Robust Stabilization of Systems with Time varying Input Delay using PI State feedback Controller

Thesis submitted in partial fulfillment of the requirements for the degree of

Master of Technology

in

Electrical Engineering

(Specialization: Control & Automation)

by

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May 2013

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Certificate

This is to certify that the work in the thesis entitled ***Robust Stabilization of Systems with Time varying Input Delay using PI State feedback Controller*** by ***Rallapati Aditya*** is a record of an original research work carried out by him under my supervision and guidance in partial fulfillment of the requirements for the award of the degree of Master of Technology with the specialization of Control & Automation in the department of Electrical Engineering, National Institute of Technology Rourkela. Neither this thesis nor any part of it has been submitted for any degree or academic award elsewhere.

Place: NIT Rourkela
Date: 21 May 2013

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Acknowledgment

I am grateful to numerous local and global peers who have contributed towards shaping this thesis. At the outset, I would like to express my sincere thanks to Prof. Sandip Ghosh for his advice during my thesis work. As my supervisor, he has constantly encouraged me to remain focused on achieving my goal. His observations and comments helped me to establish the overall direction of the research and to move forward with investigation in depth. He has helped me greatly and been a source of knowledge.

I extend my thanks to our HOD, Prof. A. K. Panda for his valuable advices and encouragement.

I am really thankful to my all friends. My sincere thanks to everyone who has provided me with kind words, a welcome ear, new ideas, useful criticism, or their invaluable time, I am truly indebted.

I must acknowledge the academic resources that I have got from NIT Rourkela. I would like to thank administrative and technical staff members of the Department who have been kind enough to advise and help in their respective roles.

Last, but not the least, I would like to dedicate this thesis to my family, for their love, patience, and understanding.

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Abstract

Time delays are often encountered in many practical systems, especially in the networked control systems and many industrial processes which also includes computation delay. The presence of input delays causes system instability and degrades system performance. For a nominal system with input delay, one may transform the system using a reduction method to a non-delay form and then can design a controller using techniques that are available for systems without time-delays. However, for uncertain systems, this reduction method does not transform the system into a non-time-delayed one. Due to this reason, one need to analyze such uncertain systems using analysis that are available for time-delay systems and one attempts to exploit the benefit of using the reduction method. It is studied in this work that using simple state feedback controller over the transformed model does not yield much benefit for uncertain systems. Various choices of Lyapunov-Krasovskii functional has been made to verify stability of the transformed system and establishing the above fact. At the end, it is observed that not using the transformation method but by using a PI-type state feedback controller for the non-transformed system does yield more benefit in controller design in the sense that the guaranteed robustness margin is improved considerably.

Contents

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Symbols & Abbreviations

$\in$	:	Belongs to
$\mathbb{R}$	:	The set of real numbers
$\mathbb{R}^n$	:	The set of n component real vectors
$\mathbb{R}^{n\times m}$	:	The set of n by m real matrices
IQC	:	Integral Quadratic Constraint
LMI	:	Linear Matrix Inequality
NCS	:	Networked Control System

Chapter 1

Introduction

Chapter 1

Introduction

Time delay systems are those for which the future evolution of the state variables not only depends on their current state value, but also on their past values. Time delays are often encountered in practical systems, especially in the networked control systems [1]. Input delays are occurred in feedback control systems because of the transmission of the measured information in process plants which also includes computation delay. The presence of these delays will degrade the system performance and also causes system instability. The stability analysis of such kind of systems is one of the emerging areas of research. So one of the challenging issue is stability analysis and control design for time delay systems. The robust stability criteria for such kind of systems can be analyzed based on Lyapunov-Krasovskii theorem.

This chapter presents the introduction to functional differential equations and a brief description of time delay systems.

1.1 Time Delay Systems

Dynamic systems are represented with ordinary dierential equations in the form of

$$\dot{x}(t) = f(t, x(t)) \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ are the state variables and the differential equations characterize the state variable evolution with respect to time. Once the initial condition is known, by using the current state variable, any future state of the system $t_0 \leq t < \infty$ for any t_0 can be determined completely.

But many dynamical systems in practice can not be exactly modeled by an ordinary differential equation. For many systems the future evolution of state variable $x(t)$ depends both on the current value $x(t_0)$ and also on their past values $x(\phi), t_0 - \tau \leq \phi \leq t_0$, such systems are called as time delay systems. Considering the transfer delays of sensor-to-controller τ_1 and controller-to-actuator τ_2, a system can be described as

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t - \tau) \tag{1.2}$$

where $\hat{A}$ and $\hat{B}$ are constant matrices, $\tau = \tau_1 + \tau_2$. If we take the parameter uncertainties into account, a more general form of (2) is given by

$$\dot{x}(t) = (\hat{A} + \Delta\hat{A}(t))x(t) + (\hat{B} + \Delta\hat{B}(t))u(t - \tau) \tag{1.3}$$

where $\Delta\hat{A}(t)$ and $\Delta\hat{B}(t)$ denote parameter uncertainties, such as additive unknown internal or external noise, non linarities and poor plant knowledge, etc.

1.2 Systems with input delay

Input delays which are occurred because of transmission of measured information in process plants in feedback control systems encountered in many practical applications [11], [3]. If the presence of input delays are not considered in the controller design, it leads to instability of the system and also deterioration in system performance. The stability analysis becomes even complicated because of parametric uncertainities of the system and the infinite dimensional nature of the system due to delay. However the complexity becomes more severe when the delay is time time-varying [12], [14]. So the controller has to be designed for the robust stabilization of uncertain time delay systems with time varying input delay. Due to the infinite dimensional nature of the problem, controller design for the time delay system has become a challenging task.

1.3 Review on controller design for input delay systems

For uncertain systems with control input delay, in the past several decades memory less controllers or memory controllers have been designed using Razumikhin method [17], [8], [4], [18], [5] an integral quadratic constraint (IQC) method [16] or a reduction method. The problem of stabilizing uncertain dynamical systems with multiple input delays is considered by introducing a new stabilizing controller which employs the predictor with in the min-max frame work in [4]. It was reported in [4] that this combination extends the system to which min-max control can be applied to uncertain systems with no current control and multiple input delays and the analysis discussed in [4] is based on Razumkhin theorem which was applied for uncertain systems containing both state delay and input delay and also time varying uncertain systems with state delays.

A stabilization for a type of linear uncertain systems with time latency is considered and the control is proposed based on the optimal control for its delay free systems with quadratic performance index, a delay dependent stability criterion based on Lyapunov functional is discussed for the asymptotic stabilization of time-latency system in [8]. Feedback control based on receding horizon method was proposed in [18] for linear systems with control input delay. An open-loop optimal control strategy is derived and is then transformed to closed loop control through receding horizon concept and control laws of [18] are perhaps some of the easiest ways of stabilizing a linear system with control input delay.

A robust stabilization approach is propose in [5] by applying the reduction method to multiple input-delayed systems with parametric uncertainties by designing a robust stabilizing controller baseb on Lyapunov approach of stability and by solving the convex problems in terms of linear matrix inequalities. Based on the Riccati-equation approach, observer-based feedback control laws for linear dynamic systems with state delay are proposed in [10]. Two alternative methods for designing observer-based H_∞ control laws whose gain matrices are obtained in terms of solutions of a pair of Riccatic-like equations are proposed in [10].

A simple delay-dependent stability criteria for linear systems with time-varying delay with polytopic-type uncertainties are presented in [7] where the analysis was done in such a way that to construct a parameter-dependent Lyapunov functional for the system, a new method of dealing the system without uncertainties is derived first in which the derivative terms of the state in the derivative of Lyapunov functional are retained and some free weighting matrices are used to express the relationships among the system variables which results in the absence of Lyapunov matrices in any product terms of the system matrices in the derivative of the Lyapunov functional.

For the memory less controller design, based on a first-order transformation Razumikhin method can handle a system with a fast time-varying delay while the IQC method is only applicable for a system with a constant delay [10]. Employing the reduction method, a delayed feedback control design method was proposed in [1]. The advantage of this method is that the controller design problem of the original system can be reduced to that of a non-delayed system. The robustness analysis of this kind of delayed feedback controller was investigated for uncertain systems with input delay based on a Lyapunov-Krasovskii approach and a linear matrix inequality technique.

However, the drawback of controller design method based on the reduction method is that the exact value of the time delay must be known in advance, which therefore limits the application to many real engineering systems. The frequency domain analysis like frequency sweeping and matrix pencil methods are giving sufficient conditions for the systems with commensurate delays [13]. But the time domain approaches have advantages like handling of time varying uncertainities and non linearities compared to frequency domain analysys [19], [20]. The controller proposed in [1] is based on LMI approach could able to stabilize the system over some delay period and uncertain range but the Lyapunov-krasvoskii functional considered for stabilization is very complex. Various controller design methods have been proposed for the robust stabilization of uncertain systems with time varying input delay which could able to stabilize the system over some delay period and with some robustness. And the research on the controller design for the robust stabilization of time delay systems has drawn more attention in the recent years.

1.4 Some mathematical tools

Lemma 1.4.1 *(Matrix lemma [11]):*
If $X, Y \in \mathbb{R}^{n\times n}$ and for a positive definite matrix P

$$2X^TY \leq X^TP^{-1}X + Y^TPY \tag{1.4}$$

Lemma 1.4.2 *(Schur-complement):*
If $Q < 0$ and $Q + RS^{-1}R^T < 0$ then

$$\begin{bmatrix} Q & R \\ R^T & -S \end{bmatrix} < 0 \tag{1.5}$$

Lemma 1.4.3 *(Jensen's inequlality [15]):*
For $0 < R, R^T = R, 0 \leq \alpha < \beta, 0 < \gamma = \beta - \alpha$ the following bounding holds:

$$-\int_{t-\beta}^{t-\alpha} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \leq \gamma^{-1}\begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix}^T \begin{bmatrix} -R & R \\ * & -R \end{bmatrix} \begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix} \tag{1.6}$$

An equivalent representation of this [15] using free variable matrices as

$$\begin{aligned} -\int_{t-\beta}^{t-\alpha} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \quad \leq \quad & \begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix}^T \left\{ \begin{bmatrix} M + M^T & -M + N^T \\ * & -N - N^T \end{bmatrix} \right. \\ & \left. +\gamma \begin{bmatrix} M \\ N \end{bmatrix} R^{-1} \begin{bmatrix} M \\ N \end{bmatrix}^T \right\} \begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix} \end{aligned} \tag{1.7}$$

Where M,N are free weighting matrices such that $M = M^T = -N =$
$-N^T = -\gamma^{-1}R$.

$$-\int_{t-\bar{\tau}}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta = -\int_{t-\tau(t)}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - \int_{t-\bar{\tau}}^{t-\tau(t)} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \tag{1.8}$$

$$-\bar{\tau}^{-1}\int_{t-\tau(t)}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \leq \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix}^T \left\{ \bar{\tau}^{-1}\begin{bmatrix} M_1+M_1{}^T & -M_1+N_1{}^T \\ * & -N_1-N_1{}^T \end{bmatrix} + \sigma \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} R_1{}^{-1} \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}^T \right\} \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} \quad (1.9)$$

and

$$-\bar{\tau}^{-1}\int_{t-\bar{\tau}}^{t-\tau(t)}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \leq \begin{bmatrix} x(t-\tau(t)) \\ x(t-\bar{\tau}) \end{bmatrix}^T \left\{ \bar{\tau}^{-1}\begin{bmatrix} M_2+M_2{}^T & -M_2+N_2{}^T \\ * & -N_2-N_2{}^T \end{bmatrix} + (1-\sigma) \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} R_1{}^{-1} \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}^T \right\} \begin{bmatrix} x(t-\tau(t)) \\ x(t-\bar{\tau}) \end{bmatrix} \quad (1.10)$$

where

$$\sigma = \frac{\tau(t)}{\bar{\tau}}, 0 \leq \sigma \leq 1. \quad (1.11)$$

1.5 Objectives

For the Robust stabilization of time delay systems, state feed back control design method is proposed based on reduction method and the stability criteria is derived in terms of Linear Matrix Inequality (LMI) approach by choosing different Lyapunov-Krasovskii functional than the existing ones in literature and tuning parameters. A PI-type state feedback controller design method for the robust stabilization of systems with input delay is also proposed which leads to cosiderable amount of robustness of the system with input delay. The derived criterions with the proposed controllers leads to improvement of the robustness than the available ones in the literature.

1.6 Thesis Structure

The rest of the thesis is organized as follows:

- **Chapter 2:** This chapter presents different types of functions for stability analysis, some theorems for stability and different approaches of stability analysis of Time-delay systems.

- **Chapter 3:** This chapter presents static state feedback controller design methods for the Robust stabilization of Time-delay systems using various-Lyapunov polynomials, approximations and tuning parameters and their respective conservativeness with a numerical example is also discussed and compared.

- **Chapter 4:** This chapter presents the robust stabilization of time-delay systems using PI-type state feedback controller design and also it's simulation results.

- **Chapter 5:** This chapter presents the discussion and conclusion with the proposed controller design approaches.

Chapter 2

Stability of Time Delay Systems

Chapter 2

Stability of Time-delay Systems

Delays are known to have the effects on stability and system performance. This chapter presents different approaches available for stability analysis, existing theorems of Time-delay systems.

