INTERVAL NONLINEAR EIGENVALUE PROBLEMS

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Declaration

I hereby certify that the work which is being presented in this thesis entitled ”Interval Nonlinear Eigenvalue Problem” in partial fulfilment of the award of the degree of Master of science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my own work carried out under the supervision of Dr. S.Chakraverty.

The matter embodied in this has not been submitted by me for the award of any other degree.

(Satyabrata Sadangi) Date:

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Contents

Abstract

Nonlinear eigenvalue Problems are currently receiving much attention because of its extensive applications in areas such as the dynamic analysis of mechanical systems, acoustics and fluid mechanics etc. These eigenvalue problems arise in various other applications too.

Open literatures reveal that nonlinear eigenvalue problems are solved by various methods when the matrices involved are having crisp or exact elements. But in actual practice the elements of the matrices may not be crisp. Those may be uncertain due to error in the experiments or observations etc. As such, in this study we have considered the uncertainty in term of intervals.

Accordingly this thesis investigates a new form of interval nonlinear eigenvalue problem using interval computation. Here the degree of above mentioned nonlinear eigenvalue problem is reduced to standard linear eigenvalue problem and the procedure is applied to various example problems including an application problem of structural mechanics. Also data from Harwell-Boeing collection matrix market have been used for investigation of dynamic analysis of structural engineering. Corresponding plots and Tables are given to understand the problem showing the efficacy and powerfulness of the method.

Chapter 1

Introduction

Eigenvalue problems occur in various branches of engineering and science. These eigenvalue problems may come into picture when a physical system is modelled mathematically. Few of the different standard problems in this respect may be mentioned as the natural frequencies and mode shapes in vibration problems, the principal axes in elasticity and dynamics, the Markov chain in stochastic modeling, queueing theory and the analytical hierarchy process for decision making etc. Sometimes we get generalized eigenvalue problem instead of standard eigenvalue problem. The problems which involve nonlinearity may transform to nonlinear eigenvalue problem. The standard form of a nonlinear eigenvalue problem is F(λ)X = 0, where F :C$\rightarrow$ $C^{m\times n}$ is a given matrix-valued function and $\lambda \in$ C and the nonzero vector X $\in C^n$ are the eigenvalues and eigenvectors, respectively. In dynamic problems of structural engineering, one often encounter problems where matrix is non-symmetric and we have to determine the value of a scalar λ which satisfies the equation AX=λ X . Since A is the real non-symmetric matrix, the eigenvalue λ is generally assumed by complex constant number $\lambda = \lambda_r + i\lambda_y$, where λ_r and λ_y are respectively the real and imaginary parts of the complex eigenvalue λ for identical structural system. Characterization of the set of eigenvalues of a general interval matrix A$'$ (where the elements of A$'$ are all intervals) is introduced and method to find eigenpair (λ , X) of A$'$X = λ X, is presented. Therefore right and left eigenvalues of the interval matrix are found.

In view of the above, one may have the quadratic form of nonlinear eigenvalue problem and it is called nonlinear quadratic eigenvalue problem. These are excellently surveyed by

[Mehrmann and Voss 2004, Tisseur and Meerbergen 2001]. They have discussed quadratic eigenvalue problem for application point of view, its mathematical properties and variety of numerical solution technique. Further [Betcke,Higham,Mehrmann,Schroder and Tissur 2011] presented a collection of nonlinear eigenvalue problem which contains problems from real-life application according to their structural properties. Recently Pseudospectra of large polynomial eigenvalue problem have been investigated by [Wang,Wang,Zhong 2011], which projects to reduce the size of nonlinear eigenvalue problem into generalized eigenvalue problem by using generalized Arnoldi method.

Here we have investigated interval nonlinear eigenvalue problems of various degree. In the above literature, authors have used only crisp matrices. Interval nonlinear eigenvalue problems are often highly structured and it is important to take account of the structure both in developing the theory and in designing the numerical methods. We therefore provide a thorough study of the above problems systematically for better understanding of the methods.

