

# Generalized Maxwell Love numbers

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By elementary methods, I study the Love numbers of a homogeneous, incompressible, self-gravitating sphere characterized by a generalized Maxwell rheology, whose mechanical analogue is represented by a finite or infinite system of classical Maxwell elements disposed in parallel. Analytical, previously unknown forms of the complex shear modulus for the generalized Maxwell body are found by algebraic manipulation, and studied in the particular case of systems of springs and dashpots whose strength follows a power-law distribution. We show that the sphere is asymptotically stable for any choice of the mechanical parameters that define the generalized Maxwell body and analytical forms of the Love numbers are always available for generalized bodies composed by less than five classical Maxwell bodies. For the homogeneous sphere, “real” Laplace inversion methods based on the Post–Widder formula can be applied without performing a numerical discretization of the  $n$ -th derivative, which can be computed in a “closed-form” with the aid of the Faà di Bruno formula.

## I. INTRODUCTION

Love numbers, named after A. E. H. Love [23, 24], represent a fundamental tool in geophysics. From a physical standpoint, Love numbers basically represent properly normalized displacements and gravity potential variations in response to impulsive perturbations of a given harmonic degree. Since Love numbers for elastic Earth models can be easily generalized to the case of a linear viscoelastic rheology, they are useful to describe the response of the Earth on a broad spectrum of time-scales. As a consequence, using the Love numbers technique, it is possible to address a number of relevant problems which range from

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post-glacial deformations (see e. g., [41] and references therein) to isostatic sea level variations [45], from post-seismic deformations [26, 31] to planetary tides [11], and from Earth rotation instabilities [19, 28] to the problem of dynamic compensation of internal mass heterogeneities [35, 42].

For an elastic, homogeneous, isotropic, incompressible and self-gravitating sphere, extremely simple analytical forms exist for the Love numbers [19, 28], obtained from the solution of the Navier–Cauchy equilibrium equations by an harmonic analysis of stress, displacement fields, and incremental gravity potential [10, 22]. The classic solutions provided by Lamé [20] and Thomson [48] for the elastic compressible sphere and by Darwin [8] for the viscous incompressible sphere have been later generalized to the viscoelastic, homogeneous sphere [29, 54, 55] making use of the elastic–viscoelastic correspondence principle [5, 21].

Amongst the existing closed-forms for the Love numbers, the one pertaining to the homogeneous Maxwell sphere has played a fundamental role during the past decades [29], since the assumption a Maxwell viscoelastic rheology largely explains some of the geophysical observations accompanying post-glacial rebound and long-term mantle dynamics [30]. Current investigations in the field of global geodynamics, however, are performed using multi-stratified Earth models compatible with seismological evidence, in which the equilibrium equations are generally solved assuming a complex viscosity profile [46], whose depth-dependence is varied until surface observations (geodetically observed deformations, relative sea level and gravity field variations) are satisfactorily reproduced [30].

In this work, we go back to the homogeneous, incompressible and self-gravitating, viscoelastic sphere (hereinafter  $H$ -sphere), to discuss some aspects that have been apparently unnoticed so far, possibly because of the large success of the simple (but simultaneously realistic) Maxwell rheology, and of the ensuing numerical applications to multi-layered models. In particular, we extend the Love numbers formalism to the case of generalized (discrete) Maxwell bodies (hereinafter GMBs), whose properties are of particular interest in various fields of physics [5, 25] and geophysics [33]. In general, a GMB results from the one-dimensional arrangement of various classical Maxwell bodies (CMBs), whose material parameters are chosen so that to reproduce physical (or geophysical) observations [5]. Here we limit our attention to discrete finite or infinite GMBs obtained by elementary parallel arrangement of CMBs, and we address the problem of Laplace inversion of the

so-generalized Love numbers. More complex combinations of CMBs, such as the ladder networks, provide fractional constitutive relationships [38] which are of particular interest in the theory of electromagnetic systems [14]. Love numbers spectra corresponding to these arrangements will be considered elsewhere, in view of possible geophysical applications.

