HARMONIC FUNCTIONS

A THESIS

submitted by

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for

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of

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under the supervision

of

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DECLARATION

I declare that the topic "**HARMONIC FUNCTIONS** " for completion for my master degree has not been submitted in any other institution or university for the award of any other degree or diploma.

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THESIS CERTIFICATE

This is to certify that the project report entitled **HARMONIC FUNCTIONS** submitted by **Subhadarshan Sahoo** to the National Institute of Technology Rourkela, Orissa for the partial fulfilment of requirements for the degree of master of science in Mathematics is a bonafide record of review work carried out by her under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

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(Subhadarshan Sahoo)

ABSTRACT

In this thesis, we study the following topics in complex analysis:-

(a) Some basic results like Lebesegue's covering lemma, Maximum modulus theorem and Schwarz lemma.

(b)The basic principle of Normal Family; results like Roche's theorem, Hurwitz theorem and the Montel's theorem.

(c) The basic theory regarding Harmonic functions.

Moreover, in this thesis we plan to focous on the advance theory of Harmonic functions. We study Poisson kernel and it's properties and finally we give the detail prove of the Harnack's theorem.

Contents

NOTATION

English Symbols

$\mathbb{R}$	the set of real number.
$\mathbb{C}$	the complex plane.
$\mathbb{D}$	the unit disk $\{z \in \mathbb{C} : \vert z\vert < 1\}$.
$\overline{B}(a, R)$	the closed ball center at a and radius R.
$\mathbb{H}(G)$	set of analytic functions in G.
$A \subset B$	A is a proper subset of B.

CHAPTER 1

INTRODUCTION

In this thesis, we plan to focus mainly on the theory of Harmonic functions. In Chapter 2, we discuss some basic and standard results on complex analysis and we study Some basic results like Lebesegue's covering lemma, Maximum modulus theorem and Schwarz lemma. In the beginning of chapter 3, we focus the spaces of continuous function. Then we study on the Normal family, which will help us to understand the Montel's Theorem. In chapter 4, we plan to study the harmonic functions, Poisson kernel and the Harnack's theorem.

CHAPTER 2

REVIEW OF SOME IMPORTANT RESULTS IN COMPLEX ANALYSIS

In this chapter, we wish to revise some important results of Complex analysis. We start with the *Connectedness* and *Compactness* of metic space. Next, we focus on some results regarding compactness and connectedness of metic space. We also study the following basic results: *Heine-Borel theorem*, *Lebesgue covering lemma* and *schwarz's lemma*.

DEFINITION 2.1 (Connectedness). A metric space (X, d) is connected if the only subset of X which are both open and closed are ϕ and X. If $A \subset X$ then A is connected subset of X if the metric space (A, d) is connected.

EXAMPLE 2.2. :The set of real number i.e. $\mathbb{R}$ is connected.

PROPOSITION 2.3. *A set $X \subset \mathbb{R}$ is connected iff X is an interval.*

PROOF. suppose $X = [a, b]$, where a,b$\in \mathbb{R}$. Let $A \subset X$ be an open subset of X such that $a \in A$ and $A \neq X$. We will show that A cannot also be closed and hence X must be connected. Since A is open and $a \in A$ there is an $\epsilon > 0$ such that $[a, a+\epsilon] \subset A$. Let r $=$ $\sup\{\epsilon : [a, a+\epsilon) \subset A\}$
Claim: $[a, a+\epsilon) \subset A$. Infact, if $a \leq x < a+r$ then, putting h $= a+r-x > 0$, by the definition of supremum there is an ϵ with $r-h < \epsilon < r$ and $[a, a+\epsilon) \subset A$. But $a \leq x = a+(r-h) < a+\epsilon$ implies $x \in A$ and our claim is established.
However, $a+r \in A:$ for if, on the contrary, $a+r \notin A$ then, by the openness of A, there is a $\delta > 0$ with $[a+r, a+r+\delta) \subset A$, contradicting the definition of r. Now if A were also closed then $a+r \in B = X-A$ which is open. Hence we could find a $\delta > o$ such

that $(a+r-\delta, a+r] \subset B$, contradicting the above claim. Similarly we can prove this for other intervals. $\square$

DEFINITION 2.4 (Compactness). A subset K of a metric space X is compact if for every collection $\mathbb{G}$ of open sets in X with property $K \subset (G : G \in \mathbb{G})$. A collection of set $\mathbb{G}$ satisfying above condition is called a cover of K. If each member of $\mathbb{G}$ is an open set it is called an open cover of K.

EXAMPLE 2.5. :

(1) The empty set and finite sets are compact.

(2) The set $D = \{z \in \mathbb{C} : |z| = 1\}$ is not compact.

PROPOSITION 2.6. *Let K be a compact subset of X; then :*

(a) K is closed;

(b) If F is closed and $F \subset K$ then F is compact.

PROOF. To prove part (a) we will show that $K = \overline{K}$. Let $x_0 \in \overline{K}$. So $\exists\ B(x_0; \epsilon) \cap K \neq \phi$ for each $\epsilon > 0$. Let $G_n = X - \overline{B}(x_0; \frac{1}{n})$ and suppose that $x_0 \notin K$. Then each G_n is open and $K \subset \bigcup\limits_{n=1}^{\infty} G_n$. Since K is compact there is an integer m such that $K \subset \bigcup\limits_{n=1}^{m} G_n$. But $G_1 \subset G_2 \subset ...$ so that $K \subset G_m = X - \overline{B}(x_0; \frac{1}{m})$.But this gives that$B(x_0; \frac{1}{m}) \cap K = \phi$, a contradiction. Thus $K = \overline{K}$.

To prove part (b) let $\mathbb{G}$ be an open cover of F. Then, since F is closed. $\mathbb{G} \cup \{X - f\}$ is an open cover of K. Let $G_1, ..., G_n$ be sets in$\mathbb{G}$ such that $K \subset G_1 \cup ... \cup G_n \cup (X - F)$. Clearly, $F \subset G_1 \cup ... \cup G_n$ and so F is compact. $\square$

COROLLARY 2.7. *Every compact metric space is complete.*

DEFINITION 2.8 (sequentially compact). A metric space (X, d) is sequentially compact if every sequence in X has a convergent subsequence .