Interval Arithmetic

Basic Definitions

The interval arithmetic is an extension of ordinary arithmetic[Chiao,1999]. We shall denote A^I by a closed interval of the form,

$$A^I = [\underline{a}, \bar{a}] = \{a | \underline{a} \leq a \leq \bar{a}, \underline{a}, \bar{a} \in R\}$$

where $\underline{a}$ is the left element and $\bar{a}$ is the right element of the interval A^I. Next we define the center and radius of A^I respectively as follows,

$$\text{center}: \ a^c = \tfrac{1}{2}(\underline{a} + \bar{a}) \ ,$$
$$\text{radius} \ \triangle a = \tfrac{1}{2}(\bar{a} - \underline{a})$$

thus,

$$A^I = [a^c - \triangle a, a^c + \triangle a]$$

The absolute value is defined by the following equation and can be written in terms of center and radius.

$$\begin{aligned}|A^I| &\triangleq max(|\underline{a}|, |\bar{a}|) \\ &= |a^c + \triangle a|\end{aligned}$$

The transitive order relation $<$ of real numbers can be extended to the intervals as follows:

$$A^I < B^I \equiv \bar{a} < \underline{b}$$

the other relations $\leq, >, \geq$ are defined similarly. For the last case we say that A^I and B^I are not comparable. However the maximum and minimum can be defined on non-comparable intervals is

$$\begin{aligned}\max(A^I, B^I) &\triangleq [max(\underline{a}, \underline{b}), max(\bar{a}, \bar{b})] \\ \min(A^I, B^I) &\triangleq [min(\underline{a}, \underline{b}), min(\bar{a}, \bar{b})]\end{aligned}$$

In general,

$$max_i\ (A_i^I) = [max_i(\underline{a}_i), max(\bar{a}_i)],\ min_i\ (A_i^I) = [min_i(\underline{a}_i), min(\bar{a}_i)],$$

Thus the order relation of the interval can be defined by the min and max operators as follows :

$$A^I \leq B^I \Longleftrightarrow max(A^I, B^I) = B^I$$

Let A^I and B^I be two intervals and * be one of the binary operators (+,-,*,/). The interval arithmetic of two intervals is a set given by

$$A^I * B^I = \{a * b | a \in A^I, b \in B^I\}$$

Note that B^I should not contain 0 if * = / , by the definition of A^I and B^I one can easily obtain that

$$\begin{aligned}A^I + B^I &= (a^c + b^c) + (\triangle a + \triangle b)[-1, 1], \\ A^I - B^I &= (a^c - b^c) + (\triangle a + \triangle b)[-1, 1], \\ A^I \times B^I &= [min(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}), max(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b})], \\ A^I/B^I &\triangleq [\underline{a}, \bar{a}] \times [\tfrac{1}{\bar{b}}, \tfrac{1}{\underline{b}}] = [min(\tfrac{\underline{a}}{\bar{b}}, \tfrac{\underline{a}}{\bar{b}}, \tfrac{\bar{a}}{\underline{b}}, \tfrac{\bar{a}}{\underline{b}}), max(\tfrac{\underline{a}}{\bar{b}}, \tfrac{\underline{a}}{\bar{b}}, \tfrac{\bar{a}}{\underline{b}}, \tfrac{\bar{a}}{\underline{b}})],\ \text{if } 0 \notin B^I\end{aligned}$$

Chapter 2

Nonlinear eigenvalue problems

A nonlinear eigenvalue problem is a generalization of ordinary eigenvalue problem which depends on the nonlinearity of the eigenvalues. Specifically it refers to equations of the form: A(λ)X = 0, where X is a nonlinear eigenvector and A is a matrix-valued function of the scalar λ, for nonlinear eigenvalue. Generally, A(λ) could be a linear map, but commonly it is a finite-dimensional square matrix. For example, an ordinary linear eigenvalue problem $Bv = \lambda v$, where B is a square matrix corresponds to $A(\lambda) = B - \lambda I$, where I is the identity matrix.

Quadratic eigenvalue problem

Definition 2.0.1. *When A is a polynomial then the eigenvalue problem is called polynomial eigenvalue problem. In particular, when the polynomial has degree two then it is called a quadratic eigenvalue problem and can be written in the form:*

$$A(\lambda) = (A_2\lambda^2 + A_1\lambda + A_0)X = 0$$

where A_0, A_1, A_2 are constant square matrices.

Cubic eigenvalue problem

Definition 2.0.2. *If the polynomial matrix is of degree three then it is called cubic eigenvalue problem and this may be written as:*

$$A(\lambda) = (A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0)X = 0$$

where A_0, A_1, A_2, A_3 are constant square matrices.

$\mathbf{n^{th}}$ degree Eigenvalue Problem

Definition 2.0.3. *We may write n^{th} degree polynomial eigenvalue problem in the following form:*

$$A(\lambda) = (A_n\lambda^n + A_{n-1}\lambda^{n-1} + \cdots + A_1\lambda + A_0)X = 0$$

where $A_i, i = 0, 1, 2, \cdots, n$ are constant square matrices.

