Variance reduction in a stochastic volatility scenario

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Abstract. This paper investigates the use of a variance reduction, called importance sampling, for Monte Carlo methods in the case of the stochastic volatility model for option pricing introduced by Hobson and Rogers (1998). We briefly recall that a European call option contract gives the right, but not the obligation, to buy a specific amount of a given stock or index at a specified price (strike price) in a specified time (maturity); we show some evidence on the call options on MIB30 Italian Index to verify the performance of the importance sampling in a complete stochastic volatility model. In Monte Carlo method the price of a call option is obtained as the average value of the simulations of a large number of independent, uniform variates (prices) by means of pseudo-random number generators. It is shown, finally, that variance is dramatically reduced meaning that numerical techniques introduced for variance reduction have still a lot to say.

§1. Introduction

Variance reduction methods represent a relevant contribution of stochastic calculus in finance because they improve convergence in Monte Carlo simulations in a stochastic volatility environment. The pricing of options often requires the application of Monte Carlo simulation methods because in many cases it does not exist a closed formula in a stochastic volatility framework and the numerical solution of the associated partial differential equation may be too complicated. The main goal of the present paper is to apply one of the most common methods for variance reduction, called importance sampling (more details in [9] and [10]), to a complete model for stochastic volatility introduced by Hobson and Rogers in1998 (see [7]). We find that the variance reduction allows for a better performance of the model in terms of the Monte Carlo approximations of option prices.

This paper is the fourth of a series of papers devoted to the analysis of Hobson and Rogers model, a model that we consider relevant because of its property of maintaining completeness of the market, of taking advantage of the recent past history of a stock price and of capturing stylized facts like kurtosis and skewness in distributions.

In Figà-Talamanca and Guerra (see [1]) we compare Hobson and Rogers model with Garch model and bivariate stochastic volatility models showing the high performance of the complete one in many cases: several maturities, many types of options, shorter or longer time to maturity and moneyness values far or near to one.

In Guerra and Sorini (see [6]) we test the robustness of some numerical procedures adopted to calibrate the Hobson and Rogers model: the numerical scheme to solve all the stochastic differential equations involved, the quadrature formula to approximate the integral and the number of simulations in Monte Carlo method. Recently in Figà-Talamanca and Guerra (see [2]) we introduce an econometric procedure to evaluate the feedback parameter of Hobson and Rogers model and we test the relevance of the feedback in pricing some new dataset of options traded in financial markets. We expect to conclude our research in the Hobson and Rogers environment with the research now in progress that is about the computation of Value at Risk and of hedging strategies within the cited model.

Contents of paper are organized as follows: in section 2 we present briefly a general framework of variance reduction methods and in particular the importance sampling approach that we apply to Hobson and Rogers model in section 3. Some empirical tests are performed via some data time series of call options on MIB30 Italian index traded in 2002; selected results are collected in section 4. In section 5 we present closing observations trying to focus on the challenging aspects of the paper.

§2. Variance Reduction Method

When dealing with variance reduction methods it is possible to distinguish procedures into two classes: in the first one the methodology mainly consists in changing the underlying probability measure using the Girsanov transformation, in the second class the variance reduction takes advantage of the Monte-Carlo integration theory. Here we focus our attention on the second class because in a stochastic volatility scenario we generally compute option prices via Monte Carlo method but a summary of the most common techniques can be found in [5].

The general framework was mainly introduced by Newton (see [9]) and it can be described as follows. Given an Ito process X that satisfies the stochastic equation:

$$(2.1) \qquad X_t = X_0 + \int_0^t b\left(s, X_s\right) ds + \sum_{j=1}^m \int_0^t \sigma^j\left(s, X_s\right) dW_s^j$$

a time discrete approximation Y of X to evaluate functional of the following form:

$$(2.2) \qquad \mathbb{E}\left[f\left(X_T\right)\right]$$

where f is a given function, can be deduced.

The expectation $\mathbb{E}\left[f\left(X_T\right)\right]$ can be computed on a fixed finite time interval $[0, T]$ via a Monte Carlo method.