2.1 Lyapunov Approaches

For the systems without delays, Lyapunov method is the effective method for analyzing and determining the stability of the time delay system. For a delay free system, $x(t)$ needs to specify the future evolution of the system beyond t, Lyapunov method needs to construct a Lyapunov function $V(t, x(t))$, which is a potential measure quantifying the state $x(t)$ deviation from trivial solution.

2.1.1 Lyapunov-Krasovskii Theorem

Lyapunov-Krasovskii Theorem: the system $F(x_t, t)$ is said to be asymptotically stable if there exist a continuous functional $V(t, \varphi); \mathbb{R} \times \ell \to \mathbb{R}^+$, which is positive-definite, decreasing, admitting an infinitesimal upper limit and its derivative $\dot{V}(t, x_t)$ along the motions is negative definite over a neighborhood of origin [13].

For a time-delay system the state required for the future evolution of the states is $x(t)$ in the interval $[t - \tau, t]$, i.e. x_t. The corresponding Lyapunov function for the time-delay systems is a functional $V(t, x_t)$ depending on x_t which should measure the deviation of x_t from the trivial solution. This kind of functional is called as Lyapunov-Krasovskii functional.

This functional requires the state variable $x(t)$ in the delay period $[t-\tau, t]$ which necessitates the modification of functionals and it makes this theorem rather difficult. This difficulty may be some times solved by Razumikhin theorem which involves only functions rather than functionals. But with this Razumikhin therorem, the robust stability criteria can be derived only for systems with fast time-varying delays but not for commensurate delays. As the Lyapunov-Krasovskii functional approach considers additional information on the delay [16], [20], the results obtained with Lyapunov-Krasovskii functional approach are less conservatine compared to Razumikhin approach.

2.2 Delay-independent Stability Analysis

2.2.1 Lyapunov-Razumikhin Approach

This section presents the delay dependent stability criteria with Lyapunov-Razumikhin approach.
Consider the system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) \tag{2.1}$$

Where A and B are matrices of appropriate dimensions.

Stability independent of delay may be obtained by means of Lyapunov- Razumikhin approach using the Lyapunov function

$$V(x) = x^T(t)Px(t) \tag{2.2}$$

Where P is a real symmetric matrix.

According to the Razmikhin theorem, a time-delay system with maximum time-delay τ is asymptotically stable if there exist a bounded quadratic Lyapunov function V such that for some $\varepsilon > 0$, it should satisfy

$$V(x) \geq \varepsilon \parallel x\|^2 \tag{2.3}$$

And the derivative along the system trajectory has to satisfy

$$\dot{V}(x(t)) \leq -\varepsilon \parallel x\|^2 \tag{2.4}$$

when

$$V(x(t+\xi)) \leq pV(x(t)), -\tau \leq \xi \leq 0 \tag{2.5}$$

for any constant $p > 1$.

The derivative of the Lyapunov function can be obtained as

$$\dot{V}(x(t)) \leq 2x^T(t)P[Ax(t) + Bx(t-\tau)] \tag{2.6}$$

For any $p > 1$, we can conclude that for any $m > 0$

$$\dot{V}(x(t)) \leq 2x^T(t)P[Ax(t)+Bx(t-\tau)]+m[px^T(t)Px(t)-x^T(t-\tau)Px(t-\tau)] \tag{2.7}$$

$$\dot{V}(x(t)) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} PA + A^T P + mpP & PB \\ B^T P & -mP \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \tag{2.8}$$

Which implies the necessary condition for stability for $m > 0$ with this approach is

$$\begin{bmatrix} PA + A^T P + mpP & PB \\ B^T P & -mP \end{bmatrix} < 0 \tag{2.9}$$

2.2.2 Lyapunov-Krasovskii Approach

This section presents the delay dependent stability criteria with Lyapunov-Krasovskii approach [13].

Consider the system (2.1)

$$\dot{x}(t) = Ax(t) + Bx(t-\tau)$$

Where A and B are matrices of appropriate dimensions .

Stability independent of delay may be obtained by means of Lyapunov-Krasovskii approach using the Lyapunov function

$$V(x_t) = x^T(t)Qx(t) + \int_{t-\tau}^{t} x^T(\phi)Rx(\phi)d\phi \tag{2.10}$$

The derivative of $V(x_t)$ can be obtained as

$$\dot{V}(x_t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} QA + A^TQ + R & QB \\ B^TQ & -R \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \tag{2.11}$$

For the system to be asymptotically stable, according to Lyapunov-Krasovskii approach $\dot{V}(x_t)$ should be negative definite. Which implies system (2.1) is asymptotically stable if there exist real, symmetric matrices $Q > 0$, $R > 0$ such that

$$\begin{bmatrix} QA + A^TQ + R & QB \\ B^TQ & -R \end{bmatrix} < 0 \tag{2.12}$$

2.3 Delay-dependent Stability Analysis

This section presents the model transformation technique and also the conditions for stability by using both Lyapunov- Razumikhin approach and Lyapunov-Krasovskii approach.

Consider the system with delay (2.1)

$$\dot{x}(t) = Ax(t) + Bx(t-\tau)$$

Where A and B are matrices of appropriate dimensions.

The system (2.1) can also be represented as

$$\dot{x}(t) = (A + B)x(t) + B(x(t - \tau) - x(t)) \tag{2.13}$$

The second term of Equ (2.13) is the disturbance of the nominal system given by

$$\dot{x}(t) = (A + B)x(t) \tag{2.14}$$

As the delay τ increases, the system becomes unstable and its performance degrades and we can show it by means of Model transformation [13].

2.3.1 Model Transformation

Consider System (2.1) with the initial condition [13]

$$x_0 = \Psi, \Psi \in \mathbb{C}([-\tau, 0], \mathbb{R}^n \tag{2.15}$$

Its well known from the Leibniz rule that

$$x(t) - x(t - \tau) = \int_{-\tau}^{0} \dot{x}(t + \theta)d\theta \tag{2.16}$$

$$x(t - \tau) = x(t) - \int_{-\tau}^{0} [Ax(t + \theta) + Bx(t - \tau + \theta)]d\theta \tag{2.17}$$

By using (2.17), equ (2.13) can be re-written as

$$\dot{x}(t) = [A + B]x(t) + B\int_{-\tau}^{0} [-Ax(t + \theta) - Bx(t - \tau + \theta)]d\theta \tag{2.18}$$

With the initial condition

$$x(\theta) = \varphi(\theta), -\tau \leq \theta \leq \tau. \tag{2.19}$$

It is observed that by using Model Transformation [13], the system described by (2.1) with its initial condition is incorporated into the system described by (2.18) followed by its intial condition (2.19) and also the stability of the system (2.18) & (2.19) guarantees the stability of the system (2.1) & (2.15).

2.3.2 Condition for stability using Razumikhin Theorem

With the delay-dependent stability criteria with explicit model transformation, the system specified by (2.1) is asymptotically stable if there exist scalars $m_0 > 0, m_1 > 0$ and real symmetric matrices $Q > 0$, S_0, S_1 such that [13]

$$Q(A+B) + (A+B)^T Q + \tau(S_0 + S_1) < 0, \tag{2.20}$$

$$\begin{bmatrix} m_0 Q - R_0 & -QBA \\ -A^T B^T & -m_0 Q \end{bmatrix} < 0 \tag{2.21}$$

and

$$\begin{bmatrix} m_1 Q - S_1 & -QB^2 \\ -(B^2)^T Q & -m_1 Q \end{bmatrix} < 0. \tag{2.22}$$

2.3.3 Condition for stability using Lyapunov-Krasovskii Theorem

With the delay-dependent stability criteria with explicit model transformation, the system specified by (2.1) is asymptotically stable if there exist symmetric matrices Q, M_0, M_1, N_0 and N_1 such that [13]

$$Q > 0 \tag{2.23}$$

$$\begin{bmatrix} Z & -QBA & -QB^2 \\ -A^T B^T Q & -N_0 & 0 \\ -(B^2)^T Q & 0 & -N_1 \end{bmatrix} < 0 \tag{2.24}$$

where

$$Z = \tau^{-1}[Q(A+B) + (A+B)^T Q] + N_0 + N_1. \tag{2.25}$$

2.4 Reduction method for Nominal systems

This section gives the links between the stability analysis of time-delay systems and the way of transformation of the state-space representation

Consider the system with input delay [3]

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t-\tau), x(t) \in \mathbb{R}^n \tag{2.26}$$

New variable is introduced with the following transformation

$$z\left(t\right) = x\left(t\right) + \int\limits_{t-\tau}^{t} e^{A(t-s-\tau)} B_1 u\left(s\right) ds \tag{2.27}$$

Which will reduce the system (2.26) to a system free of delay as

$$\dot{z}(t) = Az(t) + e^{-A\tau} Bu(t), z(t) \in \mathbb{R}^n \tag{2.28}$$

And for this kind of delay free system desigining a classical feedback controller is straight forward provided that the pair $\left(A, e^{-A\tau}B\right)$ is stabilizable.

2.5 Discussion

Razumikhin approach and Lyapunov-Krasovskii approach of stability are two different time domain approaches of stability which have the advantages of easy handling of non linearities, time-vbarying uncertainties over frequency domain analysis [13]. Robust stability criteria for systems with fast time-varying delay, not for commensurate delay using Razumikhin approach. But as the Lyapunov-Krasovskii functional approach considers additional information on the delay [16], [20], the results obtained with Lyapunov-Krasovskii functional approach are less conservatine compared to Razumikhin approach.

Chapter 3

Static State feedback Stabilization of Systems with input delay

Chapter 3

Static State feedback stabilization of Systems with input-delay

This chapter presents static state feedback controller design methods for the Robust stabilization of Time-delay systems using various Lyapunov functionals, approximations involved demonstrating the intrieacies in the design methods. For clear understanding of the proposed controller design, delayed feed back control design propsed in [1] is discussed in section 3.1 of this chapter. Then several other choices of L-K functionals have been studied exploiting the transformation approach discussed in section 2.4.

3.1 Controller design for the stabilization of time-delay systems

This section presents the brief study of the controller design approach proposed in [1].

System description:
Consider a system with uncertainity and time-varying input delay

$$\dot{x}\,(t) = (A + \Delta A\,(t))\,x\,(t) + (B_0 + \Delta B_0\,(t))\,u\,(t) + (B_1 + \Delta B_1\,(t))\,u(t - \tau(t)), t \geq 0 \tag{3.1}$$

$$x(0) = x_0, u(t) = \Phi(t), t\epsilon[-\tau, 0] \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and control respectively and $\Delta A(t), \Delta B_0(t), \Delta B_1(t)$ are time-varying uncertain matrices satisfying

$$\begin{bmatrix}\Delta A(t) & \Delta B_0(t) & \Delta B_1(t)\end{bmatrix} = DF(t)\begin{bmatrix}E_a & E_0 & E_1\end{bmatrix} \tag{3.3}$$

whereD, E_a, E_0, E_1 are real constant matrices and $F(t)$ is a unknown time-varying matrix such hat $F^T(t)F(t) \leq I$ and $0 \leq \tau(t) \leq \bar{\tau}$ is the time delay and $\tau(t)$ is a continuous function satisfying $\tau(t) \in [\tau_0 - \delta, \tau_0 + \delta]$, where τ_0, δ are known constants and $\tau_0 \geq \delta$ and μ is the rate of change of delay which is also represented as d_τ.

When $\tau(t)$ is time-invariant and it′s exact value is known,the robust stabilization control problem can be solved by using reduction method.

Assumption 1. The pair (A, B) is stabilizable, where $B = B_0 + e^{-A\tau_0}B_1$

Assumption 2. The full state variable $x(t)$ is available for measurement.

By using the transformation (2.27) employed in reduction method which is discussed in section 2.4 and also with the following Leibniz′s rule

$$z(t-\tau(t)) - z(t-\tau_0) = \int_{t-\tau_0}^{t-\tau(t)} \dot{z}(s)ds \tag{3.4}$$

The system (3.1) is transformed as

$$\begin{aligned}\dot{z}(t) = (A + \Delta A(t))z(t) + (B + \Delta B_0(t))u(t) + (B_1 + \Delta B_1(t))u(t-\tau(t))\\ -B_1u(t-\tau_0) - \Delta A \int_{t-\tau_0}^{t} e^{A(t-s-\tau_0)}B_1u(s)ds, \qquad t \geq 0\end{aligned} \tag{3.5}$$

The nominal system of (3.5) is

$$\dot{z}(t) = Az(t) + Bu(t) \tag{3.6}$$

and the other parts can be considered as a perturbation of the nominal system. Then, if (A, B) is stabilizable, system (3.5) can also be stabilized when the effect of the perturbation on the system is limited.

The objective is to design a linear control law as

$$u(t) = Kz(t) \tag{3.7}$$

where K is a state feedback gain matrix.

