right\rangle &= -i \xi m_W \frac{g}{2} \; \begin{matrix} 1 \\ \pm 1 \end{matrix} \\
\left\langle \bar{X}^\pm \Phi^\pm X_A \right\rangle &= -i \xi m_W e \\
\left\langle \bar{X}^\pm \Phi^\pm X_Z \right\rangle &= -i \xi m_W e \frac{1}{\tan 2\theta_W}
\end{aligned}
\tag{C.77}
$$

Appendix D

The Background Field gauge

The background formulation of field theory is an useful tool, which allows to take full advantage of local symmetries. The point is that one can separate the field in a "classical" (we will see below in which sense) and a quantum part; to be definite one finds that for gauge theories the original symmetry is implemented via two different transformations, and the "classical" symmetry can be implemented regardless from the quantization method.

In the following we will summarize known results about this formulation and its implementation in the Standard Model (see for instance [110] and references therein).

D.1   Generalities

To be definite, let us consider the theory of a single scalar field ϕ, described by some lagrangian $\mathcal{L}(\phi,\ \partial_\mu\phi)$. The Green functions of the theory can be obtained by the generating functional

$$Z\left[J\right] = e^{iW[J]} = \int \mathcal{D}\phi \exp\left[i\int dx\mathcal{L} + J\phi\right]$$

where W is the generator of connected Green functions. Let us consider instead the same theory in presence of a fixed background field, that is,

$$e^{iW[\phi^{\mathrm{cl}},J]} = \int \mathcal{D}\phi \exp\left[i\int dx\mathcal{L}\left(\phi + \phi^{\mathrm{cl}}\right) + J\phi\right] \tag{D.1}$$

Suppose then to choose J as a function of ϕ^{cl}, in such a manner that the condition

$$\frac{d}{d\phi^{\mathrm{cl}}(x)}W\left[\phi^{\mathrm{cl}}, J\right] = -J(x)$$

is enforced. Explicitly one has then

$$\frac{\partial W}{\partial \phi^{\mathrm{cl}}(x)} + \int dy \frac{\partial J(y)}{\partial \phi^{\mathrm{cl}}(x)} \frac{\partial W}{\partial J(y)} = -J\left(\phi^{\mathrm{cl}}, x\right) \tag{D.2}$$

The sense is clear in the zero loop ($\hbar \to 0$) limit: in this case the expression for W is the extremum of the integrand under the variation of ϕ, that is

$$W\left[\phi^{\mathrm{cl}}, J\right] = extr_\phi \int dx\mathcal{L}\left(\phi + \phi^{\mathrm{cl}}\right) + J(x)\phi(x)$$

which means that a solution is given by

$$\phi = 0$$

$$J_0 = -\frac{\delta\mathcal{L}\left[\phi^{\mathrm{cl}}\right]}{\delta\phi^{\mathrm{cl}}}$$

$$W_0\left[\phi^{\mathrm{cl}}, J\left(\phi^{\mathrm{cl}}\right)\right] = \int dx\mathcal{L}\left(\phi^{\mathrm{cl}}\right) \tag{D.3}$$

Now equation D.2 can be iterated

$$J = J_0 + \hbar J_1 + \hbar^2 J_2 + \dots$$

$$W = W_0 + \hbar W_1 + \hbar^2 W_2 + \dots$$

$$J_1\left(\phi^{\mathrm{cl}}\right) = -\frac{d}{d\phi^{\mathrm{cl}}}W_1\left[\phi^{\mathrm{cl}}, J_0\right] + O\left(\hbar\right)^2 + \dots \tag{D.4}$$