LEMMA 2.9 (Lebesgue's covering Lemma). *If (X, d) is sequentially compact and $\mathbb{G}$ is an open cover of X then there is an $\epsilon > 0$ such that if x is in X, there is a set G in $\mathbb{G}$ with $B(x; \epsilon) \subset G$.*

Proof. We will prove this lemma by method of contradiction. Suppose that $\mathbb{G}$ is an open cover of X and no such $\epsilon > 0$ can be found. In particularly, for every integer n there is a point x_n in X such that $B(x_n; \frac{1}{n})$ is not contained in any set G in $\mathbb{G}$. Since X is sequentially compact there is a point x_0 in X and a subsequence $\{x_{n_k}\}$ such that $\lim_{n\to\infty} x_{n_k} = x_0$. Let $G_0 \in \mathbb{G}$ such that $x_0 \in G_0$ and choose $\epsilon > 0$ such that $B(x_0; \epsilon) \subset G_0$. Now let N be such that $d(x_0, x_{n_k}) < \dfrac{\epsilon}{2}\ \forall\ n_k \geq N$. Let n_k be any integer larger than both N and $\dfrac{2}{\epsilon}$ and let $y \in B(x_{n_k}; \dfrac{1}{n_k})$. Then by triangle inequality

$$d(x,y) = d(x_0, x_{n_k}) + d(x_{n_k}, y) < \frac{\epsilon}{2} + \frac{1}{n_k} < \epsilon$$

That is $B(x_{n_k}; \frac{1}{n_k}) \subset B(x_0, \epsilon) \subset G_0$. Which is a contradiction to fact that x_n has a convergent subsequence x_0 in X. So our choice of x_{n_k} is wrong. □

Remarks 2.10. The following are two common misinterpretation of Lebesgue's covering Lemma:

(1) This lemma gives one $\epsilon > 0$ such that for any x, $B(x; \epsilon)$ is contained in some member of $\mathbb{G}$.

(2) Also it is believed that for the $\epsilon > 0$ obtained in the lemma, $B(x; \epsilon)$ is contained in each G in $\mathbb{G}$ such that $x \in G$.

Theorem 2.11. *Let (X, d) be a metric space; then the following are equivalent statement:*

(a) X is compact;

(b) Every infinite set in X has a limit point;

(c) X is sequentially compact;

(d) X is complete and for every $\epsilon > 0$ there are a finite number of points $x_1, ..., x_n$ in X such that $X = \bigcup_{k=1}^{n} B(x_k; \epsilon)$.

Theorem 2.12 (Heine-Borel Theorem). *A subset K of $\mathbb{R}^n (n \geq 1)$ is compact iff K is closed and bounded.*

Theorem 2.13 (The Maximum principle). *Let $\Omega \subset \mathbb{C}$ and suppose α is in the interior of Ω. We can therefore, choose a positive number ξ such that $B(\alpha, \xi) \subset \Omega$, it readily follows that there is a point ξ in Ω with $|\xi| > |\alpha|$ i.e if α is a point in Ω with $|\xi| > |\alpha|$ for each ξ in the set Ω then α belongs to $\partial\Omega$.*

Theorem 2.14 (Maximum Modulus theorem). *If f is analytic in a region G and a is a point in G with $|f(a)| \geq |f(z)|$ $\forall z$ in G then f must be a constant function.*

Theorem 2.15 (Schwarz's lemma). *Let $\mathbb{D} = \{z : |z| < 1\}$ and suppose f is analytic on $\mathbb{D}$ with*
(a) $|f(z)| \leq 1$ for z in $\mathbb{D}$.
(b) $f(0) = 0$.
Then $|f^{'}(0)| \leq 1$ and $|f(z)| \leq |z|$ $\forall z \in \mathbb{D}$. Moreover if $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \neq 0$ then there is a constant c, $|c| < 1$ such that $f(w) = cw$ $\forall$ w in $\mathbb{D}$.

Proof. Let define $g : \mathbb{D} \to \mathbb{C}$ by

$$g(z) = \frac{f(z)}{z} \Rightarrow f'(0) = g(0) \ for \ z \neq 0,$$

then g is analytic in $\mathbb{D}$. According to Maximum Modulus theorem for $|z| \leq r$ and $0 < r < 1$, we have $|g(z)| = \dfrac{|f(z)|}{|z|} \leq r^{-1}$, ($\because$ $|f(z)| \leq 1$ $\forall z \in \mathbb{D}$). As $r \check{\to} 1$, we have $|f(z)| \leq |z|$ $\forall$ $z \in \mathbb{D}$ and $|f'(0)| = |g(0)| \leq 1$. If $|f(z)| \leq |z|$ for some z in $\mathbb{D}$, $z = 0$ or $|f'(0)| = 1$, then $|g|$ assumes its maximum value inside $\mathbb{D}$. Then again by applying maximum modulus theorem, $|g(z)| \equiv c$ for some constant c with $c = 1$, since $|g(z)| = \dfrac{|f(z)|}{|z|} = c$, so we have $f(z) = cz$ $\forall$ $z \in \mathbb{D}$. □

CHAPTER 3

NORMAL FAMILY AND MONTEL'S THEOREM

In this chapter, we discuss mainly about the spaces of analytic functions, *Normal family* and the famous *Montel' Theorem.* In the beginning of this chapter, we focus on spaces of continuous function.

1. Spaces of Continuous functions

PROPOSITION 3.1. *If G is open in $\mathbb{C}$. then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{i=1}^{n} K_i$. Moreover, the sets $\{K_n\}$ can be chosen to satisfy the following conditions:*

(a) $K_n \subset intK_{n+1}$.

(b) $K \subset G$ *and* K *compact implies* $K \subset k_n$ *for some* n.