When applying Euler scheme as follows:

$$(2.3) \qquad \bar{X}_{p+1}^h = \bar{X}_p^h + b\left(\bar{X}_p^h\right) h + \sigma\left(\bar{X}_p^h\right) \left(W\left((p+1)h\right) - W\left(ph\right)\right)$$

where h is a discretization step, a large number N of independent realizations of the Gaussian sequence

(2.4) $$W\left((p+1)h\right) - W\left(ph\right) = \left(\triangle_{p+1}^{h}W, p \in \mathbb{N}\right)$$

has to be simulated.

For each discretization step we get N independent realizations of $\bar{X}_{p}^{h}$, that are denoted by $\bar{X}_{p}^{h}\left(\omega_i\right)$, and we compute:

(2.5) $$\widetilde{\mu} = \frac{1}{N}\sum_{i=1}^{N} f\left(\bar{X}_{p}^{h}\left(\omega_i\right)\right).$$

A well known result for the estimation in (2.2) is that the estimation error:

$$\widetilde{\mu} = \eta_N - \mathbb{E}\left[f\left(X_T\right)\right]$$

is asymptotically Gaussian distributed with mean zero and variance:

$$Var\left(\widetilde{\mu}\right) = \frac{1}{N} Var f\left(\bar{X}_{p}^{h}\left(\omega_i\right)\right)$$

The length of the confidence interval decreases only with order $N^{-\frac{1}{2}}$ as $N \to +\infty$, so to obtain small confidence interval it is necessary to begin with a small variance in the random variable $f\left(\bar{X}_{p}^{h}\left(\omega_i\right)\right)$; a good way to proceed is to fix the variance of $f\left(\bar{X}_{p}^{h}\left(\omega_i\right)\right)$ to a value which is as close as possible to the variance of $f\left(X_T\right)$.

It often happens that the variance of $f\left(X_T\right)$ is very large, this fact depends only on the form of f and on the stochastic differential equation and it suggests the possibility of constructing estimators with the same expectation and smaller variance.

By the strong law of large numbers the expression (2.5) gives an approximate value of $\mathbb{E}\left[f\left(\bar{X}_{p}^{h}\left(\omega_i\right)\right)\right]$. The choice of the number of the independent realizations to perform the Monte Carlo computation depends on the desired accuracy. In practice a good way to proceed is to estimate the maximal value of the variance of $f\left(\bar{X}_{p}^{h}\right)$ by simulating a small number N_0 of sample paths of $\left(\bar{X}_{p}^{h}\right)$ and then to simulate $N - N_0$ other samples to approximate $\mathbb{E}\left[f\left(\bar{X}_{p}^{h}\right)\right]$ with a better accuracy. But this is just the critical point of the procedure because Monte Carlo algorithms converges slowly and in practice N must be large, in particular when the variance of $f\left(X_t\right)$ increases with t. Any Monte Carlo method for evaluating the expectation of functional of the solution of a general SDE produce two kinds of error: the first error depends on the discretization scheme and it can be reduced by applying an higher order of convergence scheme like Milstein or Kloeden and Platen (more and rich details in [8]). The second error is implicit in the Monte Carlo procedure.

Newton in 1994 (see [9]) develops several variance reduction techniques for the Monte Carlo integration of functional of the solutions of Ito's SDE; in particular he deepens importance sampling (see [10]) that allows variance reduction arising from the Monte Carlo integration of $f : \mathbb{R}.^d \times C\left([0,T]; \mathbb{R}.^d\right) \rightarrow \mathbb{R}$ with respect to the multidimensional Wiener measure.

Fouque and Tullie (see [4]) apply the importance sampling developed by Newton in the case of European call options in a stochastic volatility scenario with two sources of randomness. Fouque and Tullie use an approximation of the option price obtained by a fast mean-reversion expansion and they find out that the method is numerically efficient in particular when a skew is present in the time series. The authors suggest many interesting generalizations to Asian options like barrier options and to models that account for jumps in volatility or in the underlying. Fouque and Han (see [3]) investigate in a very recent paper the importance sampling under two-factor stochastic volatility model and for Asian options.