The important result is that the functional $W\left[\phi^{\mathrm{cl}}, J\left(\phi^{\mathrm{cl}}\right)\right]$ so obtained concides with the generating functional $\Gamma\left(\phi^{\mathrm{cl}}\right)$ of $1PI$ Green functions of classical fields. To see this, we have the give a sense to the term "classical" we have been using until now: we will choose the condition on ϕ^{cl}

$$J\left(\phi^{\mathrm{cl}}\right) = 0$$

which substitutes the classical motion equation

$$\frac{\delta S}{\delta\phi} = 0 \tag{D.5}$$

with the perturbative form

$$\frac{\delta W}{\delta\phi^{\mathrm{cl}}} = 0 \tag{D.6}$$

By now it is almost evident the thesis: in fact

- shifting the fields one has that

$$W\left[\phi^{\mathrm{cl}}, J\right] = W\left[J\right] - \int J\phi^{\mathrm{cl}}$$

which means that one has the Legendre transform of $W\left[J\right]$, that is Γ

- The equation of motion implies

$$\frac{\partial}{\partial\phi^{\mathrm{cl}}}\Gamma\left(\phi^{\mathrm{cl}}\right) = 0$$

so the functional Γ is an effective lagrangian containing the quantum fluctuations, which allows to compute Green fucntions by a tree level expansion in terms of irreducible quantum kernels.

What about gauge theories? One has the following relevant consequences:

1. the gauge fixing needed to compute quantum fluctuations is largely arbitrary: by choosing it invariant under gauge transformations of the classical fields one ensures that the effective lagrangian Γ is invariant under the same classical transformations.

2. As the amplitudes are built of irreducible functions derived by Γ and connected by classical lines, the gauge fixing needed to determine the classical propagator is completely arbitrary, in the sense that any gauge dependence in it is canceled by the effect of Ward Identities for the irreducible vertex parts.

Having laid down the basics, we will work now on our preferred model.

D.2   Compact Notation

We will follow the notation used by Van Damme in [24, 31]; let us first consider the gauge+higgs sector:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - (D_\mu \Phi)^\dagger (D_\mu \Phi) - V\left(\Phi^\dagger \Phi\right) \\
G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + f^{abc} G^a_\mu G^b_\nu \\
D_\mu \Phi &= \partial_\mu \Phi + G^a_\mu \left(T^a \phi\right) \\
D_\mu G^a_\nu &= \partial_\mu G^a_\nu + f^{abc} G^b_\mu G^c_\nu \\
(T^a)^\dagger &= -T^a \\
\left[T^a, T^b\right] &= f^{abc} T^c \\
(D_\mu \Phi)^\dagger &= \partial_\mu \Phi^\dagger - G^a_\mu \Phi^\dagger T^a
\end{aligned}
\tag{D.7}
$$

where the index a will run over the Lie generators of the symmetry group, even for non-simple groups as is the Standard Model case: all couplings are contained in the definitions for the generators itself. The field Φ will in general be a multiplet of complex fields. Simmetry transformations are of the form

$$
\begin{aligned}
\delta G^a_\mu &= -\partial_\mu \Lambda^a + f^{abc} \Lambda^b G^c_\mu \\
\delta \Phi &= \Lambda^a T^a \Phi
\end{aligned}
\tag{D.8}
$$

where in our notation Λ are real parameters.

We want now to perform the shift $\Phi \to \Phi + \Phi^{\text{cl}}$, $G_\mu \to G_\mu + G^{\text{cl}}_\mu$. It is a matter of algebra to show then that

$$
\begin{aligned}
G^a_{\mu\nu} &\to D^{\text{cl}}_\mu G^a_\nu - D^{\text{cl}}_\nu G^a_\mu + f^{abc} G^b_\mu G^c_\nu + G^{a\,\text{cl}}_{\mu\nu} \\
D^{\text{cl}}_\mu G^a_\nu &= \partial_\mu G^a_\nu + f^{abc} G^{b\,\text{cl}}_\mu G^c_\nu \\
-(D_\mu \Phi)^\dagger (D_\mu \Phi) &\to -\left(D^{\text{cl}}_\mu \left(\Phi + \Phi^{\text{cl}}\right)\right)^\dagger D^{\text{cl}}_\mu \left(\Phi + \Phi^{\text{cl}}\right) \\
&\qquad -G^a_\mu \left[\left(D^{\text{cl}}_\mu \left(\Phi + \Phi^{\text{cl}}\right)\right)^\dagger T^a \left(\Phi + \Phi^{\text{cl}}\right)\right. \\
&\qquad\qquad \left. -\left(\Phi + \Phi^{\text{cl}}\right)^\dagger T^a D^{\text{cl}}_\mu \left(\Phi + \Phi^{\text{cl}}\right)\right] \\
&\qquad + G^a_\mu G^b_\mu \left(\Phi + \Phi^{\text{cl}}\right)^\dagger T^a T^b \left(\Phi + \Phi^{\text{cl}}\right)
\end{aligned}
\tag{D.9}
$$

where we have put in evidence the quantum gauge fields.

