# DELAMINATION EFFECT ON RESPONSE OF A COMPOSITE BEAM BY WAVELET SPECTRAL FINITE ELEMENT METHOD 

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## Certificate

This is to certify that the thesis entitled "DELAMINATION EFFECT ON RESPONSE OF A COMPOSITE BEAM BY WAVELET SPECTRAL FINITE ELEMENT METHOD" submitted by Mr. VENNA VENKATESWARAREDDY, bearing Roll No. 210CE2030 in partial fulfilment of the requirements for the award of Master of Technology Degree in Civil Engineering with specialization in Structural Engineering during 2010-2012 session at the National Institute of Technology, Rourkela is an authentic work carried out by him under my supervision.

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## PREFACE

Transform methods are very useful to solve the ordinary and partial differential equations. Fourier and Laplace transforms are the most commonly used transforms. Wavelet transforms are most popular with electrical and communication engineers to analyse the signals. From last few years, Wavelet transforms are in use for structural engineering problems, like solution of ordinary and partial differential equations. Dynamical problems in structural engineering fall under two categories, one involving low frequencies (structural dynamics problems) and the other involving high frequencies (wave propagation problems).

Spectral Finite Element (SFE) method is a transform method to solve the high frequency excitation problems which are encountered in structural engineering. SFE based on Fourier transforms has high limitations in handling finite structures and boundary conditions. SFE based with wavelet transforms is a very good tool to analyse the dynamical problems and eliminate many limitations.

In this project, a model for embedded de-laminated composite beam is developed using the wavelet based spectral finite element (WSFE) method for the de-lamination effect on response using wave propagation analysis. The simulated responses are used as surrogate experimental results for the inverse problem of detection of damage using wavelet filtering. The technique used to model a structure that, through width de-lamination subdivides the beam into base-laminates and sub-laminates along the line of de-lamination. The base-laminates and sublaminates are treated as structural waveguides and kinematics are enforced along the connecting line. These waveguides are modeled as Timoshenko beams with elastic and inertial coupling and the corresponding spectral elements have three degrees of freedom, namely axial, transverse and
shear displacements at each node. The internal spectral elements in the region of de-lamination are assembled assuming constant cross sectional rotation and equilibrium at the interfaces between the base-laminates and sub-laminates. Finally, the redundant internal spectral element nodes are condensed out to form two-noded spectral elements with embedded de-lamination. The response is being obtained by coding programs in MATLAB.

## CONTENTS

CHAPTERPAGE No.
CERTIFICATE ..... ii
ACKNOWLEDGEMENT ..... iii
PREFACE ..... iv
CONTENTS ..... vi
LIST OF FIGURES ..... ix
LIST OF TABLES ..... xii
ABBREVATIONS ..... xii
SYMBOLS ..... XV
1 INTRODUCTION .....  1
1.1 Introduction ..... 2
1.2 Solution methods for structural dynamic problems .....  2
1.3 Solution methods for wave propagation problems ..... 4
2 LITERATURE REVIEW ..... 7
2.1 Natural frequency-based methods ..... 8
2.2 Mode shape-based methods ..... 12
2.3 Curvature/strain mode shape-based methods ..... 18
2.4 Other methods based on modal parameters ..... 20
3 TRANSFORM METHODS ..... 22
3.1 Laplace transforms. ..... 23
3.2 Fourier transforms ..... 25
3.2.1 Continuous Fourier transforms (CFT). ..... 25
3.2.2 Fourier Series (FS) ..... 28
3.2.3 Discrete Fourier Transform (DFT) ..... 30
3.3 Wavelet transforms ..... 31
3.3.1 Types of wavelets ..... 32
3.3.1.1 Morlet's Wavelet ..... 32
3.3.1.2 Mexican Hat wavelet ..... 34
3.3.1.3 The Continuous Wavelet Transform ..... 34
3.3.1.4 Discrete Wavelet Transform ..... 35
3.3.2 Multi-resolution analysis ..... 36
4 WAVELET INTEGRALS ..... 39
4.1 Evaluation of wavelet integrals ..... 40
4.2 Construction of Daubechies compactly supported wavelets ..... 40
4.3 Scaling function $\varphi(t)$ and Wavelet function $\psi(t)$ ..... 44
4.4 Connection Coefficients $\left(\Omega_{n}^{r}\right)$ ..... 49
4.5 Moment of scaling functions ..... 49
4.6 Impact load ..... 52
5 WAVELET BASED SPECTRAL FINITE ELEMENT FORMULATION ..... 53
5.1 Reduction of wave equations to ordinary differential equations ..... 54
5.1.1 Wavelet Extrapolation Technique. ..... 57
5.1.2 Calculation of wave numbers and wave amplitudes ..... 60
5.2 Spectral finite element formulation ..... 62
5.3 Modeling of de-lamination in composite beam. ..... 67
6 NUMERICAL EXAMPLES AND CONCLUSIONS ..... 72
6.1 Wave responses to the impulse load. ..... 74
7 REFERENCES ..... 82-91