Applying the control law to the system (3.5), we can write

$$\begin{aligned}\dot{z}(t) = (A + BK + \Delta A(t) + \Delta B_0(t)K)z(t) + (B_1 + \Delta B_1(t))Kz(t - \tau(t))\\ -B_1Kz(t - \tau_0) - \Delta A(t) \int_{t-\tau_0}^{t} e^{A(t-s-\tau_0)} B_1Kz(s)ds, \qquad t \geq 0\end{aligned}$$

$$\begin{aligned}\dot{z}(t) = \bar{A}\,(t)\,z\,(t) + \bar{B}_1(t)\ Kz(t - \tau(t)) - B_1\ Kz(t - \tau_0)\\ -\Delta A\,(t) \int_{t-\tau_0}^{t} e^{A(t-s-\tau_0)} B_1Kz\,(s)\,ds, \quad t \geq 0\end{aligned} \tag{3.8}$$

Where

$\bar{A}\,(t) = \ A + BK + \Delta A\,(t) + \Delta B_0(t)K$ and $\bar{B}_1\,(t) = B_1 + \Delta B_1\,(t)$

3.1.1 Stability Analysis of the System

Lemma 3.1.1 *([1]): Consider the closed loop system (3.1)-(3.2). For given scalars* $\tau_0,\ \delta$ *and feedback gain matrix* K *the system is asymptotically stable if there exist matrices* $P_k\,(k = 1, 2, 3)\,, N_i$ *and* $M_i\ \ (i = 1, 2, 3, 4)\,, T > 0, R > 0$ *and* $S > 0$ *and scalars* $\varepsilon_j > 0(j = 1, 2, 3)$ *such that*

$$S = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \tau_0 P_3^T & \delta N_1 & [E_a + E_0K]^T & 0 \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & 0 & \delta N_2 & 0 & [E_1K]^T \\ * & * & \Omega_{33} & \Omega_{34} & -\tau_0 P_3^T & \delta N_3 & 0 & 0 \\ * & * & * & \Gamma_{44} + 2\delta R & -\tau_0 P_3^T & \delta N_4 & 0 & 0 \\ * & * & * & * & -\tau_0 S & 0 & 0 & 0 \\ * & * & * & * & * & -\delta R & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_3 I \end{bmatrix} < 0 \tag{3.9}$$

and

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0 \tag{3.10}$$

Proof. According to Lyapunov stability criteria the system (3.1) is asymptotically stable if there exist a continuous positive valued functional V and its derivative is negative definite, i.e. $\dot{V} < 0$. For the purpose, a Lyapunov-Krasovskii functional is constructed as

$$\mathrm{V}(\mathrm{t}, z_t) = V_1(\mathrm{t}, z_t) + V_2(\mathrm{t}, z_t) \tag{3.11}$$

where

$$V_1(t, z_t) = z^T(t)P_1z(t) + 2z^T(t)P_2\int_{t-\tau_0}^{t} z(s)ds + \int_{t-\tau_0}^{t} z^T(s)dsP_3\int_{t-\tau_0}^{t} z(s)ds \tag{3.12}$$

and

$$\begin{aligned} V_2(t, z_t) = &\int_{t-\tau_0}^{t} z^T(s)Tz(s)ds + \int_{t-\tau_0}^{t}\int_{s}^{t} z^T(v)sz(v)dvds + \int_{t-\tau_0-\delta}^{t-\tau_0}\int_{s}^{t-\tau_0} \dot{z}^T(v)s\dot{z}(v)dvds + \\ &\delta\int_{t-\tau_0}^{t} \dot{z}^T(s)R\dot{z}(s)ds + \int_{t-\tau_0}^{t-\tau_0+\delta}\int_{s}^{t-\tau_0+\delta} \dot{z}^T(v)R\dot{z}(v)dvds + \delta\int_{t-\tau_0+\delta}^{t} \dot{z}^T(s)R\dot{z}(s)ds + \\ &\int_{t-\tau_0}^{t}\int_{s}^{t} z^T(v)\tau_0\varepsilon_1^{-1}K^TB_1^Te^{A^T(s-v)}E_a^TE_ae^{A(s-v)}B_1Kz(v)dvds \end{aligned} \tag{3.13}$$

It is well known from (3.4) that $z(t-\tau(t))-z(t-\tau_0) = \int_{t-\tau_0}^{t-\tau(t)} \dot{z}(s)ds$, by combining this with the system dynamics (3.8) we can obtain

$$\begin{aligned} &z^T(t)N_1 + z^T(t-\tau(t))N_2 + z^T(t-\tau_0)N_3 + \dot{z}^T(t)N_4 \\ &\{z(t-\tau(t)) - z(t-\tau_0) - \int_{t-\tau_0}^{t-\tau(t)} \dot{z}(s)ds\} = 0 \end{aligned} \tag{3.14}$$

and the following is also true.

$$\begin{aligned} &[z^T(t)M_1 + z^T(t-\tau(t))M_2 + z^T(t-\tau_0)M_3 + \dot{z}^T(t)M_4] \\ &\{-\bar{A}(t)z(t) - \bar{B}_1(t)Kz(t-\tau(t)) + B_1Kz(t-\tau_0) + \\ &\Delta A(t)\int_{t-\tau_0}^{t} e^{A(t-s-\tau_0)}B_1Kz(s)ds + \dot{z}(t)\} = 0 \end{aligned} \tag{3.15}$$

Using the Lemma 1.4.1 the following inequalities are derived for uncertain terms in (3.15):

$$\begin{aligned}2e^T(t)MDF(t)E_a\int\limits_{t-\tau_0}^{t} e^{A(t-s-\tau_0)}B_1Kz(s)ds \leq e^T(t)\varepsilon_1 MDD^TM^Te(t)+\\ \int\limits_{t-\tau_0}^{t} z^T(v)\tau_0\varepsilon_1^{-1}K^TB_1^Te^{A^T(s-v)}E_a^TE_ae^{A(s-v)}B_1Kz(s)ds\end{aligned} \tag{3.16}$$

$$\begin{aligned}2e^T(t)MDF(t)[E_a+E_0K]z(t) &\leq e^T(t)\varepsilon_2 MDD^TM^Te(t)\\ &+z^T(t)\varepsilon_2^{-1}[E_a+E_0K]^T[E_a+E_0K]z(t)\end{aligned} \tag{3.17}$$

$$\begin{aligned}2e^T(t)MDF(t)E_1Kz(t-\tau(t)) &\leq e^T(t)\varepsilon_3 MDD^TM^Te(t)\\ &+z^T(t-\tau(t))\varepsilon_3^{-1}K^T{E_1}^TE_1Kz(t-\tau(t))\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}2[z^T(t)N_1+z^T(t-\tau(t))N_2+z^T(t-\tau_0)N_3+\dot{z}^T(t)N_4]\int\limits_{t-\tau_0}^{t-\tau(t)}\dot{z}(s)ds\\ \leq \delta e^T(t)NR^{-1}N^Te(t)+\int\limits_{t-\tau_0}^{t-\tau(t)}\dot{z}^T(s)R\dot{z}(s)ds\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}e^T(t) &= \begin{bmatrix}z^T(t) & z^T(t-\tau(t)) & z^T(t-\tau_0) & \dot{z}^T(t)\end{bmatrix},\\ M^T &= \begin{bmatrix}{M_1}^T & {M_2}^T & {M_3}^T & {M_4}^T\end{bmatrix},\\ N^T &= \begin{bmatrix}{N_1}^T & {N_2}^T & {N_3}^T & {N_4}^T\end{bmatrix}\end{aligned} \tag{3.20}$$

using (3.16) to (3.19) in (3.15), one can write:

$$\dot{V}(t,z_t) \leq \left[e^T(t)\quad \int\limits_{t-\tau_0}^{t} z^T(s)ds\right]\bar{\Omega}\left[e^T(t)\quad \int\limits_{t-\tau_0}^{t} z^T(s)ds\right]^T \tag{3.21}$$

where

$$\bar{\Omega} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & P_3^T \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & 0 \\ * & * & \Omega_{33} & \Omega_{34} & -P_3^T \\ * & * & * & \Omega_{44}+2\delta R & P_2 \\ * & * & * & * & \frac{-s}{\tau_0} \end{bmatrix}$$

$$+ \begin{bmatrix} \varepsilon_2^{-1}[E_a+E_0K]^T[E_a+E_0K] & 0 & 0 & 0 & 0 \\ * & \varepsilon_3^{-1}K^TE_1^TE_1K & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}$$

$$+\delta \begin{bmatrix} N & 0 \end{bmatrix} R^{-1} \begin{bmatrix} N^T & 0 \end{bmatrix} \tag{3.22}$$

Applying the schur compliment (1.5) using Lemma 1.4.2 to (3.22) it can be shown that (3.9) implies $\bar{\Omega} < 0$ which implies

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \tau_0 P_3^T & \delta N_1 & [E_a+E_0K]^T & 0 \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & 0 & \delta N_2 & 0 & [E_1K]^T \\ * & * & \Omega_{33} & \Omega_{34} & -\tau_0 P_3^T & \delta N_3 & 0 & 0 \\ * & * & * & \Omega_{44}+2\delta R & \tau_0 P_2 & \delta N_4 & 0 & 0 \\ \tau_0 P_3 & 0 & -\tau_0 P_3 & \tau_0 P_2^T & -\tau_0 s & 0 & 0 & 0 \\ \delta {N_1}^T & \delta {N_2}^T & \delta {N_3}^T & \delta {N_4}^T & 0 & -\delta R & 0 & 0 \\ [E_a+E_0K] & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\ 0 & E_1K & 0 & 0 & 0 & 0 & 0 & -\varepsilon_3 I \end{bmatrix} < 0 \tag{3.23}$$

where,

$$\Omega_{11} = P_2 + {P_2}^T - M_1(\mathrm{A}+\mathrm{BK}) - (\mathrm{A}+\mathrm{BK})^T {M_1}^T + \mathrm{T} + \tau_0 \mathrm{S}$$
$$+ \tau_0 \varepsilon_1^{-1} K^T B_1^T \int_{-\tau_0}^{t} e^{A^T s} E_a^T E_a e^{As} ds + (\varepsilon_1+\varepsilon_2+\varepsilon_3)\, \mathrm{M_1 DD^T\, {M_1}^T}$$

$$\Omega_{12} = N_1 - M_1 B_1 K - (A+BK)^T {M_2}^T + (\varepsilon_1+\varepsilon_2+\varepsilon_3) M_1 DD^T {M_2}^T$$

$$\Omega_{13} = -P_2 - N_1 + M_1 B_1 K - (A+BK)^T {M_3}^T + (\varepsilon_1+\varepsilon_2+\varepsilon_3) M_1 DD^T {M_3}^T$$

$$\Omega_{14} = P_1 + M_1 + (\varepsilon_1+\varepsilon_2+\varepsilon_3) M_1 DD^T {M_4}^T$$

$$\Omega_{22} = N_2 + {N_2}^T - M_2 B_1 K - (M_2 B_1 K)^T + (\varepsilon_1+\varepsilon_2+\varepsilon_3) M_2 DD^T {M_2}^T$$

$$\Omega_{23} = -N_2 + {N_3}^T + M_2 B_1 K - (M_3 B_1 K)^T + (\varepsilon_1+\varepsilon_2+\varepsilon_3) M_2 DD^T {M_3}^T$$

$$\Omega_{24} = M_2 + {N_4}^T - (M_4 B_1 K)^T + (\varepsilon_1+\varepsilon_2+\varepsilon_3) M_2 DD^T {M_4}^T$$

$$\Omega_{33} = -T - N_3 - {N_3}^T + M_3 \mathrm{B_1 K} + (M_3 \mathrm{B_1 K})^T + (\varepsilon_1+\varepsilon_2+\varepsilon_3) M_3 DD^T {M_3}^T$$

$$\Omega_{34} = M_3 - {N_4}^T - (M_4 \mathrm{B_1 K})^T + (\varepsilon_1+\varepsilon_2+\varepsilon_3) M_3 DD^T {M_4}^T$$

$$\Omega_{44} = M_4 + {M_4}^T + (\varepsilon_1+\varepsilon_2+\varepsilon_3) M_4 DD^T {M_4}^T$$

where * represents symmetric component •

3.1.2 Controller design

Theorem 3.1.2 *([1]): For the given scalars $\rho_l > 0\,(l = 2, 3, 4)\,, \tau_0$ and δ if there exist matrices $\widetilde{P}_k\,(k = 1, 2, 3)\,, \widetilde{N}_k\,(k = 1, 2, 3, 4)\,, \tilde{T} > 0, \tilde{R} > 0, \tilde{S} > 0,$ Y and a non singular matrix X and scalars $\varepsilon_j > 0(j = 1, 2, 3)$ such that*

$$\eta = \begin{bmatrix} \eta_{11} & \eta_{12} \\ \eta_{12}^T & \eta_{22} \end{bmatrix} < 0 \tag{3.24}$$

where

$$\eta_{11} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ * & * & \Gamma_{33} & \Gamma_{34} \\ * & * & * & \Gamma_{44} + 2\delta R \end{bmatrix} \tag{3.25}$$

$$\eta_{12} = \begin{bmatrix} \tau_0 \tilde{P}_3^T & \delta \tilde{N}_1 & X E_a^T + Y^T E_0^T & 0 & Y^T B_1^T \\ 0 & \delta \tilde{N}_2 & 0 & Y^T E_1^T & 0 \\ -\tau_0 \tilde{P}_3^T & \delta \tilde{N}_3 & 0 & 0 & 0 \\ \tau_0 \tilde{P}_2 & \delta \tilde{N}_4 & 0 & 0 & 0 \end{bmatrix} \tag{3.26}$$

$$\eta_{22} = \begin{bmatrix} -\tau_0 \tilde{S} & 0 & 0 & 0 & 0 \\ * & -\delta \tilde{R} & 0 & 0 & 0 \\ * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & -\tau_0^{-1} \varepsilon_1 W \end{bmatrix} \tag{3.27}$$