Suppose now that the potential V is of the Higgs form, thus giving rise to a symmetry breaking: in this notation we may then write that the classical field acquires an expectation value, thus giving rise to a shift

$$\Phi^{\mathrm{cl}} = \Phi^{\mathrm{cl}} + \varepsilon .$$

This gives rise, thanks to the last term of D.9 to a mass matrix term for the gauge fields:

$$M_{ab} = -\varepsilon^{\dagger} \left(T^{a}T^{b} + T^{b}T^{a} \right) \varepsilon . \tag{D.10}$$

The Goldstone theorem states that this mass matrix will have some zero eigenvalue, so at least one (as in the Standard Model) gauge field will remain massless: this will enforce us to introduce some gauge fixing to work in perturbation theory.

They appear in the quadratic part of the lagrangian also a set of terms involving gauge and Higgs fields, which would lead to mixed propagators:

$$-G_{\mu}^{a} \left[\partial_{\mu}\Phi T^{a}\varepsilon - \varepsilon^{\dagger}T^{a}\partial_{\mu}\Phi \right] . \tag{D.11}$$

The resulting complications in the Feynman rules can be avoided by choosing, following t'Hooft as we did in App C, an appropriate non-linear form for the gauge fixing term:

$$\mathcal{L}_{g.f.} = -\frac{1}{2\xi} \left| D_{\mu}^{\mathrm{cl}} G_{\mu}^{a} + \xi \left(\Phi^{\dagger}T^{a}\Phi^{\mathrm{cl}} - \Phi^{\dagger\mathrm{cl}}T^{a}\Phi \right) \right|^{2} \tag{D.12}$$

as the classical field acquires a "vev" ε, the resulting quadratic term combines with the one in D.11 to form a total derivative. It is important to observe that this gauge fixing is invariant under the so called type II transformations

$$\begin{array}{ll} \delta G_{\mu} = \Lambda \wedge G_{\mu} & \delta G_{\mu}^{\mathrm{cl}} = \Lambda \wedge G_{\mu}^{\mathrm{cl}} - \partial_{\mu}\Lambda \\ \delta \Phi = \Lambda^{a}T^{a}\Phi & \delta \Phi^{\mathrm{cl}} = \Lambda^{a}T^{a}\Phi^{\mathrm{cl}} \end{array} \tag{D.13}$$

this results in the classical invariance for Green functions of classical fields: in fact the accompanying transformation of quantum variables amounts to a change of variable in the functional integral.

Finally after the shift the mixing between non-physical Higgs and gauge fields disappears, and the following quadratic lagrangian results for the gauge fields

$$\mathcal{L}_{2\mathrm{gauge}} = -\frac{1}{4} \left(\partial_{\mu}G_{\nu}^{a} - \partial_{\nu}G_{\mu}^{a} \right)^{2} - \frac{1}{2\xi} \left(\partial_{\mu}G_{\mu}^{a} \right)^{2} - \frac{1}{2}G_{\mu}^{a}G_{\mu}^{b}\mathcal{M}_{ab} \tag{D.14}$$

while for the (quantum) Higgs fields one has

$$\mathcal{L}_{2\mathrm{higgs}} = - \left(\partial_{\mu}\Phi \right)^{\dagger} \partial_{\mu}\Phi - \frac{1}{2}\xi \left[\Phi^{\dagger}T^{a}\varepsilon - \varepsilon^{\dagger}T^{a}\Phi \right]^{2} - V_{2}\left(\varepsilon, \Phi \right) \tag{D.15}$$

and the last term results from the symmetry breaking potential.

D.3   Application to Standard Model

Let us write down a transcription table to use the results of the preceding section in the Standard Model.

