Stabilization of Discrete-Time Systems with Time-Varying Delay
Using Simple Lyapunov-Krasovskii Functional

A THESIS SUBMITTED IN
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CERTIFICATE

This is to certify that the thesis entitled "Stabilization of Discrete-Time Systems with Time-Varying Delay Using Simple Lyapunov-Krasovskii Functional" submitted by Mr Umesh Mahapatra in partial fulfillment of the requirements for the award of Master of Technology Degree in Electrical Engineering at National Institute of Technology, Rourkela is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University/ Institute for the award of any degree or diploma.

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DATE: _____________

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ABSTRACT

This thesis studies stabilization of discrete time-delay systems based on Lyapunov-Krasovskii functional. Time-delays frequently occur in various practical systems, such as networked control systems, chemical processes, neural networks, and long transmission lines in pneumatic systems.

The different phenomena that cause time-delay are: (a) Time needed to transport mass, energy or information; (b) Time lags get accumulated in great number of low-order systems connected in series; and (c) Sensors, such as analyzers; controllers need some time to implement a complicated control algorithm or process. The presence of delay causes in general performance degradation and may lead to instability within the system

First, stability of networked control systems has been studied. Two available stability criteria for linear discrete time systems with interval like time varying delay have been considered. Also a numerical example has been solved and the results of both the stability criteria have been compared.

Static output-feedback stabilization of discrete-time system with time-varying delay is studied next. Two stabilization approaches based on the above discussed stability criteria are studied. A numerical example has been solved and simulation results have been obtained for both the approaches.

Simulation output indicates that the given stabilization approaches effectively stabilize the system and their performance in terms of achievable delay margin is compared.

Chapter 1

INTRODUCTION

1 Introduction

In this chapter, we define stability of different operational systems. Some examples of time-delay system are discussed. Standard results involving LMIs are also analyzed.

1.1 Time-Delay System

Time delays appear in many industrial processes, economical and biological systems as indicated below:

Examples of time-delay systems

a. Regenerative chatter in metal cutting [1]: In regenerative chatter, the surface generated by the previous pass becomes the upper surface of the chip on the subsequent pass. Any imprecision in machining the desired chip thickness results in an additional force to be encountered by the tooth in the subsequent pass. This instability ultimately leads to increased tool wear, undesirable surface quality, and reduced productivity.

b. Internal combustion engine[1]: In this system, the crankshaft rotation is modeled as

$$J\dot{\omega}(t) = T_d(t-h) - T_f(t) - T_l(t)$$

where T_d is the developed torque, which is delayed by h seconds due to engine cycle delay. This delay is caused by time taken in fuel-air mixing, ignition delay, etc. T_l represents the load, T_f the friction, J the moment of inertia and ω the angular velocity of the crankshaft.

c. Feedback control system [2]: Feedback control systems as shown in Fig. 1.1 often function in presence of delays, primarily due to the time it takes to acquire the information required for decision-making, to create the control decisions, and to execute these decisions.

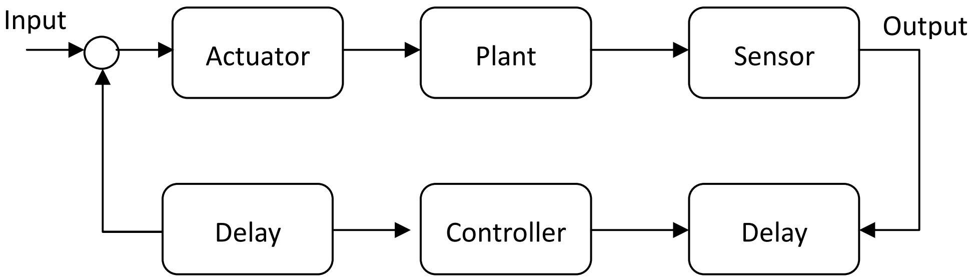


Fig. 1.1 Feedback Control System

d. Networked control system [3-5]: Network Control Systems (NCSs) as shown in Fig. 1.2 are spatially distributed systems in which the communication between sensors, actuators and controllers occurs through a shared digital communication network. Networks enable remote data transfers and data exchanges among users, reduce the complexity in wiring connections and provide ease in maintenance.

The presence of a communication network in a control loop induces many imperfections such as varying transmission delays, varying sampling/transmission intervals, packet loss, which can degrade the control performance significantly and even lead to instability.

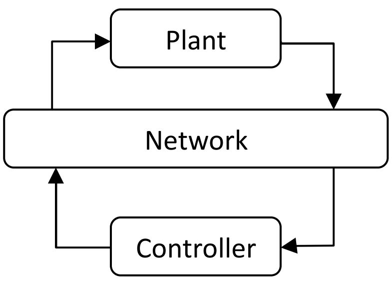


Fig. 1.2 Networked Control Systems

e. Traffic-flow models [6]: In traffic-flow models finite reaction times and estimation capabilities which impair the human driving performance and stability must be considered. The delay due to finite reaction times and estimation capabilities are critical in analyzing traffic-flow stability.

f. Material distribution and supply-chain systems [7-9]: Supply networks are an ensemble of interconnections of customers, suppliers, manufacturing units, companies, and sources that share products and information to regulate inventories and respond to customer demands. Material deliveries, information flow and decision-making result in delays which lead to poor performance, synchronization problems and fluctuations in inventory levels resulting in major economical losses.

1.2 Stability and stabilization

Time-delays frequently occurs in various practical systems, such as networked control systems, chemical processes, neural networks, and long transmission lines in pneumatic systems. Since time-delay is an important source of instability and poor performance, considerable attention has been paid to the problem of stability analysis for continuous time-delay system [10]–[14].

Stability Definition: [15]
A time-delay system:

$$\dot{x}(t) = Ax(t) + A_d x(t-d)$$

Necessary condition for stability

At d=0, $A + A_d$is Hurwitz

With $d\uparrow$

Delay-independent stability: System never becomes unstable i.e. system is stable till $d\rightarrow$infinity. A must be Hurwitz.

Delay-dependent stability: System becomes unstable after a certain value of d say $\bar{d}$.

Delay-dependent stability:

When a particular stability condition is derived which depends on the size of the delay factor, the obtained result is delay dependent stability.

Delay-independent stability:

When derived stability condition does not depend upon delay size, we eventually get delay independent stability condition.

Delay dependent stabilization provides an upper bound of the delay such that the closed loop system is stable for any delay less than the upper bound.

For a linear discrete-time system with a constant delay, by using state augmentation method and introducing some new variables, one can transform the system to an equivalent one without a time-delay, for which to be asymptotically stable the existence of a quadratic Lyapunov function is a sufficient and necessary condition. This augmentation of the system is, however, inappropriate for systems with unknown delays and for systems with time-varying delay which are the subject analysis in this work.

Interest in understanding the effects of delays and designing stabilizing controllers that account for delays is also increasing with the complexity of control systems. In particular, the effect of delays becomes more pronounced in interconnected and distributed systems, where multiple sensors, actuators, and controllers introduce multiple deterministic and stochastic delays. In interconnected systems, delays may arise from the availability of shared communication networks, such as the internet and wireless networks illustrated in Fig. 1.2.

1.3 Some Standard Results Involving LMIs

Linear Matrix Inequalities (LMIs) and LMI techniques have emerged as powerful design tools in areas ranging from control engineering to system identification and structural design.

Lemma1.1 [16]**:** For any positive definite matrix$W$, two positive integers r and $r_0$satisfying $r \geq r_0 \geq 1$, vector function$x(i)$, we have

$$\left(\sum_{i=r_0}^{r} x(i)\right)^T W \left(\sum_{i=r_0}^{r} x(i)\right) \leq \tilde{r} \sum_{i=r_0}^{r} x(i)^T W x(i)$$
$$where\ \tilde{r} = r - r_0 + 1$$

<u>Proof:</u>

$$\left(\sum_{i=r_0}^{r} x(i)\right)^T W \left(\sum_{i=r_0}^{r} x(i)\right) = \frac{1}{2}\sum_{i=r_0}^{r} \sum_{j=r_0}^{r} 2x^T(i)Wx(j)$$
$$= \frac{1}{2}\sum_{i=r_0}^{r} \sum_{j=r_0}^{r} 2\left(W^{\frac{1}{2}}x(i)\right)^T \left(W^{\frac{1}{2}}x(j)\right)$$
$$\leq \frac{1}{2}\sum_{i=r_0}^{r} \sum_{j=r_0}^{r} \left(x^T(i)Wx(i) + x^T(j)Wx(j)\right)$$
$$= r - r_0 + 1\sum_{i=r_0}^{r} x^T(i)Wx(i)$$

$$X^T Y + Y^T X \leq X^T Z X + Y^T Z^{-1} Y$$
$$X = W^{\frac{1}{2}}x(i)Y = W^{\frac{1}{2}}x(j)Z = I$$
$$2\left(W^{\frac{1}{2}}x(i)\right)^T \left(W^{\frac{1}{2}}x(j)\right) = \left(W^{\frac{1}{2}}x(i)\right)^T \left(W^{\frac{1}{2}}x(j)\right) + \left(W^{\frac{1}{2}}x(j)\right)^T \left(W^{\frac{1}{2}}x(i)\right)$$
$$\leq \left(W^{\frac{1}{2}}x(i)\right)^T \left(W^{\frac{1}{2}}x(i)\right) + \left(W^{\frac{1}{2}}x(j)\right)^T \left(W^{\frac{1}{2}}x(j)\right)$$
$$= x^T(i)Wx(i) + x(j)Wx(j)$$

This completes the proof. ■

Elimination of matrix variables in LMIs [1]

Proposition 1.1: There exists a matrix X such that

$$\begin{pmatrix} P & Q & X \\ Q^T & R & V \\ X^T & V^T & S \end{pmatrix} > 0 \qquad (1.1)$$

if and only if

$$\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} > 0 \qquad (1.2)$$

$$\begin{pmatrix} R & V \\ V^T & S \end{pmatrix} > 0 \qquad (1.3)$$

Proof: Left multiply 1.1 by $\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -V^T R^{-1} & I \end{pmatrix}$ and right multiply by its transpose, to show that (1.1) is equivalent to $\begin{pmatrix} P & Q & X - QR^{-1}V \\ Q^T & R & 0 \\ X^T - V^T R^{-1} Q^T & 0 & S - V^T R^{-1} V \end{pmatrix} > 0$. But the above is clearly satisfied for $X = QR^{-1}V$ if 1.2 and 1.3 are satisfied in view of Schur complement.

Schur Complement [1]: For matrices A, B, C the inequality $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} > 0$ is equivalent to the following two inequalities

$$A > 0$$
$$C - B^T A^{-1} B > 0$$

1.4 Outline of Thesis

The Thesis is organized into 4 different chapters as indicated below:

<u>Chapter1</u>

This chapter includes a literature survey of time-delay system. Stability and stabilization of time-delay system has been discussed. Some preliminaries of LMIs have been explained.

<u>Chapter2</u>

In this chapter two stability criteria for discrete-time system with time-varying delay has been studied. A numerical example has been solved and the results of both the criteria have been compared. Stability of uncertain time-delay system is studied using this stability criterion.

<u>Chapter3</u>

In this chapter stability of networked control system is studied. Delay-independent stability analysis is done and a numerical example is solved.

<u>Chapter4</u>

In this chapter stabilization of discrete-time system with time-varying delay is done by static output-feedback controller. A numerical example is solved and the controller gain is found out. Simulation results are obtained both for fixed delay and variable delay.

