BLASIUS FLOW OF A VISCOUS FLUID REVISITED

A DISSERTATION

Submitted in partial fulfilment of the requirements for the award of the degree

of

Master of Science in Mathematics

by

SATYAJIT KUMURA

DEPARTMENT OF MATHEMATICS
NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA
ROURKELA, ODISHA – 769 008 (INDIA)

MAY 2014

CANDIDATE'S DECLARATION

I hereby certify that the work which is being presented in the dissertation entitled **BLASIUS FLOW OF A VISCOUS FLUID REVISITED** in partial fulfillment of the requirement for the award of the Degree of Master of Science and submitted in the Department of Mathematics of the National Institute of Technology Rourkela, Rourkela is an authentic record of my own work carried out during a period from August 2013 to May 2014 under the supervision of Dr. Bikash Sahoo, Assistant Professor, Department of Mathematics of National Institute of Technology Roourkela, Roourkela.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute.

(SATYAJIT KUMURA)

This is to certify that the above statement made by the candidate is correct to the best of our knowledge.

Date:

(BIKASH SAHOO)
Supervisor

The Viva-Voce Examination of **Mr. Satyajit Kumura**, has been held on ....

ACKNOWLEDGEMENT

It is my pleasure to thank to the people, for whom this dissertation is possible. I specially like to thank my guide, Dr. Bikash Sahoo for his keen guidance and encouragement during the course of study and preparation of the final manuscript of this project. I would also like to thank the faculty members of the Department of Mathematics for their cooperation. I heartily thank to my friends, who helped me during the preparation of this project. I thank the Director, National Institute of Technology Rourkela, for providing the facilities to pursue my postgraduate degree. I am also thankful to Mr. Abhijit Das (Research Scholar, NIT Rourkela) for his true help and inspiration. I thank all my classmates and friends for making my stay memorable at National Institute of Technology Rourkela.

Finally, I thank my parents for their constant inspiration.

(SATYAJIT KUMURA)

ABSTRACT

The simplest example of the application of the boundary layer equations is afforded by the flow along a flat plate. Historically, this was the first example illustrating the application of Prandtl's boundary layer theory. The problem was discussed by Heinrich Blasius in his doctoral thesis at Göttingen. That is why the flow is widely known as the Blasius flow. Literature study reveals hardly any attention has been given to the effects of partial slip on the boundary layer flow over a flat plate. The no-slip boundary condition (the assumption that a liquid adheres to a solid boundary) is one of the central tenets of the Navier-Stokes theory. Mathematically the no-slip condition is given by $v_n = 0$ and $v_t = 0$, where v_n and v_t are the normal and the tangential component of the velocity on the wall. In certain situations, however, the assumption of no-slip does no longer apply and should be replaced by a partial slip boundary condition. Navier [7] proposed a slip boundary condition wherein the amount of relative slip depends linearly on the local shear stress. The motivation behind our study is to see the effects of slip on the boundary layer flow of a viscous fluid past an infinite plate. The present investigation is not only important because of its technological significance but also in view of the interesting mathematical features presented by the equations governing the steady, laminar flow with slip boundary conditions.

The partial slip is controlled by a dimensionless slip factor, which varies between zero (total adhesion) and infinity (full slip). The resulting third order nonlinear similarity equation has been numerically integrated using shooting method, along with fourth order Runge-Kutta method. It is observed that the horizontal component of velocity increases with an increase in slip. Thus, the boundary layer thickness decreases with an increase in slip. It is interesting to observe that the skin friction coefficient decreases exponentially with an increase in slip.

Contents

1 Introduction

One of the most successful, fascinating and useful applications of Mathematics has been in the study of motions of fluids. Newtonian fluid mechanics underwent a transformation. Primary causes for the transformation were the concept of a boundary layer put forward by Ludwig Prandtl in his seminal presentation [8] at Heidelberg, Germany during the week of 8 August 1904. The companion paper, entitled "Über Flüssigkeitsbewegung bei sehr kleiner Reibung" ("On the Motion of Fluids with Very Little Friction") was only eight pages long, but it would prove to be one of the most important fluid-dynamics papers ever written. Prandtl's paper gave the first description of the boundary layer concept and revolutionized the understanding and analysis of fluid dynamics. He theorized that an effect of friction was to cause the fluid immediately adjacent to the surface to stick to the surface, in other words, he assumed the *no-slip* condition at the surface and that frictional effects were experienced only in a boundary layer, a thin region near the surface. Outside the boundary layer, the flow was essentially the inviscid flow that had been studied over the previous two centuries. In fact, the velocity changes enormously over a very short distance normal to the surface of the body immersed in a flow. In other words, the boundary layer is a region of very large velocity gradient $(\frac{\partial u}{\partial y})$. According to Newton's shear-stress law, which states that the shear stress is proportional to the velocity gradient, the local shear stress (τ) can be very large within the boundary layer, since $\tau = \mu \frac{\partial u}{\partial y}$ can reach considerable values, even for very small viscosity μ. As a result, the skin-friction drag force exerted on the body is not negligible, contrary to what some earlier 19th century investigators believed. Indeed, for slender aerodynamic shapes, most of the drag is due to skin friction.