Given an $n-$dimensional stochastic process which dynamic is given by:

$$(2.6) \qquad\qquad dP_t = P_t\alpha(t, P_t)dt + P_t\sigma(t, P_t)dW_t$$

with the usual property for drift and diffusion coefficients and for Brownian motion under $\mathbb{P}$, it is possible to define a function:

$$u\left(t, p\right) = \mathbb{E}\left[\phi\left(P_T\right) \mid P_t = p\right]$$

where $\phi\left(p\right)$ is a real function; in the next it will be the payoff of a European call option.

A Monte Carlo simulation is based on the following approximation of the function $u\left(t, p\right)$:

$$(2.7) \qquad\qquad u\left(t, p\right) \approx \frac{e^{-r(T-t)}}{N}\sum_{i=1}^{N}\phi\left(P_T^{(i)}\right)$$

where $P^{(i)}$ are independent realizations of the process. The variance reduction method is based on the research of some functions $h\left(t, p\right)$ that produce a smaller variance for the Monte Carlo approximation we are going to describe, than the variance in (2.7).

The fundamental idea is to find a probability measure equivalent to $\mathbb{P}$, called $\mathbb{Q}$ such that the new process Q_t satisfies:

$$(2.8) \qquad\qquad \left(Q_T\right)^{-1} = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

where:

$$(2.9) \qquad\qquad Q_t = \exp\left\{\int_0^t h\left(s, P_s\right)dW_s + \frac{1}{2}\int_0^t \left\|h\left(s, P_s\right)\right\|^2 ds\right\}$$

and $h\left(t, P_t\right)$ is a square integrable $W-$adapted process.

Under the new measure $\mathbb{Q}$ the two processes P_t and Q_t can be rewritten in terms of the following Brownian motion

$$(2.10) \qquad \widetilde{W}_t = W_t + \int_0^t h\left(s, P_s\right) ds$$

and they become:

$$(2.11) \qquad dP_t = P_t\left(\alpha(t, P_t) - \sigma(t, P_t)h\left(t, P_t\right)\right) dt + P_t\sigma(t, P_t)d\widetilde{W}_t$$

$$(2.12) \qquad Q_t = \exp\left\{\int_0^t h\left(s, P_s\right) d\widetilde{W}_s - \frac{1}{2}\int_0^t \left\|h\left(s, P_s\right)\right\|^2 ds\right\}$$

The function $u\left(t, p\right)$ under $\mathbb{Q}$ becomes:

$$u\left(t, p\right) = \mathbb{E}_{\mathbb{Q}}\left[\phi\left(P_T\right) Q_T \mid P_t = p\right]$$

and it can be approximated by the following Monte Carlo simulation:

$$(2.13) \qquad u\left(t, p\right) \approx \frac{1}{N}\sum_{i=1}^N \phi\left(P_T^{(i)}\right) Q_T^{(i)}$$

Importance sampling consists finally in finding functions $h\left(t, P_t\right)$ such that

$$(2.14) \qquad \mathbb{E}_{\mathbb{Q}}\left[\phi\left(P_T\right) Q_T \mid P_t = p\right] = \mathbb{E}_{\mathbb{P}}\left[\phi\left(P_T\right) \mid P_t = p\right]$$

but a smaller variance is obtained under $\mathbb{Q}$ than under $\mathbb{P}$.

If $u\left(t, p\right)$ were known then the choice for h that gives zero variance is:

$$h = -\frac{1}{u}\sigma^T \bigtriangledown u$$

where $\bigtriangledown u$ is the gradient of u with respect to the state variable p.

§3. A Complete Stochastic Volatility Model

An option contract gives the right, but not the obligation, to buy (call option) or sell (put option) a specific amount of a given stock, commodity, currency, index, or debt, at a specified price (strike price) at a specified time (fixed maturity for European options). Each option has a buyer, called the holder, and a seller, known as the writer. If the option contract is exercised, the writer is responsible for fulfilling the terms of the contract by delivering the shares to the appropriate party. In the case of a security that cannot be delivered such as an index, the contract is settled in cash. For the holder, the potential loss is limited to the price paid to acquire the option. When an option is not exercised, it expires. No shares change hands and the money spent

to purchase the option is lost. For the buyer, the upside is unlimited. The costs of trading options is higher on a percentage basis than trading the underlying stock because they protect from market risks.