Chapter 2

STABILITY ANALYSIS OF DISCRETE TIME-DELAY SYSTEMS

2 Stability of time-delay system

The stability of discrete-time system with time-varying delay is studied. Two stability criteria have been studied. A numerical example is solved and the results of both the stability criteria are compared. Stability of uncertain time-delay system is studied.

2.1 Choice of Lyapunov-Krasovskii functional

An appropriate Lyapunov–Krasovskii functional (LKF) is used for deriving delay-dependent stability criterion. It is known that the existence of a Complete Quadratic LKF (CQLKF) is a necessary and sufficient condition for asymptotic stability of the time-delay system [17-18]. Using the CQLKF, one can obtain the Maximum Allowable Upper Bound (MAUB) of delay which is very close to the analytical delay limit for stability [19]. However; the CQLKF requires solution of partial differential equations, yielding infinite dimensional LMIs for numerical synthesis.

By defining new Lyapunov functions and by making use of novel techniques to achieve delay dependence, several results have been obtained for the stability analysis of discrete-time systems with a time-varying delay in the state [16, 20-22]. The merit of the condition in [20] lies in their less conservativeness, which is achieved by avoiding the utilization of bounding inequalities for cross products. A sum inequality has been established in [16] to derive a less-conservative criterion which are dependent on the lower and upper bounds of the time-varying delay. The summation limits in the Lyapunov functional in [20] have been taken to the upper and lower bound of the delay as compared to [16] where average of the upper and lower bound of the delay have been taken. The number of variables in the LMI derived in [20] is greater than that of [16]. New delay-range-dependent stability criteria are developed by using a finite sum inequality approach [21] which has fewer matrix variables and can provide less conservative results.

Special forms of Lyapunov–Krasovskii functional lead to simpler delay-independent and less conservative delay-dependent conditions. In the past few years, there have been various approaches to reduce the conservatism of delay-dependent conditions by choosing new Lyapunov–Krasovskii functionals.

In order to obtain some less conservative stability conditions, interval delay-dependent LKFs (IDDLKF) have been considered [23]. By checking the variation of the IDDLKF defined on the subintervals, some new delay-dependent stability criteria are derived [23].

Problem Formulation:

Consider the system with a time varying delay described by

$$\left\{ \begin{array}{l} x(k+1) = Ax(k) + A_1 x(k-d(k)) \\ x(k) = \Phi(k), -h_2 \le k \le 0 \end{array} \right\} \tag{2.1}$$

where $x(k)$ is the state; A and A_1 are known real constant matrices; $\Phi(k)$ is the initial condition; $d(k)$ is the time varying delay satisfying $h_1 \le d(k) \le h_2$ with h_1 and h_2 nonnegative integers.

We will study the asymptotical stability of this discrete-time system with time-varying delay. We will first define a Lyapunov-Krasovskii functional $V(k)$, and then calculate $\Delta V(k)$. If $\Delta V(k) \le 0$, then $V(k)$ is the Lyapunov function and the system is asymptotically stable. Then we will solve the LMIs to find the upper bound of delay for which the system is stable.

2.2 Stability Criteria 1 [23]

In this section, we will study two stability criteria for system (2.1). We now state the first delay-dependent stability criterion.

Proposition2.1: Given two nonnegative integers h_1 and h_2 satisfying $0 < h_1 < h_2$, the system (2.1) is asymptotically stable if there exist matrices $P > 0, Q_i > 0\ (i = 1,2,3), Z_j > 0\ (j = 1,2), \bar{Y} = \begin{bmatrix} Y_1^T & Y_2^T & 0 & 0 & 0 & 0 \end{bmatrix}^T$ and $\bar{W} = \begin{bmatrix} W_1^T & W_2^T & 0 & 0 & 0 & 0 \end{bmatrix}^T$

such that $\begin{bmatrix} -Z_2 & \bar{Y}^T & X \\ \bar{Y} & \bar{\Psi} & \bar{W} \\ X^T & \bar{W}^T & -Z_2 \end{bmatrix} < 0$ (2.2)

with

$$h_{12} = h_2 - h_1$$
$$\Upsilon_{11} = A^T PA - P + Q_1 + Q_2 + (1 + h_{12})Q_3 - Z_1$$
$$\Upsilon_{15} = h_1(A - I)^T Z_1, \Upsilon_{16} = h_{12}(A - I)^T Z_2$$
$$\bar{\Psi} = \begin{bmatrix} \Upsilon_{11} & A^T PA_1 & Z_1 & 0 & \Upsilon_{15} & \Upsilon_{16} \\ * & A_1^T PA_1 - Q_3 & 0 & 0 & h_1 A_1^T Z_1 & h_{12} A_1^T Z_2 \\ * & * & -Q_1 - Z_1 & 0 & 0 & 0 \\ * & * & * & -Q_2 & 0 & 0 \\ * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & -Z_2 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & -\bar{Y} + \bar{W} & \bar{Y} & -\bar{W} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\bar{Y} + \bar{W} & \bar{Y} & -\bar{W} & 0 & 0 \end{bmatrix}^T$$

<u>Proof:</u> Define a Lyapunov functional as

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) + V_5(k) + V_6(k) \quad (2.3)$$

where

$$V_1(k) = x^T(k)Px(k)$$
$$V_2(k) = \sum_{j=1}^{2}\sum_{i=k-h_j}^{k-1} x^T(i)Q_j x(i)$$
$$V_3(k) = \sum_{i=k-d(k)}^{k-1} x^T(i)Q_3 x(i)$$
$$V_4(k) = \sum_{j=-h_2+1}^{-h_1}\sum_{i=k+j}^{k-1} x^T(i)Q_3 x(i)$$
$$V_5(k) = \sum_{j=-h_1}^{-1}\sum_{i=k+j}^{k-1} h_1\Delta x^T(i)Z_1\Delta x(i)$$
$$V_6(k) = \sum_{j=-h_2}^{-h_1-1}\sum_{i=k+j}^{k-1} h_{12}\Delta x^T(i)Z_2\Delta x(i)$$

For system (2.1), one can write the following:

$$\begin{aligned}\Delta V_1(k) &= V_1(k+1) - V_1(k)\\ &= x^T(k+1)Px(k+1) - x^T(k)Px(k)\\ &= \left[Ax(k) + A_1x(k-d(k))\right]^T P\left[Ax(k) + A_1x(k-d(k))\right] - x^T(k)Px(k)\end{aligned}$$

Similarly, the following can be written corresponding to $V_2(k)$:

$$\begin{aligned}\Delta V_2(k) = V_2(k+1) - V_2(k) &= \sum_{i=k+1-h_1}^{k} x^T(i)(Q_1+Q_2)x(i) + \sum_{i=k+1-h_2}^{k} x^T(i)(Q_1+Q_2)x(i)\\ &\quad - \sum_{i=k-h_1}^{k-1} x^T(i)(Q_1+Q_2)x(i) + \sum_{i=k-h_2}^{k-1} x^T(i)(Q_1+Q_2)x(i)\\ &= \sum_{i=1}^{3} x^T(k)Q_i x(k) - \sum_{i=1}^{2} x^T(k-h_i)Q_i x(k-h_i) - x^T(k)Q_3x(k).\end{aligned}$$

Further,

$$\begin{aligned}\Delta V_3(k) = V_3(k+1) - V_3(k) &= \sum_{i=k+1-d(k+1)}^{k} x^T(i)Q_3x(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Q_3x(i)\\ &= \sum_{i=k+1-d(k+1)}^{k-1} x^T(i)Q_3x(i) + x^T(k)Q_3x(k)\\ &\quad - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Q_3x(i) - x^T(k-d(k))Q_3x(k-d(k)),.\end{aligned}$$

$$\Delta V_4(k)=V_4(k+1)-V_4(k)=\sum_{j=-h_2+1}^{-h_1}\sum_{i=k+1+j}^{k}x^T(i)Q_3x(i)-\sum_{j=-h_2+1}^{-h_1}\sum_{i=k+j}^{k-1}x^T(i)Q_3x(i)$$

$$=h_{12}x^T(k)Q_3x(k)-\sum_{i=k+1-h_2}^{k-h_1}x^T(i)Q_3x(i),$$

$$\Delta V_5(k)=V_5(k+1)-V_5(k)=\sum_{j=-h_1}^{-1}\sum_{i=k+1+j}^{k}h_1\Delta x^T(i)Z_1\Delta x(i)-\sum_{j=-h_1}^{-1}\sum_{i=k+j}^{k-1}h_1\Delta x^T(i)Z_1\Delta x(i)$$

$$=h_1^2\Delta x^T(k)Z_1\Delta x(k)-\sum_{i=k-h_1}^{k-1}h_1\Delta x^T(i)Z_1\Delta x(i),$$

$$\Delta V_6(k)=V_6(k+1)-V_6(k)=\sum_{j=-h_2}^{-h_1-1}\sum_{i=k+1+j}^{k}h_{12}\Delta x^T(i)Z_2\Delta x(i)-\sum_{j=-h_2}^{-h_1-1}\sum_{i=k+j}^{k-1}h_{12}\Delta x^T(i)Z_2\Delta x(i)$$

$$=h_{12}^2\Delta x^T(k)Z_2\Delta x(k)-\sum_{i=k-h_2}^{k-h_1}\Delta x^T(k)Z_2\Delta x(k).$$

Combining the above, one gets

$$\begin{aligned}
\Delta V(k)&=\Delta V_1(k)+\Delta V_2(k)+\Delta V_3(k)+\Delta V_4(k)+\Delta V_5(k)+\Delta V_6(k)\\
&=\left[Ax(k)+A_1x(k-d(k))\right]^T P\left[Ax(k)+A_1x(k-d(k))\right]-x^T(k)Px(k)\\
&+\sum_{i=1}^{3}x^T(k)Q_ix(k)-\sum_{i=1}^{2}x^T(k-h_i)Q_ix(k-h_i)-x^T(k-d(k))Q_3x(k-d(k))\\
&+\sum_{i=k+1-d(k+1)}^{k-1}x^T(i)Q_3x(i)-\sum_{i=k+1-d(k)}^{k-1}x^T(i)Q_3x(i)+h_{12}x^T(k)Q_3x(k)\\
&-\sum_{i=k+1-h_2}^{k-h_1}x^T(i)Q_3x(i)+\Delta x^T(k)\left(h_1^2Z_1+h_{12}^2Z_2\right)\Delta x(k)\\
&-\sum_{i=k-h_1}^{k-1}h_1\Delta x^T(i)Z_1\Delta x(i)-\sum_{i=k-h_2}^{k-h_1-1}h_{12}\Delta x^T(i)Z_2\Delta x(i).
\end{aligned}\tag{2.4}$$

Now, using Lemma1.1 one obtains

$$-\sum_{i=k-h_1}^{k-1} h_1 \Delta x^T(i) Z_1 \Delta x(i) \le -\left(\sum_{i=k-h_1}^{k-1} \Delta x(i)\right)^T Z_1 \left(\sum_{i=k-h_1}^{k-1} \Delta x(i)\right)$$

$$\sum_{i=k-h_1}^{k-1} \Delta x(i) = x(k) - x(k-h_1)$$

$$-\sum_{i=k-h_1}^{k-1} h_1 \Delta x^T(i) Z_1 \Delta x(i) \le -\left(x(k) - x(k-h_1)\right)^T Z_1 \left(x(k) - x(k-h_1)\right) \qquad (2.5)$$

Note that

$$\sum_{i=k+1-d(k+1)}^{k-1} x^T(i) Q_3 x(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i) Q_3 x(i) \le \sum_{i=k+1-h_2}^{k-h_1} x^T(i) Q_3 x(i) \qquad (2.6)$$

$$2\zeta^T(k) Y \left[x(k-h_1) - x(k-d(k)) - \sum_{i=k-d(k)}^{k-h_1-1} \Delta x(i) \right] = 0 \qquad (2.7)$$

$$2\zeta^T(k) W \left[x(k-d(k)) - x(k-h_2) - \sum_{i=k-h_2}^{k-d(k)-1} \Delta x(i) \right] = 0 \qquad (2.8)$$

where $Y = \begin{bmatrix} Y_1^T & Y_2^T & 0 & 0 \end{bmatrix}^T$ $and W = \begin{bmatrix} W_1^T & W_2^T & 0 & 0 \end{bmatrix}^T$ are constant matrices of appropriate dimensions and $\zeta(k) = \begin{bmatrix} x^T(k) & x^T(k-d(k)) & x^T(k-h_1) & x^T(k-h_2) \end{bmatrix}^T$.