Prandtl showed that for the boundary layer, the Navier-Stokes equations can be reduced to a simpler form, applicable only to the boundary layer. The results, called the boundary layer equations, are similar to Navier-Stokes in that each system consists of coupled, nonlinear partial differential equations. The major mathematical breakthrough, however, is that the boundary layer equations exhibit a completely different mathematical behavior than the Navier-Stokes equations. The Navier-Stokes equations have what mathematicians called elliptic behavior. That is to say, the complete flow field must be solved simultaneously, in accord with the specific boundary conditions defined along the entire boundary of the flow. In contrast, the boundary layer equations have parabolic behavior, which affords tremendous analytical and computational simplification. They can be solved step-by-step by marching downstream from where the flow encounters a body, subject to specified inflow conditions at the encounter and specified boundary conditions at the outer edge of the boundary layer. The systematic calculation yields the flow variables in the boundary layer, including the velocity gradient at the wall surface. The shear stress at the wall, hence the skin friction drag on the surface, is obtained directly from those velocity gradients. Such step-by-step solutions for the boundary layer flows began within a few years of Prandtl's 1904 presentation, carried out mainly by his students at the University of Göttingen. With those solutions, it became possible to predict with some accuracy, the skin friction drag on a body, the locations of the flow separation on the surface and given those locations, the form drag and the pressure drag due to the flow separation.

In his 1905 paper, short as it was, Prandtl gave the boundary layer equations for steady two-dimensional flow, suggested some solution approaches for those equations, made a rough calculation of friction drag on a flat plate, and discussed aspects of boundary layer separation under the influence of an adverse pressure gradient. Those were all pioneering contributions. Despite the important work, Prandtl's research group at Göttingen paid little attention, especially outside of Germany. It surfaced again in 1908 when Prandtl's student Heinrich Blasius, published in the respected journal *Zeitschrift für Mathematik und Physik (ZAMP)*, his paper "Boundary Layers in Fluids with Little Friction", which discussed $2D$ boundary layer flows over a flat plate and a circular cylinder. Blasius study was based on the conventional no-slip boundary conditions. The no-slip boundary condition (the assumption that a liquid adheres to a solid boundary) is one of the central tenets of the Navier-Stokes theory. Mathematically the no-slip condition is given by $v_n = 0$ and $v_t = 0$, where v_n and v_t are the normal and the tangential component of the velocity on the wall. In certain situations, however, the assumption of no-slip does no longer apply and should be replaced by a partial slip boundary condition. Navier [7] proposed a slip boundary condition wherein the amount of relative slip depends linearly on the local shear stress. The equations of motion are still valid for these flows, but the boundary conditions have to be changed appropriately. This is a condition which was discovered empirically, and which is satisfied well within the framework of continuum mechanics. Literature survey reveals that some rarefied gases and most of the non-Newtonian fluids exhibit the slip boundary conditions. Although the Navier condition looks simple, analytically it is much more difficult than the no-slip condition and only a few simple exact slip flow solutions have been found. A precise discussion regarding different kinds of possible slip boundary conditions is provided in the Ph.D. thesis submitted by Sahoo [9].

A numerical approximation to the Blasius flow problem was given by L.Howarth [5] in 1938. R.Fazio [3] reformulated the problem as a free boundary value problem and obtained the solution numerically. S.J Liao [6] obtained an explicit, totally analytic approximation solution for Blasius viscous flow problem by using Homotopy Analysis Method (HAM) and this analytic solution converges to L.Howarth numerical result. Subsequently, analytical solution was represented by Jihuan HE [4] via the variational iteration method. Lei Wang [10] obtained a new algorithm for solving classical Blasius equation by the Adomian decomposition method. R. Cortell [2] obtained an effective numerical solution to the Blasius flow problem. Abdul Aziz [1] numerically obtained the solution for boundary layer flow over a flat plat with slip flow and constant heat flux surface condition. Xu and Guo [11] solved the Blasius flow and its variation and obtain a semi-analytical solution. From the literature survey it is clear that hardly any attention has been given to to study the effects of slip on the viscous boundary layer flow near an infinite plate. The present study is an attempt to fill this gap.