(c) *Every component of* $\mathbb{C}_\infty - K_n$ *contains a component of* $\mathbb{C}_\infty - \mathbb{G}$.

PROOF. (a) For each positive integer n, let $K_n = \{z : |z| < n\} \cap \{z : d(z, \mathbb{C} - G) \geq \frac{1}{n}\}$. Since K_n is bounded and it is intersection of two closed subsets of $\mathbb{C}$. So K_n is compact. Now consider the set $S = \{z : |z| < n+1\} \cap \{z : d(z, \mathbb{C} - \mathbb{G}) \geq \frac{1}{n+1}\}$ is open. Hence $K_n \subset S$ and $S \subset K_{n+1}$. So $K_n \subset$ int K_n. G is an open set, so $G = \bigcup_{n=1}^{\infty} K_n$. Then we can get $G = \bigcup_{n=1}^{\infty} \text{int}K_n$.
(b) If K is compact subset of G, then the set intK_n form an open cover of K. So $K \subset K_n$ for some n.
(c) Now we ant to prove that every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$. The unbounded component of $\mathbb{C}_\infty - K_n$ must contain ∞. So the component of $\mathbb{C}_\infty - G$

which contains ∞. Also the unbounded component contains $\{z : |z| > n\}$. So if $\mathbb{D}$ is a bounded component, it contains a point z with $d(z, \mathbb{C} - G) < \frac{1}{n}$. According to definition this gives a point w in $\mathbb{C} - G$ with $|z - w| < \frac{1}{n}$. But then $z \in B\left(w; \frac{1}{n}\right) \subset \mathbb{C}_\infty - K_n$; since disks are connected and z is in the component $\mathbb{D}$ of $\mathbb{C}_\infty - k_n$, $B\left(w, \frac{1}{n}\right) \subset \mathbb{D}$. If $\mathbb{D}_1$ is the component of $\mathbb{C}_\infty - \mathbb{D}$ that contains w it follows that $\mathbb{D}_1 \subset \mathbb{D}$. □

PROPOSITION 3.2. $\mathbb{C}(G, \Omega)$ *is a metric space.*

PROOF. According to above theorem we have $G = \bigcup_{n=1}^{\infty} k_n$ where k_n is compact and $k_n \subset int k_{n+1}$. Define $\rho_n(f, g) = \sup\{d(f(z), g(z)) : z \in k_n\}$ for all functions $f, g \in C(G, \Omega)$.

$$\rho(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f, g)}{1 + \rho_n(f, g)}\right) \tag{3.1}$$

Now, first we have to show that the series in (3.1)is convergent, let $t = \rho_n(f, g)$, then $\frac{t}{1+t} \leq 1$. So the series in (3.1) dominated by the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, which is a convergent series . Now we have to show that ρ is a metric on $C(G, \Omega)$. It can be easily shown that $\rho(f, g) > 0$, $\rho(f, g) = 0 \Leftrightarrow f = g$, $\rho(f, g) = \rho(g, f)$. Now only we have to establish the triangle inequality condition, i.e to show that $\rho(f, g) \leq \rho(f, h) + \rho(h, g)$. Since $\rho_n(f, g)$ is a metric space, so we have

$$\begin{aligned}
&\rho_n(f, g) \leq \rho_n(f, h) + \rho_n(h, g) \\
\Rightarrow\ &\frac{\rho_n(f, g)}{1 + \rho_n(f, g)} \leq \frac{\rho_n(f, h) + \rho_n(h, g)}{1 + \rho_n(f, h) + \rho_n(h, g)} \leq \left(\frac{\rho_n(f, h)}{1 + \rho_n(f, h)}\right) + \left(\frac{\rho_n(h, g)}{1 + \rho_n(h, g)}\right) \\
\Rightarrow\ &\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f, g)}{1 + \rho_n(f, g)}\right) \leq \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f, h)}{1 + \rho_n(f, h)}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(h, g)}{1 + \rho_n(h, g)}\right) \\
\Rightarrow\ &\rho(f, g) \leq \rho(f, h) + \rho(h, g)
\end{aligned}$$

So $C(G, \Omega)$ is a metric space. □

LEMMA 3.3. *Let the metric* ρ *be defined as* (3.1). *If* $\epsilon > 0$ *is given then there is a* $\delta > 0$ *and a compact set* $K \subset G$ *such that for f and g in* $C(G, \Omega)$, $\sup\{d(f(z), g(z)) : z \in$

$K\} < \delta \Rightarrow \rho(f,g) < \epsilon$. Conversely, if $\delta > 0$ and a compact set K are given, there is an $\epsilon > 0$ such that for f and g in $C(G,\Omega)$, $\rho(f,g) < \epsilon \Rightarrow \sup\{d(f(z),g(z)) : z \in \ K\} < \delta$.

PROOF. First we want to prove $\sup\{d(f(z),g(z)) : z \in K\} < \delta \Rightarrow \rho(f,g) < \epsilon$.
Let $\epsilon > 0$ is fixed and p be a positive number such that $\sum_{n=p+1}^{\infty} \left(\frac{1}{2}\right)^n < \left(\frac{1}{2}\right)\epsilon$ and Put $K = K_n$. Choose $\delta > 0$ such that $0 \le t \le \delta$ gives $\frac{t}{1+t} < \frac{1}{2}\epsilon$. Let $f, g \in C(G,\Omega)$ such that $\sup\{d(f(z),g(z)) : z \in K\} < \delta$. Since $K_n \subset K_p$ for $1 \le n \le p$, $0 < \rho_n(f,g) < \delta \ for \ 1 \le n \le p$. This gives $\frac{\rho_n(f,g)}{1+\rho_n(f,g)} < \left(\frac{1}{2}\right)\epsilon$. for $1 \le n \le p$ .Here,

$$\begin{aligned}
\rho(f,g) &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right) \\
&= \sum_{n=1}^{p} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right) + \sum_{n=p+1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right) \\
&< \sum_{n=1}^{p} \left(\frac{1}{2}\right)^n \left(\frac{\epsilon}{2}\right) + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Now, we want to prove $\rho(f,g) < \epsilon \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \delta$.
Let K and δ are given, Since $G = \bigcup_{n=1}^{\infty} k_n = \bigcup_{n=1}^{\infty} intK_n$ and K is compact there is an integer $p \ge 1$ such that $K \subset K_p$; this gives $\rho_p(f,g) \ge \sup\{d(f(z),g(z)) : z \in K\}$. Choose $\epsilon > 0$ such that $0 \le s \le 2^p\epsilon$.