The following correspondence holds:

$$G_\mu^a T^a = -ig\vec{W}_\mu \cdot \frac{\vec{\sigma}}{2} - ig' B_\mu \frac{Y}{2} \tag{D.16}$$

giving so $f^{abc} = g\epsilon^{abc}$ for $\{a, b, c\} \in \{1, 2, 3\}$, and zero otherwise.

On the other hand it is easier to perform already at this stage the Weinberg's rotation which gives diagonal mass matrices, and the choice of charged states; so we use the following "vector" of group generators:

$$\{T^a\} = \left\{-\frac{ig}{\sqrt{2}}\sigma^+, -\frac{ig}{\sqrt{2}}\sigma^-, -ieQ, -ieQ'\right\} \tag{D.17}$$

where $\sigma^\pm$ are the raising/lowering operators in the $SU(2)$ group, and Q, Q' are defined as in Sec. C.1.5.

We need then to write down the gauge fixing term in this notation: first note that the "vev" has the form

$$\varepsilon = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

so it results that

$$\begin{array}{ll} T^+\varepsilon = \begin{pmatrix} \frac{v}{\sqrt{2}} \\ 0 \end{pmatrix} & T^-\varepsilon = 0 \\ (-ieQ)\varepsilon = 0 & (-ieQ')\varepsilon = \frac{ie}{2\sin_W\cos_W}\varepsilon \end{array} \tag{D.18}$$

Next let us write down the type II covariant derivatives for the gauge fields:

$$\begin{aligned} D_\mu^{\mathrm{cl}} W_\nu^\pm &= \partial_\mu W_\nu^\pm \mp ig\left(W_\mu^{3\mathrm{cl}}W_\nu^\pm - W_\nu^3 W_\mu^{\pm\mathrm{cl}}\right) \\ &= \partial_\mu W_\nu^\pm \mp ie\left(A_\mu^{\mathrm{cl}}W_\nu^\pm - A_\nu W_\mu^{\pm\mathrm{cl}}\right) \\ &\quad \mp ig\cos_W\left(Z_\mu^{\mathrm{cl}}W_\nu^\pm - Z_\nu W_\mu^{\pm\mathrm{cl}}\right) \\ D_\mu^{\mathrm{cl}} A_\nu &= \partial_\mu A_\nu - ie\left(W_\mu^{+\mathrm{cl}}W_\nu^- - W_\mu^{-\mathrm{cl}}W_\nu^+\right) \\ D_\mu^{\mathrm{cl}} Z_\nu &= \partial_\mu Z_\nu - ig\cos_W\left(W_\mu^{+\mathrm{cl}}W_\nu^- - W_\mu^{-\mathrm{cl}}W_\nu^+\right) \end{aligned} \tag{D.19}$$

We will need also the gauge fixing term, which is given by

$$\begin{aligned} \mathcal{L}_{\mathrm{g.f.}} &= -\frac{1}{\xi}\left|D_\mu^{\mathrm{cl}}W_\mu^+ + \xi\left(-\frac{ig}{\sqrt{2}}\right)\left(\Phi^\dagger\sigma^-\Phi^{\mathrm{cl}} - \Phi^{\dagger\mathrm{cl}}\sigma^-\Phi\right)\right|^2 \\ &\quad - \frac{1}{2\xi}\left[D_\mu^{\mathrm{cl}}A_\mu + \xi\left(-ie\right)\left(\Phi^\dagger Q\Phi^{\mathrm{cl}} - \Phi^{\dagger\mathrm{cl}}Q\Phi\right)\right]^2 \\ &\quad - \frac{1}{2\xi}\left[D_\mu^{\mathrm{cl}}Z_\mu + \xi\left(-ie\right)\left(\Phi^\dagger Q'\Phi^{\mathrm{cl}} - \Phi^{\dagger\mathrm{cl}}Q'\Phi\right)\right]^2 \ . \end{aligned} \tag{D.20}$$