The paper is organized into four sections. After reviewing in Section II A the properties of discrete GMBs composed by a finite number of CMBs, we consider the complex shear modulus of a discrete, infinite GMB with mechanical parameters distributed according to a power-law, also giving – apparently for the first time – closed forms for the viscoelastic material functions in terms of classic special functions, reported in Section II B. Then, in Section II C, the Love numbers for the  $H$ -sphere are generalized to a rheology described by finite GMBs, also discussing their Laplace-inversion by means of traditional methods. In the final part (Section II D), we address the problem of Laplace inversion of the generalized Love numbers by means of Post’s formula [32]. Seen the simple structure of Love numbers in the Laplace domain (this is a consequence of the geometrical simplicity of the  $H$ -sphere), “closed forms” are available for the second-order Bell polynomials that enter the Faà di Bruno formula [17], hence, in principle, the  $n$ -derivative of the Love numbers – required in Post’s formula – is available analytically.

## II. RESULTS

### A. Discrete GMBs

The classical Maxwell body (CMB) is a simple mechanical system composed by a spring connected in series with a dashpot [5, 25, 33]. The quasi-static creep or relaxation of the CMB can be studied in the Laplace-transformed domain introducing the complex shear modulus

$$\tilde{\mu}(s) = \frac{\mu s}{s + \mu/\eta}, \quad (1)$$

where  $s = x + iy$  is the complex Laplace variable and the material parameters  $\mu$  ( $\mu > 0$ ) and  $\eta$  ( $\eta > 0$ ) represent the rigidity and the viscosity of the spring and of the dashpot, respectively. The ratio

$$\tau = \frac{\eta}{\mu} \quad (2)$$

is Maxwell relaxation time of the CMB.

Function  $\tilde{\mu}(s)$  fully describes the response of GMB, expressed by the stress–strain relationship [5]. The creep compliance  $J(s)$  and relaxation modulus  $G(s)$ , which represent the response of the GMB to a unit stress and strain, respectively, are in fact related to  $\tilde{\mu}(s)$  by  $G(s) = 2\tilde{\mu}(s)/s$  and  $J(s) = 1/2s\tilde{\mu}(s)$  (see e. g. [25]). Functions  $J(s)$  and  $G(s)$ , also referred to as material functions of the CMB, are not independent one from each other, being linked by the reciprocity relation  $J(s)G(s) = 1/s^2$  (e. g. [25]).

By the combination rule for mechanical analogues [5, 34], the complex shear modulus of a discrete GMB composed by  $N$  CMBs disposed in parallel is

$$\tilde{\mu}(s) \equiv \sum_{n=1}^N \tilde{\mu}_n(s) \quad (3)$$

where, from (1), the complex shear modulus of the of the  $n$ -th CMB is

$$\tilde{\mu}_n(s) = \frac{\mu_n s}{s + \mu_n/\eta_n}, \quad (4)$$

with rigidity  $\mu_n > 0$  and viscosity  $\eta_n > 0$ . The constant

$$\tau_n = \frac{\eta_n}{\mu_n} \quad (5)$$

represents the Maxwell relaxation time of the  $n$ -th CMB component (hereinafter, it will be assumed that times  $\tau_n$ 's are distinct).

In terms of  $\tau_n$ , the complex shear modulus of a  $N$ -elements GMB reads

$$\tilde{\mu}(s) = \sum_{n=1}^N \frac{\mu_n s}{s + 1/\tau_n}, \quad (6)$$

showing that  $\tilde{\mu}(0) = 0$  and that  $\tilde{\mu}(s)$  has exactly  $N$  isolated poles for  $s \in \mathbb{R}^-$ , located at  $s_n = -1/\tau_n$ . From

$$\frac{\partial \tilde{\mu}(s)}{\partial s} = \sum_{n=1}^N \frac{\mu_n/\tau_n}{(s + 1/\tau_n)^2} \quad (7)$$

and by the positivity of  $\mu_n$  and  $\eta_n$ , it follows that  $\tilde{\mu}(s)$  is strictly monotonic for  $s \in \mathbb{R}$ . These properties show that the  $N$  zeros of  $\tilde{\mu}(s)$  are interlacing the poles in  $s \in \mathbb{R}_0^-$  [25].

Since the  $k$ -th derivative of the complex shear modulus is

$$\tilde{\mu}^{(k)}(s) = (-1)^{k+1} k! \sum_{n=1}^N \frac{\mu_n/\tau_n}{(s + 1/\tau_n)^{k+1}}, \quad (8)$$

$\tilde{\mu}(s)$  is a  $C^\infty$  function for  $s \in \mathbb{R}_0^+$  (i. e., it is infinitely differentiable along the real positive axis) which ensures the applicability of the “real” Post–Widder Laplace inversion method

to the Love numbers problem for the homogeneous sphere, as we will discuss in Section IID below. In addition, since

$$(-1)^k \tilde{\mu}^{(k)}(s) \leq 0, \quad s \in \mathbb{R}_0^+, \quad (9)$$

we note that  $\tilde{\mu}(s)$  is a completely monotonic function (e. g. [25]).