Option pricing models in continuous time are generally introduced in a simple setting in which only one risky asset (a stock) is available. They describe the stock price dynamics by means of a stochastic differential equation (SDE) of the following type:

$$(3.15) \qquad\qquad dP_t = \alpha(t, P_t)dt + \sigma(t, P_t)dW_t$$

where P_t is the stock price at time t and σ is the so called volatility of the price process.

Each model is characterized by the functional form of α and σ; for example, in Black and Scholes model, α and σ are constant. Note that in most cases the stochastic differential equation in (3.15) is autonomous (i.e. α and σ do not depend explicitly on time).

The models which take into account random volatility can be divided into two broad classes: the level dependent approach, where volatility is considered as a function of the stock price level P_t, (i.e. $\sigma(t, P_t) = \sigma(P_t)$), and the exogenous stochastic volatility (SV) approach, where another SDE is introduced to describe the randomness of the volatility process, which is driven by a second source of risk.

Hobson and Rogers model (see [7]) can be viewed as a good compromise between the level-dependent and the SV approaches; the random volatility σ may depend on the entire path of the stock price P_t by introducing a feedback effect into the volatility process itself. By this way the volatility process becomes more flexible without any need of new source of randomness.

In addition, the Hobson and Rogers market is complete: every contingent claim is attainable in the market and its price is obtained by equilibrium or no-arbitrage arguments. Roughly speaking, market completeness is guaranteed when the number of stocks and sources of randomness is the same.

More precisely, Hobson and Rogers specify the volatility as a function of a vector of state variables $\left(X_t^{(1)}, X_t^{(2)}, ..., X_t^{(m)}\right)$, which are called offsets, defined as follows:

$$(3.16) \qquad\qquad X_t^{(m)} = \int\limits_0^\infty \nu e^{-\nu u} \left(Z_t - Z_{t-u}\right)^m du$$

where Z_t is the discounted log-price process defined as:

$$(3.17) \qquad\qquad Z_t = \log\left\{e^{-rt}P_t\right\}$$

and ν is a positive constant parameter referred to as the feedback parameter. A crucial point in the calibration of Hobson and Rogers model is the estimation of the feedback parameter ν: Figà-Talamanca and Guerra (see [2]) suggest an econometric procedure for the estimation of the feedback parameter based on the analysis of the autocorrelation function of the quadratic yields and we present an extensive analysis on calibration of Hobson and Rogers model showing that it has a high capability in fitting real markets. The empirical evidence is based on the time series of call

options on CAC40 index in 1999 (MONEP French market) and on MIB30 index in 2002 (IDEM Italian market).

The capability of Hobson and Rogers model to capture option prices is highlighted in Figà-Talamanca and Guerra (see [1]) in terms of comparison with some important stochastic volatility models like GARCH models and stochastic volatility models with two sources of risk.

The state variables in (3.16) represent the exponentially weighted moments of the historical logreturns according to different time scales. In practice the logreturn for the time interval $(t - u, t)$ becomes less significant in the definition of the offsets as long as the time-lag u increases; the feedback parameter ν represents the rate at which past information on logreturns is actually discounted.

The process Z_t solves the following SDE:

$$(3.18) \qquad dZ_t = \alpha(X_t^{(1)}, X_t^{(2)}, ..., X_t^{(m)})dt + \sigma(X_t^{(1)}, X_t^{(2)}, ..., X_t^{(m)})dW_t$$

where $\alpha\,()$ and $\sigma\,()$ are Lipschitz functions and $\sigma\,()$ is strictly positive.