$$\begin{aligned}
\Rightarrow \frac{s}{1-s} < \frac{2^p\epsilon}{1-2^p\epsilon} = \delta \Rightarrow \frac{s}{1-s} < \delta \\
and \ for \ 0 \le t \le \delta \Rightarrow \frac{t}{1+t} < \frac{s}{1+s} = 2^p\epsilon \\
So \ if \ \rho_p(f,g) < \epsilon \Rightarrow \frac{\rho_p(f,g)}{1+\rho_p(f,g)} < 2^p\epsilon \\
\Rightarrow \rho_p(f,g) < \delta \Rightarrow \sup\{d(f(z),g(z)) : z \in K\} < \delta
\end{aligned}$$

□

2. Normal Family

DEFINITION 3.4 (Normal family). A set $\mathbb{F} \subset C(G, \Omega)$ is normal if each sequence in $\mathbb{F}$ has a subsequence which converges to a function f in $C(G, \Omega)$.

PROPOSITION 3.5. *A set $\mathbb{F} \subset C(G, \Omega)$ is normal iff for every compact set $K \subset G$ and $\delta > 0$ there are function $f_1, ..., f_n$ in $\mathbb{F}$ such that for f in $\mathbb{F}$ there is at least one k, $1 \leq k \leq n$, with $sup\{d(f(z), f_k(z)) : z \in K\} < \delta$.*

PROOF. Suppose $\mathbb{F}$ is normal and let K and $\delta > 0$ be given.By lemma 3.3 there is an $\epsilon > 0$ such that $\rho(f, g) < \epsilon \Rightarrow \sup\{d(f(z), g(z)) : z \in \ K\} < \delta$ holds. But since $\bar{\mathbb{F}}$ is compact $\mathbb{F}$ is totally bounded. So there are $f_1, ..., f_n$ in $\mathbb{F}$ such that $\mathbb{F} \subset \bigcup\limits_{k=1}^{n} \{f : \rho(f, f_k) < \epsilon\}$ But from the choice of ϵ this gives

$$\mathbb{F} \subset \bigcup_{k=1}^{n} \{f : d(f(z), f_k(z)) < \delta, z \in K\}$$

that is $\mathbb{F}$ satisfies the condition of proposition.

Conversely, $\mathbb{F}$ satisfied the stated property. From this, it is follows that $\overline{\mathbb{F}}$ also satisfies this condition, assume that $\mathbb{F}$ is closed. But since $C(G, \Omega)$ is complete. And again using 3.3 it is follow that $\mathbb{F}$ is totally bounded . From theorem 2.11 $\mathbb{F}$ is compact and therefore normal. □

DEFINITION 3.6 (Equicontinuous at a point). A set $\mathbb{F} \subset C(G, \Omega)$ is equicontinuous at a point $z_0 \in G$ iff for every $\epsilon > 0$ such that for $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \epsilon$ for every f in $\mathbb{F}$.

DEFINITION 3.7 (Equicontinuous over a set). $\mathbb{F}$ is equicontinuous over a set $E \in G$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for z and z_0 in $\mathbb{F}$ and $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \epsilon \ \forall \ f \in \mathbb{F}$.

PROPOSITION 3.8. *Suppose $\mathbb{F} \subset C(G, \Omega)$ is equicontinuous at each point of G; then $\mathbb{F}$ is equicontinuous over each compact subset of G.*

PROOF. Let $K \subset G$ be compact and fix $\epsilon > 0$. Then for each w in K there is a $\omega_w > 0$ such that $d(f(w^{'}), f(w)) < \frac{\epsilon}{2}$ $\forall$ $f \in \mathbb{F}$ whenever $|w = w^{'}| < \delta_w$. Now $\{B(w; \delta_w) : w \in K$ from an cover of K; by lemma 2.9 there is a $\delta > 0$ such that each $z \in K, B(z; \delta)$ is contained in one of the sets of the cover so if z and z' are in K and $|z - z'| < \delta$ there is a w in K with $z' \in B(z; \delta) \subset B(w, \delta_w)$. That is, $|z - w| < \delta_w$ and $|z - z^{'}| < \delta_w$. This gives $d(f(z), f(w)) < \frac{\epsilon}{2}$ and $d(f(z'), f(w)) < \frac{\epsilon}{2}$. So that

$$\begin{aligned} d(f(z), f(z')) &< d(f(z), f(w)) + d(f(z'), f(w)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

and $\mathbb{F}$ is equicontinuous over K. □

THEOREM 3.9 (Arzela-Ascoli theorem). *A set* $\mathbb{F} \subset C(G, \Omega)$ *is normal iff the following two conditions are satisfied:*

(a) For each $z \in G$,$\{f(z) : f \in \mathbb{F}\}$ *has compact closure in* Ω.

(b) $\mathbb{F}$ *is equicontinuous at each point of* $\mathbb{G}$.

3. Montel's Theorem

In this section we discuss about some results in spaces of holomorphic functions, which help for proving famous Montel's theorem.

THEOREM 3.10 (Rouche's Theorem). *Suppose f and g are meromorphic in a neighborhood of* $\overline{B}(a; R)$ *with no zeros or poles on the circle* $\gamma = \{z : |z - a| = R\}$. *If* z_f, z_g (p_f,p_g) *are the number of zeros(poles) of* f, g *inside* γ *counted according to their multiplicities and if* $|f(z) + g(z)| < |f(z)| + |g(z)|$ *on* γ, *then* $Z_f - P_f = Z_g - P_g$.