The first two elements of (2.7) and (2.8) can be written as

$$2\zeta^T(k) Y \left[x(k-h_1) - x(k-d(k)) \right] = 2\zeta^T(k) Y \left\{ \begin{bmatrix} 0 & 0 & I & 0 \end{bmatrix} \zeta - \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} \zeta \right\}$$

$$= \zeta^T(k) \left\{ \begin{bmatrix} 0 & 0 & Y_1 & 0 \\ 0 & 0 & Y_2 & 0 \\ Y_1^T & Y_2^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & Y_1 & 0 & 0 \\ Y_1^T & Y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \zeta = \zeta^T(k) \begin{bmatrix} 0 & -Y_1 & Y_1 & 0 \\ -Y_1^T & -Y_2 & Y_2 & 0 \\ Y_1^T & Y_2^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \zeta$$

$$2\zeta^T(k) W \left[x(k-d(k)) - x(k-h_2) \right] = 2\zeta^T(k) W \left\{ \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} \zeta - \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix} \zeta \right\}$$

$$= \zeta^T(k) \left\{ \begin{bmatrix} 0 & W_1 & 0 & 0 \\ W_1^T & W_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & W_1 \\ 0 & 0 & 0 & W_2 \\ 0 & 0 & 0 & 0 \\ W_1^T & W_2^T & 0 & 0 \end{bmatrix} \right\} = \zeta^T(k) \begin{bmatrix} 0 & W_1 & 0 & -W_1 \\ W_1^T & W_2 & 0 & -W_2 \\ 0 & 0 & 0 & 0 \\ -W_1^T & -W_2^T & 0 & 0 \end{bmatrix} \zeta$$

From (2.1), we have

$$\Delta x(k) = x(k+1) - x(k) = (A - I)x(k) + A_1 x(k - d(k))$$

Let

$$\Psi = \begin{bmatrix} \begin{Bmatrix} A^T PA - P + Q_1 + Q_2 \\ +(1+h_{12})Q_3 - Z_1 \end{Bmatrix} & A^T PA_1 & Z_1 & 0 \\ * & A_1^T PA_1 - Q_3 & 0 & 0 \\ * & * & -Q_1 - Z_1 & 0 \\ * & * & * & -Q_2 \end{bmatrix}$$

$$+\begin{bmatrix} A-I & A_1 & 0 & 0 \end{bmatrix}^T (h_1^2 Z_1 + h_{12}^2 Z_2)\begin{bmatrix} A-I & A_1 & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & -Y+W & Y & -W \end{bmatrix} + \begin{bmatrix} 0 & -Y+W & Y & -W \end{bmatrix}^T$$

Then it is derived from (2.4)-(2.8) that

$$\begin{aligned}
\Delta V(k) &\le \left[Ax(k) + A_1 x(k-d(k)) \right]^T P \left[Ax(k) + A_1 x(k-d(k)) \right] - x^T(k)Px(k) \\
&+ \sum_{i=1}^{3} x^T(k) Q_i x(k) - \sum_{i=1}^{3} x^T(k-h_i) Q_i x(k-h_i) - x^T(k-d(k)) Q_3 x(k-d(k)) \\
&+ h_{12} x^T(k) Q_3 x(k) - (x(k) - x(k-h_1))^T Z_1 (x(k) - x(k-h_1)) \\
&+ \Delta x^T(k)(h_1^2 Z_1 + h_{12}^2 Z_2)\Delta x(k) - \sum_{i=k-h_2}^{k-h_1-1} h_{12} \Delta x^T(i) Z_2 \Delta x(i) \\
&+ 2\zeta^T(k) Y \left[x(k-h_1) - x(k-d(k)) - \sum_{i=k-d(k)}^{k-h_1-1} \Delta x(i) \right] \\
&+ 2\zeta^T(k) W \left[x(k-d(k)) - x(k-h_2) - \sum_{i=k-h_2}^{k-d(k)-1} \Delta x(i) \right] \qquad (2.9) \\
&= \zeta^T(k)\Psi\zeta(k) - 2\zeta^T(k) Y \sum_{i=k-d(k)}^{k-h_1-1} \Delta x(i) - 2\zeta^T(k) W \sum_{i=k-h_2}^{k-d(k)-1} \Delta x(i) - \sum_{i=k-d(k)}^{k-h_1-1} \Delta x^T(i) h_{12} Z_2 \Delta x(i) \\
&- \sum_{i=k-h_2}^{k-d(k)-1} \Delta x^T(i) h_{12} Z_2 \Delta x(i)
\end{aligned}$$

$$=\frac{1}{h_{12}}\sum_{i=k-d(k)}^{k-h_1-1}\left[\zeta^T(k)\Psi\zeta(k)-h_{12}\zeta^T(k)Y\Delta x(i)-h_{12}\Delta x^T(i)Y^T\zeta(k)-h_{12}\Delta x^T(i)Z_2\Delta x(i)\right]$$
$$+\frac{1}{h_{12}}\sum_{i=k-h_2}^{k-d(k)-1}\left[\zeta^T(k)\Psi\zeta(k)-h_{12}\zeta^T(k)W\Delta x(i)-h_{12}\Delta x^T(i)W^T\zeta(k)-h_{12}\Delta x^T(i)Z_2\Delta x(i)\right]$$

$$\Delta V(k)\le\frac{1}{h_{12}}\sum_{i=k-d(k)}^{k-h_1-1}\begin{bmatrix}\zeta(k)\\-h_{12}\Delta x(i)\end{bmatrix}^T\begin{bmatrix}\Psi & Y\\ Y^T & -Z_2\end{bmatrix}\begin{bmatrix}\zeta(k)\\-h_{12}\Delta x(i)\end{bmatrix}$$
$$+\frac{1}{h_{12}}\sum_{i=k-h_2}^{k-d(k)-1}\begin{bmatrix}\zeta(k)\\-h_{12}\Delta x(i)\end{bmatrix}^T\begin{bmatrix}\Psi & W\\ W^T & -Z_2\end{bmatrix}\begin{bmatrix}\zeta(k)\\-h_{12}\Delta x(i)\end{bmatrix} \tag{2.10}$$

$$\begin{bmatrix}-Z_2 & \bar{Y}^T & X\\ \bar{Y} & \bar{\Psi} & \bar{W}\\ X^T & \bar{W}^T & -Z_2\end{bmatrix}<0 \tag{2.2}$$

By proposition1.1, there exists X of appropriate dimensions such that (2.2) holds if and only if

$$\begin{bmatrix}\bar{\Psi} & \bar{Y}\\ \bar{Y}^T & -Z_2\end{bmatrix}<0,\begin{bmatrix}\bar{\Psi} & \bar{W}\\ \bar{W}^T & -Z_2\end{bmatrix}<0$$

which are equivalent to

$$\begin{bmatrix}\Psi & Y\\ Y^T & -Z_2\end{bmatrix}<0,\begin{bmatrix}\Psi & W\\ W^T & -Z_2\end{bmatrix}<0 \tag{2.11}$$

Therefore if condition (2.2) is satisfied, the condition (2.11) is satisfied. By(2.10), $\Delta V(k)\le 0$from which it is concluded that the system described by (2.1) is asymptotically stable. This completes the proof. ■

Numerical Example:

Consider the system (2.1) with $A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, A_1 = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}$

$$\begin{bmatrix} -Z_2 & \bar{Y}^T & X \\ \bar{Y} & \bar{\Psi} & \bar{W} \\ X^T & \bar{W}^T & -Z_2 \end{bmatrix} < 0 \qquad (2.2)$$

Substitute $\bar{\Psi}$ in (2.2) we get

$$\begin{bmatrix} -Z_2 & Y_1^T & Y_2^T & 0 & 0 & 0 & 0 & X \\ * & \begin{Bmatrix} A^T PA - P + Q_1 + Q_2 \\ +(1+h_{12})Q_3 - Z_1 \end{Bmatrix} & \begin{Bmatrix} A^T PA_1 - Y_1 \\ +W_1 \end{Bmatrix} & Z_1 + Y_1 & -W_1 & \{h_1 A^T Z_1 - h_1 Z_1\} & \{h_{12} A^T Z_2 - h_{12} Z_2\} & W_1 \\ * & * & \begin{Bmatrix} A_1^T PA_1 - Q_3 - Y_2 \\ +W_2 - Y_2^T + W_2^T \end{Bmatrix} & Y_2 & -W_2 & h_1 A_1^T Z_1 & h_{12} A_1^T Z_2 & W_2 \\ * & * & * & -Q_1 - Z_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -Q_2 & 0 & 0 & 0 \\ * & * & * & * & * & -Z_1 & 0 & 0 \\ * & * & * & * & * & * & -Z_2 & 0 \\ * & * & * & * & * & * & * & -Z_2 \end{bmatrix} < 0$$

This LMI is solved, and for different value of h_1, the admissible upper bound h_2 of the delay is found out.