To rewrite this expression in terms of component fields, we need the value of the following terms

$$
\begin{aligned}
\Phi^\dagger \sigma^- \Phi^{\mathrm{cl}} - \Phi^{\dagger\mathrm{cl}} \sigma^- \Phi &= \frac{1}{\sqrt{2}} \left[\phi^{+\mathrm{cl}} \left(H - i\chi \right) - \phi^+ \left(H^{\mathrm{cl}} + v - i\chi^{\mathrm{cl}} \right) \right] \\
\Phi^\dagger Q \Phi^{\mathrm{cl}} - \Phi^{\dagger\mathrm{cl}} Q \Phi &= \phi^{+\mathrm{cl}} \phi^- - \phi^{-\mathrm{cl}} \phi^+ \\
\Phi^\dagger Q' \Phi^{\mathrm{cl}} - \Phi^{\dagger\mathrm{cl}} Q' \Phi &= \frac{1}{\sin_W \cos_W} \left[\left(\frac{1}{2} - \sin_W^2 \right) \left(\phi^- \phi^{+\mathrm{cl}} - \phi^{-\mathrm{cl}} \phi^+ \right) \right. \\
&\quad \left. + \frac{i}{2} \left(\left(H^{\mathrm{cl}} + v \right) \chi - H \chi^{\mathrm{cl}} \right) \right] \ .
\end{aligned}
\tag{D.21}
$$

To write down the gauge fixing lagrangian in expanded form it is easier to consider separately the different terms

Quadratic part

as expected, we have

$$
\mathcal{L}_{\mathrm{g.f.2}} = -\frac{1}{\xi} \left[\left| \partial_\mu W_\mu^+ + i\xi M_W \phi^+ \right|^2 + \frac{1}{2} \left(\partial_\mu A_\mu \right)^2 + \frac{1}{2} \left(\partial_\mu Z_\mu + \xi M_Z \chi \right)^2 \right]
\tag{D.22}
$$

Trilinear in gauge fields

$$
\begin{aligned}
\mathcal{L}_{\mathrm{g.f.3g}} &= -\frac{ig}{\xi} \left[W_\mu^{3\mathrm{cl}} \left(\partial_\nu W_\nu^+ W_\mu^- - \partial_\nu W_\nu^- W_\mu^+ \right) - W_\mu^3 \left(\partial_\nu W_\nu^+ W_\mu^{-\mathrm{cl}} - \partial_\nu W_\nu^- W_\mu^{+\mathrm{cl}} \right) \right] \\
&\quad - \frac{ig}{\xi} \left(\partial_\mu W_\mu^3 \right) \left(W_\nu^{-\mathrm{cl}} W_\nu^+ - W_\nu^{+\mathrm{cl}} W_\nu^- \right)
\end{aligned}
\tag{D.23}
$$

Quadrilinear in gauge fields

$$
\begin{aligned}
\mathcal{L}_{\mathrm{g.f.4g}} &= -\frac{g^2}{\xi} \left(W_\mu^{3\mathrm{cl}} W_\mu^+ - W_\mu^3 W_\mu^{+\mathrm{cl}} \right) \left(W_\nu^{3\mathrm{cl}} W_\nu^- - W_\nu^3 W_\nu^{-\mathrm{cl}} \right) \\
&\quad - \frac{e^2 + g^2 \cos_W}{2\xi} \left(W_\mu^{+\mathrm{cl}} W_\mu^- - W_\mu^{-\mathrm{cl}} W_\mu^+ \right) \left(W_\nu^{-\mathrm{cl}} W_\nu^+ - W_\nu^{+\mathrm{cl}} W_\nu^- \right)
\end{aligned}
\tag{D.24}
$$

Trilinear in Higgs field

$$
\begin{aligned}
\mathcal{L}_{\mathrm{g.f.3H}} =& \\
& \xi \frac{g}{2} \left(\frac{gv}{2} \right) \left[H \left(\phi^{+\mathrm{cl}} \phi^- + \phi^{-\mathrm{cl}} \phi^+ \right) + i\chi \left(\phi^+ \phi^{-\mathrm{cl}} - \phi^- \phi^{+\mathrm{cl}} \right) \right] \\
&+ \xi \frac{g}{2\cos_W} \left(\frac{gv}{2\cos_W} \right) i\chi \left[\left(1 - 2\sin_W^2 \right) \left(\phi^- \phi^{+\mathrm{cl}} - \phi^+ \phi^{-\mathrm{cl}} \right) + i \left(H^{\mathrm{cl}} \chi - H \chi^{\mathrm{cl}} \right) \right]
\end{aligned}
\tag{D.25}
$$