PROOF. If $\lambda = \frac{f(z)}{g(z)}$ and if λ is a positive real number, then this inequality becomes $\lambda + 1 < \lambda + 1$. This is a contradiction, hence the meromorphic function $\frac{f}{g}$ maps γ onto $\Omega = \mathbb{C} - [0, \infty)$. If l is a branch of the logarithm on Ω, then $l\left(\frac{f(z)}{g(z)}\right)$ is well-defined primitive for $\left(\frac{f}{g'}\right)\left(\frac{f}{g^{-1}}\right)$ in a neighborhood of γ.Thus

$$
\begin{aligned}
0 &= \frac{1}{2\pi i}\int_\gamma (f/g)'(f/g)^{-1} \\
&= \frac{1}{2\pi i}\int_\gamma \left[\frac{f'}{f} - \frac{g'}{g}\right] \\
&= (Z_f - P_f) - (Z_g - P_g).
\end{aligned}
$$

So we have $Z_f - P_f = Z_g - P_g$. □

THEOREM 3.11 (Hurwitz's Theorem). *Let G be a region and suppose the sequence $\{f_n\}$ in $\mathbb{H}(G)$ converges to f. If $f \not\equiv 0$, $\overline{B}(a;R)$ and $f(z) \neq 0$ for $|z-a| = R$, then there is an integer N such that for $n \geq N$, f and $\{f_n\}$ have the same number of zeros in $B(a;R)$.*

PROOF. Let G be a region and $\{f_n\}$ in $\mathbb{H}(G)$ converges to f. Since $f(z) \neq 0 \;\forall\; |z-a| = R$, let $\delta = \inf\{|f(z)| : |z-a| = R\} > 0$. But $\{f_n\} \to f$ uniformly on $|z| : |z-a| = R$. So there is an integer N such that if $n \geq N$ and $|z-a| = R$, then

$$|f(z) - f_n(z)| < \frac{1}{2}\delta < |f(z)| \leq |f(z)| + |f_n(z)|.$$

According to Rouche's theorem f and $\{f_n\}$ have same number of zeros in $B(a;R)$. □

DEFINITION 3.12 (Locally bounded). A set $\mathbb{F} \subset \mathbb{H}(G)$ is locally bounded if for each point a in G there are constants M and $r > 0$ such that for all f in $\mathbb{F}$, $|f(z)| \leq M$, for $|z-a| < r$ that is $\sup\{f(z) : |z-a| < r, f \in \mathbb{F}\} < \infty$.

THEOREM 3.13 (Montel's Theorem). *A family $\mathbb{F}$ in $\mathbb{H}(G)$ is normal iff $\mathbb{F}$ is locally bounded.*

PROOF. Suppose $\mathbb{F}$ is normal but fails to be locally bounded; then there is a compact set $K \in G$ such that $\sup\{|f(z)| : z \in k, f \in \mathbb{F}\} = \infty$. that is, there is a sequence $\{f_n\}$ in $\mathbb{F}$ such that $\sup\{|f(z)| : z \in k\} \geq n$. Since $\mathbb{F}$ is normal there is a function f in $\mathbb{H}(G)$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$. But this gives that $\sup\{|f_{n_k}(z) - f(z)| : z \in k\} \to 0$ as $K \to \infty$. If $|f(z)| \leq M$ for z in K,

$$n_k \leq \sup\{|f_{n_k}(z) - f(z)| : z \in k\} + M.$$

Since the right hand side converges to M, so this is a contradiction. So $\mathbb{F}$ is locally bounded.

conversely, suppose $\mathbb{F}$ is locally bounded. Here we use Arzela-Ascoli theorem to show $\mathbb{F}$ is normal and from (a) of theorem 3.9 the first condition is satisfied. Now only we have to prove $\mathbb{F}$ is equicontinuous at each point of G. Let fix a point $a \in G$ and $\epsilon > 0$, so according to hypothesis $\exists\ r > 0$ and $M > 0$ such that $\overline{B}(a;r) \subset G$ and $|f(z)| \leq M$ $\forall z \in \overline{B}(a;r)$ and $\forall\ f \in \mathbb{F}$. Let $|z - a| < \frac{1}{2}r$ and $f \in \mathbb{F}$; then using Cauchy's formula with $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, we get

$$\begin{aligned} |f(a) - f(z)| &\leq \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)}dw\right| \\ &\leq \frac{1}{2\pi}|a-z|\left|\int_\gamma \frac{f(w)}{(w-a)(w-z)}dw\right| \qquad (3.2) \end{aligned}$$

At $w = a$

$$\left|\lim_{w\to a} \frac{f(w)(w-a)}{(w-a)(w-z)}\right| = \frac{M}{(1/2)r} = \frac{2M}{r} \qquad (3.3)$$

At $w = z$

$$\left|\lim_{w\to z} \frac{f(w)(w-z)}{(w-a)(w-z)}\right| = \frac{M}{(1/2)r} = \frac{2M}{r} \qquad (3.4)$$

According to Cauchy's formula and from (3.3) and (3.4), we get

$$\int_\gamma \frac{f(w)}{(w-a)(w-z)}dw = 2\pi\left(\frac{2M}{r} + \frac{2M}{r}\right) = \frac{8M\pi}{r} \qquad (3.5)$$

Putting the value of (3.5) in (3.2) , we get

$$|f(a) - f(z)| = \frac{1}{2\pi}|a-z|\frac{8M\pi}{r} = |a-z|\frac{4M}{r}$$

Let $\delta = \min\left\{\frac{1}{2r}, \frac{r\epsilon}{4M}\right\}$. So $|a - z| < \delta$. So $|f(a) - f(z)| < \epsilon$ $\forall f \in \mathbb{F}$. Hence it is proved. $\square$

CHAPTER 4

HARMONIC FUNCTIONS

1. Preliminaries

Definition 4.1.