2.3 Stability Criteria 2 [23]

Proposition2.2: For given two nonnegative integers h_1 and h_2 satisfying $0 < h_1 < h_2$, the system (2.1) is asymptotically stable if there exist matrices $P>0, Q_i>0$(i=1, 2, 3) and $Z_j > 0(j=1,2)$ such that

$$\Theta - [0 \quad I \quad 0 \quad -I \quad 0 \quad 0]^T Z_2 [0 \quad I \quad 0 \quad -I \quad 0 \quad 0] < 0 \tag{2.12}$$

$$\Theta - [0 \quad -I \quad I \quad 0 \quad 0 \quad 0]^T Z_2 [0 \quad -I \quad I \quad 0 \quad 0 \quad 0] < 0 \tag{2.13}$$

where

$$\Theta = \begin{bmatrix} \begin{Bmatrix} A^T PA - P + Q_1 \\ +Q_2 + (1+h_{12})Q_3 \\ -Z_1 \end{Bmatrix} & A^T PA_1 & Z_1 & 0 & h_1 (A-I)^T Z_1 & h_{12} (A-I)^T Z_2 \\ * & \begin{Bmatrix} A_1^T PA_1 \\ -Q_3 - 2Z_2 \end{Bmatrix} & Z_2 & Z_2 & h_1 A_1^T Z_1 & h_{12} A_1^T Z_2 \\ * & * & \begin{Bmatrix} -Q_1 - Z_1 \\ -Z_2 \end{Bmatrix} & 0 & 0 & 0 \\ * & * & * & -Q_2 - Z_2 & 0 & 0 \\ * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & -Z_2 \end{bmatrix}$$

Proof: Use the same Lyapunov functional as (2.3) to calculate $\Delta V(k)$ like (2.4). We have that

$$-\sum_{i=k-h_2}^{k-h_1-1} h_{12} \Delta x^T(i) Z_2 \Delta x(i) = -\sum_{i=k-h_2}^{k-d(k)-1} h_{12} \Delta x^T(i) Z_2 \Delta x(i) - \sum_{k-d(k)}^{k-h_1-1} h_{12} \Delta x^T(i) Z_2 \Delta x(i)$$

$$= -\sum_{i=k-h_2}^{k-d(k)-1} (h_2 - d(k)) \Delta x^T(i) Z_2 \Delta x(i)$$

$$-\sum_{i=k-h_2}^{k-d(k)-1} (d(k) - h_1) \Delta x^T(i) Z_2 \Delta x(i) - \sum_{i=k-d(k)}^{k-h_1-1} (d(k) - h_1) \Delta x^T(i) Z_2 \Delta x(i)$$

$$-\sum_{i=k-d(k)}^{k-h_1-1} (h_2 - d(k)) \Delta x^T(i) Z_2 \Delta x(i)$$

Letting $\beta - (d(k) - h_1)/h_{12}$, one obtains

$$
\begin{aligned}
-\sum_{i=k-h_2}^{k-d(k)-1}(d(k)-h_1)\Delta x^T(i)Z_2\Delta x(i) &= -\beta\sum_{i=k-h_2}^{k-d(k)-1}h_{12}\Delta x^T(i)Z_2\Delta x(i)\\
&\le -\beta\sum_{i=k-h_2}^{k-d(k)-1}(h_2-d(k))\Delta x^T(i)Z_2\Delta x(i)
\end{aligned}
$$

and

$$
-\sum_{i=k-d(k)}^{k-h_1-1}(h_2-d(k))\Delta x^T(i)Z_2\Delta x(i) = -(1-\beta)\sum_{i=k-d(k)}^{k-h_1-1}h_{12}\Delta x^T(i)Z_2\Delta x(i)
$$

$$
\le -(1-\beta)-\sum_{i=k-d(k)}^{k-h_1-1}(d(k)-h_1)\Delta x^T(i)Z_2\Delta x(i)
$$

Applying Lemma 1.1, we have

$$
\begin{aligned}
-\sum_{i=k-h_2}^{k-h_1-1}h_{12}\Delta x^T(i)Z_2\Delta x(i) \le &-(x(k-d(k))-x(k-h_2))^T Z_2(x(k-d(k))-x(k-h_2))\\
&-(x(k-h_1)-x(k-d(k)))^T Z_2(x(k-h_1)-x(k-d(k)))\\
&-\beta(x(k-d(k))-x(k-h_2))^T Z_2(x(k-d(k))-x(k-h_2))\\
&-(1-\beta)(x(k-h_1)-x(k-d(k)))^T Z_2(x(k-h_1)-x(k-d(k)))
\end{aligned}
\tag{2.14}
$$

Let

$$
\Phi_1 = \Phi-\begin{bmatrix}0 & -I & I & 0\end{bmatrix}^T Z_2\begin{bmatrix}0 & -I & I & 0\end{bmatrix}
$$

$$
\Phi_2 = \Phi-\begin{bmatrix}0 & I & 0 & -I\end{bmatrix}^T Z_2\begin{bmatrix}0 & I & 0 & -I\end{bmatrix}
$$

where

$$\Phi = \begin{bmatrix} \begin{Bmatrix} A^T PA - P \\ +Q_1 + Q_2 \\ +(1+h_{12})Q_3 \\ -Z_1 \end{Bmatrix} & A^T PA_1 & Z_1 & 0 \\ * & \begin{Bmatrix} A_1^T PA_1 - Q_3 \\ -2Z_2 \end{Bmatrix} & Z_2 & Z_2 \\ * & * & \begin{Bmatrix} -Q_1 - Z_1 \\ -Z_2 \end{Bmatrix} & 0 \\ * & * & * & -Q_2 - Z_2 \end{bmatrix}$$

$$+\begin{bmatrix} A-I & A_1 & 0 & 0 \end{bmatrix}^T \left(h_1^2 Z_1 + h_{12}^2 Z_2\right) \begin{bmatrix} A-I & A_1 & 0 & 0 \end{bmatrix} \qquad (2.15)$$

Then from (2.4)-(2.6) and (2.14) we have

$$\begin{aligned} \Delta V(k) \le\ & x^T(k)[A^T PA - P + Q_1 + Q_2 + (1+h_{12})Q_3 - Z_1 + \\ & (A-I)^T (h_1^2 Z_1 + h_{12}^2 Z_2)(A-I)]x(k) + 2x^T(k)[A^T PA_1 \\ & +(A-I)^T (h_1^2 Z_1 + h_{12}^2 Z_2)A_1]x(k-d(k)) + 2x^T(k)Z_1 x(k-h_1) \\ & +x^T(k-d(k))[A_1^T PA_1 - Q_3 - 2Z_2 + A_1^T (h_1^2 Z_1 + h_{12}^2 Z_2)A_1]x(k-d(k)) \\ & +2x^T(k-d(k))Z_2 x(k-h_1) + 2x^T(k-d(k))Z_2 x(k-h_2) \\ & -x^T(k-h_1)(Q_1 + Z_1 + Z_2)x(k-h_1) - x^T(k-h_2)(Q_2 + Z_2)x(k-h_2) \\ & -\beta(x(k-d(k)) - x(k-h_2))^T Z_2 (x(k-d(k)) - x(k-h_2)) \\ & -(1-\beta)(x(k-h_1) - x(k-d(k)))^T Z_2 (x(k-h_1) - x(k-d(k))) \\ =\ & \zeta^T(k)[(1-\beta)\Phi_1 + \beta\Phi_2]\zeta(k) \end{aligned} \qquad (2.16)$$

On the other hand, from (2.12) and (2.13) it follows $\Phi_1 < 0$ and $\Phi_2 < 0$. This implies $(1-\beta)\Phi_1 + \beta\Phi_2 < 0$. By (2.16) $\Delta V(k) \le 0$, hence the system (2.1) is asymptotically stable. This completes the proof. ■

2.4 Stability Criteria 3 [28]

Proposition 2.3: The system (2.1) with time-varying delay is asymptotically stable if there exists n×n matrices $P > 0, Q > 0, R, S$ *and* T satisfying the following LMI

$$\begin{bmatrix} G & -R+S^T & -R+T^T & (A+A_1)^T \\ * & -Q-S-S^T & -S-S^T & 0 \\ * & * & -T-T^T & -A_1^T P \\ * & * & * & -P \end{bmatrix} < 0 \qquad (2.17)$$

where $G = -P + (h_2 - h_1 + 1)Q + R + R^T$.

Proof: Let $y(k) = x(k+1) - x(k)$ and $\eta(k) = \sum_{m=k-d(k)}^{k-1} y(m)$. Choose a Lyapunov functional candidate $V(k) = V_1(k) + V_2(k) + V_3(k)$ where

$$V_1(k) = x^T(k)Px(k)$$
$$V_2(k) = \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i)$$
$$V_3(k) = \sum_{j=-h_2+2}^{-h_1+1} \sum_{i=k+j-1}^{k-1} x^T(i)Qx(i)$$

Define $\Delta V(k) = V(k+1) - V(k)$.

For system (2.1), one can write the following:

$$\Delta V_1(k) = x^T(k+1)Px(k+1) - x^T(k)Px(k)$$
$$= x^T(k)\left[(A+A_1)^T P(A+A_1) - P\right]x(k) + \sum_{m=k-d(k)}^{k-1} -2x^T(k)(A+A_1)^T PA_1 y(m)$$
$$+\left[A_1 \sum_{m=k-d(k)}^{k-1} y(m)\right]^T P\left[A_1 \sum_{m=k-d(k)}^{k-1} y(m)\right]$$
$$= x^T(k)[(A+A_1)^T P(A+A_1) - P] - 2x^T(k)(A+A_1)^T A_1\eta(k) + \eta^T(k)A_1^T PA_1\eta(k) \qquad (2.18)$$

Similarly, the following can be written corresponding to $V_2(k)$:

$$\Delta V_2(k) = \sum_{i=k+1-d(k+1)}^{k} x^T(i)Qx(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i)$$

$$= x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k))$$

$$+ \sum_{i=k-d(k+1)+1}^{k-1} x^T(i)Qx(i) - \sum_{i=k-d(k)+1}^{k-1} x^T(i)Qx(i)$$

As

$$\sum_{i=k-d(k+1)+1}^{k-1} x^T(i)Qx(i) = \sum_{i=k-h_1+1}^{k-1} x^T(i)Qx(i) + \sum_{i=k-d(k+1)+1}^{k-h_1} x^T(i)Qx(i)$$

$$\leq \sum_{i=k-d(k)+1}^{k-1} x^T(i)Qx(i) + \sum_{i=k-h_2+1}^{k-h_1} x^T(i)Qx(i)$$

we have

$$\Delta V_2(k) \leq x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) + \sum_{i=k-h_2+1}^{k-h_1} x^T(i)Qx(i) \tag{2.19}$$

Note that

$$\Delta V_3(k) = \sum_{j=-h_2+2}^{-h_1+1} \left[x^T(k)Qx(k) - x^T(k+j-1)Qx(k+j-1) \right]$$

$$= (h_2 - h_1)x^T(k)Qx(k) - \sum_{i=k-h_2+1}^{k-h_1} x^T(i)Qx(i) \tag{2.20}$$

Also we have $x(k) - x(k-d(k)) - \eta(k) = 0$

For any appropriately dimensioned matrices R, S and T, we have the following equation

$$2\left[x^T(k)R + x^T(k-d(k))S + \eta(k)^T T \right] \times \left[x(k) - x(k-d(k)) - \eta(k) \right] = 0. \tag{2.21}$$

It follows by adding (2.18), (2.19), (2.20) and (2.21) that

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k)$$
$$\le x^T(k)\left[(A+A_1)^T P(A+A_1) - P\right]x(k) - 2x^T(k)(A+A_1)^T PA_1\eta(k)$$
$$+\eta(k)^T A_1^T PA_1\eta(k) + x^T(k)Qx(k) - x^T(k-d(k))Qx(k-d(k)) + (h_2-h_1)x^T(k)Qx(k)$$
$$+2x^T(k)Rx(k) + x^T(k)[-2R+2S^T]x(k-d(k)) + x^T(k)[-2R+2T^T]\eta(k)$$
$$-2x^T(k-d(k))Sx(k-d(k)) + x^T(k-d(k))[-2S-2T^T]\eta(k) - 2\eta(k)^T T\eta(k)$$

$$= x^T(k)[(A+A_1)^T P(A+A_1) - P + (h_2-h_1+1)Q + 2R]x(k) + x^T(k)[-2R+2S^T]x(k-d(k))$$
$$+x^T(k)[-2(A+A_1)^T PA_1 - 2R + 2T^T]\eta(k) + x^T(k-d(k))[-Q-2S]x(k-d(k))$$
$$+x^T(k-d(k))[-2S-2T^T]\eta(k) + \eta(k)^T[A_1^T PA_1 - 2T]\eta(k)$$

$$= \xi(k)^T \Omega\xi(k)$$

where we define $\xi(k)^T = \begin{bmatrix} x^T(k) & x^T(k-d(k)) & \eta(k)^T \end{bmatrix}$ and

$$\Omega = \begin{bmatrix} \Omega_{11} & -R+S^T & -(A+A_1)^T PA_1 - R + T^T \\ * & -Q-S-S^T & -S-T^T \\ * & * & A_1^T PA_1 - T - T^T \end{bmatrix} \text{ where}$$

$$\Omega_{11} = (A+A_1)^T P(A+A_1) - P + (h_2-h_1+1)Q + R + R^T .$$

From this it follows that the inequality $\Omega < 0$ guarantees that $\Delta V(k) \le 0$ for all non-zero$\xi(k)$. Hence, $\Omega < 0$ guarantees that the system given in (2.1) is asymptotically stable. By schur complement, $\Omega < 0$ is equivalent to LMI (2.17). This completes the proof of Proposition 2.3. ■