## LIST OF FIGURES

Figure 3.1: (a) Morlet's wavelet: real part ..... 33
(b) Morlet's wavelet Imaginary part ..... 33
Figure 3.2: Mexican Hat wavelet ..... 34
Figure 4.1: (a) Scaling function for $\mathrm{N}=2$ ..... 45
(b) Wavelet function for $\mathrm{N}=2$ ..... 45
Figure 4.2: (a) Scaling function for $\mathrm{N}=4$ ..... 46
(b) Wavelet function for $\mathrm{N}=4$ ..... 46
Figure 4.3: (a) Scaling function for $\mathrm{N}=12$ ..... 47
(b) Wavelet function for $\mathrm{N}=12$ ..... 47
Figure 4.4: (a) Scaling Function for $\mathrm{N}=22$ ..... 48
(b) Wavelet Function for $\mathrm{N}=22$ ..... 48
Figure 4.5: Discredited impulse time signal. ..... 52
Figure 5.1: Composite beam element with nodal forces and nodal displacements. ..... 62
Figure 5.2: Graphite-Epoxy [0] 8layered cantilever beam ..... 67

Figure 5.3: (a) Cross section at the end................................................................. 68
(b) Cross section at the de-lamination....................................................... 68

Figure 5.4: Modeling of an embedded de-lamination with base and sub laminates68

Figure 5.5: Representation of the base and sub base laminates by spectral elements....

Figure 5.6: Force balance at the interface between base and sub laminate elements

Figure 6.1: Axial tip velocity of undamaged $\left[0_{8}\right]$ composite beam.

Figure 6.2: Transverse tip velocity of undamaged $\left[0_{8}\right]$ composite beam75

Figure 6.3: Axial tip velocities for undamaged and different de-lamination lengths
( $L_{d}=10 \mathrm{~mm}$ and 20 mm ) along the centerline of $\left[0_{8}\right]$ layered composite beam........... 76

Figure 6.4: Transverse tip velocities for undamaged and different de-lamination lengths

$$
\left(L_{d}=10 \mathrm{~mm} \text { and } 20 \mathrm{~mm} \text { ) along the centerline of }\left[0_{8}\right] \text { layered composite beam......... } 76\right.
$$

Figure 6.5: Axial tip velocities for undamaged and different de-lamination lengths ( $L_{d}=30 \mathrm{~mm}$ and 40 mm ) along the centerline of $\left[0_{8}\right]$ layered composite beam........ 77

Figure 6.6: Transverse tip velocities for undamaged and different de-lamination lengths ( $L_{d}=30 \mathrm{~mm}$ and 40 mm ) along the centerline of $\left[0_{8}\right]$ layered composite beam........ 77

Figure 6.7: Axial tip velocities for undamaged and different position ( 0.25 m and 0.35 m ) of 20 mm de-lamination from free end and along the centerline of $\left[0_{8}\right]$
layered composite beam

Figure 6.8: Transverse tip velocities for undamaged and different position ( 0.25 m and 0.35 m )
of 20 mm de-lamination from free end and along the centerline of $\left[0_{8}\right]$
composite beam 78

Figure 6.9: Axial tip velocities for undamaged and different depths above the centerline (h1=h, h1=h/2 and h1=h/4) of 20 mm de-lamination of $\left[0_{8}\right]$ layered composite beam 79