Proof. The analysis was done based on the control law $u(t) = YX^{-T}[x(t) + \int_{t-\tau_0}^{t} e^{A(t-s-\tau_0)} B_1 u\,(s)\,ds]$ and by defining $X=M_1^{-1}$,then Pre, post-multiplying both sides of (3.9) $diag\left\{X \quad X \quad X \quad X \quad X \quad X \quad I \quad I\right\}$ and it's transpose and by defining $\widetilde{P}_k = XP_kX^T$ (k=1,2,3), $\widetilde{N}_i = XN_iX^T(i = 1, 2, 3, 4), \tilde{T} = XTX^T, \tilde{R} = XRX^T$, $\tilde{S} = XSX^T$, & $K = YX^{-T}$ then the condition for the control $u(t) = Kz(t)$ to guarantee the asymptotic stability of the closed loop system is obtained as

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \tau_0 \tilde{P}_3^T & \delta \tilde{N}_1 & X E_a^T + Y^T E_0^T & 0 & Y^T B_1^T \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & 0 & \delta \tilde{N}_2 & 0 & Y^T E_1^T & 0 \\ * & * & \Gamma_{33} & \Gamma_{34} & -\tau_0 \tilde{P}_3^T & \delta \tilde{N}_3 & 0 & 0 & 0 \\ * & * & * & \Gamma_{44} + 2\delta R & \tau_0 \tilde{P}_2 & \delta \tilde{N}_4 & 0 & 0 & 0 \\ * & * & * & * & -\tau_0 \tilde{S} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\delta \tilde{R} & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & * & -\tau_0^{-1} \varepsilon_1 W \end{bmatrix} < 0 \tag{3.28}$$

where,

$\Gamma_{11} = \tilde{P}_2 + \tilde{P}_2^T - A\,X^T - BY - XA^T - Y^T\,B^T + + \tau_0 + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\,DD^T$

$\Gamma_{12} = \tilde{N}_1 - \mathrm{B_1Y} - \rho_2 X\mathrm{A^T} - \rho_2\mathrm{Y^T}\ \mathrm{B^T} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_2\ DD^T$

$\Gamma_{13} = \tilde{P}_2 - \tilde{N}_1 + \mathrm{B_1Y} - \rho_3 X\mathrm{A^T} - \rho_3\mathrm{Y^T}\ \mathrm{B^T} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_3\ DD^T$

$\Gamma_{14} = \widetilde{\mathrm{P_1}} + \ \mathrm{X^T} - \rho_4 X\mathrm{A^T} - \rho_4\mathrm{Y^T}\ \mathrm{B^T} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_4\ DD^T$

$\Gamma_{22} = \tilde{N}_2 + \tilde{N}_2\ ^T - \rho_2\mathrm{B_1Y} - \rho_2\mathrm{Y^T}\ \mathrm{B_1}^\mathrm{T} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\ \rho_2{}^2 DD^T$

$\Gamma_{23} = -\tilde{N}_2 + \ \tilde{N}_3^T - \rho_2\mathrm{B_1Y} - \rho_3\mathrm{Y^T}\ \mathrm{B_1}^\mathrm{T} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_2\rho_3\ DD^T$

$\Gamma_{24} = \ \tilde{N}_4^T\ + \rho_2\ \mathrm{X^T} - \rho_4\mathrm{Y^T}\ \mathrm{B_1}^\mathrm{T} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_2\rho_4\ DD^T$

$\Gamma_{33} = \widetilde{T} - \ \tilde{N}_3 - \ \tilde{N}_3^T + \rho_3\mathrm{B_1Y} + \rho_3\mathrm{Y^T}\ \mathrm{B_1}^\mathrm{T} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\ \rho_3{}^2 DD^T$

$\Gamma_{34} = \ -\tilde{N}_4^T\ + \rho_3\ \mathrm{X^T} + \rho_4\mathrm{Y^T}\ \mathrm{B_1}^\mathrm{T} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_3\rho_4\ DD^T$

$\Gamma_{44} = \ \rho_4 X + \rho_4\ \mathrm{X^T} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\ \rho_4{}^2 DD^T$

$\mathrm{W}^{-1} \geq \int_{-\tau_0}^{0} \mathrm{e^{A^Ts}E_a{}^TE_ae^{As}ds}$ •

The LMI's obtained which discribes the stability conditions of the system are solved using Robust control tool box in MATLAB to obtain the robustness of this method.

3.2 Alternate approach (Transformation using $\bar{\tau}$)

In the previous section, it was observed that the Lyapunov-krasovskii functional considered for the stability analysis is lengthy and complex. In this section the transformation used in the process of reducing the system to system to a system free of delays is modified in such a way that the lower bound of the integral limit in the transformation is changed to $t - \bar{\tau}$ with an idea that the total delay range can be included as $\bar{\tau}$ is the maximum value of the delay. Then with the knowledge of the previous section, in this section static state feedback stabilization controller design is proposed with a different Lyapunov-Krasovskii functional with an idea that each term will handle the respective uncertainities in the system.

System description: Consider the system (3.1)-(3.2)

The robust stabilization control problem can be solved by reduction method using the following transformation

$$z(t) = x(t) + \int_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})} B_1 u(s)\, ds \tag{3.29}$$

where $\bar{\tau}$ is the maximum value of the delay. By applying the the transformation (3.29), and the control law (3.7), the system can be written as

$$\dot{z}(t) = (A + BK + \Delta A(t) + \Delta B_0(t)K)z(t) + (B_1 + \Delta B_1(t))Kz(t - \tau(t)) \\ -B_1Kz(t-\bar{\tau}) - \Delta A(t)\int_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})}B_1Kz(s)ds, t \geq 0 \quad (3.30)$$

$$\dot{z}(t) = \bar{A}(t)\,z(t) + \bar{B}_1(t)\ \ Kz(t-\tau(t)) - B_1\ Kz(t-\bar{\tau}) \\ -\Delta A(t)\int_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})}B_1Kz(s)\,ds, t \geq 0 \quad (3.31)$$

where $\bar{A}(t) = A + BK + \Delta A(t) + \Delta B_0(t)K$ and $\bar{B}_1(t) = B_1 + \Delta B_1(t)$.

3.2.1 Stability Analysis

Lemma 3.2.1 *Consider the closed loop system (3.1)-(3.2). For given scalars τ_0, δ and feedback gain matrix K th system is asymptotically stable if there exist matrices $P, Q_k\,(k=1,2)\,, N_i,\ M_i\ , R_i\,(i=1,2) > 0$ and scalars $\varepsilon_j > 0 (j=1,2,3)$ such that*

$$\begin{bmatrix} b_{11} + \varepsilon_2^{-1}[E_a + E_0K]^T[E_a + E_0K] & b_{12} & b_{13} & b_{14} & M_1 \\ * & b_{22} + \varepsilon_3^{-1}[E_1K]^T[E_1K] & b_{23} & b_{24} & N_1 \\ * & * & b_{33} & b_{34} & 0 \\ * & * & * & b_{44} & 0 \\ * & * & * & * & -R_1 \end{bmatrix} < 0 \quad (3.32)$$

and

$$\begin{bmatrix} b_{11} + \varepsilon_2^{-1}[E_a + E_0K]^T[E_a + E_0K] & b_{12} & b_{13} & b_{14} & 0 \\ * & b_{22} + \varepsilon_3^{-1}[E_1K]^T[E_1K] & b_{23} & b_{24} & M_2 \\ * & * & b_{33} & b_{34} & N_2 \\ * & * & * & b_{44} & 0 \\ * & * & * & * & -R_1 \end{bmatrix} < 0 \quad (3.33)$$

Proof. Now, Lyapunov-Krasovskii functional is choosen as:

$$V(t,z_t) = z^T(t)Pz(t) + \int_{t-\bar{\tau}}^{t} z^T(\theta)Q_1 z(\theta)d\theta + \int_{t-\tau(t)}^{t} z^T(\theta)Q_2 z(\theta)d\theta$$
$$+\bar{\tau}^{-1}\int_{t-\bar{\tau}}^{t}\int_{\theta}^{t} \dot{z}^T(\phi)R_1\dot{z}(\phi)d\phi d\theta + \bar{\tau}^{-1}\int_{t-\bar{\tau}}^{t}\int_{\theta}^{t} \dot{z}^T(\phi)R_2\dot{z}(\phi)d\phi d\theta \tag{3.34}$$
$$+\int_{t-\bar{\tau}}^{t}\int_{s}^{t} z^T(v)\bar{\tau}\varepsilon_1^{-1}K^T B_1^T e^{A^T(s-v)}E_a^T E_a e^{A(s-v)}B_1 K z(v)dvds$$

The first term in (3.34) is to handle the first term associated with the uncertain terms in (3.31). The second term and third terms in (3.34) provides a measure of the signal energy during the delay period while the fourth and fifth terms in (3.31) provide the measure of the energy corresponding to the difference between the instantaneous feedback signal and the delayed one. The sixth term is to handle the respective terms associated with uncertainty of the system.

The derivative of $V(t,z_t)$ is obtained as

$$\dot{V}(t,z_t) = 2z^T(t)P\dot{z}(t) + z^T(t)[Q_1+Q_1]z(t) - z^T(t-\bar{\tau})Q_1 z(t-\bar{\tau})$$
$$-(1-\mu)z^T(t-\tau(t))Q_2 z(t-\tau(t)) + \dot{z}^T(t)[R_1+R_2]\dot{z}(t)$$
$$-\bar{\tau}^{-1}\left\{\int_{t-\bar{\tau}}^{t} \dot{z}^T(\theta)R_1\dot{z}(\theta)d\theta\right\} - \bar{\tau}^{-1}\left\{\int_{t-\bar{\tau}}^{t} \dot{z}^T(\theta)R_2\dot{z}(\theta)d\theta\right\} + \tag{3.35}$$
$$z^T(t)\bar{\tau}\varepsilon_1^{-1}K^T B_1^T \int_{t-\bar{\tau}}^{t} e^{A^T(s-t)}E_a^T E_a e^{A(s-t)}dsB_1 K z(t)$$
$$-\int_{t-\bar{\tau}}^{t} z^T(s)\bar{\tau}\varepsilon_1^{-1}K^T B_1^T e^{A^T(t-s-\bar{\tau})}E_a^T E_a e^{A(t-s-\bar{\tau})}B_1 K z(s)ds$$

Using Lemma 1.4.3 (1.6) to (1.11) to re write the terms associated with R_1 and R_2 in (3.35) using (1.8) and (1.6) respectively and then add the following equality

(3.36) to (3.35).

$$\begin{aligned}2[z^T(t)C_1 + z^T(t-\tau(t))C_2 + z^T(t-\bar{\tau})C_3 + \dot{z}^T(t)C_4]\\ \{-\bar{A}(t)z(t) - \bar{B}_1(t)Kz(t-\tau(t)) + B_1Kz(t-\bar{\tau})\\ +\Delta A(t)\int_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})}B_1Kz(s)ds + \dot{z}(t)\} = 0\end{aligned} \tag{3.36}$$

Then by using Lemma 1.4.1 the following inequalities are derived.

$$\begin{aligned}2\xi^T(t)CDF(t)E_a\int_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})}B_1Kz(s)ds \le \xi^T(t)\varepsilon_1 CDD^TC^T\xi(t)\\ +\int_{t-\bar{\tau}}^{t} z^T(v)\bar{\tau}\varepsilon_1^{-1}K^TB_1^Te^{A^T(s-v)}E_a^TE_ae^{A(s-v)}B_1Kz(s)ds\end{aligned} \tag{3.37}$$

$$\begin{aligned}2\xi^T(t)CDF(t)[E_a + E_0K]z(t) \le \xi^T(t)\varepsilon_2 CDD^TC^T\xi(t)\\ +z^T(t)\varepsilon_2^{-1}[E_a+E_0K]^T[E_a+E_0K]z(t)\end{aligned} \tag{3.38}$$

$$\begin{aligned}2\xi^T(t)CDF(t)E_1Kz(t-\tau(t)) \le \xi^T(t)\varepsilon_3 CDD^TC^T\xi(t)\\ +z^T(t-\tau(t))\varepsilon_3^{-1}K^T{E_1}^TE_1Kz(t-\tau(t))\end{aligned} \tag{3.39}$$

where

$$\xi^T(t) = \begin{bmatrix} z^T(t) & z^T(t-\tau(t)) & z^T(t-\bar{\tau}) & \dot{z}^T(t)\end{bmatrix}, C^T = \begin{bmatrix} {C_1}^T & {C_2}^T & {C_3}^T & {C_4}^T\end{bmatrix}. \tag{3.40}$$

The derivative of $V(t, z_t)$ is obtained as

$$\dot{V}(t,z_t) \le \xi^T(t)\left\{\Theta + \sigma\phi_1R_1^{-1}{\phi_1}^T + (1-\sigma)\phi_2R_1^{-1}{\phi_2}^T\right\}\xi(t) \tag{3.41}$$

where

$$\phi_1 = \begin{bmatrix} M_1\\ N_1\\ 0\\ 0\end{bmatrix}, \phi_2 = \begin{bmatrix} 0\\ M_2\\ N_2\\ 0\end{bmatrix} \tag{3.42}$$

Accordind to the Lyapunov-Krasovskii theorem in section 2.1.1, for asymptotic stability of the system, one requires

$$\Theta + \sigma\phi_1 R_1^{-1}{\phi_1}^T + (1-\sigma)\phi_2 R_1^{-1}{\phi_2}^T < 0 \tag{3.43}$$

Adding $\pm\sigma\Theta$ to the L.H.S, we obtain

$$\sigma(\Theta + \phi_1 R_1^{-1}{\phi_1}^T) + (1-\sigma)(\Theta + \phi_2 R_1^{-1}{\phi_2}^T) < 0 \tag{3.44}$$