2 Mathematical formulation

We consider the steady flow of an incompressible viscous fluid past an infinite plate. The surface of the plate admits partial slip. The leading edge of the plate is at $x = 0$ and the plate coincides with $y = 0$. The flow far away from the plate is uniform and is in the direction parallel to the plate.

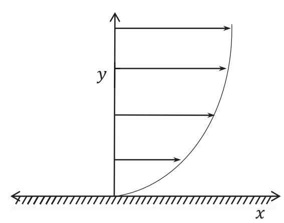

Figure 1: Schematic diagram of Blasius flow domain

2.1 Governing differential equations

The Navier-Stokes equation along x-axis is given be

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.1}$$

and along y-axis is given by

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2.2}$$

The equation of continuity is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.3}$$

The boundary conditions are

$$u = \lambda^* \tau_{xy}, \quad v = 0, \quad \text{at } y = 0$$
$$u \to U_\infty, \quad \text{as } y \to \infty$$

2.2 Boundary layer approximation

We now see the familiar strategy in boundary layer theory, which is to scale the cross-stream distance by a much smaller length scale, and adjust that length scale in order to achieve a balance between convection and diffusion. The dimensionless x co-ordinate is defined as $x^* = (x/L)$, while the dimensionless y co-ordinate is defined as $y^* = (y/\delta)$, where the length δ is determined by a balance between convection and diffusion. In momentum boundary layers, it is also necessary to scale the velocity components and the pressure. In the stream wise direction, the natural scale for the velocity is the free-stream velocity U_∞, so we define a scaled velocity in the x direction as $u_x^* = (u_x/U_\infty)$. The scaled velocity in the y direction is determined from the mass conservation condition, when the above equation is expressed in terms of scaled variables $x^* = (x/L)$, $y^* = (y/\delta)$ and $u_x^* = (u_x/U_\infty)$, and multiplied throughout by (L/U_∞), we obtain,

$$\frac{\partial u_x^*}{\partial x^*} + \frac{L}{\delta U_\infty}\frac{\partial u_y}{\partial y^*} = 0 \tag{2.4}$$

The above equation (2.4) indicates that the approximate scaled velocity in the y direction is $u_y^* = (u_y/(U_\infty \delta/L))$. Note that the magnitude of the velocity u_y in the cross-stream y direction, $(U_\infty \delta/L)$ is small compared to that in the stream wise direction. This is a feature common to all the boundary layers in incompressible flows.

Next we turn to the x momentum equation, (2.1). When it is expressed in terms of scaled spatial and velocity coordinates, and divided throughout by $(\rho U_\infty^2/L)$, we obtain,

$$u_x^*\frac{\partial u_x^*}{\partial x^*} + u_y^*\frac{\partial u_y^*}{\partial y^*} = -\frac{1}{\rho U_\infty^2}\frac{\partial p}{\partial x^*} + \frac{\mu}{\rho U_\infty L}\left(\frac{L^2}{\delta^2}\right)\left(\frac{\partial^2 u_x^*}{\partial {y^*}^2} + \frac{\delta^2}{L^2}\frac{\partial^2 u_x^*}{\partial {x^*}^2}\right) \tag{2.5}$$

The above equation indicates that it is appropriate to define the scaled pressure as $p^* = (p/\rho U_\infty^2)$. Also note that the factor $(\mu/\rho U_\infty L)$ on the right side of the equation (2.5) is the inverse of the Reynolds number based on the free stream velocity and the length of the plate. In the right side of the equation (2.5), we can also neglect the streamwise gradient $(\partial^2 u_x^*/\partial x_*^2)$, since this is multiplied by the factor $(\delta/L)^2$, which is small in the limit $(\delta/L) \ll 1$. With this simplifications, equation (2.5) reduces to,

$$u_x^*\frac{\partial u_x^*}{\partial x^*} + u_y^*\frac{\partial u_y^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + Re^{-1}\left(\frac{L^2}{\delta^2}\right)\frac{\partial^2 u_x^*}{\partial {y^*}^2} \tag{2.6}$$