Figure 6.10: Transverse tip velocities for undamaged and different depths above the centerline ( $\mathrm{h} 1=\mathrm{h}, \mathrm{h} 1=\mathrm{h} / 2$ and $\mathrm{h} 1=\mathrm{h} / 4$ ) of 20 mm de-lamination of $\left[0_{8}\right]$ layered composite beam .80

Figure 6.11: Axial tip velocities for different orientation $\left([0]_{8},[30]_{8},[45]_{8}\right.$, and $\left.[60]_{8}\right)$ of 20 mm de-lamination along the centerline of composite beam...................... 80

Figure 6.12: Transverse tip velocities for different orientation $\left([0]_{8},[30]_{8},[45]_{8}\right.$, and $\left.[60]_{8}\right)$ of 20 mm de-lamination along the centerline of composite beam.

## LIST OF TABLES

## Table 4.1: Filter Coefficients for $\mathrm{N}=22$ <br> 44

Table4. 2: First order $\left(\Omega^{1}\right)$ and Second order $\left(\Omega^{2}\right)$ connection coefficients for $N=22 \ldots \ldots \ldots \ldots . .50$

Table 4.3: Moment of Scaling Functions for $\mathrm{N}=22$51

Table 6.1: Material properties of the Graphite- Epoxy composite beam.............................. 73

# ABBREVATIONS 

| ANN | Artificial Neural Networks |
| :---: | :---: |
| AWCD | Approximate Waveform Capacity Dimension |
| CFT | Continuous Fourier Transform |
| CWT | Continuous Wavelet Transform |
| DI | Damage Index |
| DOF | Degrees of Freedom |
| DST | Discrete Fourier Transform |
| FD | Fractal Dimension |
| FE | Finite Element |
| FEM | Finite Element Method |
| FFT | Fast Fourier transform |
| FIR | Finite Impulse Responce |
| FS | Fourier Series |
| FSFE | Fourier Transform based Spectral Finite Element |
| GFD | Generalized Fractal Dimension |


| MDLAC | Multiple Damage Location Assurance Criterion |
| :---: | :---: |
| MDOF | Multi Degree of freedom |
| MRA | Multiresolution Analysis |
| MSA | Multiscale Analysis |
| MSC | Mode Shape Curvature |
| NN | Neural Network |
| ODE | Ordinary Differential Equations |
| PDE | Partial Differential Equations |
| PEP | Polynomial Evaluation Problem |
| SCFEM | Super Convergent Finite Element Method |
| SCCM | Spectral Centre Correction Method |
| SDI | Single Damage Indicator |
| SQP | Sequential Quadratic Programming |
| SVD | Singular Value Decomposition |
| SWT | Stationary Wavelet Transform |
| ULS | Uniform Load Surface |
| WSFEM | Wavelet based Spectral Finite Element Method |

## SYMBOLS

| $a$ | Scaling parameter |
| :---: | :---: |
| $a_{k}$ | Filter coefficients |
| $A_{i j}$ | In-plane laminate maduli coefficients |
| b | Translation parameter |
| $2 b$ | Width of the beam |
| $B_{i j}$ | In-plane/ flexure coupling laminate moduli coefficients |
| $c_{l}$ | Constant coefficients |
| $c_{j, k}$ | Approximation coefficients |
| d | Rectangular pulse width |
| $d_{j, k}$ | Detailed coefficients |
| $D_{i j}$ | Flexural laminate stiffness coefficient. |
| $E_{11}$ | Young's modulus of fibre along the longitudinal axis |
| $E_{22}$ | Young's modulus of fibre across the longitudinal axis |
| $F(s)$ | Function in terms of the Laplace s |
| $F(t)$ | Function in terms of time t |


| $\widehat{F}_{j}{ }^{i}$ | Nodal force vector of the $i^{\text {th }}$ element at $j^{\text {th }}$ node |
| :---: | :---: |
| $\{\hat{F}\}$ | Global Force vector |
| $G_{i j}$ | Shear modulus of composite in $i j^{\text {th }}$ plane |
| 2h | Depth of the beam |
| $i$ | Imaginary unit $\sqrt{-1}$ |
| $I_{0}, I_{1}$, and $I_{2}$ | Inertial constants |
| $k$ | Wave number. |
| $\left[K_{D}^{e}\right]$ | Elemental Dynamic Stiffness Matrix |
| $[\overline{\bar{K}}]$ | Global Dynamic Stiffness Matrix |
| $L$ | Length of the beam |
| $L_{1}$ | Distance between the free end and edge of the de-lamination |
| $L_{d}$ | De-lamination length |
| M | Vanishing moment |
| $M(x, t)$ | Moment in spatial and temporal dimensions |
| $M_{j}$ | Transformed moment |
| $n$ | Number of sampling points |
| N | Order of Debauchies |