Since $\sigma = \frac{\tau(t)}{\bar{\tau}}, 0 \le \sigma \le 1.$ the equ (3.44) holds good if

$$\Theta + \phi_l R_1^{-1}{\phi_l}^T < 0, l = 1, 2. \tag{3.45}$$

Applying schur compliment Lemma 1.4.2, one obtains

$$\begin{bmatrix} \Theta & \phi_1 \\ {\phi_1}^T & -R_1 \end{bmatrix} < 0, l = 1, 2 \tag{3.46}$$

This can be written as:

$$\begin{bmatrix} b_{11} + \varepsilon_2^{-1}[E_a + E_0K]^T[E_a + E_0K] & b_{12} & b_{13} & b_{14} & M_1 \\ * & b_{22} + \varepsilon_3^{-1}[E_1K]^T[E_1K] & b_{23} & b_{24} & N_1 \\ * & * & b_{33} & b_{34} & 0 \\ * & * & * & b_{44} & 0 \\ * & * & * & * & -R_1 \end{bmatrix} < 0 \tag{3.47}$$

and

$$\begin{bmatrix} b_{11} + \varepsilon_2^{-1}[E_a + E_0K]^T[E_a + E_0K] & b_{12} & b_{13} & b_{14} & 0 \\ * & b_{22} + \varepsilon_3^{-1}[E_1K]^T[E_1K] & b_{23} & b_{24} & M_2 \\ * & * & b_{33} & b_{34} & N_2 \\ * & * & * & b_{44} & 0 \\ * & * & * & * & -R_1 \end{bmatrix} < 0 \tag{3.48}$$

where,

$b_{11} = Q_1 + Q_2 + \bar{\tau}^{-1}(M_1 + M_1{}^T) - \bar{\tau}^{-2}R_2 - C_1\,(\mathrm{A} + \mathrm{BK}) - (\mathrm{A} + \mathrm{BK})^T\ C_1{}^T$
$+ \bar{\tau}\varepsilon_1^{-1}K^T B_1^T \int\limits_{-\bar{\tau}}^{t} e^{A^T s}E_a^T E_a e^{As}ds + (\varepsilon_1{+}\varepsilon_2{+}\varepsilon_3)\ C_1\mathrm{DD}^{\mathrm{T}}\ C_1{}^{\mathrm{T}}$

$b_{12} = \bar{\tau}^{-1}(-M_1 + N_1{}^T) - C_1B_1K - (A + BK)^T C_2{}^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)C_1DD^T C_2{}^T$

$b_{13} = \bar{\tau}^{-2}R_2 + C_1B_1K - (A + BK)^T C_3{}^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)C_1DD^T C_3{}^T$

$b_{14} = P + C_1 + M_1B_1K - (A + BK)^T C_4{}^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)C_1DD^T C_4{}^T$

$b_{22} = \bar{\tau}^{-1}(-N_1 - N_1{}^T) + \bar{\tau}^{-1}(M_2 + M_2{}^T) - (1 - \mu)Q_2 - C_2B_1K - (C_2B_1K)^T +$
$(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)C_2DD^T C_2{}^T$

$b_{23} = \bar{\tau}^{-1}(-M_2 + N_2{}^T) + C_2B_1K - (C_3B_1K)^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)C_2DD^T C_3{}^T$

$b_{24} = C_2 - (C_4B_1K)^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)C_2DD^T C_4{}^T$

$b_{33} = \bar{\tau}^{-1}(-N_2{-}N_2{}^T){-}\bar{\tau}^{-2}R_2{-}Q_1{+}C_3\mathrm{B}_1\mathrm{K}{+}\ (C_3\mathrm{B}_1\mathrm{K})^T{+}(\varepsilon_1{+}\varepsilon_2{+}\varepsilon_3)C_3DD^T\ C_3{}^T$

$b_{34} = C_3 + (C_4\mathrm{B}_1\mathrm{K})^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)C_3DD^T\ C_4{}^T$

$b_{44} = R_1 + R_2 + C_4 + C_4{}^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)C_4DD^T\ C_4{}^T$

Where * represents symmetric component•

3.2.2 Controller design

Theorem 3.2.2 *for the given scalars* $\rho_l\,(l = 2, 3, 4)\,, \tau_0$ *such that* $C_l = \rho_l C_1, (l = 2, 3, 4)$ *if there exist matrices* $\widetilde{P}, \widetilde{Q}_k\,(k = 1, 2)\,, \widetilde{N}_k\,(k = 1, 2)\,, \widetilde{M}_k\,(k = 1, 2)\,,$
$\widetilde{R}_1 > 0, \widetilde{R}_2 > 0,$ *Y and a non singular matrix X and scalars* $\varepsilon_j > 0(j = 1, 2, 3)$ *such that*

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & XM_1X^T & XE_a^T + Y^T E_0^T & 0 & Y^T B_1^T \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & XN_1X^T & 0 & Y^T E_1^T & 0 \\ * & * & \Gamma_{33} & \Gamma_{34} & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -XR_1X^T & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & * & -\bar{\tau}^{-1}\varepsilon_1 W \end{bmatrix} < 0 \tag{3.49}$$

and

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & 0 & XE_a^T + Y^T E_0^T & 0 & Y^T B_1^T \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & XM_2X^T & 0 & Y^T E_1^T & 0 \\ * & * & \Gamma_{33} & \Gamma_{34} & XN_2X^T & 0 & 0 & 0 \\ * & * & * & \Gamma_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & -XR_1X^T & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & * & -\bar{\tau}^{-1}\varepsilon_1 W \end{bmatrix} < 0 \tag{3.50}$$

where,

$\Gamma_{11} = \tilde{Q}_1 + \tilde{Q}_2 + \bar{\tau}^{-1}\left(\tilde{M}_1 + \tilde{M}_1^T\right) - \bar{\tau}^{-2}\widetilde{R_2} - A\ X^T - BY - XA^T - \ Y^T\ B^T$
$+ (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\ DD^T$

$\Gamma_{12} = \bar{\tau}^{-1}\left(-\tilde{M}_1 + \tilde{N}_1^T\right) - B_1 Y - \rho_2 X A^T - \rho_2 Y^T\ B^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_2\ DD^T$

$\Gamma_{13} = \bar{\tau}^{-2}\widetilde{R}_2 + B_1 Y - \rho_3 X A^T - \rho_3 Y^T\ B^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_3\ DD^T$

$\Gamma_{14} = +\ X^T - \rho_4 X A^T - \rho_4 Y^T\ B^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_4\ DD^T$

$\Gamma_{22} = \bar{\tau}^{-1}\left(-\widetilde{N}_1 - \widetilde{N}_1^T\right) + \bar{\tau}^{-1}\left(\widetilde{M}_1 + \widetilde{M}_1^T\right) - (1-\mu)\widetilde{Q}_2 - \rho_2 B_1 Y - \rho_2 Y^T\ {B_1}^T$
$+ (\varepsilon_1 + \varepsilon_2 + \ \varepsilon_3)\ {\rho_2}^2 DD^T$

$\Gamma_{23} = \bar{\tau}^{-1}\left(-\widetilde{M}_2 + \widetilde{N}_2^T\right) + \rho_2 B_1 Y - \rho_3 Y^T\ {B_1}^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_2\rho_3\ DD^T$

$\Gamma_{24} = \rho_2\ X^T - \rho_4 Y^T\ {B_1}^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_2\rho_4\ DD^T$

$\Gamma_{33} = \ \bar{\tau}^{-1}\left(-\widetilde{N}_2 - \ \widetilde{N}_2^T\right) - \bar{\tau}^{-2} - \widetilde{Q}_1 + \rho_3 B_1 Y + \rho_3 Y^T\ {B_1}^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\ {\rho_3}^2 DD^T$

$\Gamma_{34} = \ \rho_3\ X^T + \rho_4 Y^T\ {B_1}^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\rho_3\rho_4\ DD^T$

$\Gamma_{44} = \ \widetilde{R}_1 + \widetilde{R}_2 + \ \rho_4 X + \rho_4\ X^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\ {\rho_4}^2 DD^T$

$W^{-1} \geq \int_{-\bar{\tau}}^{0} e^{A^T s}{E_a}^T E_a e^{As} ds.$

Proof.With the stability conditions obtained the controller can be designed with the analysis was done based on the control law $u(t) = YX^{-T}[x(t) + \int_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})}B_1$ $u\,(s)\,ds]$ by defining $X = \mathrm{C_1}^{-1}$,then Pre, post-multiplying each of both sides of (3.47), (3.48) respectively with $diag\left\{X \quad X \quad X \quad X \quad I \quad I\right\}$ and it's transpose and by defining $\widetilde{\mathrm{Q}}_k = \mathrm{XQ_kX^T}$ $(k = 1, 2)$, $\widetilde{\mathrm{M}}_i = \mathrm{XM_iX^T}$, $\widetilde{\mathrm{N}}_i = \mathrm{XN_iX^T}$, $\widetilde{\mathrm{R}}_i = \mathrm{XR_iX^T}(i = 1, 2)$ and $\mathrm{K{=}YX^{-T}}$ then the conditions for the control to guarantee the asymptotic stability of the closed loop system are obtained as (3.49) and (3.50)•

3.2.3 Numerical Example

Consider the system (3.1) with uncertainty and time-varying input delay

$\dot{x}\,(t) = (A + \Delta A\,(t))\,x\,(t) + (B_0 + \Delta B_0\,(t))\,u\,(t) + (B_1 + \Delta B_1\,(t))\,u(t - \tau(t)), t \geq 0$

$x(0) = x_0, u(t) = \phi(t), t \in [-0.2, 0]$ with
$A = \begin{bmatrix} 0 & 1 \\ -1.25 & -3 \end{bmatrix}, \Delta A = \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix}, \quad |q| \leq \gamma, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
For the time delay τ (t) , γ_{max} is defined as the maximum allowable value of γ $0 \leq \tau$ (t) ≤ 0.2 and $\dot{\tau}$ (t) $\leq d_\tau < 1$. It was reported in [5] that 1.2499 and 0.2499 were thae values of γ_{max} provide by the methods [9], [10] respectively. By solving the LMIs in Robust control tool box in MATLAB now here in this case with the controller propose in this section, with ρ_2= 0.3, ρ_3= 0.3, ρ_4= 0.3 and with the controller gain $K = [-12.195 \quad -3.366]$ it was found that $\gamma_{max} = 5.0951$ which shows the results obtained with this are still less conservative.

3.3 Stabilization of systems with input-delay

In conventional approaches LyapunovKrasovskii functionals are defined to deal with both time varying delay and fixed delay in (3.30). Here by using the Leibniz rule (3.4) the system is simplified in such a way that the third term compensate the effect of second term in (3.30) which is more beneficiary compared to the conventional approaches. This section presents the stabilization of such system which is modified by using Leibniz rule with a different Lyapunov-Krasovskii functional and the results obtained with controller proposed in this section are still les conservative than the one discussed in the previous section which was shown by means of a numerical exmple.
Consider the system with time-delay

$$\dot{z}(t) = (A + BK + \Delta A(t) + \Delta B_0(t)K)z(t) + (B_1 + \Delta B_1(t))Kz(t - \tau(t)) \\ -B_1Kz(t - \bar{\tau}) - \Delta A(t) \int_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})} B_1Kz(s)ds, t \geq 0 \quad (3.51)$$

By using the following equation

$$z(t - \tau(t)) - z(t - \bar{\tau}) = \int_{t-\bar{\tau}}^{t-\tau(t)} \dot{z}(s)ds, \quad (3.52)$$

the system can be re-written as

$$\dot{z}(t) = (A + BK + \Delta A(t) + \Delta B_0(t)K)z(t) + B_1K \int_{t-\bar{\tau}}^{t-\tau(t)} \dot{z}(\theta)d\theta +$$
$$\Delta B_1(t)Kz(t-\tau(t)) - \Delta A(t) \int_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})}B_1Kz(s)ds, t \geq 0 \qquad (3.53)$$

3.3.1 Stability Analysis

Lemma 3.3.1 *: For the given scalars $\tau_0$and matrix K the above system is asymptotically stable if ther exist matrices $P, Q_1, R_1 > 0$ and scalars$\varepsilon_j > 0 (j = 1, 2, 3)$ such that*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ * & a_{22} & a_{23} \\ * & * & a_{33} \end{bmatrix} < 0 \qquad (3.54)$$

Proof. For the Robust stabilization of the (3.51) the Lyapunov-Krasovskii functional can be constructed as

$$V(t, z_t) = z^T(t)Pz(t) + \int_{t-\tau(t)}^{t} z^T(\theta)Q_1z(\theta)d\theta + \int_{t-\bar{\tau}}^{t}\int_{\theta}^{t} \dot{z}^T(\phi)R_1\dot{z}(\phi)d\phi d\theta +$$
$$\int_{t-\bar{\tau}}^{t}\int_{s}^{t} z^T(v)\bar{\tau}\varepsilon_1^{-1}K^TB_1^Te^{A^T(s-v)}E_a^TE_ae^{A(s-v)}B_1Kz(v)dvds \qquad (3.55)$$