Interval quadratic eigenvalue problem

Definition 2.0.4. *Let us consider A is a polynomial matrix whose entries are intervals. For such interval matrices the eigenvalue problem is called interval polynomial eigenvalue problem. In particular, when interval polynomials have degree two then it is called an interval quadratic eigenvalue problem and one can write in the form:*

$$A(\lambda) = ([(A_2)^l, (A_2)^r]\lambda^2 + [(A_1)^l, (A_1)^r]\lambda + [(A_0)^l, (A_0)^r])X = 0$$

where $[A_0^l, A_0^r], [A_1^l, A_1^r]$ and $[A_2^l, A_2^r]$ are constant interval square matrices. Here $[A_0^l, A_0^r]$ means the elements of this interval matrix are of the form $(\underline{a}_0, \bar{a}_0)$ etc.

Interval cubic eigenvalue problem

Definition 2.0.5. *If the interval polynomial matrices have degree three then it is called interval cubic eigenvalue problem, and this can be written in the form:*

$$A(\lambda) = ([(A_3)^l, (A_3)^r]\lambda^3 + [(A_2)^l, (A_2)^r]\lambda^2 + [(A_1)^l, (A_1)^r]\lambda + [(A_0)^l, (A_0)^r])X = 0$$

where $[A_0^l, A_0^r], [A_1^l, A_1^r], [A_2^l, A_2^r], [A_3^l, A_3^r]$ are constant interval square matrices.

Interval $\mathbf{n^{th}}$ degree Eigenvalue Problem

Definition 2.0.6. *We may write the $\boldsymbol{n^{th}}$ degree interval eigenvalue problem in the following form :*

$$A(\lambda) = ([(A_n)^l, (A_n)^r]\lambda^n + [(A_{n-1})^l, (A_{n-1})^r]\lambda^{n-1} + \cdots + [(A_1)^l, (A_1)^r]\lambda^1 + [(A_0)^l, (A_0)^r]X) = 0$$

where $[A_i^l, A_i^r], i = 1, 2, \cdots, n$ are all constant interval square matrices.

2.1 Method of solution

Method of solution for quadratic eigenvalue problem

Quadratic eigenvalue problem may be converted into an ordinary linear generalized eigenvalue problem by defining a new vector $Y = \lambda X$ [Wang et.al., 2011]. In terms of X and Y, the quadratic eigenvalue problem becomes:

$$\begin{pmatrix} -A_0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \lambda \begin{pmatrix} A_1 & A_2 \\ I & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \tag{2.1.1}$$

where I is the identity matrix. Generally, if A is a polynomial matrix of degree d, then one can convert the nonlinear eigenvalue problem into a linear generalized eigenvalue problem of d times the size, besides converting them to ordinary eigenvalue problems, which only works if A is polynomial.

Method of solution for cubic eigenvalue problem

Cubic eigenvalue problem can also be converted into an ordinary linear generalized eigenvalue problem of thrice the size by defining a new vectors $Y = \lambda X$ and $Z = \lambda Y$. In terms of X,Y and Z, the cubic eigenvalue problem becomes:

$$\begin{pmatrix} -A_0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \lambda \begin{pmatrix} A_1 & A_2 & A_3 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{2.1.2}$$

where I is the identity matrix.

Method of solution for $\mathbf{n^{th}}$ degree eigenvalue problem

n^{th} degree eigenvalue problem can also be converted into an ordinary linear generalized eigenvalue problem by defining a new vectors $X = \begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{pmatrix}$. Now the n^{th} degree eigenvalue problem becomes:

$$\begin{pmatrix} -A_0 & 0 & . & . & . & 0 \\ 0 & I & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & I \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{pmatrix} = \lambda \begin{pmatrix} A_1 & A_2 & . & . & . & A_n \\ I & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & I & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{pmatrix} \tag{2.1.3}$$

where I is the identity matrix.

2.2 Interval eigenvalue problem

Method of solution for interval quadratic eigenvalue problem

Interval quadratic eigenvalue problem may be converted into an ordinary linear interval generalized eigenvalue problem and now it becomes:

$$\begin{pmatrix} -(A_0^l, A_0^r) & (0,0) \\ (0,0) & (I,I) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \lambda \begin{pmatrix} (A_1^l, A_1^r) & (A_2^l, A_2^r) \\ (I,I) & (0,0) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \tag{2.2.1}$$

where I is the identity matrix.