From the above equation, it is clear that a balance is achieved between convection and diffusion only for $(\delta/L) \sim \mathrm{Re}^{-1/2}$ in the limit of the right Reynolds number. This indicates that the boundary layer thickness is $\mathrm{Re}^{-1/2}$ smaller than the length of the plate. Without loss of generality, we substitute $\delta = \mathrm{Re}^{-1/2} L$ in equation (2.6), to get the scaled momentum equation in the streamwise direction,

$$u_x^*\frac{\partial u_x^*}{\partial x^*} + u_y^*\frac{\partial u_y^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u_x^*}{\partial {y^*}^2} \tag{2.7}$$

Next we analyse the momentum equation in the cross-stream direction, (2.2). This equation is expressed in terms of the scaled spatial co-ordinates, velocities and pressure, to obtain,

$$\frac{\rho U_\infty^2 \delta}{L^2}\left(u_x^*\frac{\partial u_y^*}{\partial x^*}+u_y^*\frac{\partial u_y^*}{\partial y^*}\right)=-\frac{\rho U_\infty^2}{L^2}\frac{\partial p^*}{\partial y^*}+\frac{\mu U_\infty}{\delta L}\left(\frac{\partial^2 u_y^*}{\partial y^{*2}}+\left(\frac{\delta}{L}\right)^2\frac{\partial^2 u_y^*}{\partial x^{*2}}\right) \quad (2.8)$$

By examining all the terms in the above equation, it is easy to see that the largest terms is the pressure gradient in the cross-stream direction. We divide throughout by the pre-factor of this term, and substitute $(\delta/L) = Re^{-1/2}$, to obtain,

$$Re^{-1}\left(u_x^*\frac{\partial u_y^*}{\partial x^*}+u_y^*\frac{\partial u_y^*}{\partial y^*}\right)=-\frac{\partial p^*}{\partial y^*}+Re^{-1}\left(\frac{\partial^2 u_y^*}{\partial y^{*2}}+Re^{-1}\frac{\partial^2 u_y^*}{\partial x^{*2}}\right) \quad (2.9)$$

In the limit Re $\gg$ 1, the above momentum conservation equation reduces to ,

$$\frac{\partial p^*}{\partial y^*}=0 \quad (2.10)$$

Thus the pressure gradient in the cross-stream direction is zero in the leading approximation, and the pressure at any streamwise location in the boundary layer is same as that in the free-stream at that same stream-wise location. This is a salient feature of the flow in a boundary layers. Thus the above scaling analysis has provided us with the simplified 'boundary layer equations' (2.4), (2.7) and (2.10), in which we neglect all terms that are O(1) in an expansion in the parameter $\mathrm{Re}^{-1/2}$. Expressed in dimensional form, the mass conservation equation is (2.3), while the approximate momentum conservation equations are,

$$\rho\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u_x}{\partial y^2} \quad (2.11)$$

$$\frac{\partial p}{\partial y}=0 \quad (2.12)$$

From the equation (2.12), the pressure is only a function of the leading edge of the plate x, and not a function of cross-stream distance y. Therefore, the pressure at a displacement x from the leading edge is independent of the normal distance form the plate y. However, in the limit $y \to \infty$, we know that the free stream velocity U_∞ is a constant, and the pressure is a constant independent of x. This implies that the pressure is also independent of x. as well, and the term $(\partial p/\partial x)$ in equation (2.11) is equal to zero. With this, equation (2.11) simplifies to,

$$\rho\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=\mu\frac{\partial^2 u_x}{\partial y^2} \quad (2.13)$$

This has to be observed, together with the mass conservation condition equation (2.3), to obtain the velocity profile.

2.3 Similarity transformation

We look for a similarity solution for the above equation, under the assumption that the stream function of a location (x, y) depends only on the distance x from the leading edge of the plate and the cross-stream distance y and not on the total length of the plate. The justification for that momentum is being conserved downstream by the flow, and so condition at a trailing edge of the plate at $x = L$ should not affect the velocity profile upstream of this location. While scaling the spatial coordinate and velocities in equations (2.4), (2.7), and (2.10), we have used the dimensionless coordinate

$$y^* = \frac{y}{Re^{-1/2}L} = \frac{y}{(\nu L/U_\infty)^{1/2}} \tag{2.14}$$

Since we have made the assumption that the only length scale in the problem is the distance from the leading edge x, it is appropriate to define the similarity variable using x instead of L in equation (2.14),