The limit of (6) for  $N \mapsto \infty$  is not straightforward. For instance, it is clear that an infinite GMB composed of identical springs ( $\mu_n = \mu_0$ ) and dashpots ( $\eta_n = \eta_0$ ) combined in parallel does not have a finite complex shear modulus (i. e., series (6) is divergent). This shows that finite values of  $\tilde{\mu}(s)$  can be obtained only with appropriate combinations of elastic and viscous elements, with varying strengths. A case study will be investigated in the next section.

### B. A power-law, discrete GMB

We consider, as a case study, the response of a GMB with moduli following a power-law distribution, with

$$\mu_n = \frac{\mu^*}{n^p}, \quad p \in \mathbb{N}, \quad \mu^* > 0, \quad (10)$$

$$\eta_n = \frac{\eta^*}{n^q}, \quad q \in \mathbb{N}, \quad \eta^* > 0, \quad (11)$$

where  $\mu^*$  and  $\eta^*$  are a reference rigidity and viscosity, whose ratio defines the time constant

$$\tau^* = \frac{\eta^*}{\mu^*}. \quad (12)$$

The two-parameters GMBs described by (10) and (11) are particularly useful since closed-form expression are available for the complex shear modulus in the case  $N = \infty$ , as we will show below. This implies, in particular, a closed-form for the material functions  $J(s)$  and  $G(s)$ , which are generally not available for finite arrangements of mechanical analogues. At the same time, a power-law distribution of material parameters is sufficiently general to be potentially useful for numerical applications in physics and geophysics. An example has been recently given by Spada [46], who has employed this distribution to study the Love numbers of a multi-stratified Earth model and has anticipated one of these analytical forms in the particular case ( $p = 0, q = 2$ ). Here the mathematical aspects

presented in [46] are considered more in detail and extended to any value of the integer exponents  $p$  and  $q$ . The case  $(p, q) \in \mathbb{R}$  will be investigated in a follow-up study.

It is now convenient to normalize the complex shear modulus

$$m(s) \equiv \frac{\tilde{\mu}(s)}{\mu^*}, \quad (13)$$

which, using (10) and (11) with (3) and (4), gives

$$m(z; p, q) = \sum_{n=1}^N \frac{z}{n^p z + n^q}, \quad (14)$$

with

$$z \equiv s\tau^* \in \mathbb{C}. \quad (15)$$

For finite values of  $N$  and arbitrary distribution of moduli, the series (14) cannot be summed to provide a closed-form complex shear modulus. However, a general result that can be easily established valid for all  $N$  values (including  $N = \infty$ ), is

$$m\left(\frac{1}{z}; q, p\right) = \frac{1}{z} m(z; p, q), \quad (16)$$

showing that the modulus of a given GMB can be obtained from that of a complementary GMB, in which springs (with distribution determined by  $p$ ) and dashpots ( $q$ ) are interchanged. As a consequence of the symmetry-duality relationship (16), the summation of (14) can be limited to  $p \leq q$ .

For a GMB composed by an infinite number of CMBs, the normalized complex shear modulus is

$$M(z; p, q) = \lim_{N \rightarrow \infty} m(z; p, q), \quad (17)$$

with  $m(z; p, q)$  given by (14). Hence we are interested in the study of the series

$$M(z; p, q) = \sum_{n=1}^{\infty} \frac{z}{n^p z + n^q}, \quad p, q \in \mathbb{N}, \quad (18)$$

for which the conditions of convergence (divergence) are the same as for the series  $\sum_{n=1}^{\infty} 1/(n^p z + n^q)$ . Since  $1/|n^p z + n^q| < 1/n^q$  and  $\sum_{n=1}^{\infty} 1/n^q$  is convergent for  $q \geq 2$ , by the Weierstrass M-test for the series of complex functions (see e. g. [13]), the (uniform) convergence of (18) in this range of  $q$  values is proved. By a similar argument, it can be easily shown that a further condition of convergence is  $p \geq 2$ . Hence, we conclude that

sufficient condition for the uniform convergence in the whole complex plane of  $M(z; p, q)$  is

$$(p, q) \in \mathbb{A}, \quad \mathbb{A} = \{p \geq 2\} \cup \{q \geq 2\}. \quad (19)$$

The divergence of (18) for  $(p, q) \notin \mathbb{A}$  can be shown in a straightforward way.