Hobson and Rogers compute the option price by numerically solving a PDE subject to some boundary condition. To reduce the complexity of the model, volatility is allowed to depend only on the first order offset (hence $X_t^{(1)} = X_t$) with the following functional dependence:

$$(3.19) \qquad \sigma\left(X_t\right) = \inf\left\{\eta\sqrt{1 + \epsilon X_t^2}, M\right\}$$

where M is large constant that guarantees the function σ to be bounded. Hobson and Rogers show that the simple setting (3.19) accounts for the possibilities of smiles and skews in the implied volatility structure; in particular, the size of the smile in the term structure of volatility is directly related to the parameter ϵ and inversely to the parameter ν (larger values of ν are associated with a shorter half-life lookback period). The estimation of the two other parameters η and ϵ relies on the minimization of the mean quadratic error between the volatility specification in (3.19) and a filtered time series for the volatility process; different results are achieved according to different filtering methodologies as in Figà-Talamanca and Guerra (see [2]).

In Hobson and Rogers model, the importance sampling technique to reduce variance, described in the previous section, can be applied as follows. Equation (3.18) can be rewritten in the following way:

$$(3.20) \qquad dZ_t = \alpha X_t dt + \eta\sqrt{1 + \epsilon X_t^2}dW_t$$

where X_t is the offset that in this simpler setting becomes an autonomous diffusion satisfying the SDE:

$$(3.21) \qquad dX_t = \left(\alpha(X_t) - \nu X_t\right)dt + \sigma(X_t)dW_t$$

with

$$(3.22) \qquad Z_t = \log\left\{e^{-rt}P_t\right\} = Z_0 + (X_t - X_0) + \nu\int_0^t X_u du$$

The price of a call option at time zero is computed by:

$$(3.23) \qquad P(0, p) = \mathbb{E}_{\mathbb{P}} \left[e^{-rT} \phi(p) \mid P_0 = p \right]$$

where $\phi(p) = (p - K)^+$.

It is then possible to apply the property in (2.10) to obtain $\widetilde{W}_t$ that admits the density $(Q_T)^{-1} = \frac{d\mathbb{Q}}{d\mathbb{P}}$.; under the new measure (3.23) becomes:

$$(3.24) \qquad P(0, p) = \mathbb{E}_{\mathbb{Q}} \left[e^{-rT} \phi(p) Q_T \mid P_0 = p \right]$$

First of all it is necessary to find an approximation of $P(t, p)$, then it is possible to determine $h(t, p)$ in order to approach the quantity in (3.24) via Monte Carlo simulations described in (2.13).

In Guerra and Sorini (see [6]) a robustness analysis is carried on with respect of several aspects of numerical procedures implemented in the calibration of the model. One of these aspects is to find out the minimum good number of simulations to perform Monte Carlo method: they find that this number is equal to 10000.

In details P can be determined in many ways, Fouque and Tullie suggest the small noise expansion and the fast mean-reversion expansion but they do not show any empirical example.

A second way to compute option price is based on the solution of a Partial Differential Equation associated to the model, we only give a flavour of the latter method because we do not show any empirical evidence by numerically solving a PDE; the following transformations must be introduced:

$$P = e^Z$$

$$U = Z - X$$

with the interest rate r taken to be zero and the strike price K taken to be 1, the option pricing problem must be solved for

$$V = V(Z_t, U_t, T - t; \eta, \epsilon, \nu)$$

where V satisfies the PDE:

$$(3.25) \qquad V_t = \frac{1}{2} \sigma (z - u)^2 [V_{zz} - V_z] + \nu (z - u) V_u$$

subject to the boundary condition

$$V(Z, U, 0) = \left(e^Z - 1 \right)^+ = \left(e^{\ln\left(Pe^{-r}\right)} - 1 \right)^+ = (P - 1)^+$$

A numerical solution of (3.25) must be implemented because of the difficulty to derive analytical properties of the explicit solution. In Hobson and Rogers paper it is not written how they estimate their option prices, probably they apply a finite difference scheme or use numerical integration to solve (3.25).

§4. Evidence on IDEM Market

In IDEM market, options on the MIB30 Italian index are of the European type and six maturities are available.

The estimates of the feedback that we finally take as an input for calculating model option prices are obtained mostly from the naive estimation results for the autoregressive order.