$$\eta = \frac{y}{(\nu x/U_\infty)^{1/2}} \tag{2.15}$$

This scaling implies that the thickness of the boundary layer at a distance x from the upstream edge of the plate is proportional to $(\nu x/U_\infty) = xRe_x^{-1/2}$, where $Re_x = (U_\infty x/\nu)$ is the Reynolds number on the distance from the upstream edge. It is appropriate to scale the velocity in the x direction by the free stream velocity U_∞, while the scaling in the y direction is obtained by replacing L by x in equation (2.4),

$$u_x^* = \frac{u_x}{U_\infty} \tag{2.16}$$

$$u_y^* = \frac{u_y}{(\nu U_\infty/x)^{1/2}} \tag{2.17}$$

where u_x^* and u_y^* are only functions of the similarity variable. It is convenient to express the velocity components in terms of stream function $\psi(x, y)$ for an incompressible flow, since the mass conservation condition is identically satisfied when the velocity is expressed in terms of the stream function. The components of the velocity are related to the stream function by,

$$\begin{aligned} u_x^* &= \frac{1}{U_\infty}\frac{\partial \psi}{\partial y} \\ &= \frac{1}{(\nu x U_\infty)^{1/2}}\frac{\partial \psi}{\partial \eta} \end{aligned} \tag{2.18}$$

The above equation indicates that it is appropriate to define the scaled stream function ψ^*, often expressed in literature as $F(\eta)$.

$$\psi^*(\eta) = F(\eta) = \frac{\psi}{(\nu x U_\infty)^{1/2}} \tag{2.19}$$

where $F(\eta)$ is a dimensionless function of the similarity variable η. The streamwise velocity can then be expressed in terms of the scaled stream function as,

$$\begin{aligned} u_x &= \frac{\partial \psi}{\partial y} \\ &= U_\infty \frac{\partial F}{\partial \eta} \end{aligned} \tag{2.20}$$

The cross-stream velocity is given by,

$$\begin{aligned} u_y &= -\frac{\partial \psi}{\partial x} \\ &= \frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{1/2}\left(\eta\frac{dF}{d\eta} - F\right) \end{aligned} \tag{2.21}$$

Equation (2.11) also contains derivatives of the streamwise velocity, which can be expressed in terms of the similarity variable η as,

$$\frac{\partial u_x}{\partial x} = -\frac{U_\infty \eta}{2x}\frac{d^2F}{d\eta^2} \tag{2.22}$$

$$\frac{\partial u_x}{\partial y} = \frac{U_\infty}{(\nu x/U_\infty)^{1/2}}\frac{d^2F}{d\eta^2} \tag{2.23}$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{U_\infty^2}{\nu x}\frac{d^3F}{d\eta^3} \tag{2.24}$$

Equations (2.21) to (2.24) are inserted into the equation (2.11) to obtain, after some simplification,

$$\frac{d^3F}{d\eta^3} + \frac{1}{2}F\frac{d^2F}{d\eta^2} = 0 \tag{2.25}$$

This is the Blasius boundary layer equation for the stream function for the flow past a flat plate. This equation has to be solved, subject to the appropriate boundary conditions, which are as follows. At the surface of the plate, the no-slip condition requires that the velocity components u_x and u_y are zero. Since u_x is given by equation (2.20), the condition $u_x = \lambda^* \tau_{xy}$ at $y = 0$ reduces to

$$\frac{dF}{d\eta} = \lambda\frac{d^2F}{d\eta^2} \text{ at } y = 0 \ (\eta = 0) \tag{2.26}$$

Using equation (2.21) for the cross-stream velocity u_y, along with condition (2.26) at the surface, the condition $u_y = 0$ at $y = 0$ reduces to,

$$F = 0 \text{ at } y = 0 \ (\eta = 0) \tag{2.27}$$

Finally, we require that the velocity u_x is equal to the free stream velocity U_∞ in the limit y $\rightarrow \infty$. Using equation (2.20) for u_x, we obtain,

$$\frac{dF}{d\eta} = 1 \text{ at } y \rightarrow \infty \ (\eta \rightarrow \infty) \tag{2.28}$$

2.4 Skin friction coefficient

The boundary layer normally generates a drag on the plate as a result of the viscous stresses which are developed at the wall. This drag is normally referred to as skin friction. Skin friction occurs from the interaction amid the fluid and the skin of the body, and is directly associated to the wetted surface, the area of the facade of the body that is in contact with the fluid.