Quadrilinear in Higgs field

$$
\begin{aligned}
\mathcal{L}_{\text{g.f.4H}} = \\
-\xi \left(\frac{g}{2}\right)^2 \Bigg\{ \phi^{+\text{cl}}\phi^{-\text{cl}}\left(H^2 + \chi^2\right) + \phi^+\phi^-\left(H^2 + \chi^2\right)^{\text{cl}} \\
- \left(\phi^{+\text{cl}}\phi^- + \phi^+\phi^{-\text{cl}}\right)\left(HH^{\text{cl}} + \chi\chi^{\text{cl}}\right) - i\left(\phi^{+\text{cl}}\phi^- - \phi^{-\text{cl}}\phi^+\right)\left(H\chi^{\text{cl}} - H^{\text{cl}}\chi\right) \Bigg\} \\
+\xi\frac{e^2}{2}\left[\phi^{+\text{cl}}\phi^- - \phi^{-\text{cl}}\phi^+\right]^2 \\
+\frac{\xi}{2}\left(\frac{e}{2\sin_W\cos_W}\right)^2\left[\left(1 - 2\sin_W^2\right)\left(\phi^-\phi^{+\text{cl}} - \phi^+\phi^{-\text{cl}}\right) + i\left(H^{\text{cl}}\chi - H\chi^{\text{cl}}\right)\right]^2 \quad \text{(D.26)}
\end{aligned}
$$

D.3.1   Modifications to Feynman Rules

By expanding this expression in terms of component fields we obtain, as already anticipated

- a quadratic correction to the gauge lagrangian which cancels the Higgs-gauge mixing

- a gauge dependent correction to the cubic and quartic coupling of gauge fields (one or two classical legs)

- a gauge dependent correction to the mass matrix of the would-be Goldstone bosons $\phi^\pm, \chi$

- gauge independent corrections (proportional to the "vev") to the cubic interaction of gauge fields and Higgs fields

- a gauge dependent correction to the interaction of Higgs fields, (one or two classical legs)

In this Appendix we shall limit ourselves to the vertices relevant for the calculation $Z \rightarrow b\bar{b}$. Then we need to reconsider only the vertices of the χ^{cl} with $2, 3$ quantum Higgs fields.

Relevant χ^{cl}-quantum vertices

$$
\begin{aligned}
\left\langle \chi^{\text{cl}}\chi H \right\rangle &= -2i\lambda\sqrt{\frac{-\mu^2}{\lambda}} + i\xi\frac{g}{2\cos_W}\left(\frac{gv}{2\cos_W}\right) \\
&= -\frac{i}{2}gm_W\left(\frac{m_H}{m_W}\right)^2 + i\xi\frac{g}{2\cos_W}m_Z \\
\left\langle \chi^{\text{cl}}\chi\phi^+\phi^- \right\rangle &= -2i\lambda \\
&= -\frac{i}{4}g^2\left(\frac{m_H}{m_W}\right)^2 \quad \text{(D.27)}
\end{aligned}
$$

It results that only the trilinear vertex H, χ, χ takes a gauge correction. For completeness we list in the following the other relevant vertices, which are the same for quantum or classical

fields

$$\begin{aligned}
\langle \chi t\bar{t} \rangle &= \frac{g_t}{\sqrt{2}}\gamma_5 = \frac{g}{2}\frac{m_t}{m_W}\gamma_5 \\
\langle \chi b\bar{b} \rangle &= -\frac{g_b}{\sqrt{2}}\gamma_5 = -\frac{g}{2}\frac{m_b}{m_W} \\
\langle \bar{t}\phi^+ b \rangle &= ig_t P_L = \frac{ig}{\sqrt{2}}\frac{m_t}{m_W}P_L \\
\langle \bar{b}\phi^- t \rangle &= ig_t P_R = \frac{ig}{\sqrt{2}}\frac{m_t}{m_W}P_R \\
\langle \bar{t}Ht \rangle &= -\frac{ig_t}{\sqrt{2}} = -\frac{ig}{2}\frac{m_t}{m_W} \qquad \text{(idem for } b)
\end{aligned}$$

$$(D.28)$$

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