2.5 Numerical Example

Consider the system (2.1) with $A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, A_1 = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}$

$$\Theta - \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 \end{bmatrix}^T Z_2 \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 \end{bmatrix} < 0 \qquad (2.12)$$
$$\Theta - \begin{bmatrix} 0 & -I & I & 0 & 0 & 0 \end{bmatrix}^T Z_2 \begin{bmatrix} 0 & -I & I & 0 & 0 & 0 \end{bmatrix} < 0 \qquad (2.13)$$

SubstituteΘ in(2.12) and (2.13) we have

$$\begin{bmatrix} \begin{Bmatrix} A^T PA - P \\ +Q_1 + Q_2 \\ +(1+h_{12})Q_3 \\ -Z_1 \end{Bmatrix} & A^T PA_1 & Z_1 & 0 & \begin{Bmatrix} h_1 A^T Z_1 \\ -h_1 Z_1 \end{Bmatrix} & \begin{Bmatrix} h_{12} A^T Z_2 \\ -h_{12} Z_2 \end{Bmatrix} \\ * & \begin{Bmatrix} A_1^T PA_1 - Q_3 \\ -3Z_2 \end{Bmatrix} & Z_2 & 2Z_2 & h_1 A_1^T Z_1 & h_{12} A_1^T Z_2 \\ * & * & \begin{Bmatrix} -Q_1 - Z_1 \\ -Z_2 \end{Bmatrix} & 0 & 0 & 0 \\ * & * & * & -Q_2 - 2Z_2 & 0 & 0 \\ * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & -Z_2 \end{bmatrix} < 0$$

$$\begin{bmatrix} \begin{Bmatrix} A^T PA - P \\ +Q_1 + Q_2 \\ +(1+h_{12})Q_3 \\ -Z_1 \end{Bmatrix} & A^T PA_1 & Z_1 & 0 & \begin{Bmatrix} h_1 A^T Z_1 \\ -h_1 Z_1 \end{Bmatrix} & \begin{Bmatrix} h_{12} A^T Z_2 \\ -h_{12} Z_2 \end{Bmatrix} \\ * & \begin{Bmatrix} A_1^T PA_1 - Q_3 \\ -3Z_2 \end{Bmatrix} & 2Z_2 & Z_2 & h_1 A_1^T Z_1 & h_{12} A_1^T Z_2 \\ * & * & \begin{Bmatrix} -Q_1 - Z_1 \\ -2Z_2 \end{Bmatrix} & 0 & 0 & 0 \\ * & * & * & -Q_2 - Z_2 & 0 & 0 \\ * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & -Z_2 \end{bmatrix} < 0$$

The above 2 LMIs are solved and for different value of h_1, the admissible upper bound h_2 of the delay is found out and compared with proposition 2.1.

h_1	6	7	10	15	20	25	30
Proposition2.1	18	18	20	23	27	31	35
Proposition2.2	16	16	18	21	25	30	34

Table2.1: Admissible upper bound h_2 for various h_1

The stability problem has been investigated for linear discrete-time systems with interval-like time-varying delays. Two delay-dependent stability criteria have been studied by utilizing novel techniques to estimate the forward difference of a new Lyapunov functional. Proposition 2.1 can lead a slightly upper bound of the delay than proposition2.2.

2.6 Uncertain Time-Delay Systems

Due to the fact that almost all existing physical and engineering systems are subjected to uncertainties, due to component aging; parameter variations or modeling errors, the concepts of robustness, robust performance and robust design have recently become common phrases in engineering literature and constitute an integral part of control systems research. By incorporating the uncertainties in the modeling of time-delay systems, we naturally obtain uncertain time-delay system (UTDS).

The system contains some elements that are uncertain. In the following system, the uncertain element is the sampler. Suppose the sampling time T=1 sec. But instead of being sampled at T=1 sec, it is being sampled at T=0.9 sec.

Consider the process model $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u$

with the state feedback gain $K = -[3.75 \quad 11.5]$.

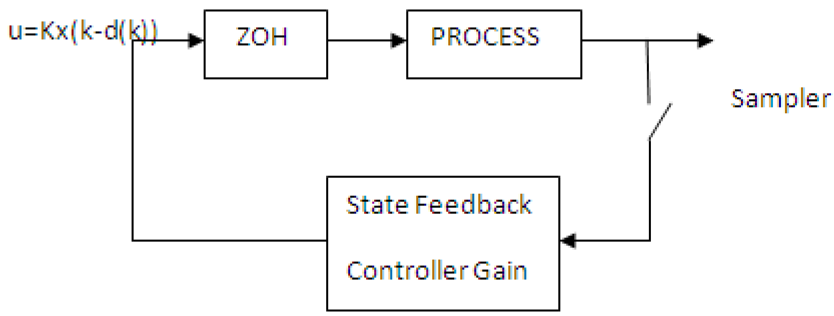


Fig. 2.1: Uncertain time-delay system

Using standard techniques from digital control, the maximum constant sampling interval for which the closed loop system remains stable is 1.7394sec.

Discretizing the above system for T=1sec, we have

$$x(k+1)=\begin{bmatrix}1 & 0.95\\ 0 & 0.9\end{bmatrix}x(k)+\begin{bmatrix}0.048\\ 0.095\end{bmatrix}u(k)$$

Due to delay because of uncertainty in sampling, we have

$$u(k)=Kx(k-d(k))$$

The system can now be written as

$$x(k+1)=\begin{bmatrix}1 & 0.95\\ 0 & 0.9\end{bmatrix}x(k)+\begin{bmatrix}-0.18 & -0.552\\ -0.356 & -1.092\end{bmatrix}x(k-d(k))$$

which is similar to system (2.1)

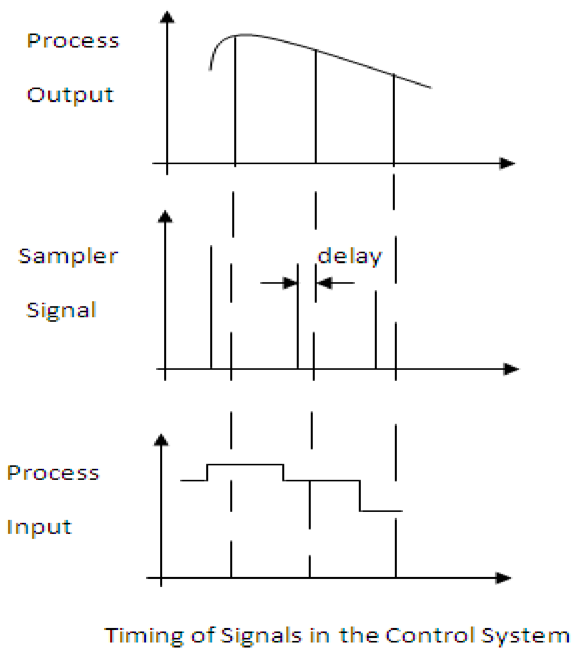


Fig. 2.2 Timing Diagram of signals of the system given in Fig. 2.1

Finding the delay dependent stability using proposition 2.1 for various values of sampling time T, we have

T(sec)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
h_2	0.5717	0.4756	0.3781	0.3031	0.2352	0.1721	0.1061	0.0228
T+h_2	1.5717	1.5756	1.5781	1.6031	1.6352	1.6721	1.7061	1.7228

Table2.2: Upper bound h_2 of delay for various sampling time T.

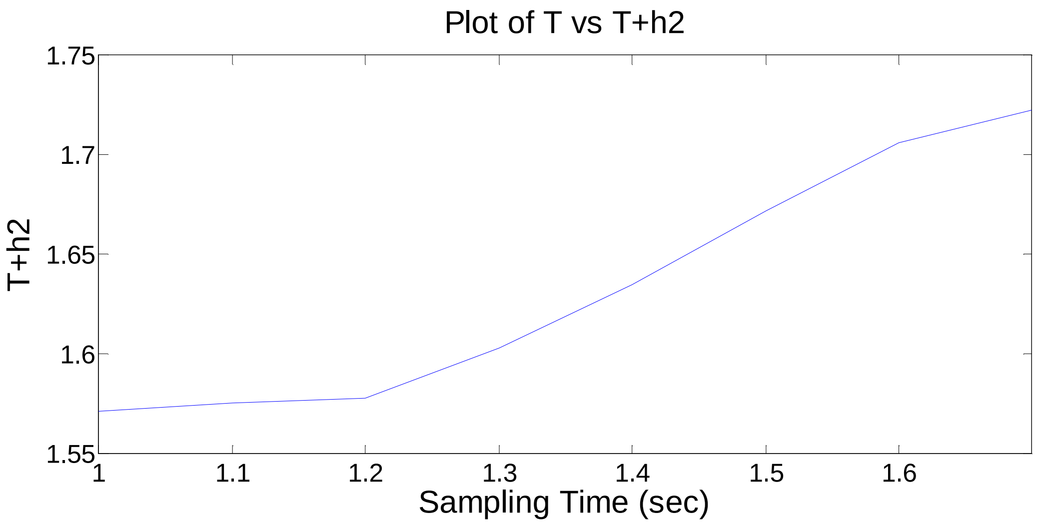


Fig2.3: Plot of T vs T+h_2

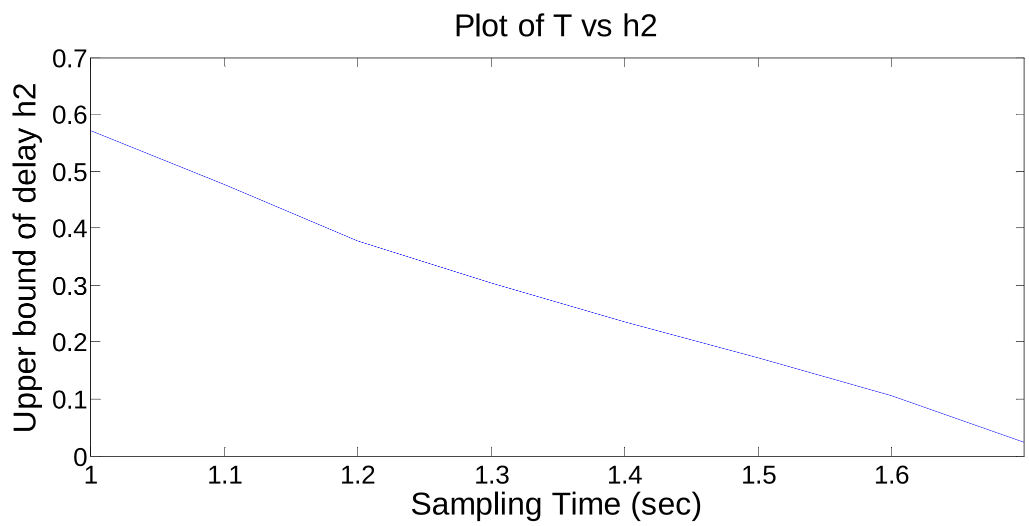


Fig2.4: Plot of T vs h_2

Conclusion:

Stability of uncertain time-delay system is studied using proposition 2.1. T+$h_2$increases with sampling time T. The upper bound of delay h_2 decreases with sampling time T. The tolerable range of delay decreases with sampling time T.