Method of solution for interval cubic eigenvalue problem

Interval cubic eigenvalue problem may also be converted into an interval ordinary linear generalized eigenvalue problem. Then the cubic eigenvalue problem becomes:

$$\begin{pmatrix} -(A_0^l, A_0^r) & (0,0) & (0,0) \\ (0,0) & (I,I) & (0,0) \\ (0,0) & (I,I) & (0,0) \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} =$$

$$\lambda \begin{pmatrix} (A_1^l, A_1^r) & (A_2^l, A_2^r) & (A_3^l, A_3^r) \\ (I,I) & (0,0) & (0,0) \\ (0,0) & (I,I) & (0,0) \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{2.2.2}$$

where I is the identity matrix.

Method of solution for interval $\mathbf{n^{th}}$ degree eigenvalue problem

Interval n^{th} degree eigenvalue problem may similarly be converted into an ordinary interval linear generalized eigenvalue problem. Accordingly, the n^{th} degree interval eigenvalue problem in matrix form may easily be represented as:

$$\begin{pmatrix} -(A_0^l, A_0^r) & (0,0) & . & . & . & (0,0) \\ (0,0) & (I,I) & . & . & . & (0,0) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ (0,0) & (0,0) & . & . & . & (I,I) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{pmatrix} =$$

$$\lambda \begin{pmatrix} (A_1^l, A_1^r) & (A_2^l, A_2^r) & . & . & . & (A_n^l, A_n^r) \\ (I,I) & (0,0) & . & . & . & (0,0) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ (0,0) & (0,0) & . & . & (I,I) & (0,0) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{pmatrix} \tag{2.2.3}$$

where I is the identity matrix.

It may be noted that the above converted linear interval eigenvalue problems may now be handled by using any known methods available in open literature.

Chapter 3

Numerical Examples and results

3.1 Crisp eigenvalue problem

Crisp quadratic eigenvalue problem

Example 1 : Let us consider a 3×3 quadratic matrix polynomial $Q(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$, [Tissure and Meerbergen, 2001], where

$$A_2 = \begin{pmatrix} 0 & 6 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_1 = \begin{pmatrix} 1 & -6 & 0 \\ 2 & -7 & 0 \\ 0 & 0 & 0 \end{pmatrix} A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using equation 2.1.1 we may get the eigenvalues for this example problem and are given in Table 3.1.

Table 3.1: Eigenvalues for Example 1

Eigenvalues	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
	-1.0000i	+1.0000i	1.0000	0.5000	0.3333	∞

Corresponding eigenvectors may be found and are presented in Table 3.2.

Table 3.2: Eigenvectors for Example 1

Eigenvectors	x_1	x_2	x_3	x_4	x_5	x_6
	0	0	0	1	1	1
	0	0	1	1	1	0
	1	1	0	0	0	0

One may see that these eigenvalues and eigenvectors satisfy the quadratic matrix polynomial $Q(\lambda)X = 0$

Crisp cubic eigenvalue Problem

Example 2: Let us consider a 3 ×3 cubic matrix polynomial $A(\lambda) = (A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0)X = 0$ where

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} A_2 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The obtained eigenvalues for this example problem using equation 2.1.2 are obtained and are incorporated in Table 3.3.

Table 3.3: Eigenvalues for Example 2

Eigenvalues	
λ_1	-1.3247
λ_2	-0.4656
λ_3	1.0000
λ_4	1.2328+0.7926i
λ_5	1.2328+0.7926i
λ_6	0.6624+0.5623i
λ_7	0.6624-0.5623i
λ_8	∞
λ_9	∞

3.2 Interval eigenvalue problem

Interval quadratic eigenvalue problem

Example 3: Let us consider a 3 × 3 interval quadratic polynomial

$$A(\lambda) = [(A_2)^l, (A_2)^r]\lambda^2 + [(A_1)^l, (A_1)^r]\lambda + [(A_0)^l, (A_0)^r]X = 0$$

where $A_2 = \begin{pmatrix} (0,0) & (5.9,6.1) & (0,0) \\ (0,0) & (5.9,6.1) & (0,0) \\ (0,0) & (0,0) & (0.9,1.1) \end{pmatrix}$

$A_1 = \begin{pmatrix} (0.9,1.1) & (-5.9,-6.1) & (0,0) \\ (1.9,2.1) & (-6.9,-7.1) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{pmatrix}$ $A_0 = \begin{pmatrix} (0.9,1.1) & (0,0) & (0,0) \\ (0,0) & (0.9,1.1) & (0,0) \\ (0,0) & (0,0) & (0.9,1.1) \end{pmatrix}$