In Figà-Talamanca and Guerra (see [2]) we find out that the so-called E-CEMY calibration gives the best results with respect of the other estimation approach; it consists in obtaining a value of the feedback via an econometric procedure and then in calibrating the Hobson and Rogers volatility model finding the estimation of two additional parameters.

Estimation on historical time series is considered with daily observation frequency from December 1996 to March 2002 for the MIB30 Index.

The estimation results are then used to compute model option prices on January 23rd, February 20th and March 20th,2002. A comparison of model and market prices provides a test of Hobson and Rogers model performance in the computation of option prices and thus in fitting the implied volatility smile shown by market data.

	Strike	Market	BS(relative error)	E-CEMY(relative error)
	30000	1907	0.035	0.019
	31000	1211	0.070	0.163
	31500	771	0.321	0.156
	32000	531	0.476	0.150
(A)	32500	340	0.735	0.115
	33000	201	1.116	0.251
	33500	120	1.604	0.174
	34000	65	2.381	0.221
	35000	26	2.908	0.175
	Variance		**1.0611**	**0.0042**

In Table A we collect call options on MIB30 index traded on January, 23rd, 2002, with a time to maturity of 23 days, a feedback parameter $\nu = 12.6$ corresponding to a lookback period of 750 observations. Nine strike prices were available and we report market prices together with Black and Scholes relative errors and E-CEMY relative errors again. The variance for each method of pricing is also reported. Note that we obtain E-CEMY prices via Monte Carlo method with 10000 simulations and we apply the variance reduction in this case.

	Maturity	Variance	Variance Reduction
	30	0.0458	0.0234
(B)	58	0.0134	0.0065
	93	0.0224	0.0089
	184	0.00282	0.00013

In Table (B) we show the effects of the variance reduction for call options on MIB30 index traded on March, 20th, 2002, with four different maturities written in days and a feedback parameter $\nu = 12.6$.

	Feedback	Variance	Variance Reduction
	22.9	0.0903	0.00676
	19.38	0.0852	0.00432
(C)	15.75	0.0768	0.00211
	14	0.0767	0.00194
	12.6	0.0616	0.00086
	11.45	0.0348	0.00064
	10.96	0.0092	0.00031

In Table (C) we show the effects of the variance reduction for call options on MIB30 index traded on February, 20th, 2002, with 23 days to maturity and all the values of the feedback parameter. On this day fifteen strike prices were available ranging from 28500 to
36000.

	Feedback	Variance	Variance Reduction
	22.9	0.0976	0.0323
	19.38	0.0675	0.00762
(D)	15.75	0.0573	0.00537
	14	0.0467	0.00308
	12.6	0.0213	0.00113
	11.45	0.0098	0.00021
	10.96	0.0065	0.00015

In Table (D) we show the effects of the variance reduction for call options on MIB30 index traded on January, 23rd, 2002, with 51 days to maturity and all the values of the feedback parameter. On this day twelve strike prices sharing the same maturity were available, ranging from 31000 to 41000.

In Figure (1) all empirical results are collected for all strike prices in the three trading days in 2002: January, 23rd, February, 20th, March, 20th. The variance and reduced variance are the mean values in all cases depending on the different values of time to maturity ranging from one months to a period longer than one year. Obviously the sets of time to maturity are disjoint and < 90 days means a period from 60 to 90 days to maturity. As observed before the variance reduces as long as time to maturity increases.

§5. Conclusions

In the stochastic volatility environment of a complete market importance sampling can be used to reduce sensibly the variance in Monte Carlo simulations for options prices. In the case of call option prices on MIB30 Italian index traded in 2002 we show that the variance of relative errors can be reduced in all cases by varying the time to maturity, the feedback value (measuring the look back period) and the trading day. More tests on data are important to validate the methodology; in particular by choosing different kind of options and more liquid financial markets. Numerical techniques for variance reduction have probably to be improved but preliminary results are still satisfactory.

Figure 1: All matuirites and all strikes for three trading days; feedback corresponding to 750 past observations.

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