$$C_f = \tau_{xy}|_{y=0} \tag{2.29}$$

$$\tau_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{2.30}$$

Where,

$$\frac{\partial u_x}{\partial y}|_{\eta=0} = \frac{U}{\sqrt{\frac{\nu}{U}}}F^{''}(0) \tag{2.31}$$

$$\frac{\partial u_y}{\partial x}|_{\eta=0} = 0 \tag{2.32}$$

Using equations (2.31) and (2.32) in (2.30), we have

$$\begin{aligned} C_f = \tau_{xy}|_{\eta=0} &= \mu\left(\frac{U}{\sqrt{\frac{\nu}{U}}}F^{''}(0)\right) \\ &= \frac{U^2\rho}{\sqrt{Re}}F''(0). \end{aligned} \tag{2.33}$$

3 Numerical tools

Solving a non-linear differential equation analytically is very complicated and there are very limited number of methods available to solve it. But there are huge number of problems do not have a analytical solution. So, here we adopt some numerical techniques to solve our problem.

3.1 Shooting method

This is an initial value problem method. Here we add sufficient number of conditions at one point and adjust these conditions until the required conditions are satisfied at other end. The convergence of this method is depends open the initial guess.

In order to solve the boundary value problem by shooting method, the system of differential equations are written as the corresponding system of first ordered equations and are solved by any of the method used for solving the initial value problems. Here we have adopt the 4^{th} ordered Runge-Kutta method to solve the system of corresponding 1st ordered initial value problem.

3.2 Formulation

Blasius Equation

$$F^{'''} + \frac{1}{2}FF^{''} = 0$$

subject to,

$$F(0) = 0,\ F^{'}(0) = \lambda F''(0),\ F_{\infty} = 1$$

Let, $u_1 = F$, $u_2 = F^{'}$ and $u_3 = F^{''}$

$$\begin{aligned}
\frac{du_1}{dx} &= F^{'} = u_2 \\
&= f_1(t, u_1, u_2, u_3) \\
\frac{du_2}{dx} &= F^{''} = u_3 \\
&= f_2(t, u_1, u_2, u_3) \\
\frac{du_3}{dx} &= F^{'''} \\
&= -\frac{1}{2}FF^{''} \\
&= f_3(t, u_1, u_2, u_3)
\end{aligned}$$

Subject to

$$\begin{aligned}
u_1(0) &= F(0) \\
u_2(0) &= F^{'}(0) \\
u_3(0) &= F^{''}(0)
\end{aligned}$$

3.3 Runge-Kutta method

The formula for the 4^{th} Runge-Kutta method to solve the single initial value problem.

$$\frac{dy}{dx} = f(x, y);\ \ f(x_0) = y_0$$

is given by

$$y_{j+1} = y_j + h\frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Where,

$$\begin{aligned}
K_1 &= f\left(x_j,\ y_j\right),\\
K_2 &= f\left(x_j + \frac{h}{2},\ y_j + \frac{k_1}{2}\right),\\
K_3 &= f\left(x_j + \frac{h}{2},\ y_j + \frac{k_2}{2}\right),\\
K_4 &= f\left(x_j + h,\ y_j + k_3\right)
\end{aligned}$$

where,h is the step length.

Similarly, the 4^{th} ordered classical Runge-Kutta method for the system of equations

$$\frac{d\mathbf{Y}}{d\mathbf{X}} = \mathbf{f}\left(x, y_1, y_2, ...y_n\right)\ ;\ \mathbf{Y}\left(\mathbf{X}_0\right) = \mathbf{Y}_0$$

may be written as

$$\mathbf{Y}_{j+1} = \mathbf{Y}_j + h\frac{1}{6}\left(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4\right)$$

where,

$$\mathbf{K}_1 = \begin{pmatrix} k_{11} \\ k_{21} \\ \vdots \\ k_{n1} \end{pmatrix}, \mathbf{K}_2 = \begin{pmatrix} k_{12} \\ k_{22} \\ \vdots \\ k_{n2} \end{pmatrix}, \mathbf{K}_3 = \begin{pmatrix} k_{13} \\ k_{23} \\ \vdots \\ k_{n3} \end{pmatrix}, \mathbf{K}_4 = \begin{pmatrix} k_{14} \\ k_{24} \\ \vdots \\ k_{n4} \end{pmatrix}$$

and,

$$k_{i1} = f_i\left(x_j,\ y_{1j},\ y_{2j},\ .\ .\ .\ ,\ y_{nj}\right)$$

$$k_{i2} = f_i\left(x_j + \frac{h}{2},\ y_{1j} + \frac{k_{11}}{2},\ y_{2j} + \frac{k_{21}}{2},\ .\ .\ .\ ,\ y_{nj} + \frac{k_{n1}}{2}\right)$$

$$k_{i3} = f_i\left(x_j + \frac{h}{2},\ y_{1j} + \frac{k_{12}}{2},\ y_{2j} + \frac{k_{22}}{2},\ .\ .\ .\ ,\ y_{nj} + \frac{k_{n2}}{2}\right)$$

$$k_{i4} = f_i\left(x_j + h,\ y_{1j} + k_{13},\ y_{2j} + k_{23},\ .\ .\ .\ ,\ y_{nj} + k_{n3}\right)$$