| $P_{j}$ | Transformed axial force |
| :---: | :---: |
| $P(x, t)$ | Axial force in spatial and temporal dimensions |
| $\left[R_{1}\right]$ | Amplitude ratio matrix |
| $S_{1}$ and $S_{2}$ | Transformation matrices |
| $t$ | Time |
| $t_{f}$ | Final time |
| $u(x, t)$ | Axial displacement in spatial and temporal dimensions |
| $\widehat{u}_{j}^{i}$ | Nodal displacement vectors of the $i^{\text {th }}$ element at $j^{\text {th }}$ node |
| $\{\hat{u}\}$ | Global displacement matrix |
| $V_{j}$ | Sequence of nested subspaces |
| $V(x, t)$ | Transverse force in spatial and temporal dimensions |
| $V_{j}$ | Transformed transverse force |
| $w(x, t)$ | Transverse displacement in spatial and temporal dimensions |
| $z$ | Thickness of each ply of laminate |
| $\pi$ | pi |
| $\omega$ | Circular frequency |
| $\omega_{n}$ | Discrete circular frequency in $\mathrm{rad} / \mathrm{sec}$ |


| $\varphi(t)$ | Scaling functions |
| :---: | :---: |
| $\psi(t)$ | Wavelet functions |
| $\psi^{*}(x)$ | Mother wavelet |
| $\varphi^{\prime}$ | First derivative of scaling function |
| $\varphi^{\prime \prime}$ | Second derivative of scaling function |
| $\oplus$ | Direct sum |
| $\perp$ | Orthogonal to each other |
| $\Omega_{n}^{i}$ | $i^{\text {th }}$ derivative of $n^{\text {th }}$ Connection coefficients |
| $\mu_{k}$ | $k^{\text {th }}$ moment of scaling function |
| $\mu s$ | Time in micro second |
| $\emptyset(x, t)$ | Shear displacement in spatial and temporal dimensions |
| $\rho$ | Mass density |
| $\Gamma^{1}$ | First order connection coefficient matrix |
| $\Gamma^{2}$ | Second order connection coefficient matrix |
| $\Phi$ | Eigenvector matrix of $\Gamma^{1}$ |
| $\Pi$ | Diagonal matrix containing corresponding eigenvalues $-i \omega_{n}$ |
| $\left[\Theta_{1}\right]$ and $\left[\Theta_{2}\right]$ | Diagonal matrices |

## CHAPTER 1

## INTRODUCTION

## CHAPTER-1

## INTRODUCTION

### 1.1 Introduction

Over the past few decades the composite materials are extensively used in many engineering fields such as civil, mechanical and aerospace engineering etc. In-plane properties of the composite material are much higher than its transverse tensile and inter-laminar shear strength. Due to less strength in transverse direction composite structures are very much prone to defects like matrix cracking, fibre fracture, fibre de-bonding, de-lamination/inter-laminar debonding, of which de-lamination is most common, easily exposed to damage and it may increase, thus reducing the life of the structure. Structural components are often subjected to damage which can potentially reduce the safety. It is very important to find the weakest location and to detect damage at the earliest possible stage to avoid brittle failure in future. This technique (WSFEM) is based on the response-based approach since the response data are directly related to damage. This approach is therefore fast and inexpensive.

### 1.2 Solution methods for structural dynamic problems

Generally dynamic analysis of the structure can be done by Finite Element Method (FEM). In structural engineering, dynamic analysis of structures can be divided into two categories, one is related with the low frequency loading categorized as structural dynamic problems and another related with high frequency loading categorized as wave propagation problems. Most of the structures of dynamic analysis come under structural dynamics. In structural dynamics problems, the solution can be determined either by system parameters such as natural frequencies and mode shapes or in terms of simulated response of the system to the
external excitation such as initial displacements support motion and applied load etc. The first few mode shapes and natural frequencies are sufficient to analyze the performance of the structure.