Chapter 3

STABILITY ANALYSIS OF NETWORKED CONTROL SYSTEM

3 Stability of Networked Control System

This chapter focuses on stability analysis of networked control systems.

3.1 Networked Control System

A Networked Control System (NCS) is a control system wherein the control loops are closed through a real-time network. A key feature of NCS is that control and feedback signals are exchanged among the system's components in the form of information packages through a network.

The functionality of a typical NCS consists of four basic elements

i. Sensors, to acquire information,
ii. Controllers, to provide decision and commands,
iii. Actuators, to perform the control commands and
iv. Communication network, to enable exchange of information.

The use of a communication network in the feedback path has several advantages, such as re-configurability, low installation cost, and easy maintenance. Because of the use of communication networks in control systems, network-induced delay occurs which is a potential source of instability and poor performance of NCSs.

Problem formulation:

Plant transfer function=1/s. Due to delay input to plant$u = Kx(k-h)$

Discretizing the above system, we have

$$x(k+1) = Ax(k) + A_d x(k-h)$$
$$A = 1, A_d = K$$

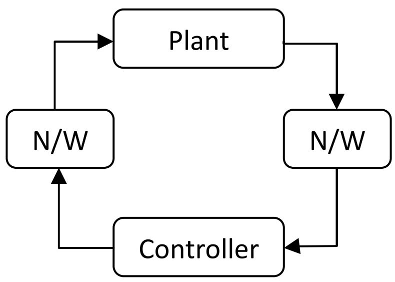


Fig. 3.1: Networked Control System

3.2 Stability Analysis

Now we will study the asymptotical stability of the above system.

Define a Lyapunov functional $V(k)=V_1(k)+V_2(k)+V_3(k)$

$$V_1(k)=x^T(k)Px(k)$$
$$V_2(k)=\sum_{j=1}^{h}x^T(k-j)Q_1x(k-j)$$
$$V_3(k)=\sum_{j=-h}^{-1}\sum_{i=k+j}^{k-1}h\Delta x^T(i)Z_1\Delta x(i)$$

<u>Calculation of $\Delta V(k)$</u>

To Calculate $\Delta V_1(k)$

$$\Delta V_1(k)=V_1(k+1)-V_1(k)=[Ax(k)+A_dx(k-h)]^TP[Ax(k)+A_dx(k-h)]-x^T(k)Px(k)$$

To Calculate $\Delta V_2(k)$

$$\begin{aligned}\Delta V_2(k)&=V_2(k+1)-V_2(k)\\&=\sum_{j=1}^{h}x^T(k+1-j)Q_1x(k+1-j)-\sum_{j=1}^{h}x^T(k-j)Q_1x(k-j)\\&=x^T(k)Q_1x(k)-x^T(k-h)Q_1x(k-h)\end{aligned}$$

To Calculate $\Delta V_3(k)$

$$\begin{aligned}\Delta V_3(k)&=V_3(k+1)-V_3(k)\\&=\sum_{j=-h}^{-1}\sum_{i=k+1+j}^{k}h\Delta x^T(i)Z_1\Delta x(i)-\sum_{j=-h}^{-1}\sum_{i=k+j}^{k-1}h\Delta x^T(i)Z_1\Delta x(i)\\&=h^2\Delta x^T(k)Z_1\Delta x(k)-\sum_{i=k-h}^{k-1}h\Delta x^T(i)Z_1\Delta x(i)\end{aligned}$$

Then it is derived that

$$\begin{aligned}\Delta V(k)&=\Delta V_1(k)+\Delta V_2(k)+\Delta V_3(k)\\&=[Ax(k)+A_dx(k-h)]^TP[Ax(k)+A_dx(k-h)]-x^T(k)Px(k)\end{aligned}$$
$$+x^T(k)Q_1x(k)-x^T(k-h)Q_1x(k-h)+h^2\Delta x^T(k)Z_1\Delta x(k)-\sum_{i=k-h}^{k-1}h\Delta x^T(i)Z_1\Delta x(i)$$

By using Lemma1.1, we have

$$\Delta V(k) \le [Ax(k) + A_d x(k-h)]^T P[Ax(k) + A_d x(k-h)] - x^T(k)Px(k)$$
$$+x^T(k)Q_1 x(k) - x^T(k-h)Q_1 x(k-h) + h^2 \Delta x^T(k) Z_1 \Delta x(k)$$
$$-[x(k) - x(k-h)]^T Z_1 [x(k) - x(k-h)]$$

$$= \zeta^T(k) \Psi \zeta(k)$$

$$\zeta(k) = [x(k) \quad x(k-h)]$$

If $\Psi < 0$, the system is asymptotically stable.

$$where \Psi = \begin{bmatrix} \begin{Bmatrix} A^T PA + Q_1 - Z_1 - P \\ +h^2 (A-I)^T Z_1 (A-I) \end{Bmatrix} & \begin{Bmatrix} A^T PA_d + Z_1 \\ +h^2 (A-I)^T Z_1 A_d \end{Bmatrix} \\ * & \begin{Bmatrix} A_d^T PA_d - Q_1 - Z_1 \\ +h^2 A_d^T Z_1 A_d \end{Bmatrix} \end{bmatrix} < 0$$

By schur complement, above LMI can be written as

$$\begin{bmatrix} \begin{Bmatrix} A^T PA + Q_1 \\ -P - Z_1 \end{Bmatrix} & A^T PA_d + Z_1 & (A-I)^T h^2 Z_1 \\ * & \begin{Bmatrix} A_d^T PA_d - Q_1 \\ -Z_1 \end{Bmatrix} & A_d^T h^2 Z_1 \\ * & * & -h^2 Z_1 \end{bmatrix} < 0$$

The above LMI is solved and the delay for different gain K is calculated.

Gain(K)	0.1	0.2	0.3	0.4	0.5
Delay(h)	14	7	4	3	2

Table 3.1: Value of delay (h) for different values of gain (K)

3.3 Delay- independent stability [26]

$$x(k+1) = Ax(k) + A_d x(k-h)$$

Lemma3.1: The above discrete time-delay system is asymptotically stable if there exist matrices $P > 0$and $Q > 0$such that the following linear matrix inequality (LMI) holds

$$\begin{bmatrix} Q-P & 0 & A^T P \\ * & -Q & A_d^T P \\ * & * & -P \end{bmatrix} < 0$$

Proof: Define the Lyapunov functional as

$$V(k) = x^T(k)Px(k) + \sum_{j=1}^{h} x^T(k-j)Qx(k-j)$$

$$\Delta V(k) = \begin{bmatrix} x(k) \\ x(k-h) \end{bmatrix}^T \begin{bmatrix} A^T PA - P + Q & A^T PA_d \\ * & A_d^T PA_d - Q \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-h) \end{bmatrix}$$

If $\begin{bmatrix} A^T PA - P + Q & A^T PA_d \\ * & A_d^T PA_d - Q \end{bmatrix} < 0$, then the system is asymptotically stable.

Using Schur complement, the above LMI can be written as

$$\begin{bmatrix} Q-P & 0 & A^T P \\ * & -Q & A_d^T P \\ * & * & -P \end{bmatrix} < 0$$

Example: $A = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & a \end{bmatrix}, A_d = d\begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.1 \end{bmatrix}$

where d is the adjustable parameter and system scalar parameter a takes the following values: -0.15 and 0.5.

The delay-independent asymptotic stability conditions are characterized by means of range of parameter d and are summarized as

Parameter a	-0.15	+0.50
Stability boundary	d=2.11	d=1.51

Table 3.2: Delay-independent stability

Conclusion:

Both delay-dependent and delay-independent stability analysis of networked control system is done. When a particular stability condition is derived which depends on the size of the delay factor, the obtained result is delay dependent stability. When derived stability condition does not depend upon delay size, we eventually get delay independent stability condition. Delay-dependent stability criteria are usually less conservative than the delay-independent ones especially for small time-delays.

Chapter 4

STABILIZATION OF TIME-DELAY SYSTEMS

4 Stabilization of time-delay systems

We consider the static output-feedback stabilization of discrete time system with time-varying delay. A static output-feedback controller only uses the current output as input.

$$\begin{aligned} &x(k+1) = Ax(k) + A_1 x(k-d(k)) \\ &x(k) = \varphi(k) h_1 \le d(k) \le h_2 \end{aligned} \tag{4.1}$$

The above system is considered as, in general, open loop unstable.

4.1 Stabilization Approach 1 [28]

The static output-feedback stabilization problem is to design a controller $u(k) = Ky(k)$ where K is an appropriately dimensioned matrix to be determined$y(k)$ is the measured output, $u(k)$ the controlled input.

Taking this controller and the time-varying delay system given below

$$\begin{aligned} &x(k+1) = Ax(k) + A_1 x(k-d(k)) + Bu(k) \\ &y(k) = Cx(k) + C_1 x(k-d(k)) \\ &x(k) = \varphi(k) \end{aligned}$$

the following closed-loop system is obtained

$$\left\{ \begin{aligned} &x(k+1) = \bar{A}x(k) + \bar{A}_1 x(k-d(k)) \\ &x(k) = \varphi(k) \end{aligned} \right\} \tag{4.2}$$

where $\bar{A} = A + BKC$, $\quad \bar{A}_1 = A_1 + BKC_1$

Theorem 4.1: For the system given in (4.2), a stabilizing output-feedback controller $u(k) = Ky(k)$ exists if there exists n×n matrices $P > 0, Q > 0, L > 0, R, S\ and\ T$ satisfying the following matrix inequality

$$\begin{bmatrix} G & -R+S^T & -R+T^T & (A+BKC+A_1+BKC_1)^T \\ * & -Q-S-S^T & -S-T^T & 0 \\ * & * & -T-T^T & -(A_1+BKC_1)^T \\ * & * & * & -L \end{bmatrix} < 0$$

with the side condition $PL = I$

where $G = -P + (h_2 - h_1 + 1)Q + R + R^T$.

Proof: From Proposition 2.3, the closed-loop system given in (4.2) is asymptotically stable if there exists n×n matrices P, Q, R, S and T satisfying the following matrix inequality

$$\begin{bmatrix} G & -R+S^T & -R+T^T & (\bar{A}+\bar{A}_1)^T \\ * & -Q-S-S^T & -S-S^T & 0 \\ * & * & -T-T^T & -\bar{A}_1^T P \\ * & * & * & -P \end{bmatrix} < 0$$

where $G = -P + (h_2 - h_1 + 1)Q + R + R^T$.

Pre multiply and post multiply the above inequality by $diag\left(I, \quad I, \quad I, \quad P^{-1}\right)$

together with the substitution of the matrices defined in (4.2) leads to

$$\begin{bmatrix} G & -R+S^T & -R+T^T & (A+BKC+A_1+BKC_1)^T \\ * & -Q-S-S^T & -S-T^T & 0 \\ * & * & -T-T^T & -(A_1+BKC_1)^T \\ * & * & * & -P^{-1} \end{bmatrix} < 0$$

Defining $L = P^{-1}$ this completes the proof of Theorem 4.1. ■

It should be noted that the condition obtained in Theorem 4.1 is not a strict LMI condition because of the side condition$PL = I$. However, we can solve this problem by formulating it as a sequential optimization problem subject to LMI constraints.