If G is an open subset of $\mathbb{C}$, then a function $u : G \to \mathbb{R}$ is harmonic if it has continuous second order partial derivative and it satisfies Laplace's equation, that is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Example 4.2.

(1)The real and imaginary part of any holomorphic function is a Harmonic function.

(2)The function $f(x, y) = e^x \cos y$ is a Harmonic function.

Lemma 4.3. *If v is a conjugate harmonic function of u, then u is a conjugate harmonic function of v.*

Proof. Given v is a conjugate harmonic function of u.

Claim : $-v + iu$ is analytic. We know that $f = u + iv$ is analytic.

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$Now \ \frac{\partial}{\partial x}(-v) = -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{ and } \frac{\partial}{\partial y}(-v) = -\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

Hence u is a conjugate harmonic function of v. □

Theorem 4.4. *A function f on a region G is analytic iff $Ref = u$ and $Imf = v$ are harmonic functions which satisfy Cauchy-Riemann equation.*

Theorem 4.5. *A region G is simply connected iff for each harmonic function u on G , there is a harmonic function v on G such that $f = u + iv$ is analytic on G.*

Proposition 4.6. *If $u : G \to \mathbb{R}$ is harmonic, then u is infinitely differentiable.*

Proof. Fix $z_0 = x_0 + iy_0$ in G. Let δ chosen such that $B(z_0; \delta) \subset G$.
As u has a harmonic conjugate v in $B(z_0; \delta)$. That means $f = u + iv$ is analytic.
$\Rightarrow$ It is infinitely differentiable on $B(z_0; \delta)$.
so u is infinitely differentiable. □

Theorem 4.7 (Mean value Theorem). *Let $u : G \to \mathbb{R}$ be a harmonic function and let $\overline{B}(a; r)$ be a closed disk contained in G. If γ is a circle $|z - a| = r$ then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta.$$

Proof. Let D be a disk such that $\overline{B}(a; r) \subset D \subset G$ and f be a analytic function on D such that $Ref = u$. By cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz, \ where\ \gamma = B(z; r).$$

$$\text{Let } z - a = re^{i\theta} \Rightarrow dz = ire^{i\theta} d\theta. \tag{4.1}$$

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}} d\theta.$$

$$\Rightarrow f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta.$$

so by taking the real part of equation (4.1), we get

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta.$$

□

Theorem 4.8 (Maximum Principle{*First Version*}).
Let G be a region and suppose that u is a continuous real valued function on G with the MVP. If there is a point a in G such that $u(a) \geq u(z)\ \forall\ z \in G$, then u is a constant function.

Proof. Let set A be defined by $A = \{z \in G : u(z) = u(a)\}$. As u is continuous on the set A is closed in G. If $z_0 \in A$, then we choose a r such that $\overline{B}(z_0; r) \subset G$. Suppose $\exists$ a point $b \in B(z_0; r)$ such that $u(b) \neq u(a)$; then, $u(b) < u(a)$. By continuity $u(z) < u(a) = u(z_0)\ \forall\ z$ in neighborhood of b. In particular $\rho = |z_0 - b|$ *and* $b = z_0 + \rho\ e^{i\beta},\ 0 \leq \beta < 2\pi$. So there is a proper interval I of $[0, 2\pi]$ *such that* $\beta \in I$ *and* $u(z_0 + \rho e^{i\theta}) < u(z_0)\ \forall\ \theta \in I$. So by MVP

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})d\theta < u(z_0),$$

Which is a contradiction. So $B(z_0; r) \subset A$ and A is open. so by defination 2.1 $A = G$. □

THEOREM 4.9 (Maximum Principle{*Second Version*}).

Let G be a region and let u and v be two continuous real valued functions on G that have the MVP. If for each point a in the extended boundary $\partial_\infty G$, $\limsup_{z\to a} u(z) \leq \liminf_{z\to a} v(z)$ *then either* $u(z) < v(z)\ \forall\ z \in G$ *or* $u = v$.

PROOF. Fix a in $\partial_\infty G$ and for each $\delta > 0$, let $G_\delta \cap B(a; \delta)$. then by hypothesis,

$$\begin{aligned} 0 &\geq \lim_{\delta\to 0}[sup\{u(z) : z \in G_\delta\} - inf\{v(z) : z \in G_\delta\}] \\ &= \lim_{\delta\to 0}[sup\{u(z) : z \in G_\delta\} - sup\{-v(z) : z \in G_\delta\}] \\ &\geq \lim_{\delta\to 0} sup\{u(z) - v(z) : z \in G_\delta\} \end{aligned}$$

$$so\ \limsup_{z\to a}[u(z) - v(z)] \leq 0\ for\ each\ a \in \partial_\infty G. \tag{4.2}$$

Let $v(z) = 0 \forall\ z \in G$. That is, assume $\limsup_{z\to a} u(z) \leq 0\ \forall\ a \in \partial_\infty G$. Claim: $u(z) < 0\ \forall\ z \in G$ *or* $u = 0$. If we show that $u(z) \leq 0\ \forall\ z \in G$, then by theorem 4.8 $u \equiv 0$. Suppose that u satisfies (4.2) and there is a point b in G with $u(b) > 0$. Let $\epsilon > 0$ be chosen so that $u(b) > \epsilon$ and let $B = \{z \in G : u(z) \geq \epsilon\}$. If $a \in \partial_\infty G$ then by proposition 4.6, there is a $\delta = \delta(a)$ such that $u(z) < \epsilon\ \forall\ z \in G \cap B(a; \delta)$. By lemma 2.9 a δ can be found which is independent of a.