The poles of  $M(z; p, q)$  are found at

$$z_n = -n^{q-p}, \quad (20)$$

hence they are simple and, for  $p \neq q$  they are countably infinite (in the particular case  $p = q$ , the infinite GMB degenerates into a CMB with Maxwell time  $\tau^*$ , with  $M(z; p, q)$  showing a single pole  $z_1 = -1$ ). For any  $p$  and  $q$  value, the poles are  $z_n \in \mathbb{R}^-$ , and, from the general properties of complex modulus  $\tilde{\mu}(s)$ , discussed in Section II A, they are interlaced with the zeros of  $M(z; p, q)$ . Points  $z = -\infty$  and  $z = 0$  are accumulation points of poles for  $q > p$  and  $q < p$ , respectively. It is also of interest to observe that, in the limit for  $z \mapsto \infty$ ,  $M(z; p, q)$  is only determined by the strength of the springs (this is physically sound, since the limit  $z \mapsto \infty$  corresponds to the small times limit). In fact, from (18) one obtains

$$\lim_{z \mapsto \infty} M(z; p, q) = \zeta(p), \quad (21)$$

where  $\zeta$  is Riemann zeta function [1]. Hence,  $M(z; p, q)$  is bounded at  $z = \infty$  only for  $p \in \mathbb{A}$ .

With the help of tables of series [13] and of an algebraic manipulator, it is straightforward to verify that closed-form expressions for  $M(z; p, q)$  exist in the case of discrete GMB with  $N = \infty$ . As discussed in Section II A, they can be used to obtain closed-forms for the material functions  $J(s)$  and  $G(s)$  of the GMB, which are usually not available for finite values of  $N$ . These analytical formulas are useful since they allow for a compact expression of  $M(z; p, q)$  but their complexity, also manifest from the infinite number of poles and the presence of accumulation points of poles along the real negative axis, can make the Laplace inversion of Love numbers in the time domain practically problematic, as it will be discussed in Section II C.

The closed-form expressions that can be obtained by Equation (18) involve classical special functions (the derivative of the digamma function  $\psi(k, z)$  and the Riemann zeta function  $\zeta(s)$ , respectively), as illustrated in Table I for low values of  $p$  and  $q$ . Definitions

and elementary properties of these functions are found in e. g. [1]. Compact forms of  $M(z; p, q)$  however exist also for larger values of  $p$  and  $q$ . For instance, have verified that

$$M(z; 0, q) = -\frac{z}{q} \sum_{k=1}^q \frac{\psi(0, -\xi_k)}{(1 + \xi_k)^{q-1}}, \quad q \geq 3, \quad (22)$$

with  $\psi(k, z) = \frac{d}{dz}\psi(z)$  where  $\psi(z)$  is the digamma function and  $\xi_k (k = 1, \dots, q)$  are solutions of the algebraic equation  $z + (1 + \xi)^q = 0$ . By virtue of the reciprocity relationship (16), the complex modulus  $M(z; q, 0)$  ( $q \geq 3$ ) can be easily determined from (22).

Though we have only studied function  $M(z; p, q)$  for a limited number of  $p$  and  $q$  values, we conjecture that algebraic manipulation can provide ‘closed-forms’ for any value of parameters  $p$  and  $q$ , though these formulas could be too complex (and the CPU time required for manipulation exceedingly long) for being of any practical use.

TABLE I: Normalized complex modulus  $M(z; p, q)$  (18) for some GMBs wit power-law distribution of moduli, characterized by low values of  $p$  and  $q$ , with  $(p, q) \in \mathbb{A}$  (19). Here  $\gamma$  is Euler constant ( $\gamma = 0.577215\dots$ ),  $\psi(k, z)$  is the  $k$ -th derivative of the digamma function, defined as  $\psi(z) = \Gamma'(z)/\Gamma(z)$  where  $\Gamma(z)$  is the Euler gamma function, and  $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$  is Riemann zeta function [1]. Note that  $M(z; p, q)$  obeys the reciprocity-symmetry relationship (16), valid for any  $N$ .  $z = z_n$  gives the location of the poles of  $M(z; p, q)$ . For  $(p, q) = (0, 2)$ ,  $M(z; p, q)$  can be obtained from published tables (see e. g. Equation 4./1.421 of [13]), while for  $p = q$ , it follows from the definition of Riemann function [1]. All the other forms have been obtained by algebraic manipulation.