Finite Element solution of the dynamic analysis of the structure can be obtained by two different methods [5] which are Modal Methods and Time Marching Schemes. In general, for multi degree of freedom system (MDOF), the governing wave equation is coupled with a set of ordinary differential equations (ODE). Linear transformation of the above ordinary differential equations linearly and decoupled by modal matrix are referred as modal analysis. Modal analysis is also an eigenvalue analysis and it is like one of the several numerical techniques such as matrix iteration method. Simple continuous systems like rod, beam, plates can be solved analytically. The solutions of such continuous systems are restricted though they are exact. But complicated structures can be solved by approximate techniques. In general, approximate techniques convert the continuous systems into discrete systems. These approximate techniques are grouped into two categories. In the first group, the solution in terms of known functions are assumed and they are combined linearly. For the continuous system, the governing differential equations are partial differential equations. These partial differential equations are converted to a set of ordinary differential equations by substituting the assumed solution with unknown coefficient as variables. The Rayleigh-Ritz and Galerkin methods are the examples of this method. In the other group, the dynamics of the continuous system is shown in terms of large number of discrete points on the system. The finite element technique falls under this group. In this finite element method the continuous structure is divided into number of elements and each element connects through nodes surrounded by it. The continuous system of structure can be reduced to multi degree of freedom system by expressing the dynamics in terms of the
displacements of the nodes. These displacements are approximated by some functions, the coefficients of these functions are obtained in terms of the displacements of the nodes. This finite element method can be applied to any arbitrary shapes and structure having high complexities. Other methods such as boundary element method [7,39] and meshless methods [6] and wave finite element method [66]are applied for solving structural dynamic problems.

### 1.3 Solution methods for wave propagation problems

Multi modal problems are related with wave propagation analysis, in which the extraction of eigenvalues is computationally most expensive. So Modal Methods are not suitable for multi modal problems which are having very high frequencies. In the wave propagation problems, the short term effects are critical, because the frequency of the input loading is very high. To get the accurate mode shapes and natural frequencies, the wave length and mesh size should be small. Alternatively we can use the time marching schemes under the finite element environment. In this method, analysis is performed over a small time step, which is a fraction of total time for which response histories are required. For some time marching schemes, a constraint is placed on the time step, and this, coupled with very large mesh sizes, make the solution of wave propagation problem. Wave propagation deals with loading of very high frequency content and finite element (FE) formulation for such problems is computationally prohibitive as it requires large system size to capture all the higher modes. These problems are usually solved by assuming solution to the field variables say displacements such that the assumed solution satisfies the governing wave equation as closely as possible. It is very difficult to assume a solution in time domain to solve the governing wave equation. Therefore, ignore the inertial part is ignored and the static part of the governing wave equation is solved exactly, and this solution is used to obtain mass and stiffness matrices. This method of develop of finite
element is called Super Convergent Finite Element Method (SCFEM) [20, 9, 10, 42, 11]. This method gives the smaller system size for wave propagation problems than conventional Finite Element Method (FEM).

Alternatively, the solution in frequency domain is assumed and the governing equations solve are transformed and solved exactly. This simplifies the problem by introducing the frequency as a parameter which removes the time variable from the governing equations by transforming to the frequency domain. Among these techniques, many methods are based on integral transforms [22] which include Laplace transform, Fourier transform and most recently wavelet transform. The solution of these transformed equations is much easier than the original partial differential equations. The main advantage of this system is computational efficiency over the finite element solution. These solutions in transformed frequency domain contain information of several frequency dependent wave properties essential for the analysis. The time domain solution is then obtained through inverse transform. In the frequency domain Fourier methods can be used to achieve high accuracy in numerical differentiation. One such method is FFT based spectral finite element method. The WSFE technique is very similar to the fast Fourier transform (FFT) based spectral finite element (FSFE) except that it uses compactly supported Daubechies scaling function approximation in time. In FSFEM, first the governing PDEs are transformed to ODEs in spatial dimension using FFT in time. These ODEs are then usually solved exactly, which are used as interpolating functions for FSFE formulation.