The first term in (3.55) is to handle the first term associated with uncertainity in (3.53). The second term in (3.55) provides a measure of the signal energy during the delay period while the third term which provides the measure of the energy corresponding to the difference between the instantaneous feedback signal and the delayed one. The fourth term is to handle the respective terms associated with uncertainty of the system. The derivative of (3.55) can be obtained by using the following which are derived from Lemma 1.4.1

$$2e^T(t)CDF(t)E_a \int_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})}B_1Kz(s)ds \leq \xi^T(t)\varepsilon_1CDD^TC^T\xi(t) +$$
$$\int_{t-\bar{\tau}}^{t} z^T(v)\bar{\tau}\varepsilon_1^{-1}K^TB_1^Te^{A^T(s-v)}E_a^TE_ae^{A(s-v)}B_1Kz(s)ds \qquad (3.56)$$

$$2e^T(t)CDF(t)[E_a+E_0K]z(t)\le e^T(t)\varepsilon_2 CDD^TC^Te(t) \\ +z^T(t)\varepsilon_2^{-1}[E_a+E_0K]^T[E_a+E_0K]z(t) \tag{3.57}$$

$$2e^T(t)CDF(t)E_1Kz(t-\tau(t))\le e^T(t)\varepsilon_3 CDD^TC^Te(t) \\ +z^T(t-\tau(t))\varepsilon_3^{-1}K^T{E_1}^TE_1Kz(t-\tau(t)) \tag{3.58}$$

where

$$e^T(t)=\begin{bmatrix} z^T(t) & z^T(t-\tau(t)) & \dot z^T(t)\end{bmatrix}, C^T=\begin{bmatrix}{C_1}^T & {C_2}^T & {C_3}^T\end{bmatrix}. \tag{3.59}$$

And then adding the following (3.60) to the derivative of (3.55), we obtain

$$2\,[z^T(t)C_1+z^T(t-\tau(t))C_2+\dot z^T(t)C_3]\{-\bar A(t)z(t)-B_1K\int_{t-\bar\tau}^{t-\tau(t)}\dot z(\theta)d\theta \\ -\Delta B_1(t)Kz(t-\tau(t))+\Delta A(t)\int_{t-\bar\tau}^{t}e^{A(t-s-\bar\tau)}B_1Kz(s)ds+\dot z(t)\}=0, \tag{3.60}$$

$$\dot V(t,z_t)\le\begin{bmatrix} z(t)\\ z(t-\tau(t))\\ \dot z(t)\end{bmatrix}^T\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ * & a_{22} & a_{23}\\ * & * & a_{33}\end{bmatrix}\begin{bmatrix} z(t)\\ z(t-\tau(t))\\ \dot z(t)\end{bmatrix} \tag{3.61}$$

Accordind to the Lyapunov-Krasovskii theorem proposed in section 2.1.1, for the asymptotic stability of the system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ * & a_{22} & a_{23}\\ * & * & a_{33}\end{bmatrix}<0 \tag{3.62}$$

where,

$a_{11}=Q_1-C_1\,(\mathrm{A}+\mathrm{BK})-\ (\mathrm{A}+\mathrm{BK})^T\ {C_1}^T+\varepsilon_2^{-1}[E_a+E_0K]^T[E_a+E_0K]$
$+\ \bar\tau\varepsilon_1^{-1}K^TB_1^T\int_{-\bar\tau}^{t}e^{A^Ts}E_a^TE_ae^{As}ds+(\varepsilon_1{+}\varepsilon_2{+}\varepsilon_3)\ C_1\mathrm{DD}^\mathrm{T}\ {C_1}^\mathrm{T}$

$a_{12}=-(A+BK)^T{C_2}^T+(\varepsilon_1+\varepsilon_2+\varepsilon_3)C_1DD^T{C_2}^T$

$a_{13}=P+C_1-(A+BK)^T{C_3}^T+(\varepsilon_1+\varepsilon_2+\varepsilon_3)C_1DD^T{C_3}^T$

$a_{22}=-(1-\mu)Q_1+\varepsilon_3^{-1}[E_1K]^T[E_1K]+(\varepsilon_1+\varepsilon_2+\varepsilon_3)C_2DD^T{C_2}^T$

$a_{23} = \ C_2 + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) C_2 D D^T \ {C_3}^T$

$a_{33} = \bar{\tau} R_1 + C_3 + \ {C_3}^T + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) C_3 D D^T \ {C_3}^T$

3.3.2 Controller design

Theorem 3.3.2 *For given scalars τ_0 and feedback gain matrix K the system (3.1) is asymptotically stable if ther exist matrices $\widetilde{P}, \widetilde{Q}_1, \widetilde{R}_1 > 0,\ Y$ and a non-singular matrix X and scalars $\varepsilon_j > 0 (j = 1, 2, 3)$ such that*

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & X E_a^T + Y^T E_0^T & 0 & Y^T B_1^T \\ * & \Gamma_{22} & \Gamma_{23} & 0 & Y^T E_1^T & 0 \\ * & * & \Gamma_{33} & 0 & 0 & 0 \\ * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & \bar{\tau}^{-1} \varepsilon_1 W \end{bmatrix} < 0 \tag{3.63}$$

where,

$\Gamma_{11} = \widetilde{\mathrm{Q}}_1 - \mathrm{A}\ \mathrm{X}^{\mathrm{T}} - \mathrm{BY} - \mathrm{XA}^{\mathrm{T}} - \ \mathrm{Y}^{\mathrm{T}}\ \mathrm{B}^{\mathrm{T}} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\ D D^T$

$\Gamma_{12} = -\rho_2 X \mathrm{A}^{\mathrm{T}} - \rho_2 \mathrm{Y}^{\mathrm{T}}\ \mathrm{B}^{\mathrm{T}} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \rho_2\ D D^T$

$\Gamma_{13} = \ + \ \mathrm{X}^{\mathrm{T}} - \rho_3 X \mathrm{A}^{\mathrm{T}} - \rho_3 \mathrm{Y}^{\mathrm{T}}\ \mathrm{B}^{\mathrm{T}} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \rho_3\ D D^T$

$\Gamma_{22} = -(1 - \mu) \widetilde{\mathrm{Q}}_1 + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\ {\rho_2}^2 D D^T$

$\Gamma_{23} = \rho_2\ \mathrm{X}^{\mathrm{T}} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \rho_2 \rho_3\ D D^T$

$\Gamma_{33} = \ \bar{\tau} \widetilde{\mathrm{R}}_1 + \ \rho_3 X + \rho_3\ \mathrm{X}^{\mathrm{T}} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\ {\rho_3}^2 D D^T$

$\mathrm{W}^{-1} \geq \int\limits_{-\bar{\tau}}^{0} \mathrm{e}^{\mathrm{A}^{\mathrm{T}} \mathrm{s}} {\mathrm{E}_{\mathrm{a}}}^{\mathrm{T}} \mathrm{E}_{\mathrm{a}} \mathrm{e}^{\mathrm{As}} \mathrm{ds} \bullet$

Proof. With the stability results obtained, controller can be designed based on the analysis of the control law $u(t) = Y X^{-T} [x(t) + \int\limits_{t-\bar{\tau}}^{t} e^{A(t-s-\bar{\tau})} B_1 u(s)\, ds]$, by defining $\mathrm{X} = {\mathrm{C}_1}^{-1}$,$C_l = \rho_l C_1, (l = 2, 3)$,$and matrices \widetilde{\mathrm{Q}}_1 = \mathrm{X} \mathrm{Q}_1 \mathrm{X}^{\mathrm{T}}$,$\widetilde{\mathrm{R}}_1 = \mathrm{X} \mathrm{R}_1 \mathrm{X}^{\mathrm{T}}$, $\tilde{P} = \mathrm{X P X}^{\mathrm{T}}$, then Pre, post-multiplying both sides of (3.62) respectively with $diag \left\{ X \quad X \quad X \quad I \quad I \right\}$ and it's transpose with a feedback gain $\mathrm{K} = \mathrm{Y X}^{-\mathrm{T}}$ then by proceeding with similar kind of analysis, the condition for the control to guarantee the asymptotic stability of the closed loop system is obtained as (3.63) •

3.3.3 Numerical Example

Consider the system (3.1) with uncertainty and time-varying input delay

$$\dot{x}(t) = (A + \Delta A(t))\, x(t) + (B_0 + \Delta B_0(t))\, u(t) + (B_1 + \Delta B_1(t))\, u(t - \tau(t)), t \geq 0$$

$x(0) = x_0, u(t) = \phi(t), t \in [-0.2, 0]$ with
$A = \begin{bmatrix} 0 & 1 \\ -1.25 & -3 \end{bmatrix}, \Delta A = \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix}, \quad |q| \leq \gamma, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
For the time delay τ (t) , γ_{max} is defined as the maximum allowable value of γ $0 \leq \tau$ (t) ≤ 0.2 and $\dot{\tau}$ (t) $\leq d_\tau < 1$. It was reported in [1] that 9.8615 and 9.8538 were thae values γ_{max} with $d_\tau = 0$ and $d_\tau = 0.1$ respectively. By solving the LMIs in Robust control tool box in MATLAB now here in this case with the controller propose in this section, with ρ_2 =0.3341, ρ_3=0.0964 and with the controller gains $K = [-216.5779 \quad -49.7890], with$ d$_\tau$ $= 0$ and $d_\tau = 0.1$, it was found that $\gamma_{max} = 10.0481$, $\gamma_{max} = 9.9691$ respectively. which shows the results obtained with the propsed controller are still less conservative compared with the controller propose in [1].

3.4 State feedback Controller design

In previous sections controller design methods are proposed based on reduction method. This section presents the simple state feedback controller design without any transformation for the same system discribed in the previous sections with a different Lyapunov-Krasovskii functional and the results obtained with controller proposed in this section are also still les conservative than the controllers discussed in the previous sections which was presented at the end of this section by means of a numerical exmple.
Consider the system with input delay

$$\dot{x}(t) = (A + \Delta A(t))\, x(t) + Bu(t - \tau(t)), t \geq 0 \tag{3.64}$$

$$x(0) = x_0, \ \mathrm{u}(\mathrm{t}) = \phi(\mathrm{t}), \ \mathrm{t} \in [-\tau, \ 0] \tag{3.65}$$

Under the control law

$$u(t) = Kx(t), \tag{3.66}$$

the system can be transformed as

$$\dot{x}\,(t) = (A + \Delta A\,(t))\,x\,(t) + BKx(t - \tau(t)), t \geq 0 \tag{3.67}$$

3.4.1 Stability Analysis

Lemma 3.4.1 *Consider the closed loop system (3.64). For given scalars τ_0 and feedback gain matrix K the system is asymptotically stable if ther exist matrices $P, Q_1, Q_2, R_1 > 0, N_i$ and M_i $(i = 1, 2) > 0$ and for scalar $\varepsilon > 0$ such that*

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & M_1 \\ * & d_{22} & d_{23} & d_{24} & N_1 \\ * & * & d_{33} & d_{34} & 0 \\ * & * & * & d_{44} & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{3.68}$$

and

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & 0 \\ * & d_{22} & d_{23} & d_{24} & 0 \\ * & * & d_{33} & d_{34} & M_2 \\ * & * & * & d_{44} & N_2 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{3.69}$$

Proof. For the robust stabilization of the system (3.64)-(3.65) the Lyapunov-Krasovskii functional is constructed as

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(\theta)Q_1x(\theta)d\theta + \int_{t-\bar{\tau}}^{t} x^T(\theta)Q_2x(\theta)d\theta$$
$$+\bar{\tau}\int_{t-\bar{\tau}}^{t}\int_{\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta \tag{3.70}$$

By using the following in-equality (3.71) which is derived from Lemma 1.4.1

$$2\xi^T(t)CDF(t)[E_a + E_0K]z(t) \leq \xi^T(t)\varepsilon CDD^TC^T\xi(t)$$
$$+z^T(t)\varepsilon^{-1}[E_a + E_0K]^T[E_a + E_0K]z(t) \tag{3.71}$$

where

$$\xi^T(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau(t)) & x^T(t-\bar{\tau}) & \dot{x}^T(t) \end{bmatrix}, C^T = \begin{bmatrix} {C_1}^T & {C_2}^T & {C_3}^T \end{bmatrix}. \tag{3.72}$$

then by adding the following equality (3.73) to the derivative of (3.70)

$$\begin{aligned} 2[x^T(t)C_1 + x^T(t-\tau(t))C_2 + \dot{x}^T(t)C_3]\,\{-\left(A+\Delta A\left(t\right)\right)x\left(t\right) \\ -BKx(t-\tau(t)) + \dot{x}\left(t\right)\}= 0 \end{aligned} \tag{3.73}$$

The derivative of $V(t, x_t)$ is obtained as

$$\dot{V}(t, z_t) \leq \xi^T(t)\left\{\Theta + \sigma\phi_1\bar{\tau}^2 R_1^{-1}{\phi_1}^T + (1-\sigma)\phi_2\bar{\tau}^2 R_1^{-1}{\phi_2}^T\right\}\xi(t) \tag{3.74}$$

where

$$\phi_1 = \begin{bmatrix} M_1 \\ N_1 \\ 0 \\ 0 \end{bmatrix}, \phi_2 = \begin{bmatrix} 0 \\ M_2 \\ N_2 \\ 0 \end{bmatrix} \tag{3.75}$$

Accordind to the Lyapunov-Krasovskii theorem proposed in section 2.1.1, for the asymptotic stability of the system

$$\Theta + \sigma\phi_1\bar{\tau}^2 R_1^{-1}{\phi_1}^T + (1-\sigma)\phi_2\bar{\tau}^2 R_1^{-1}{\phi_2}^T < 0 \tag{3.76}$$