$p$	$q$	$M(z; p, q)$	$-z_n$
0	2	$^{1/2}(-1 + \pi\sqrt{z} \coth \pi\sqrt{z})$	$n^2$
2	0	$^{1/2}(-z + \pi\sqrt{z} \coth \frac{\pi}{\sqrt{z}})$	$\frac{1}{n^2}$
1	2	$\gamma + \psi(0, 1 + z)$	$n$
2	1	$z(\gamma + \psi(0, 1 + \frac{1}{z}))$	$\frac{1}{n}$
$p$	$p$	$\frac{z\zeta(p)}{1 + z}$	1



### C. Generalized Love numbers for the $H$ -sphere

At a given harmonic degree  $\ell$ , the Laplace-transformed Love numbers for the  $H$ -sphere can be cast in the form

$$\tilde{L}(s) = \frac{L_f(\ell)}{1 + \lambda^2 \frac{\mu(s)}{\mu_e}}, \quad (23)$$

where  $L_f(\ell)$  is the “fluid” limit of the Love number (i. e.,  $L_f(\ell) = \lim_{s \rightarrow 0} \tilde{L}(s)$ ),  $\tilde{\mu}(s)$  is the complex shear modulus of the CMB (or more generally, of the GMB) that mimics the rheological behavior of the sphere and  $\mu_e$  is the elastic rigidity of the sphere. With appropriate functions  $L_f = L_f(\ell)$ , Equation (23) is useful to describe vertical and horizontal component of displacement, and the incremental gravitational potential, for Love numbers of both tidal and loading type [19, 28, 41]. In Equation (23), I have introduced the non-dimensional constant

$$\lambda^2 = \frac{2\ell^2 + 4\ell + 3}{\ell} \frac{\mu_e}{\rho g a}, \quad (24)$$

where  $\rho$  is the density of the sphere,  $a$  is its radius, and  $g$  is gravity at the surface ( $g = 4/3\pi G\rho a$ ,  $G$  being Newton gravity constant). At a given degree  $\ell$ ,  $\lambda^2$  is a measure of the ratio between elastic stress (governed by  $\mu_e$ ) and gravitational stress (described by  $\rho g a$ ). For the “average” Earth,  $\mu_e/\rho g a \approx 0.60$  [57].

By substitution of (4) into (23), the Love numbers for a  $H$ -sphere with a GMB rheology can be easily studied. The poles of  $\tilde{L}(s)$ , which correspond to the zeros of  $1 + \lambda^2 \tilde{\mu}(s)/\mu_e$ , are all real and negative. In fact, recalling from Section II A that  $\tilde{\mu}(s)$  is monotonic for  $s \in \mathbb{R}$ , vanishes for  $s = 0$  and has  $N - 1$  more zeros for  $z \in \mathbb{R}^-$ , the zeros of  $1 + \lambda^2 \tilde{\mu}(s)/\mu_e$  must be found for  $s \in \mathbb{R}^-$ , being shifted to the left relative to those of  $\tilde{\mu}(s)$  because of the additive term “1”.

From above, we conclude that *any* (incompressible)  $H$ -sphere with GMB rheology is stable with respect to surface or tidal loading, for perturbations of any harmonic degree and regardless of its material properties. This also holds for  $N = \infty$ , since adding more CMBs to the system would not change qualitatively the distribution of the zeros of  $\tilde{\mu}(s)$ . This stability property is certainly violated for compressible spheres of initially constant density, as clearly illustrated by [15] in the case of a simple CMB.

For a GMB composed of  $N$  elements, an analytical Laplace inversion of  $\tilde{L}(s)$  can only be obtained, in principle, for  $N \leq 4$ . This can be seen by substitution of (4) into (23).

which provides

$$F(s) \equiv \frac{\tilde{L}(s)}{L_f} = \frac{1}{1 + \lambda^2 \sum_{n=1}^N \frac{\mu'_n s}{s + 1/\tau_n}}, \quad (25)$$

where  $\mu'_n = \mu_n/\mu_e$ . Hence

$$F(s) = \frac{P(s)}{Q(s)}, \quad (26)$$

where

$$P(s) = \prod_{n=1}^N \left( s + \frac{1}{\tau_n} \right) \quad (27)$$

and

$$Q(s) = P(s) + \lambda^2 \sum_{n=1}^N \mu'_n s \left[ \prod_{\substack{n'=1 \\ n' \neq n}}^N \left( s + \frac{1}{\tau_{n'}} \right) \right] \quad (28)$$

are degree  $N$  polynomials in the variable  $s$ .