The advantages of FSFEM are, they reduces the system size and the wave characteristics can be extracted directly from such formulation. The main drawback of FSFEM is that it cannot handle waveguides of short lengths. This is because the required assumption of periodicity results in wrap around problems, which totally distorts the response. It is in such cases,
compactly supported wavelets, which have localized basis functions can be efficiently used for waveguides of short lengths. The wavelet based spectral finite element method follows an approach very similar to FSFEM, except that Daubechies scaling functions are used for approximation in time for reduction of PDEs to ODEs. The approach removes the problem associated with 'wrap around' in FSFEM and thus requires a smaller time window for the same problem. The Fourier transform is a tool widely used for many scientific purposes, but it is well suited only to the study of stationary signals where all frequencies have an infinite coherence time. The Fourier analysis brings only global information which is not sufficient to detect compact patterns.

## CHAPTER-2

## LITERATURE REVIEW

## CHAPTER-2

## LITERATURE REVIEW

This review is organized by the classification using the features extracted for damage identification, and these damage identification methods are categorized as follows:

1. Natural frequency-based methods;
2. Mode shape-based methods;
3. Curvature/strain mode shape-based methods;
4. Other methods based on modal parameters.

### 2.1 NATURAL FREQUENCY-BASED METHODS:- Natural frequency-based

 methods use the natural frequency change as the basic feature for damage identification. The choice of the natural frequency change is attractive because the natural frequencies can be conveniently measured from just a few accessible points on the structure and are usually less contaminated by experimental noise.Liang et al. [36] developed a method based on three bending natural frequencies for the detection of crack location and quantification of damage magnitude in a uniform beam under simply supported or cantilever boundary conditions. The method involves representing crack as a rotational spring and obtaining plots of its stiffness with crack location for any three natural modes through the characteristic equation. The point of intersection of the three curves gives the crack location and stiffness. The crack size is then computed using the standard relation between stiffness and crack size based on fracture mechanics. This method had been extended to stepped beams by Nandwana and Maiti [52] and to segmented beams by Chaudhari and Maiti [13] using the Frobenius method to solve Euler-Bernoulli type differential equations.

Chinchalkar [14] used a finite element-based numerical approach to mimic the semi-analytical approach using the Frobenius method. The beam is modelled using beam elements and the inverse problem of finding the spring stiffness, given the natural frequency, is shown to be related to the problem of a rank-one modification of an eigenvalue problem. This approach does not require quadruple precision computation and is relatively easy to apply to different boundary conditions. The results are compared with those from semi-analytical approaches. The biggest advantage of this method is the generality in the approach; different boundary conditions and variations in the depth of the beam can be easily modelled.

Morassi and Rollo [47] showed that the frequency sensitivity of a cracked beam-type structure can be explicitly evaluated by using a general perturbation approach. Frequency sensitivity turns to be proportional to the potential energy stored at the cracked cross section of the undamaged beam. Moreover, the ratio of the frequency changes of two different modes turns to be a function of damage location only. Morassi's method based on Euler-Bernoulli beam theory modeled crack and as a massless, infinitesimal rotational spring. The explicit expression is valid only for small defects.

Kasper et al. [30] derived the explicit expressions of wave number shift and frequency shift for a cracked symmetric uniform beam. These expressions apply to beams with both shallow and deeper cracks. But the explicit expressions are based on high frequency approximation, and therefore, they are generally inaccurate for the fundamental mode and for a crack located in a boundary-near field.

Messina et al. [40] proposed a correlation coefficient termed the multiple damage location assurance criterion (MDLAC) by introducing two methods of estimating the size of defects in a structure. The method is based on the sensitivity of the frequency of each mode to damage in
each location. 'MDLAC' is defined as a statistical correlation between the analytical predictions of the frequency changes $\delta \mathrm{f}$ and the measured frequency changes $\Delta \mathrm{f}$. The analytical frequency change $\delta \mathrm{f}$ can be written as a function of the damage extent vector $\delta \mathrm{D}$. The required damage state is obtained by searching for the damage extent vector $\delta \mathrm{D}$ which maximizes the MDLAC value. Two algorithms (i.e., first and second order methods) were developed to estimate the absolute damage extent. Both the numerical and experimental test results were presented to show that the MDLAC approach offers the practical attraction of only requiring measurements of the changes in a few of natural frequencies of the structure between the undamaged and damaged states and provides good predictions of both the location and absolute size of damage at one or more sites.