Problem Statement 1 (Static output-feedback stabilization): Find matrices $P > 0, Q > 0, L > 0, R, S, T, K$ such that the following nonlinear minimization problem can be solved

Minimize $|tr(PL) - n|$

subject to

$$\begin{bmatrix} G & -R+S^T & -R+T^T & (A+BKC+A_1+BKC_1)^T \\ * & -Q-S-S^T & -S-T^T & 0 \\ * & * & -T-T^T & -(A_1+BKC_1)^T \\ * & * & * & -L \end{bmatrix} < 0 \tag{4.3}$$

and $\begin{bmatrix} P & I \\ I & L \end{bmatrix} \geq 0$ (4.4) where $G = -P + (h_2 - h_1 + 1)Q + R + R^T$.

Trying to minimize $|tr(PL) - n|$ is easier than trying to directly solve the original problem. This is done using Algorithm 1 as indicated in the below section.

4.2 Algorithm 1 (Static output-feedback stabilization)

Step 1: Compute an initial feasible set $(P^0, Q^0, L^0, R^0, S^0, T^0)$ satisfying (4.3) and (4.4). Set $k = 0$.

Step 2: Solve the following LMI problem: find $(P_S, Q_S, L_S, R_S, S_S, T_S, K)$ which minimize $tr(PL^k + P^kL)$ subject to (4.3) and (4.4).

Step 3: If

$$\begin{bmatrix} \tilde{G} & -R_s+S_s^T & -R_s+T_s^T & (A+BKC+A_1+BKC_1)^T \\ * & -Q_s-S_s-S_s^T & -S_s-T_s^T & 0 \\ * & * & -T_s-T_s^T & -(A_1+BKC_1)^T \\ * & * & * & -P_s^{-1} \end{bmatrix} < 0$$

where $\tilde{G} = -P_s + (h_2 - h_1 + 1)Q_s + R_s + R_s^T$. Then K is the solution and stop.

Step 4: If the above condition is not satisfied, then go to Step 2.

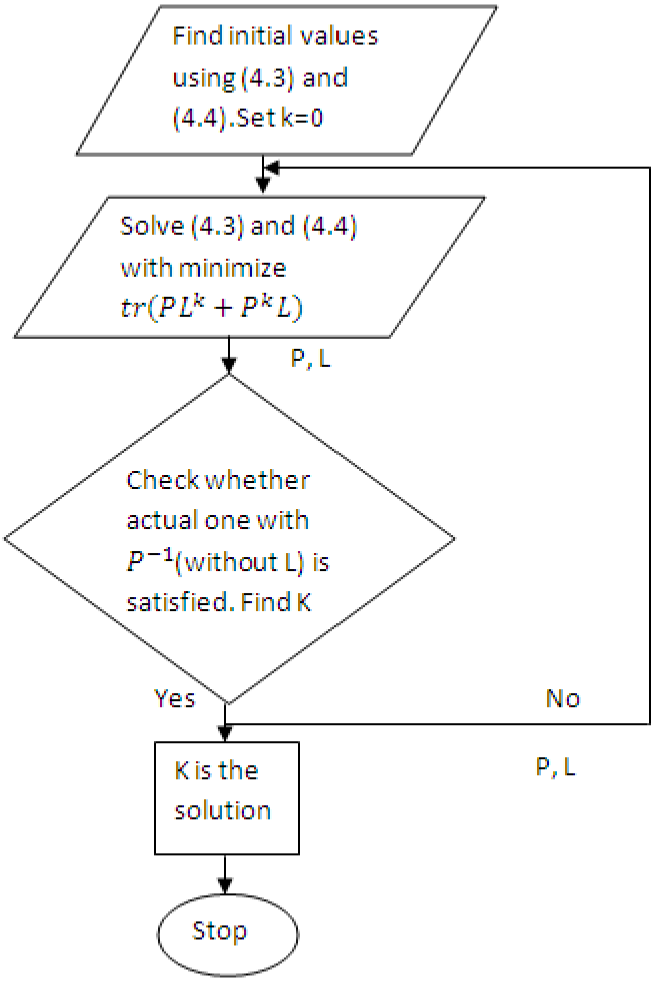


Fig 4.1: Flowchart Representation of SOFS Algorithm

4.3 Numerical Example

Consider the following discrete-time system with a time-varying delay

$$x(k+1) = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} x(k-d(k)) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(k-d(k))$$

Assume that the minimum delay bound is h_1=2.

Using Theorem 4.1 and SOFS algorithm (Section 4.2), a stabilizing controller gain is obtained as $K = -[0.1224 \quad 0.0447]$for $h_2 = 10$.

Simulation of the above system:

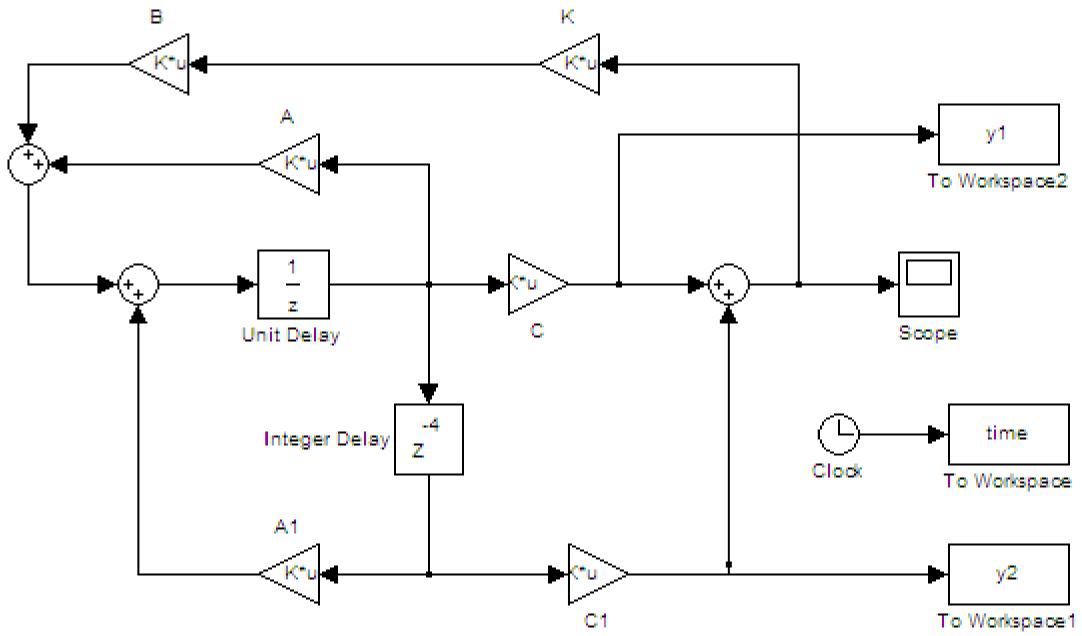
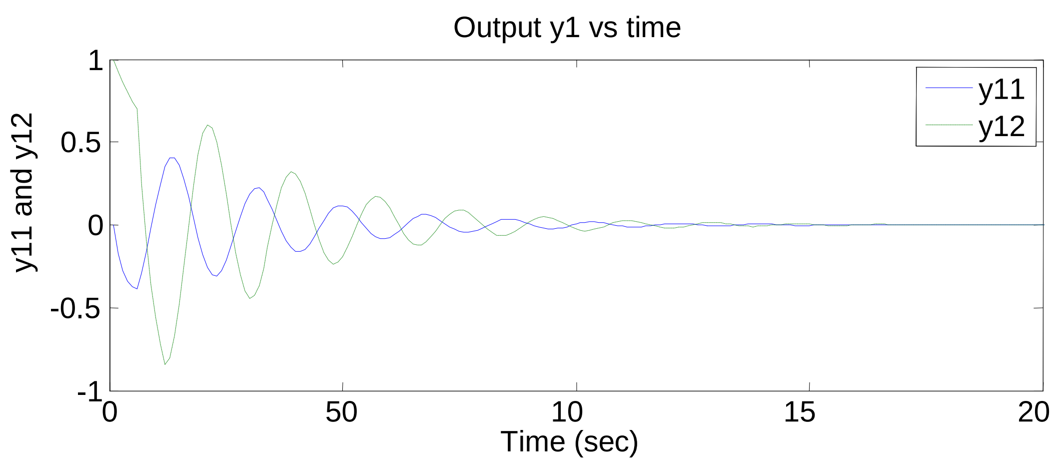


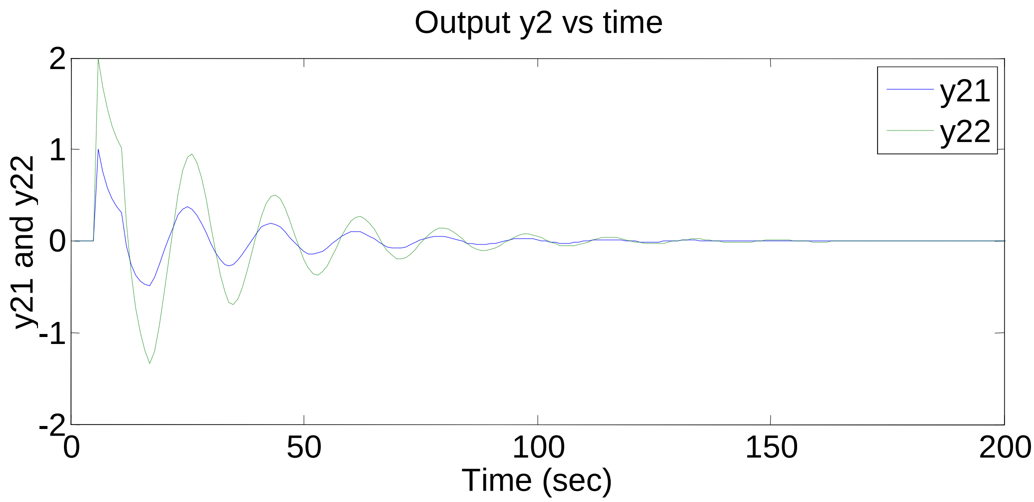
Fig 4.2 Simulation Diagram of the above system

The system given in 4.3 is simulated both for fixed delay and variable delay and simulation results were obtained.