Since $\sigma = \frac{\tau(t)}{\bar{\tau}}, 0 \leq \sigma \leq 1$.
the equ (3.71) holds good if

$$\Theta + \phi_l\left(\bar{\tau}^{-2}R_1\right)^{-1}{\phi_l}^T < 0, l = 1, 2. \tag{3.77}$$

Applying schur compliment using Lemma 2 (1.5), the condition for stability is obtained as

$$\begin{bmatrix} \Theta & \phi_1 \\ {\phi_1}^T & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0, l = 1, 2 \tag{3.78}$$

which implies

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & M_1 \\ * & d_{22} & d_{23} & d_{24} & N_1 \\ * & * & d_{33} & d_{34} & 0 \\ * & * & * & d_{44} & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{3.79}$$

and

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & 0 \\ * & d_{22} & d_{23} & d_{24} & 0 \\ * & * & d_{33} & d_{34} & M_2 \\ * & * & * & d_{44} & N_2 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{3.80}$$

where,

$d_{11} = Q_1 + Q_2 + \bar{\tau}(M_1 + {M_1}^T) - C_1A - \ A^T\ {C_1}^T + \varepsilon^{-1}E_a^T E_a + \varepsilon\ C_1\mathrm{DD}^\mathrm{T}\ {C_1}^\mathrm{T}$

$d_{12} = \bar{\tau}(-M_1 + {N_1}^T) - C_1BK - A^T{C_2}^T + \varepsilon C_1DD^T{C_2}^T$

$d_{13} = 0$

$d_{14} = P + C_1 - A^T{C_3}^T + \varepsilon C_1DD^T{C_3}^T$

$d_{22} = \bar{\tau}(-N_1 - {N_1}^T) + \bar{\tau}(M_2 + {M_2}^T) - (1-\mu)Q_1 - C_2BK - (C_2BK)^T + \varepsilon C_2DD^T{C_2}^T$

$d_{23} = \bar{\tau}(-M_2 + {N_2}^T)$

$d_{24} = C_2 - (C_3BK)^T + \varepsilon C_2DD^T{C_3}^T$

$d_{33} = \bar{\tau}(-N_2 - {N_2}^T) - Q_2$

$d_{34} = 0$

$d_{44} = \ \bar{\tau}^2R_1 + C_3 + \ {C_3}^T + \varepsilon C_3DD^T\ {C_3}^T$

3.4.2 Controller Design

Theorem 3.4.2 *For the given scalars $\rho_l\,(l = 2, 3)\,, \tau_0$ the system (3.64)-(3.65) is asymptotically stable if there exist matrices $\widetilde{P}, \widetilde{Q}_k, \widetilde{N}_k, \widetilde{M}_k\,(k = 1, 2)\,, \widetilde{R}_1 > 0, \widetilde{R}_2 > 0$, Y and a non-singular matrix X and scalar $\varepsilon\ > 0$ such that*

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & XM_1X^T & XE_a^T \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & XM_1X^T & 0 \\ * & * & \Gamma_{33} & \Gamma_{34} & 0 & 0 \\ * & * & * & \Gamma_{44} & 0 & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 & 0 \\ * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0 \quad (3.81)$$

and

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & 0 & XE_a^T \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & XM_2X^T & 0 \\ * & * & \Gamma_{33} & \Gamma_{34} & XN_2X^T & 0 \\ * & * & * & \Gamma_{44} & 0 & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_2 & 0 \\ * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0 \quad (3.82)$$

where,

$\Gamma_{11} = \widetilde{\mathrm{Q}}_1 + \widetilde{\mathrm{Q}}_2 + \bar{\tau}\left(\widetilde{\mathrm{M}}_1 + \widetilde{\mathrm{M}}_1^T\right) - \mathrm{A}\ \mathrm{X}^\mathrm{T} - \mathrm{XA}^\mathrm{T} + \varepsilon\ DD^T$

$\Gamma_{12} = \bar{\tau}\left(-\widetilde{\mathrm{M}}_1 + \widetilde{\mathrm{N}}_1^T\right) - \mathrm{BY} - \rho_2 X\mathrm{A}^\mathrm{T} + \varepsilon\rho_2\ DD^T$

$\Gamma_{13} = 0$

$\Gamma_{14} = \tilde{P} + \mathrm{X}^\mathrm{T} - \rho_3 X\mathrm{A}^\mathrm{T} + \varepsilon\rho_3\ DD^T$

$\Gamma_{22} = \bar{\tau}\left(-\widetilde{\mathrm{N}}_1 - \widetilde{\mathrm{N}}_1^T\right) + \bar{\tau}\left(\widetilde{\mathrm{M}}_2 + \widetilde{\mathrm{M}}_2^T\right) - (1-\mu)\widetilde{\mathrm{Q}}_1 - \rho_2\mathrm{BY} - \rho_2\mathrm{Y}^\mathrm{T}\ \mathrm{B}^\mathrm{T} + \varepsilon{\rho_2}^2 DD^T$

$\Gamma_{23} = \bar{\tau}\left(-\widetilde{\mathrm{M}}_2 + \widetilde{\mathrm{N}}_2^T\right)$

$\Gamma_{24} = \rho_2\ \mathrm{X}^\mathrm{T} - \rho_3\mathrm{Y}^\mathrm{T}\ \mathrm{B}^\mathrm{T} + \varepsilon\rho_2\rho_3\ DD^T$

$\Gamma_{33} = \bar{\tau}\left(-\widetilde{\mathrm{N}}_2 - \widetilde{\mathrm{N}}_2^T\right)\widetilde{\mathrm{Q}}_2$

$\Gamma_{34} = 0$

$\Gamma_{44} = \bar{\tau}^2\widetilde{\mathrm{R}}_1 + \rho_3 X + \rho_3\ \mathrm{X}^\mathrm{T} + \varepsilon{\rho_3}^2 DD^T$

Proof. The state feedback controller can be designed with the analysis done based on the control law (3.66) and by defining $\mathrm{X}={\mathrm{C}_1}^{-1}$,$C_l = \rho_l C_1, (l = 2, 3)$,then Pre, post-multiplying both sides of each of (3.79) *and* (3.80) with $diag\left\{X \quad X \quad X \quad X \quad X \quad I\right\}$ and it's transpose respectively and also by defining $\widetilde{\mathrm{Q}}_k = \mathrm{XQ_kX^T}$(k=1,2),$\widetilde{\mathrm{M}}_i = \mathrm{XM_iX^T}$, $\widetilde{\mathrm{N}}_i = \mathrm{XN_iX^T}$, $\widetilde{\mathrm{R}}_\mathrm{i} = \mathrm{XR}_i\mathrm{X^T}(\mathrm{i} = 1, 2)$, $\tilde{P} = \mathrm{XPX^T}$, and with a feedback gain $\mathrm{K=YX^{-T}}$ then the conditions for the control to guarantee the asymptotic stability of the closed loop system are obtained as (3.81) and (3.82)•

3.4.3 Numerical Example

Consider the system (3.1) with uncertainty and time-varying input delay

$$\dot{x}(t) = (A + \Delta A(t))\, x(t) + (B_0 + \Delta B_0(t))\, u(t) + (B_1 + \Delta B_1(t))\, u(t - \tau(t)), t \geq 0$$

$x(0) = x_0, u(t) = \phi(t), t \in [-0.2, 0]$ with

$$A = \begin{bmatrix} 0 & 1 \\ -1.25 & -3 \end{bmatrix}, \Delta A = \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix}, \quad |q| \leq \gamma, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For the time delay τ (t) , γ_{max} is defined as the maximum allowable value of γ $0 \leq \tau$ (t) ≤ 0.2 and $\dot{\tau}$ (t) $\leq d_\tau < 1$. It was reported in [1] that 9.8615 and 9.8538 were thae values γ_{max} with $d_\tau = 0$ and $d_\tau = 0.1$ respectively. By solving the LMIs in Robust control tool box in MATLAB now here in this case with the controller propose in this section , with ρ_2= -0.000188, ρ_3= 0.213054 and with the controller gains $K = [-14.0417 \quad -3.6489]$,with $d_\tau = 10^{19}$ (very large), it was observed that $\gamma_{max} = 10.07823$ which shows the results obtained with the propsed controller are still less conservative compared to the one proposed in [1] and also the one proosed in the previous section.

3.5 Discussion

Different Static state feed back controller design methods are proposed in each section of this chapter for the stabilization of systems with time varying input delay. With the controller proposed in section 3.2 by modifying the transformation of the system robustness is improved compared to [9] and [10]. With the controller proposed in the section 3.3 by transforming the system to another form by using Leibniz rule leads to much lesser conservative results compared to the one proposed in section 3.2. with the simple state feed back controller proposed in section 3.4 the results obtained are still lesser conservative than the one proposed in sections 3.2 and 3.3. The comparision of the performances of each controller discussed in each section of this chapter is shown in Table 3.1.

Section	Control law	No. of Tuning Parameters	γ_{max}
3.2	$u(t) = Kz(t)$	3	5.0951
3.3	$u(t) = Kz(t)$	2	10.0481
3.4	$u(t) = Kx(t)$	2	10.0782

Table 3.1: Comparision of robustness for $d_\tau = 0$

Chapter 4

PI-type State feedback Controller for Robust Stabilization of Systems with Input delay

Chapter 4

PI-type State feedback Controller for Robust Stabilization of systems with input delay

This chapter presents the robust stabilization of time-delay systems using PI-type state feedback controller design.

4.1 PI controller

A proportional-Integral controller (PI-Controller) is one of the used feedback control loop mechanism used in Industrial-control systems. A PI controller calculates an error value which is the difference between the value of processed measured variable and the desired set-point value. The controller is designed such that it attempts to minimize the error by varying control inputs of the process. The PI controller design involves two parameters, the proportional and the derivative which are denoted as P and I respectively and their respective gains are represented as K_p and K_i. P depends on the present error and I depends on the accumulation of past errors.The weighted sum of these two are used to adjust the process so as to achieve the setpoint.

The PI controller output is of the form

$$u = K_p e(t) + K_i \int e(t)dt \tag{4.1}$$

where e(t) is the difference between the value of processed measured variable and the desired set-point value.

4.1.1 Proportional term

The output value of the proportional term is proportional to the current error value.The response of proportional part can be modifies or adjusted by multiplying the error by a constant gain K_p which is called as the proportional gain constant. The output of proportional part is given by:

$$u_P = K_p e(t) \tag{4.2}$$

If K_p is very high,the system may become unstable but high K_p results in large change in the output with a given change in error. And small value of K_p leads to small output response when the input error is large.

4.1.2 Integral term

The integral part in PI controller, eliminates the steady-state error which occurs with a pure proportional controller. But as the integral term responds to accumulated errors in the past, it causes the present value to overshoot the set point value.
The output of integral part is given by:

$$u_I = K_i \int e(t)dt \tag{4.3}$$

4.2 PI Controller design for Robust Stabilzation

Now with the knowledge of the PI controller, a PI controller design method is propsed for the robust stabilization of the system with time-varying input delay. Consider the system

$$\dot{x}\,(t) = (A + \Delta A\,(t))\,x\,(t) + Bu(t - \tau(t)), t \geq 0 \tag{4.4}$$

Under the control law

$$u(t) = K_p x(t) + K_I \int_0^t x(\theta) d\theta \tag{4.5}$$

$$\dot{z}(t) = x(t) \tag{4.6}$$

State space form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A + \Delta A & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t - \tau(t)) \tag{4.7}$$

$$u(t) = \begin{bmatrix} K_P & K_I \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{4.8}$$

By using (4.8), (4.7) can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A + \Delta A & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K_P & K_I \end{bmatrix} \begin{bmatrix} x(t - \tau(t)) \\ z(t - \tau(t)) \end{bmatrix} \tag{4.9}$$

the system (4.9) can be represented as

$$\dot{\bar{x}}(t) = \hat{A}\bar{x}(t) + \hat{B}K\bar{x}(t - \tau(t)) \tag{4.10}$$

where $\dot{\bar{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix}$, $\hat{A} = \begin{bmatrix} A + \Delta A & 0 \\ 1 & 0 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $K = \begin{bmatrix} K_P & K_I \end{bmatrix}$

4.2.1 Stability Analysis

Lemma 4.2.1 *Consider the closed-loop system (4.4). For given scalars τ_0 and feedback gain matrix K the system is asymptotically stable if there exist matrices $P, Q_1, Q_2 > 0$ such that*

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & M_1 \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} & N_1 \\ * & * & \Theta_{33} & \Theta_{34} & 0 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{4.11}$$

and

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & 0 \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} & M_2 \\ * & * & \Theta_{33} & \Theta_{34} & N_2 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{4.12}$$

Proof. Now, Lyapunov-Krasovskii functional is choosen as

$$V(t,x_t) = x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(\theta)Q_1x(\theta)d\theta + \int_{t-\bar{\tau}}^{t} x^T(\theta)Q_2x(\theta)d\theta + \bar{\tau}\int_{t-\bar{\tau}}^{t}\int_{\theta}^{t} \dot{x}^T(s)R_1\dot{x}(s)dsd\theta \tag{4.13}$$

The first term in (4.13) is to handle the first term associated with uncertainity in (4.4), second term and third terms in (4.13) provides a measure of the signal energy during the delay period while the fourth term provides the measure of the energy corresponding to the difference between the signal sought for feedback, and the one available for feedback. The derivative of (4.13) is obtained as

$$\begin{aligned} \dot{v}(t,x_t) &= 2x^T(t)P\dot{x}(t) + x^T(t)[Q_1+Q_2]x(t) - (1-\mu)x^T(t-\tau(t))Q_1x(t-\tau(t)) \\ &\quad -x^T(t-\bar{\tau})Q_2x(t-\bar{\tau}) + \bar{\tau}\dot{x}^T(t)R_1\dot{x}(t) - \left\{\int_{t-\bar{\tau}}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta\right\} \end{aligned} \tag{4.14}$$