We use equation 2.2.1 to get the eigenvalues shown in Table 3.4. . One may note that the linear interval eigenvalue problem has been solved by taking $[left(L), right(R)]$ and $[right(R), left(L)]$ combinations of each of the entries in matrices A_0, A_1 and A_2. The obtained eigenvalues for this example problem are

Table 3.4: Eigenvalues for Example 3

Eigenvalues	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
Left	0-0.8182i	0+0.8182i	-1.8196	0.1959-0.1262i	0.1959+0.1262i	∞
Right	0-1.2222i	0+1.2222i	-2.9250	0.2760-0.1381i	0.2760+0.1381i	∞

Interval cubic eigenvalue problem

Example 4: Let us consider a 3×3 interval cubic polynomial

$$A(\lambda) = [(A_3)^l, (A_3)^r]\lambda^3 + [(A_2)^l, (A_2)^r]\lambda^2 + [(A_1)^l, (A_1)^r]\lambda + [(A_0)^l, (A_0)^r]X = 0$$

where

$$A_3 = \begin{pmatrix} (4.9,5.1) & (0,0) & (0,0) \\ (0,0) & (4.9,5.1) & (0,0) \\ (0,0) & (0,0) & (4.9,5.1) \end{pmatrix}$$

$$A_2 = \begin{pmatrix} (0.9,1.1) & (0,0) & (0,0) \\ (0,0) & (0.9,1.1) & (0,0) \\ (0,0) & (0,0) & (0.9,1.1) \end{pmatrix}$$

$$A_1 = \begin{pmatrix} (29.9, 30.1) & (-9.9, -10.1) & (-9.9, -10.1) \\ (-9.9, -10.1) & (29.9, 30.1) & (-9.9, -10.1) \\ (-9.9, -10.1) & (-9.9, -10.1) & (29.9, 30.1) \end{pmatrix}$$

$$A_0 = \begin{pmatrix} (14.9, 15.1) & (-4.9, -5.1) & (-4.9, -5.1) \\ (-4.9, -5.1) & (14.9, 15.1) & (-4.9, -5.1) \\ (-4.9, -5.1) & (-4.9, -5.1) & (14.9, 15.1) \end{pmatrix}$$

Again by using equation 2.2.2 and taking combination of LR and RL corresponding eigenvalues are obtained and are incorporated in Table 3.5.

Table 3.5: Eigenvalues for Example 4

Eigenvalues	Left	Right
λ_1	0.1126 + 2.6685i	0.1888 + 3.3560i
λ_2	0.1126 - 2.6685i	0.1888 - 3.3560i
λ_3	0.1119 + 2.2077i	0.1856 + 2.7739i
λ_4	0.1119 - 2.2077i	0.1856 - 2.7739i
λ_5	0.1101 + 1.6205i	0.1774 + 2.0296i
λ_6	0.1101 - 1.6205i	0.1774 - 2.0296i
λ_7	-0.3966	-0.5793
λ_8	-0.4002	-0.5956
λ_9	-0.4017	-0.6021

3.3 Application

3.3.1 Quadratic Eigenvalue Problems

Let us consider a structural dynamics [Tisseur and Meerbergen, 2006] problem. Here the involved matrices for governing differential equations are M(mass matrix), C(damping matrix), K(stiffness matrix). These matrices are real symmetric in general. When M > 0, C > 0 and K $\geq$ 0 and

$$min_{\|x\|_2=1} \; [(x^*Cx)^2 - 4(x^*Mx)(x^*Kx)] > 0$$

Then the system is said to be overdamped, Note that if a system is overdamped, the corresponding quadratic eigenvalue problem (QEP) becomes hyperbolic. In this case, it is easy to verify that all the eigenvalues are not only real but also negative. This ensures that the general solution to the equation of motion is sum of bounded exponential.