Hence, by the Heaviside expansion theorem (see e. g., [6]), the time-domain Love number for the GMB can be cast in the multi-exponential form

$$L(t) = L_e \delta(t) + \sum_{n=1}^N L_n e^{s_n t}, \quad t \geq 0, \quad (29)$$

where  $\delta$  is Dirac's delta,  $s_n$  ( $n = 1, \dots, N$ ) are the (real and negative) distinct roots of the algebraic equation

$$Q(s) = 0, \quad (30)$$

and elastic and viscoelastic components of Love number are

$$L_e = \lim_{s \rightarrow \infty} \frac{P(s)}{Q(s)}, \quad (31)$$

and

$$L_n = \frac{P(s_n)}{Q'(s_n)}, \quad n = 1, \dots, N, \quad (32)$$

respectively, where  $Q'$  is the first derivative of  $Q(s)$ . An exact solution of Equation (30) is only possible analytically for  $N \leq 4$ , since by the Abel–Ruffini “impossibility theorem”, general quintic equation cannot be solved in terms of radicals (e. g., [47]). We remark that times  $-1/s_n$  bear no obvious relationship with the time constants  $\tau_n$ , defined by (5).

The existence of closed forms for the Love numbers for GMBs with  $N \leq 4$  guarantees the possibility of obtaining analytical results for particular GMBs of great interest in

geophysics. This is the case of Burgers rheology, a four–parameters model which is traditionally represented by a CMB combined in series with a Kelvin–Voigt element (see e. g., [33]), and widely employed in the study of post–seismic deformations [26, 31], post–glacial rebound [18, 27, 37, 56] and planetary dynamics [53]. Since it has been shown that such disposition is mechanically equivalent to a four–elements GMB composed of two CMBs in parallel [27], a closed–form expression of the type (29) with  $N = 2$  is certainly possible for the Burgers  $H$ –sphere, where the explicit relationship between  $L_e$ ,  $L_n$  and  $s_n$  ( $i = 1, \dots, N$ ) and the four free parameters of the Burgers body ( $\mu_1, \mu_2, \eta_1, \eta_2$ ) can be obtained by lengthy algebra, since a quartic equation is involved.

For  $N \geq 5$ , the Laplace inversion of the Love numbers can be only performed by a numerical evaluation of the roots of polynomial  $Q(s)$  in Equation (28), again followed by the application of Heaviside expansion theorem. The multi–exponential form given by (29) is therefore still formally valid for  $N \geq 5$ , but the coefficients cannot be expressed explicitly in terms of the mechanical parameters of the GMB.

#### D. Post–Widder formula and the $H$ –sphere

The simple analytical structure of the generalized Love numbers (25) allows, at least formally, alternative approaches to the Laplace inversion, based on ”real” methods such as the Post–Widder (PW) formula [32, 51, 52] (a modern, detailed proof of Post’s inversion formula can be found in [3], with a nice comment on the ill–posedness). In numerical applications (e. g., [49]), the main advantage of PW formula is that it does not require root–finding numerical algorithms, which can become unreliable especially for large  $N$ , when equation (30) may possess densely packed (and thus numerically difficult to resolve) roots on the real negative axis [44]. In the context of this study, as we have discussed in Section II B, for  $N \mapsto \infty$ , the roots are countably infinite and accumulation points of poles appear, that enhances the numerical difficulties.

The PW formula requires the computation the derivatives  $\tilde{L}^{(n)}(s)$  along the real positive axis (hence the attribute *real*) and the evaluation of the limit of a sequence according to

$$L(t) = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \tilde{L}^{(n)}\left(\frac{n}{t}\right) \quad (33)$$

[32, 51, 52] requires  $\tilde{L}(s) \in C^\infty$  for  $s \in \mathbb{R}^+$  [6]. The convergence of sequence (33) is logarith-

mically slow, but it can be efficiently accelerated [49, 50] without seriously compromising the performance of numerical computations – at least in the geophysical applications performed so far [44, 46]. Lacking, in general, an analytical expression for  $\tilde{L}^{(n)}(s)$ , numerical application of the PW formula requires a finite-difference discretization, a noisy numerical operation that demands a multi-precision environment (a nice tool is offered by FMLIB [40]) to prevent the phenomenon of catastrophic cancellation [39]. As we will show below,  $\tilde{L}^{(k)}(s)$  can be evaluated analytically in the present context, thus avoiding numerical the discretization which constitutes a major limitation of the PW method.