That means, there is a $\delta > 0$ such that if $z \in G$ and $d(z, \partial_\infty G) < \delta$ then $u(z) < \epsilon$. Thus $B \subset \{z \in G : d(z, \partial_\infty G) \geq \delta\}$. This gives that B is a bounded plane and closed. So B is compact. So $B \neq \phi$, there is a point $z_0 \in B$ such that $u(z_0) \geq u(z)\ \forall\ z \in B$. Since $u(z) < \epsilon\ for\ z \in G - B$, it gives that u assumes a maximum value at a point in G. So

u must be constant, which is nothing but the $u(z_0)$ and positive. Which contradict (4.2). So it gives the prove of the theorem. □

2. Poisson Kernel and It's Properties

DEFINITION 4.10. The function

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

for $0 \leq r < 1$ *and* $-\infty < \theta < \infty$, is called poisson kernel.

Let $z = re^{i\theta}, 0 \leq r < 1$; then

$$\frac{1 + re^{i\theta}}{1 - re^{i\theta}} = \frac{1+z}{1-z} = (1+z)(1-z)^{-1}$$

by expanding, we get

$$\begin{aligned} = (1+z)(1+z+z^2+...) &= 1 + 2\sum_{n=1}^{\infty} z^n \\ &= 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta} \end{aligned}$$

Hence,

$$\begin{aligned} Re\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) &= 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta \\ &= 1 + 2\sum_{n=1}^{\infty} \frac{r^n(e^{in\theta} + e^{-in\theta})}{2} \\ &= P_r(\theta) \end{aligned}$$

and also

$$\frac{1 + re^{i\theta}}{1 - re^{i\theta}} = \frac{1 + re^{i\theta} - re^{-i\theta} - r^2}{|1 - re^{i\theta|^2}}$$

so that

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = Re\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) \tag{4.3}$$

PROPOSITION 4.11. *The poisson kernel satisfies followings:*

(a) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta = 1;$

(b) $P_r(\theta) > 0\ \forall\ \theta, P_r(\theta) = P_r(-\theta), P_r$ *is a periodic in* θ *with period* 2π*;*

(c) $P_r(\theta) < P_r(\delta)\ if\ 0 < \delta < |\theta| \leq \pi;$

(d) for each $\delta > 0$ *,* $\lim_{r\to 1^-} P_r(\theta) = 0$ *uniformly in* θ *for* $\pi \geq |\theta| \geq \delta$*.*

PROOF. (a) For a fixed value of r, $0 \leq r < 1$, the series

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$$

converges uniformly in θ. So

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}d\theta \\
&= \sum_{n=-\infty}^{\infty} r^{|n|}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}d\theta \\
&= \sum_{n=-\infty}^{\infty} r^{|n|}\frac{1}{2\pi}[e^{in\theta}]_{-\pi}^{\pi} \times \frac{1}{in} \\
&= \sum_{n=-\infty}^{\infty} r^{|n|}\frac{1}{2\pi}[e^{in\pi} - e^{-in\pi}] \times \frac{1}{in} \\
&= \sum_{n=-\infty}^{\infty} r^{|n|}\frac{1}{2\pi} \times \frac{1}{in} \times 2i\sin n\pi \\
&= \sum_{n=-\infty}^{\infty} r^{|n|} \times \frac{\sin n\pi}{n\pi} = 1
\end{aligned}$$

(b)

$$\begin{aligned}
P_r(\theta) = \frac{1+re^{i\theta}}{1-re^{i\theta}} &= \frac{1+re^{i\theta}-e^{-i\theta}-r^2}{|1-re^{i\theta}|^2} \\
&= Re\left(\frac{1-r^2}{|1-re^{i\theta}|^2}\right) \\
&= (1-r^2)(|1-re^{i\theta}|^{-2}) > 0,\ since\ r < 1
\end{aligned}$$

and $P_r(\theta) = P_r(-\theta)$ by equation (4.3).

(c) Let $0 < \delta < |\theta| \leq \pi$ and define $f : [\delta, \theta] \to \mathbb{R}\ by\ f(t) = P_r(t)$.

If $0 < \delta < \pi$ then $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly in θ for $\delta \leq |\theta| \leq \pi$. If we fixed δ *and* $0 < \delta < \pi$, then

$$P_r'(\theta) = \frac{2r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2} \quad < \quad 0 \; for \; \delta \leq \theta \leq \pi$$

$$\geq \quad 0 \; for \; -\pi \leq \theta \leq -\delta.$$

So $P_r(\theta)$ is increasing for $-\pi \leq \theta \leq -\delta$ and decreasing for $\delta \leq \theta \leq \pi$. That is $0 < P_r(\theta) \leq P_r(\delta) = \dfrac{1-r^2}{1-2r\cos\theta+r^2}$ when $\delta < |\theta| \leq \pi$.

(d) For proving uniform convergence of $P_r(\theta)$, we have to show that $\lim_{r\to 1^-}[sup\{P_r(\theta)\} : \delta < |\theta| \leq \pi] = 0$ by (c), $P_r(\theta) \leq P_r(\delta) \; if \; \delta < |\theta| \leq \pi$. To prove this it is sufficient to show that $\lim_{r\to 1^-} P_r(\theta) = 0$. Which is by (4.3). □

THEOREM 4.12. *Let $D = z : |z| < 1$ and suppose that $f : \partial D \to \mathbb{R}$ is a continuous function. Then there is a continuous function $u : \overline{D} \to \mathbb{R}$ such that*

(a) $u(z) = f(z) \; z \; \in \partial D$;

(b) u is harmonic in D.

Moreover u is unique and defined by the formula

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})dt \tag{4.4}$$

for $0 \leq r < 1, 0 \leq \theta \leq 2\pi$

COROLLARY 4.13. *Let $a \in \mathbb{C}$, $\rho > 0$ and suppose his continuous real valued function on $\{z : |z-a| = \rho\}$; then there is a unique continuous function $w : \overline{B}(a;\rho) \to \mathbb{R}$ such that w is harmonic on $B(a;\rho)$ and w(z) = h(z) $|z-a| = \rho$.*

PROOF. Consider $f(e^{i\theta}) = h(a+\rho e^{i\theta})$. Then by maximum principle f is continuous on ∂D. If $u : \overline{D} \to \mathbb{R}$ is continuous function such that u is harmonic in D and $u(e^{i\theta}) = f(e^{i\theta})$ then $w(z) = u\left(\dfrac{z-a}{\rho}\right)$ is the desired function on $\overline{B}(a;\rho)$. □

THEOREM 4.14. *(Converse Mean Value Theorem)*

If $u : G \to \mathbb{R}$ is continuous function which has the mean value property, then u is harmonic.