Simulation Results:

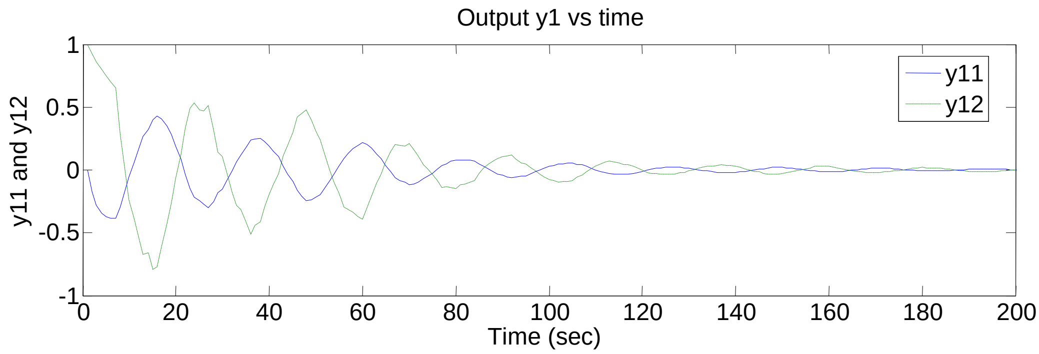


(4.3a)

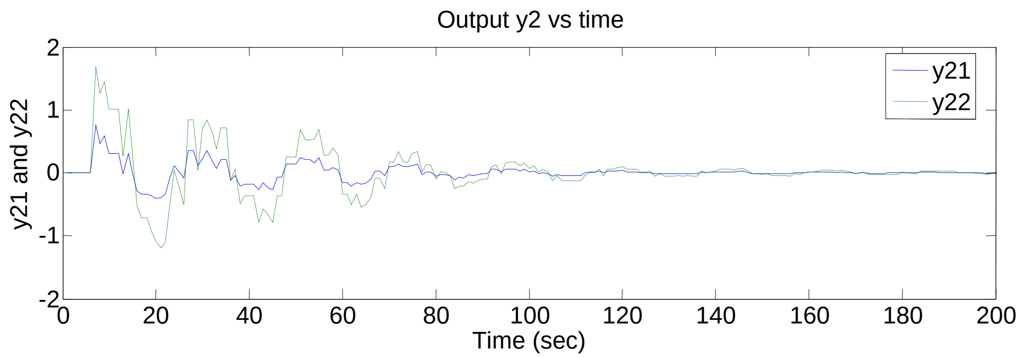


(4.3b)

Fig 4.3: Simulation results of the above system for fixed delay



(4.4a)



(4.4b)

Fig 4.4: Simulation results of the above system for variable delay

Conclusion: System is stabilized by the output-feedback controller K.

4.4 Stabilization Approach 2

Proposition2.2: For given two nonnegative integers h_1 and h_2 satisfying $0<h_1<h_2$, the system (2.1) is asymptotically stable if there exist matrices $P>0, Q_i>0$(i=1, 2, 3) and $Z_j>0(j=1,2)$ such that

$$\Theta-\begin{bmatrix}0 & I & 0 & -I & 0 & 0\end{bmatrix}^T Z_2\begin{bmatrix}0 & I & 0 & -I & 0 & 0\end{bmatrix}<0 \qquad (2.12)$$

$$\Theta-\begin{bmatrix}0 & -I & I & 0 & 0 & 0\end{bmatrix}^T Z_2\begin{bmatrix}0 & -I & I & 0 & 0 & 0\end{bmatrix}<0 \qquad (2.13)$$

where

$$\Theta=\begin{bmatrix}
\begin{Bmatrix}A^T PA-P+Q_1\\ +Q_2+(1+h_{12})Q_3\\ -Z_1\end{Bmatrix} & A^T PA_1 & Z_1 & 0 & h_1(A-I)^T Z_1 & h_{12}(A-I)^T Z_2\\
* & \begin{Bmatrix}A_1^T PA_1\\ -Q_3-2Z_2\end{Bmatrix} & Z_2 & Z_2 & h_1 A_1^T Z_1 & h_{12} A_1^T Z_2\\
* & * & \begin{Bmatrix}-Q_1-Z_1\\ -Z_2\end{Bmatrix} & 0 & 0 & 0\\
* & * & * & -Q_2-Z_2 & 0 & 0\\
* & * & * & * & -Z_1 & 0\\
* & * & * & * & * & -Z_2
\end{bmatrix}$$

Theorem 4.2: Applying Schur complement and pre-multiply and post multiply the above two inequalities by $diag(I,\ I,\ I,\ I,\ Z_1^{-1},\ Z_2^{-1}, P^{-1})$ together with the substitution of the matrices defined in (4.2) leads to

$$\Psi_1=\begin{bmatrix}
\begin{Bmatrix}Q_1+Q_2\\ +(1+h_{12})Q_3\\ -Z_1-P\end{Bmatrix} & 0 & Z_1 & 0 & \begin{Bmatrix}h_1\bar{A}^T\\ -h_1 I\end{Bmatrix} & \begin{Bmatrix}h_{12}\bar{A}^T\\ -h_{12} I\end{Bmatrix} & \bar{A}^T\\
* & -Q_3-3Z_2 & 2Z_2 & Z_2 & h_1\bar{A}_1^T & h_{12}\bar{A}_1^T & \bar{A}_1^T\\
* & * & \begin{Bmatrix}-Q_1-Z_1\\ -2Z_2\end{Bmatrix} & 0 & 0 & 0 & 0\\
* & * & * & -Q_2-Z_2 & 0 & 0 & 0\\
* & * & * & * & -N & 0 & 0\\
* & * & * & * & * & -M & 0\\
* & * & * & * & * & 0 & -L
\end{bmatrix}<0$$

$$\Psi_2 = \begin{bmatrix} \begin{Bmatrix} Q_1 + Q_2 \\ +(1+h_{12})Q_3 \\ -Z_1 - P \end{Bmatrix} & 0 & Z_1 & 0 & \begin{Bmatrix} h_1 \bar{A}^T \\ -h_1 I \end{Bmatrix} & \begin{Bmatrix} h_{12} \bar{A}^T \\ -h_{12} I \end{Bmatrix} & \bar{A}^T \\ * & \{-Q_3 - 3Z_2\} & Z_2 & 2Z_2 & h_1 \bar{A}_1^T & h_{12} \bar{A}_1^T & \bar{A}_1^T \\ * & * & \begin{Bmatrix} -Q_1 - Z_1 \\ -Z_2 \end{Bmatrix} & 0 & 0 & 0 & 0 \\ * & * & * & -Q_2 - 2Z_2 & 0 & 0 & 0 \\ * & * & * & * & -N & 0 & 0 \\ * & * & * & * & * & -M & 0 \\ * & * & * & * & * & * & -L \end{bmatrix} < 0$$

with the side condition PL=I, $Z_2M = I, Z_1N = I$.

It should be noted that the condition obtained in Theorem 4.2 is not a strict LMI condition because of the side conditions $PL = I$, $Z_2M = I, Z_1N = I$. However, we can solve this problem by formulating it as a sequential optimization problem subject to LMI constraints.

Problem Statement 2 (Static output-feedback stabilization): Find matrices P>0, $Q_i(i = 1,2,3) > 0, Z_j(j = 1,2), K$ such that the following nonlinear minimization problem can be solved

Minimize $|tr(PL) - n|$,$|tr(Z_2M) - n|$, $|tr(Z_1N) - n|$

subject to $\Psi_1 < 0$

$\Psi_2 < 0$

and $\begin{bmatrix} P & I \\ I & L \end{bmatrix} \geq 0, \begin{bmatrix} Z_2 & I \\ I & M \end{bmatrix} \geq 0, \begin{bmatrix} Z_1 & I \\ I & N \end{bmatrix} \geq 0$

Numerical Example: Consider the following discrete-time system with a time-varying delay

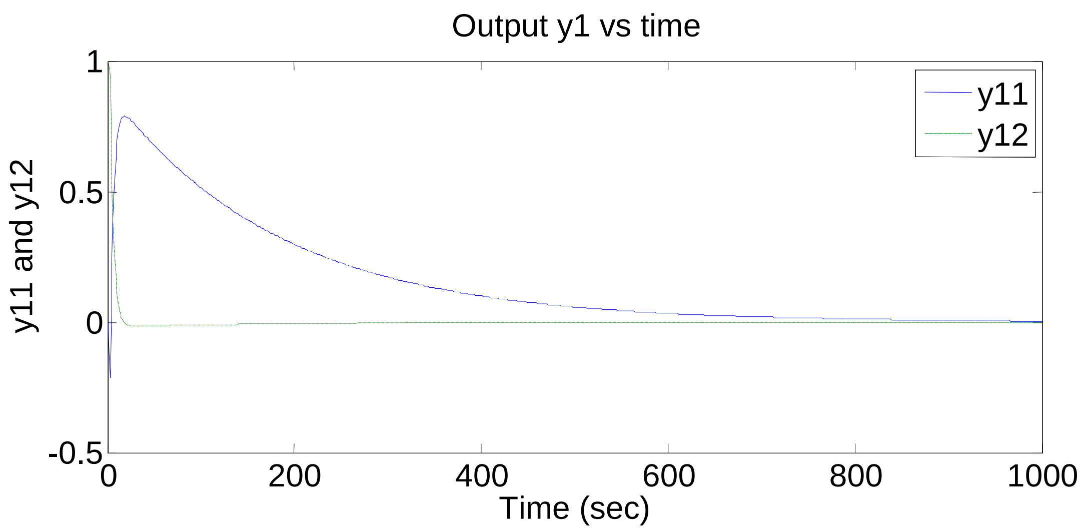
$$x(k+1)=\begin{bmatrix}0.8 & 0\\ 0.05 & 0.9\end{bmatrix}x(k)+\begin{bmatrix}-0.1 & 0\\ -0.2 & -0.1\end{bmatrix}x(k-d(k))+\begin{bmatrix}1\\ 0.5\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}x(k)+\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}x(k-d(k))$$

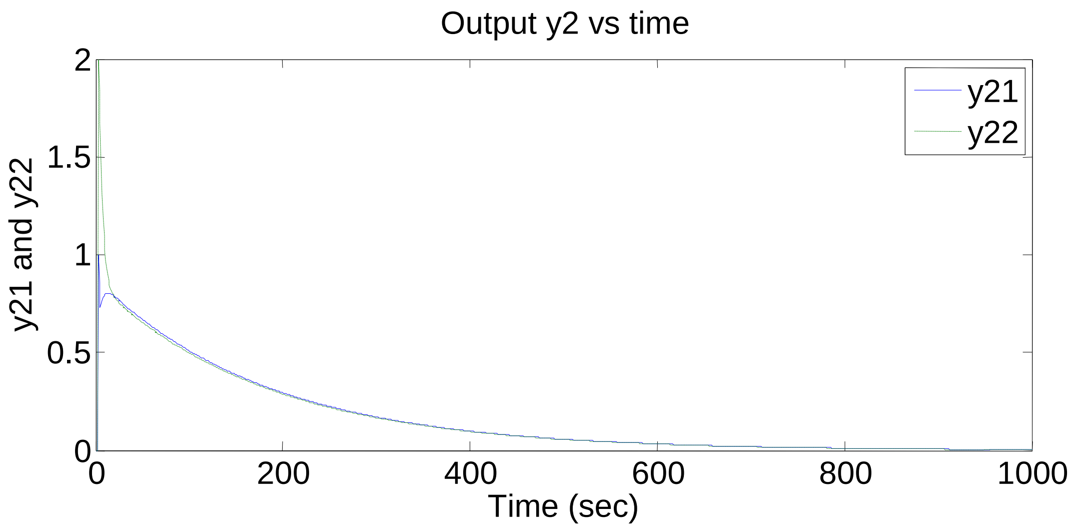
Assume that the minimum delay bound is $h_1 = 2$.

Using Theorem 4.2 and SOFS algorithm (Section 4.2), a stabilizing controller gain is obtained as $K = [0.1177 \quad 0.0583]$.

Simulation Results:



(4.5a)



(4.5b)

Fig 4.5: Simulation results of the above system

Conclusion:

System is stabilized by the output-feedback controller K. Two stabilization approaches have been studied. In earlier case we are getting negative value of gain K and the system stabilizes faster. But by proposition2.2 we are getting positive value of gain K and the system takes longer time to stabilize.

5 Contribution & Future Work

5.1 Contribution

Existing stability criteria has been studied and one stability criteria has been extended to stabilize the system.

5.2 Future Work

Stability criteria may also be used to enhance the performance of the system.

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