$i = 1(1)n$

4 Results and discussion

The above algorithm was translated to $C++$ program and was run on a core $i5$ personal computer. The step length was taken to be $h = 0.01$. Throughout the run of the program η_∞ was taken to be 10. Fig. 2 depicts the variation of horizontal component of velocity with slip. Clearly F' increases with an increase in λ and assumes its asymptotic value 1 nearer to the surface of the sheet. Thus the boundary layer thickness decrease with an increase in λ. Fig. 3 shows the variation of the stream function F with λ. It increases with an increase in λ and the profiles get flattened. Finally, Fig. 4 presents the variation of $F''(0)$ with λ. $F''(0)$ is the measure of skin friction coefficient. Slip has a significant effect on skin friction. It decreases exponentially with an increase in λ, as was expected. Tables 1, 2 and 3 tabulate the values of F', F and $F''(0)$ for different values of the slip parameter λ.

Table 1: Table of $F^{'}(\eta)$

η	λ= 0	λ= 0.5	λ= 1	λ= 1.5	λ= 2	λ= 5	λ= 10
0	0	0.159892	0.293897	0.398517	0.479324	0.71869	0.842548
0.2	0.0664079	0.223811	0.352616	0.45158	0.527178	0.747368	0.859352
0.4	0.132764	0.287479	0.410951	0.504189	0.574549	0.775628	0.875871
0.6	0.198938	0.350573	0.468464	0.555853	0.62093	0.803056	0.891833
0.8	0.26471	0.412701	0.524671	0.606054	0.665801	0.829259	0.906987
1	0.329781	0.473414	0.579059	0.654271	0.708658	0.853888	0.921116
2	0.629767	0.738457	0.80684	0.850173	0.878928	0.945879	0.972347
3	0.846045	0.905757	0.937225	0.954732	0.965296	0.98653	0.993558
4	0.955519	0.977022	0.986406	0.990971	0.993472	0.997832	0.999036
5	0.991542	0.996367	0.998107	0.998847	0.999217	0.999782	0.999911
6	0.998973	0.999636	0.999834	0.999907	0.999941	0.99999	0.999998
7	0.999923	0.999977	0.999992	0.999995	0.999998	1	1
8	0.999997	0.999999	1	1	1	1	1
9	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1

Table 2: Values of $F(\eta)$ for different λ

η	$\lambda= 0$	$\lambda= 0.5$	$\lambda= 1$	$\lambda= 1.5$	$\lambda= 2$	$\lambda= 5$	$\lambda= 10$
0	0	0	0	0	0	0	0
0.2	0.0066412	0.0383724	0.0646545	0.0850134	0.100654	0.146609	0.170192
0.4	0.0265602	0.0895082	0.141021	0.180602	0.210839	0.298919	0.343722
0.6	0.0597352	0.153326	0.22898	0.286626	0.330408	0.456805	0.520504
0.8	0.106109	0.229674	0.32832	0.402846	0.459111	0.62006	0.700401
1	0.165573	0.318313	0.438728	0.528916	0.596594	0.788404	0.88323
2	0.650027	0.930774	1.13885	1.28828	1.39722	1.69291	1.83278
3	1.39681	1.76196	2.01918	2.19808	2.32581	2.66267	2.81771
4	2.30575	2.70973	2.98591	3.17481	3.30834	3.65626	3.81472
5	3.28328	3.69887	3.97976	4.17084	4.30554	4.65538	4.81434
6	4.27963	4.6974	4.97903	5.17041	5.30525	5.6553	5.81431
7	5.27925	5.69727	5.97897	6.17038	6.30523	6.6553	6.81431
8	6.27922	6.69726	6.97897	7.17037	7.30523	7.65531	7.81432
9	7.27922	7.69726	7.97897	8.17037	8.30523	8.65531	8.81432
10	8.27923	8.69726	8.97897	9.17037	9.30523	9.65531	9.81433