Application of the PW inversion method to the Love number problem for the  $H$ -sphere is feasible, since  $\tilde{L}(s)$  is smooth (i. e.,  $\tilde{L}(s) \in C^\infty$ ) for  $s \in \mathbb{R}^+$ , being  $\tilde{\mu}(s)$  itself smooth in this interval. Writing

$$F(s) \equiv \frac{\tilde{L}(s)}{L^f(\ell)} \quad (34)$$

gives

$$F(s) = \frac{1}{1+g}, \quad (35)$$

with

$$g = g(s) \equiv \lambda^2 \frac{\tilde{\mu}(s)}{\mu_e}. \quad (36)$$

The  $n$ -th derivative of  $F(s)$ , required in Equation (33), can be expressed using the Faà di Bruno chain rule formula [17, 36] for the derivative of the composite function  $F = F(g(s))$ . Namely

$$F^{(n)}(s) = \sum_{k=0}^n F^{(k)}(g) B_{n,k}(g^{(1)}, g^{(2)}, \dots, g^{(n-k+1)}), \quad (37)$$

where, using (35), the  $k$ -th derivative of  $F$  with respect to  $g$  is

$$F^{(k)}(g) = (-1)^k \frac{k!}{(1+g)^{k+1}} \quad (38)$$

and  $B_{n,k}$  denotes the incomplete Bell polynomials (also known as second kind Bell polynomials) [2, 7], defined as

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \dots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \dots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \quad (39)$$

where the sum is over all sequences of non-negative integers  $j_1, j_2, \dots, j_{n-k+1}$  which are solutions of equations  $j_1 + j_2 + \dots = k$  and  $j_1 + 2j_2 + 3j_3 + \dots = n$ .

In the present context, a “closed–form” expression for Bell polynomials can be obtained recalling that for a GMB with  $N$  elements, the  $m$ –th derivative of  $\tilde{\mu}(s)$  is given by Equation (8). Hence, for any integer  $m$ ,

$$g^{(m)}(s) = \lambda^2(-1)^{m+1} m! \sum_{n=1}^N \frac{\mu'_n/\tau_n}{(s + 1/\tau_n)^{m+1}} \quad (40)$$

can be used in the right hand side of Equation (37), obtaining a fully explicit (but extremely complex) expression for  $F^{(n)}(s)$ . In this way, the major shortcoming of the PW formula, namely the numerical noise amplification produced by repeated differentiation of  $\tilde{L}(s)$ , can be circumvented for the  $H$ –sphere (but to the cost of very complex algebraic computations). Therefore, the generalized Love numbers for the  $H$ –sphere can be expressed as a limit of the sequence (33), which in principle may constitute an alternative to the classical root–finding approach, especially for large  $N$  values.

### III. CONCLUSIONS

Our main conclusions can be summarized as follows. *i)* In the case of discrete GMBs composed of an infinite number of CMBs disposed in parallel, analytical forms for the complex shear modulus are available in the case of material parameters distributed according to an (integer) power–law (see Equation 10). These forms involve classic special functions, and are moderately simple for low values of the powers. After algebraic manipulation of several case studies, we conjecture that analytical (but exceedingly complex) moduli can always be formally determined. *ii)* For finite GMBs composed by limited number of elements (in particular,  $N \leq 4$ ), the Love numbers of the  $H$ –sphere can be determined in closed form. These Love numbers are asymptotically stable for any value of  $N$ , provided that the  $H$ –sphere is incompressible. For  $N \geq 5$ , standard numerical instruments can be used to determine the poles of the Love numbers, which could however suffer from the presence of accumulation points for the poles. Numerical difficulties in the numerical Laplace inversion of the Love numbers are well documented even in the case such singularities do not enter into play [44]. *iii)* The extremely simple algebraic form of Love numbers for the  $H$ –sphere allows for a closed–form construction of the Bell polynomials, which enter the Faà di Bruno formula for the  $n$ –th derivative [17]. Therefore, the numerical difficulties that follow from the numerical discretization of the derivative [12–50] can be partly circumvented

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