PROOF. Let $a \in G$ and ρ chosen such that $\overline{B}(a;\rho) \subset G$.
To show u is harmonic on $\overline{B}(a;\rho)$.
By corollary 4.13 there is a continuous function $w : \overline{B}(a;\rho) \to \mathbb{R}$, which is harmonic in $B(a;\rho)$ and $w(a+\rho e^{i\theta}) = u(a+\rho e^{i\theta})$. Since $u-w$ satisfies the MVP and $(u-w)(z) = 0\ for\ |z-a| = \rho$. So by maximum principle $u \equiv w\ in\ \overline{B}(a;\rho)$. That means u must be harmonic. ☐

3. Harnack's Inequality and Harnack's Theorem

In this section we discuss about the important inequality and theorem in Harmonic functions. We start with the Harnack's Inequality. If $R > 0$ then substituting $\frac{r}{R}$ for r in (4.3), we get

$$\frac{1-(\frac{r}{R})^2}{1-2(\frac{r}{R})\cos\theta+(\frac{r}{R})^2} = \frac{R^2-r^2}{R^2-2rR\cos\theta+r^2} \tag{4.5}$$

for $0 \le r < R$ and all θ. So if u is continuous on $\overline{B}(a;r)$ and Harmonic in B(a;r), then

$$u(a+re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}\right]u(a+Re^{it})dt \tag{4.6}$$

Now from (4.5)

$$\frac{R^2-r^2}{|Re^{it}-re^{i\theta}|^2}$$

and $R-r \le |Re^{it}-re^{i\theta}| \le R+r$. Therefore

$$\frac{R-r}{R+r} \le \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} \le \frac{R+r}{R-r}.$$

DEFINITION 4.15. If $u : \overline{B}(a;r) \to \mathbb{R}$ is continuous, harmonic in B(a;R), and $u \ge 0$ then for $0 \le r < R$ and all θ

$$\frac{R-r}{R+r}u(a) \le u(a+re^{i\theta}) \le \frac{R+r}{R-r}u(a).$$

This inequality is called Harnack's Inequality. If G is an open subset of $\mathbb{C}$ then Har(G) is the space of harmonic functions on G.

THEOREM 4.16. *(Harnack's Theorem) Let G be a region. Then*

(a) The metric space Har(G) is complete.

(b) If $\{u_n\}$ is a sequence in Har(G) such that $u_1 \leq u_2 \leq ...$ then either $u_n(z) \to \infty$ uniformly on compact subsets of G or $\{u_n\}$ converges in Har(G) to a harmonic function.

PROOF. (a) To show Har(G) is complete. It is sufficient to show that it is closed subspace of $C(G,\mathbb{R})$. So let $\{u_n\}$ be a sequence in Har(G) such that $u_n \to u$ in $C(G,\mathbb{R})$. Then

$$\int_\gamma u = \lim_{n\to\infty} \int_\gamma u_n. \ where\ \gamma \in [-\pi, \pi]$$

So u has MVP. Then by theorem 4.14 u is Harmonic.

(b) Let us assume that $u_1 \geq 0$.Let u(z)=sup$\{u_n(z) : n \geq 1\}$ for each z in G. So for each z in G, there may be two possibility occures

(i) $u(z) = \infty\ or\ u(z) \in \mathbb{R}$ and

(ii) $u_n(z) \to u(z)$.

Let us define

$$A = \{z \in G : u(z) = \infty\}$$

$$B = \{z \in G : u(z) < \infty\}$$

then $G = A \cup B$ and $A \cap B = \phi$.

To show both A and B are open.

If $a \in G$, let $\mathbb{R}$ be chosen such that $\overline{B}(a;R) \subset G$. By Harnack's inequality

$$\frac{R - |z-a|}{R + |z+a|} u_n(a) \leq u_n(z) \leq R + |z-a| R - |z+a| u_n(a) \tag{4.7}$$

for all $z \in B(a;R)\ and\ n \geq 1$. If $a \in A\ then\ u_n(a) \to \infty$,the left half of (4.7) gives that $u_n(z) \to \infty \forall z \in B(a;R)$. That is $B(a;R) \subset A$ and so A is open. Similarly if $a \in B$ the right half of (4.7) gives that $u(z) < \infty\ \forall\ |z-a| < R$.That is B is open.

Since G is connected, eithe r$A = G$ or $B = G$. Suppose $A = G$; that is $u \equiv \infty$. Again if $\overline{B}(a;R) \subset G\ and\ 0 < \rho < R\ then\ M = (R-\rho)(R+\rho)^{-1}$ and (4.7) gives that $Mu_n(a) \leq u_n(z)$ for $|z-a| \leq \rho$. Hence $u_n(z) \to \infty$ uniformly for z in $\overline{B}(a;R)$.

Now suppose $B = G$, or that $u(z) < \infty \forall\ z \in G$. If $\rho < R$ then, there is a constant N, which depends only on a andρ such that $Mu_n(a) \leq u_n(z) \leq Nu_n(a)\ for\ |z-a| \leq \rho$ and all n. So if $m \leq n$

$$\begin{aligned} 0 \leq u_n(z) - u_n(z) &\leq Nu_n(a) - Mu_m(a) \\ &\leq C[u_n(a) - u_m(a)] \end{aligned}$$

for some constant C. Thus, $\{u_n(z)\}$ is uniformly cauchy sequence on $\overline{B}(a;\rho)$. From this$\{u_n\}$ is a cauchy sequence in Har(G) and from (a), it must converge to a harmonic function. Since $u_n(z) \to u(z)$, u is harmonic function. □

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