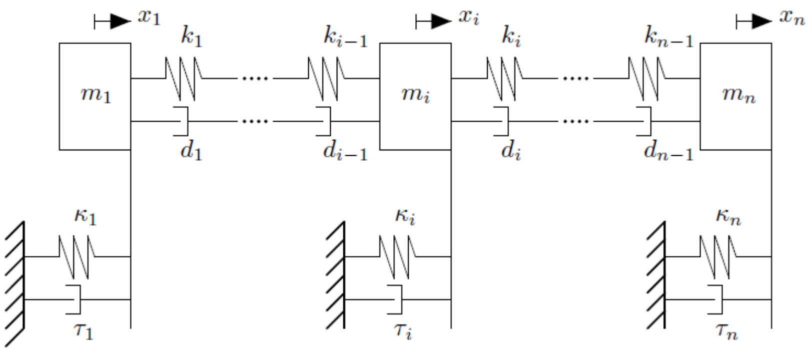

Figure 3.1: n-dimensional mass spring system [Tisseur and Meerbergen, 2006]

In Fig 3.1, we have considered the connected damped mass-spring system. The i^{th} mass of the i^{th} object is $m_i, i = 1, 2, \cdots, n$. These m_i are connected to the $(i+1)^{th}$ mass by a spring and a damper with constants k_i and d_i respectively. The i^{th} mass is also connected to the ground by a spring and a damper with constants κ_i and τ_i respectively. Here the governing differential equation for vibration of this system is a second-order differential equation, where the mass matrix is M=diag($m_1 \cdots m_n$), C is damping matrix and K is stiffness matrix and these matrices are as follow[Tisseur and Meerbergen, 2006]

$$\text{C=P diag}(d_1 \cdots d_{n-1}, 0)\ P^T\text{+diag}(\tau_1 \cdots \tau_n)$$

$$\text{K=P diag}(\kappa_1 \cdots \kappa_{n-1}, 0)\ P^T\text{+diag}(\kappa_1 \cdots \kappa_n)$$

with P=$(\delta_{ij} - \delta_{i,j+1})$, where δ_{ij} is the Kronecker delta.

In the following example we have taken the spring constant as κ for first and last ones the similar constant is $\kappa_1 = \kappa_n = 2\kappa$. The damper constant for first and last one is $\tau_1 = \tau_n = 2\tau$ where as for others it is τ . Here we assume that $m_i \equiv 1$.
Now, $M = I$, C=τ tridiag(-1,3,-1), K=κ tridiag(-1,3,-1)

.

Example 5: To solve the quadratic equation $Q(\lambda) = \lambda^2 M + \lambda C + K$ from Overdamped Systems in structural dynamics we have taken[Tisseur and Meerbergen, 2006]

$$M = I, C = \tau tridiag(-1, 3, -1), K = \kappa tridiag(-1, 3, -1),$$

n = 15 (degree of freedom) and $\tau = 10, \kappa = 5$, then the system is overdamped and so all the eigenvalues are real and nonpositive as shown in Fig 3.2. And if we will take $\tau = 3, \kappa = 5$ then the system is not overdamped i.e $M > 0$, $C > 0$ and $K > 0$, the system is stable, as shown in Fig 3.3.

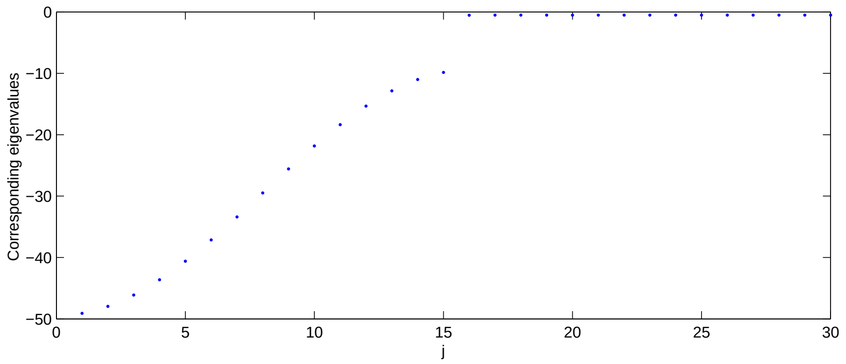

Figure 3.2: Eigenvalue distribution of the QEP for the overdamped mass-spring system with n = 15

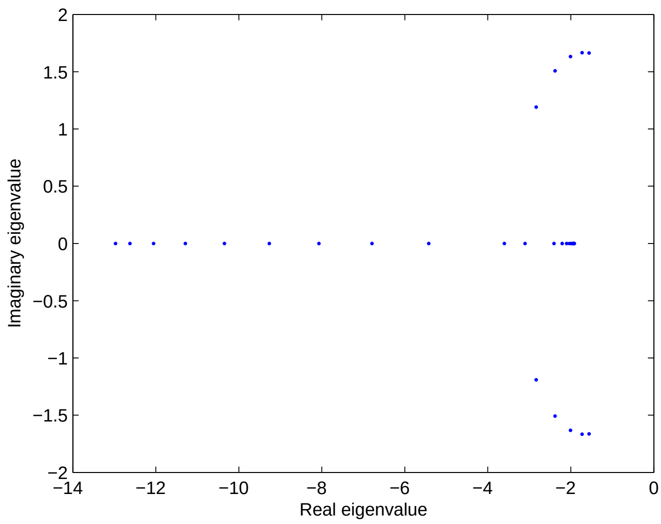

Figure 3.3: Eigenvalue in the complex plane of the QEP for the nonoverdamped mass-spring system with n = 15