Table 3: Values of $F''(0)$ for different λ

λ	$F''(0)$
0	0.332058
0.2	0.329819
0.4	0.32383
0.6	0.315217
0.8	0.30498
1	0.293897
2	0.239662
4	0.166719
6	0.126125
8	0.101095
10	0.0842548
15	0.0593911
20	0.0458262

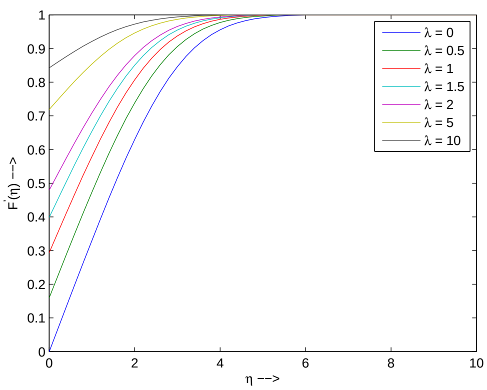

Figure 2: Variation of $F'(\eta)$ with λ.

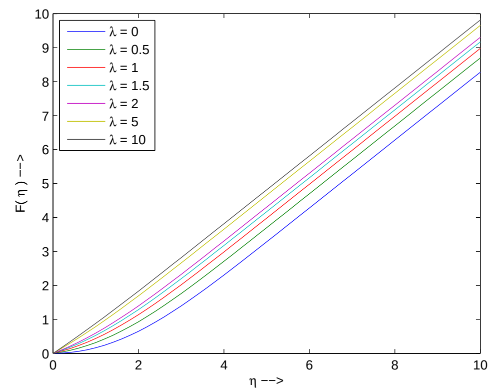

Figure 3: Variation of $F(\eta)$ with λ.

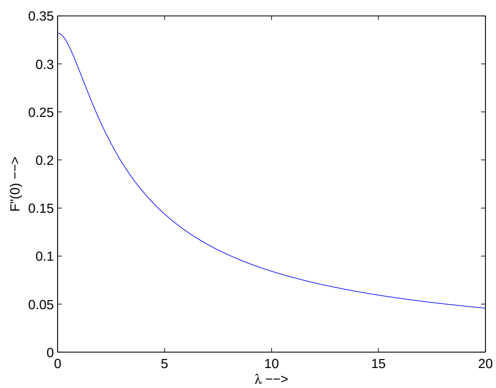

Figure 4: Variation of $F''(0)$ with λ.

5 Conclusion

In this report we obtained the numerical solution of the Blasius flow of a viscous fluid with partial slip boundary condition. It is observed that with increase in slip parameter λ the boundary layer thickness decreases and hence, the viscous drag of the fluid also decreases. For very large value of slip, the fluid behaves as if inviscid.

References

[1] A. Aziz. Hydrodynamic and thermal slip flow boundary layers over a flat plate with constant heat flux boundary condition. *Commun Nonlinear Sci Numer Simulat*, 15:573–580, 2010.

[2] R. Cortell. Numerical solution of the classical blasius flat-plate problem. *Appllied Mathematics and Computation*, 170:707–710, 2005.

[3] R. Fazio. The blasius problem formulates as a boundary value problem. *Acta Mechanica*, 95:1–7, 1992.

[4] J. HE. Approximate analytic solution of blasius equation. *Comm. Nonlin. Sc. Num. Sim.*, 4:75–78, 1999.

[5] L. Howarth. On the solution of the laminar boundary layer equations. *Proc. Roy. Soc. London A*, 164:547–579, 1938.

[6] S.J. Liao. An explicit, totally analytic approximate solution for blasius viscous flow problem. *International Journal of Non-Linear Mechanics*, 34:759–778, 1999.

[7] C.L.M.H. Navier. Sur les lois du mouvement des fluides. *Mem. Acad. R. Acad. R. Sci. Inst. Fr.*, 6:389–440, 1827.

[8] L. Prandtl. Über Flüssigkeitsbewegungen bei sehr kleiner Reibung. In *Verhandlg. III Int. Math. Cong.*, Heidelberg, August 1904.

[9] B. Sahoo. Study of some non-newtonian boundary layer flows. *Ph.D. Thesis submitted to IIT Roorkee*, 2008.

[10] L. Wang. A new algorithm for solving classical blasius equation. *Appl. Math. Comp.*, 157:1–9, 2004.

[11] D. Xu and Guo X. Application of fixed point method to obtain semi-analytical solution to blasius flow and its variation. *Applied Mathematics and Computation*, 224:791–802, 2013.