Using Lemma 1.4.3 and also by adding the following equality (4.15) to the derivative of (4.13)

$$[x^T(t)C_1 + x^T(t-\tau(t))C_2 + \dot{x}^T(t)C_3]\{-\hat{A}\bar{x}(t) - \hat{B}K\bar{x}(t-\tau(t)) + \dot{\bar{x}}(t)\} = 0 \tag{4.15}$$

we obtain

$$\dot{V}(t,z_t) \leq \xi^T(t)\left\{\Theta + \sigma\phi_1\bar{\tau}^2R_1^{-1}\phi_1{}^T + (1-\sigma)\phi_2\bar{\tau}^2R_1^{-1}\phi_2{}^T\right\}\xi(t) \tag{4.16}$$

where

$\xi^T(t) = \begin{bmatrix} \bar{x}^T(t) & \bar{x}^T(t-\tau(t)) & \bar{x}^T(t-\bar{\tau}) & \dot{\bar{x}}^T(t) \end{bmatrix}, C^T = \begin{bmatrix} {C_1}^T & {C_2}^T & {C_3}^T \end{bmatrix},$

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} \\ * & * & \Theta_{33} & \Theta_{34} \\ * & * & * & \Theta_{44} \end{bmatrix} \tag{4.17}$$

and

$$\phi_1 = \begin{bmatrix} M_1 \\ N_1 \\ 0 \\ 0 \end{bmatrix}, \phi_2 = \begin{bmatrix} 0 \\ M_2 \\ N_2 \\ 0 \end{bmatrix} \tag{4.18}$$

According to the Lyapunov-Krasovskii theorem proposed in section 2.1.1, for the asymptotic stability of the system

$$\Theta + \sigma\phi_1\bar{\tau}^2 R_1^{-1}{\phi_1}^T + (1-\sigma)\phi_2\bar{\tau}^2 R_1^{-1}{\phi_2}^T < 0 \tag{4.19}$$

Since $\sigma = \frac{\tau(t)}{\bar{\tau}}, 0 \leq \sigma \leq 1$, the equ (4.19) holds good only if

$$\Theta + \phi_l(\bar{\tau}^{-2}R_1)^{-1}{\phi_l}^T < 0, l = 1, 2 \tag{4.20}$$

which implies

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & M_1 \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} & N_1 \\ * & * & \Theta_{33} & \Theta_{34} & 0 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{4.21}$$

and

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & 0 \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} & M_2 \\ * & * & \Theta_{33} & \Theta_{34} & N_2 \\ * & * & * & \Theta_{44} & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{4.22}$$

where,

$\Theta_{11} = Q_1 + Q_2 + (M_1 + M_1{}^T) - C_1\hat{A} - \ \hat{A}^T\ C_1{}^T$

$\Theta_{12} = (-M_1 + N_1{}^T) - C_1\hat{B}K - \hat{A}^T C_2{}^T$

$\Theta_{13} = 0$

$\Theta_{14} = P + C_1 - \hat{A}^T C_3{}^T$

$\Theta_{22} = (-N_1 - N_1{}^T) + (M_2 + M_2{}^T) - (1-\mu)Q_1 - C_2\hat{B}K - (C_2\hat{B}K)^T$

$\Theta_{23} = (-M_2 + N_2{}^T)$

$\Theta_{24} = C_2 - (C_3\hat{B}K)^T$

$\Theta_{33} = (-N_2 - N_2{}^T) - Q_2$

$\Theta_{34} = 0$

$\Theta_{44} = \ \bar{\tau}R_1 + C_3 + \ C_3{}^T$

4.2.2 Controller design

Theorem 4.2.2 *For the given scalars $\rho_l\,(l = 2, 3)\,, \tau_0$ and a feedback gain matrix K, the system (4.4) is asymptotically stable if there exist matrices $\widetilde{P}, \widetilde{Q}_k\,(k = 1, 2)\,,$ $\widetilde{N}_k\,(k = 1, 2)\,, \widetilde{M}_k\,(k = 1, 2)\,, \widetilde{R}_1 > 0$, Y and a non-singular matrix X such that*

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & XM_1X^T \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & XN_1X^T \\ * & * & \Gamma_{33} & \Gamma_{34} & 0 \\ * & * & * & \Gamma_{44} & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{4.23}$$

and

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & 0 \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & XM_2X^T \\ * & * & \Gamma_{33} & \Gamma_{34} & XN_2X^T0 \\ * & * & * & \Gamma_{44} & 0 \\ * & * & * & * & -\bar{\tau}^{-2}R_1 \end{bmatrix} < 0 \tag{4.24}$$

where,

$\Gamma_{11} = \widetilde{\mathrm{Q}}_1 + \widetilde{\mathrm{Q}}_2 + \left(\widetilde{\mathrm{M}}_1 + \ \widetilde{\mathrm{M}}_1^T\right) - \hat{A}X^T - X\hat{A}^T$

$\Gamma_{12} = \left(-\widetilde{\mathrm{M}}_1 + \ \widetilde{\mathrm{N}}_1^T\right) - \hat{B}Y - \rho_2 X\hat{A}^T$

$\Gamma_{13} = 0$

$\Gamma_{14} = \tilde{P} + \mathrm{X}^{\mathrm{T}} - \rho_3 X \hat{A}^T$
$\Gamma_{22} = \left(-\widetilde{\mathrm{N}}_1 - \widetilde{\mathrm{N}}_1^T\right) + \left(\widetilde{\mathrm{M}}_2 + \widetilde{\mathrm{M}}_2^T\right) - (1-\mu)\widetilde{\mathrm{Q}}_1 - \rho_2 \hat{B} Y - \rho_2 \mathrm{Y}^{\mathrm{T}} \hat{B}^T$
$\Gamma_{23} = \left(-\widetilde{\mathrm{M}}_2 + \widetilde{\mathrm{N}}_2^T\right)$
$\Gamma_{24} = \rho_2\ \mathrm{X}^{\mathrm{T}} - \rho_3 \mathrm{Y}^{\mathrm{T}} \hat{B}^T$
$\Gamma_{33} = \left(-\widetilde{\mathrm{N}}_2 - \widetilde{\mathrm{N}}_2^T\right) - \widetilde{\mathrm{Q}}_2$
$\Gamma_{34} = 0$
$\Gamma_{44} = \bar{\tau}\widetilde{\mathrm{R}}_1 + \rho_3 X + \rho_3\ \mathrm{X}^{\mathrm{T}}$

Proof.The analysis was done based on the control law (4.8) by defining $\mathrm{X}=\mathrm{C_1}^{-1}$,$C_l = \rho_l C_1, (l = 2, 3)$,then Pre, post-multiplying both sides of each of (4.9) and (4.10) with $diag\left\{X \quad X \quad X \quad X \quad X\right\}$ and its transpose and by defining $\widetilde{\mathrm{Q}}_k = \mathrm{XQ_kX^T}$(k=1,2), $\widetilde{\mathrm{M}}_i = \mathrm{XM_iX^T}$, $\widetilde{\mathrm{N}}_i = \mathrm{XN_iX^T}(\mathrm{i} = 1, 2), \widetilde{\mathrm{R}_1} = \mathrm{XR_1X^T}$, $\tilde{P} = \mathrm{XPX^T}$, and with a feedback gain K then the conditions for the control to guarantee the asymptotic stability of the closed loop system are obtained as (4.21) and (4.22)

4.2.3 Numerical Example

Consider the system (3.1) with uncertainty and time-varying input delay

$$\dot{x}(t) = (A + \Delta A(t))\, x(t) + (B_0 + \Delta B_0(t))\, u(t) + (B_1 + \Delta B_1(t))\, u(t - \tau(t)), t \geq 0$$

$x(0) = x_0, u(t) = \phi(t), t \in [-0.2, 0]$ with

$$A = \begin{bmatrix} 0 & 1 \\ -1.25 & -3 \end{bmatrix}, \Delta A = \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix}, \quad |q| \leq \gamma, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For the time delay τ (t) , γ_{max} is defined as the maximum allowable value of γ $0 \leq \tau$ (t) ≤ 0.2 and $\dot{\tau}$ (t) $\leq d_\tau < 1$. It was reported in [1] that 9.8615 and 9.8538 were thae values γ_{max} with $d_\tau = 0$ and $d_\tau = 0.1$ respectively. By solving the LMIs in Robust control tool box in MATLAB now here in this case with the controller propose in this section, with ρ_2 =-0.00018, ρ_3= 0.5499, and with the controller gains $K_p = [-33.8957 \quad -4.7821]$ and $K_I = -0.0304$, with $d_\tau = 10^{19}$ (very large), $\gamma_{max} = 28.6025$, which shows that the results obtained with the propsed PI type state feedback controller improves the robustness to a larger extent than the static state feedback controller proposed in previous chapter.

4.2.4 Simulation and Results

The Simulink model of PI controller for the system (4.7) is shown in Figure 4.1 and it's simulation results are shown in Figure 4.2.

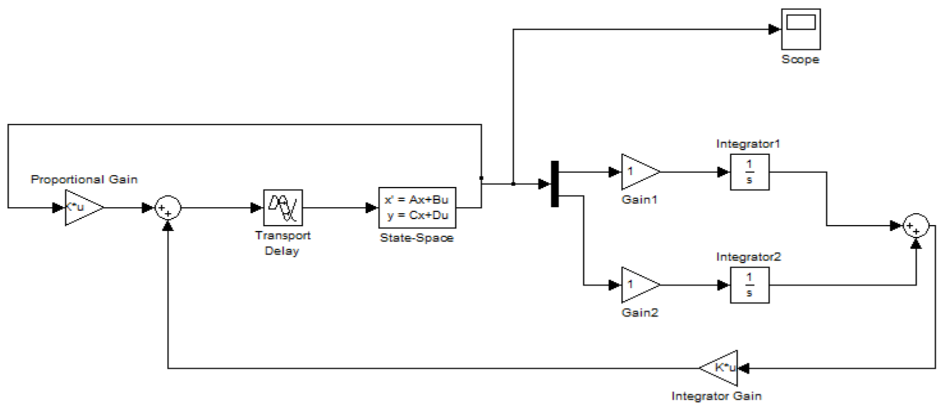

Figure 4.1: Simulink model of PI Controller

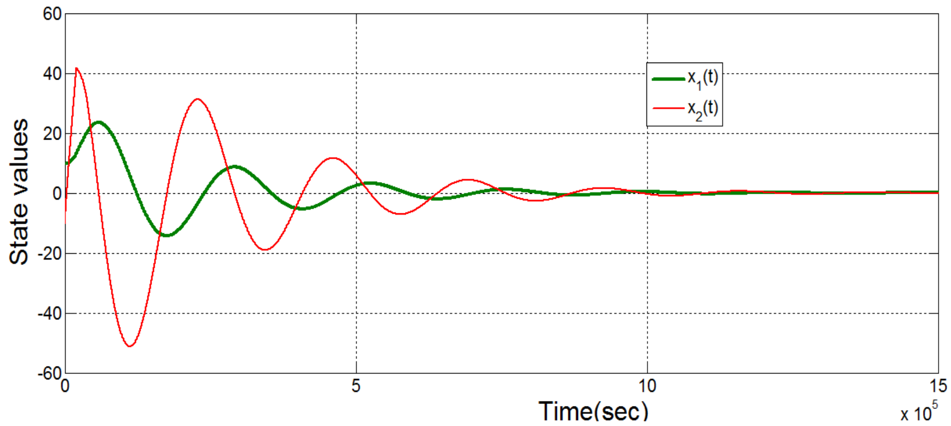

Figure 4.2: State varibles reaching steady state with the progress of time

From Figure 4.2 it is clear that the state are reaching to steady state values after some transient period. So with the proposed PI-type state feedback controller design, it is observed that with $K_p = [-33.8957 \quad -4.7821]$ and $K_I = -0.0304$, the system is stable over a large value of $\gamma_{max} = 28.6025$. Which shows that the propsed PI controller design method leads to much less conservative results than the one proposed in [1] and also the controller design methods presented in the previous chapters.

Chapter 5

Conclusions

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Conclusions

The propsed approaches which are developed for the robust stabilzation of systems with time-varying input delay improves the robustness using simple static state feedback controller, PI-type state feedback controller. From the results, it can be found that the proposed controllers lead to much less conservative comparative to the other. Various Lyapunov-Krasovskii functional are constructed to study the stability analysis for the systems with time-varying input delay. As the design of the controller only needs the information on the variation range of the time delay, make us deal with the unknown and time-varying input delay case. The linearization methods which are used in the proposed approaches to obtain the LMI′s which discribes the stability conditions are simple and easy. Hence static state feedback controller design methods based on reduction method are proposed for systems with input delay. And a PI-type state feedback controller design method is proposed for uncertain linear systems with time-varying input delay which improves the robustness of the system to a large extent.

This method can be extended by choosing appropriate Lyapunov polynomials and tuning parameters to obtain much lesser conservative results. The stabilization criteria can also be simplified by using necessary available mathematical tools to reduce the complexity. The proposed controller design methods can also be checked for the systems with multiple input delays and state delays.

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