3.3.2 Cubic eigenvalue problems

We use some structural engineering matrices from the Harwell-Boeing collection in Matrix-Market[5]. The matrices are considered as,

$$A_3 = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} A_2 = \begin{pmatrix} 9 & -3 & 0 & 0 & 0 \\ -3 & 9 & -3 & 0 & 0 \\ 0 & -3 & 9 & -3 & 0 \\ 0 & 0 & -3 & 9 & -3 \\ 0 & 0 & 0 & -3 & 9 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 6 & 0 \\ 0 & 10.5 & 0 & 0 & 0 \\ 0 & 0 & 0.015 & 0 & 0 \\ 0 & 250.5 & 0 & -280 & 33.32 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix} A_0 = \begin{pmatrix} 1 & 0 & 0 & 6 & 0 \\ 0 & 10.5 & 0 & 0 & 0 \\ 0 & 0 & 0.015 & 0 & 0 \\ 0 & 250.5 & 0 & -280 & 33.32 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix}$$

Corresponding eigenvalues may be found for this problem and are given in Table 3.6.

Table 3.6:

Eigenvalues	
λ_1	-0.0004 - 0.0408i
λ_2	-0.0004 + 0.0408i
λ_3	-0.9926
λ_4	-0.9148
λ_5	-1.3057
λ_6	-0.0249 - 0.3081i
λ_7	-0.0249 + 0.3081i
λ_8	-1.7684
λ_9	-0.2124 - 1.3226i
λ_{10}	-0.2124 + 1.3226i
λ_{11}	-0.0025 - 1.3656i
λ_{12}	-0.0025 + 1.3656i
λ_{13}	-2.6457
λ_{14}	-8.0275
λ_{15}	7.1351

3.4 Applications in interval problem

3.4.1 Interval quadratic eigenvalue problem

Here to solve the quadratic eigenvalue equation $Q(\lambda) = \lambda^2 M + \lambda C + K$ of Overdamped Systems for structural dynamics we have taken

$$M = \begin{pmatrix} (1,1) & (0,0) & (0,0) \\ (0,0) & (1,1) & (0,0) \\ (0,0) & (0,0) & (1,1) \end{pmatrix}$$

$$C = \begin{pmatrix} (29.7, 30.3) & (-10.1, -9.9) & (0,0) \\ (-10.1, -9.9) & (29.7, 30.3) & (-10.1, -9.9) \\ (0,0) & (-10.1, -9.9) & (29.7, 30.3) \end{pmatrix}$$

$$K = \begin{pmatrix} (14.7, 15.3) & (-5.1, -4.9) & (0,0) \\ (-5.1, -4.9) & (14.7, 15.3) & (-5.1, -4.9) \\ (0,0) & (-5.1, -4.9) & (14.7, 15.3) \end{pmatrix}$$

in interval term. Finally obtained interval eigenvalues in term of the left and right eigenvalues are presented in Table 3.7.

Table 3.7: M = I, τ = 10, κ = 5

Eigenvalues	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
Left	-0.5114	-0.5244	-0.5636	-15.8262	-29.8068	-43.8004
Right	-0.5003	-0.4932	-0.4731	-14.8529	-29.1756	-43.4722

3.4.2 Interval cubic eigenvalue problem

In this case, for the interval cubic eigenvalue problem the following matrices are considered. These matrices may represent dynamic analysis in structural engineering.

$$A_3 = \begin{pmatrix} (4.9, 5.1) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (4.9, 5.1) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (4.9, 5.1) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (4.9, 5.1) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (4.9, 5.1) \end{pmatrix}$$

$$A_2 = \begin{pmatrix} (8.9, 9.1) & (-2.9, -3.1) & (0, 0) & (0, 0) & (0, 0) \\ (-2.9, -3.1) & (8.9, 9.1) & (-2.9, -3.1) & (0, 0) & (0, 0) \\ (0, 0) & (-2.9, -3.1) & (8.9, 9.1) & (-2.9, -3.1) & (0, 0) \\ (0, 0) & (0, 0) & (-2.9, -3.1) & (8.9, 9.1) & (-2.9, -3.1) \\ (0, 0) & (0, 0) & (0, 0) & (-2.9, -3.1) & (8.9, 9.1) \end{pmatrix}$$

$$A_1 = \begin{pmatrix} (0.9, 1.1) & (0, 0) & (0, 0) & (5.9, 6.1) & (0, 0) \\ (0, 0) & (10.4, 10.6) & (0, 0) & (0, 0) & (0, 0) \\ (0, 0) & (0, 0) & (0.014, 0.016) & (0, 0) & (0, 0) \\ (0, 0) & (250.4, 250.6) & (0, 0) & (-279.9, -280.1) & (33.31, 33.33) \\ (0, 0) & (0, 0) & (0, 0) & (0, 0) & (11.9, 12.1) \end{pmatrix}$$

$$A_0 = \begin{pmatrix} (0.9, 1.1) & (0, 0) & (0, 0) & (5.9, 6.1) & (0, 0) \\ (0, 0) & (10.4, 10.6) & (0, 0) & (0, 0) & (0, 0) \\ (0, 0) & (0, 0) & (0.014, 0.016) & (0, 0) & (0, 0) \\ (0, 0) & (250.4, 250.6) & (0, 0) & (-279.9, -280.1) & (33.31, 33.33) \\ (0, 0) & (0, 0) & (0, 0) & (0, 0) & (11.9, 12.1) \end{pmatrix}$$

Finally one may again compute the interval eigenvalues by the mentioned procedure. The left and right eigenvalues are shown in Table 3.8.

Table 3.8:

Eigenvalues	Left	Right
λ_1	-0.0004 - 0.0392i	-0.0003 - 0.0424i
λ_2	-0.0004 + 0.0392i	-0.0003 + 0.0424i
λ_3	-0.9934	-0.9912
λ_4	-0.8890	-0.9419
λ_5	-1.2849	-1.3269
λ_6	-0.0313 - 0.2908i	-0.0186 - 0.3247i
λ_7	-0.0313 + 0.2908i	-0.0186 + 0.3247i
λ_8	-1.7433	-1.7942
λ_9	-0.2142 - 1.3134i	-0.2106 - 1.3321i
λ_{10}	-0.2142 + 1.3134i	-0.2106 + 1.3321i
λ_{11}	0.0004 - 1.3597i	-0.0057 - 1.3715i
λ_{12}	0.0004 + 1.3597i	-0.0057 + 1.3715i
λ_{13}	-2.6411	-2.6505
λ_{14}	-7.9486	-8.1090
λ_{15}	7.0698	7.2023

Chapter 4

Conclusion and Future direction

Linear eigenvalue problems are well known and there exist variety of methods to solve those. In general, application problems reduce to nonlinear eigenvalue problems but due to simplicity of the methods these are solved assuming the problem as linear. This may not represent some times the actual behavior of the system. So we need to investigate the nonlinear eigenvalue problems. Although there exist various methods to handle those but here a simple method is used to convert the nonlinear eigenvalue problem to a linear one. Then we use the standard procedure to solve the same.

Moreover it may be noted that the matrix elements actually are the essence of the system properties and characteristics and these are obtained by some experiments or observations. As such we must have errors in the measurement. So, one may not take the values of the elements as crisp or exact but it is batter to consider those as uncertain for handling those uncertainty. We have taken these uncertainties as intervals. Finally we may get interval nonlinear eigenvalue problem. To the best of the author's knowledge there exists no work related to the above. Accordingly this may be the first step to handle such problems. The methodology has been demonstrated by various example problems as discussed previously. Results are also validated by substituting the eigenvalue and eigenvectors in the original equation. Lastly various tables and graphs are incorporated to show the efficiency of the method.

One may note that simple procedure viz. combinations of left and right elements have been used in this investigation to handle the interval eigenvalue problems. However, sometimes we may get weak solution or the bounds of the interval results may be wide by following the presented algorithm. To overcome this and other complexities, we may have to develop efficient methods to handle the interval eigenvalue problems. Moreover the methodology needs to be extended for other application problems with large interval matrices. Another extension may be combinations of interval and crisp matrices. Also sensitivity and computational complexity analysis may also be done to validate the method and the problem(s). In view of the above there are many directions to which the method and the investigation may be extended in future. The value of this work is to present a new idea of analyzing interval nonlinear eigenvalue problems in an easy and efficient way.

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