Lele and Maiti[34] extended Nandwana and Maiti's method [53] to short beam, taking into account the effects of shear deformation and rotational inertia through Timoshenko beam theory. Patil and Maiti proposed a frequency shift-based method for detection of multiple open cracks in an Euler-Bernoulli beam with varying boundary conditions. This method is based on the transfer matrix method and extends the scope of the approximate method given by Liang for a single segment beam to multi-segment beams. Murigendrappa et al. [50] later applied Patil and Maiti's approach to single/multiple crack detection in pipes filled with fluid.

Morassi [45] presented a single crack identification in a vibrating rod based on the knowledge of the damage-induced shifts in a pair of natural frequencies. The analysis is based on an explicit expression of the frequency sensitivity to damage and enables non uniform bars under general boundary conditions to be considered. Some of the results are also valid for cracked beams in bending. Morassi and Rollo [47] later extended the method to the identification of two cracks of equal severity in a simply supported beam under flexural vibrations.

Kim and Stubbs [31] proposed a single damage indicator (SDI) method to locate and quantify a crack in beam-type structures by using changes in a few natural frequencies. A crack location model and a crack size model were formulated by relating fractional changes in modal energy to changes in natural frequencies due to damage. In the crack location model, the measured fractional change in the $i^{\text {th }}$ eigenvalue $\mathrm{Z}_{\mathrm{i}}$ and the theoretical (FEM based) modal sensitivity of the $\mathrm{i}^{\text {th }}$ modal stiffness with respect to the $j^{\text {th }}$ element $F_{i j}$ is defined. The theoretical modal curvature is obtained from a third order interpolation function of theoretical displacement mode shape. Then, an error index $e_{i j}$ is introduced to represent the localization error for the $i^{\text {th }}$ mode and the $j^{\text {th }}$ location. SDI is defined to indicate the damage location. While in the crack size model, the damage inflicted $a_{j}$ at predefined locations can be predicted using the sensitivity equation. The crack depth can be computed from $a_{j}$ and the crack size model based on fracture mechanics. The feasibility and practicality of the crack detection scheme were evaluated by applying the approach to the 16 test beams.

Zhong et al. [73] recently proposed a new approach based on auxiliary mass spatial probing using the spectral centre correction method (SCCM), to provide a simple solution for damage detection by just using the output-only time history of beam-like structures. A SCCM corrected highly accurate natural frequency versus auxiliary mass location curve is plotted along with the curves of its derivatives (up to third order) to detect the crack. However, only the FE verification was provided to illustrate the method. Since it is not so easy to get a high resolution natural frequency versus auxiliary mass location curve in experiment as in numerical simulation, the applicability and practicality of the method in in-situ testing or even laboratory testing are still in question.

Kim, B.H et al. [32] presented a vibration-based damage monitoring scheme to give warning of the occurrence, location, and severity of damage under temperature induced uncertainty conditions. A damage warning model is selected to statistically identify the occurrence of damage by recognizing the patterns of damage driven changes in natural frequencies of the structure and by distinguishing temperature-induced off-limits.

Jiang et al. [29] incorporated a tunable piezoelectric transducer circuitry into the structure to enrich the modal frequency measurements, meanwhile implementing a high-order identification algorithm to sufficiently utilize the enriched information. It is shown that the modal frequencies can be greatly enriched by inductance tuning, which, together with the high-order identification algorithm, leads to a fundamentally-improved performance on the identification of single and multiple damages with the usage of only lower-order frequency measurements.

In 1997 Salawu [64] presented an extensive review of publications before 1997 dealing with the detection of structural damage through frequency changes. In the conclusion of this review paper, Salawu suggested that natural frequency changes alone may not be sufficient for a unique identification of the location of structural damage because cracks associated with similar crack lengths but at two different locations may cause the same amount of frequency change.

### 2.2 MODE SHAPE-BASED METHODS:-

Compared to using natural frequencies, the advantage of using mode shapes and their derivatives as a basic feature for damage detection is quite obvious. First, mode shapes contain local information, which makes them more sensitive to local damages and enables them to be used directly in multiple damage detection. Second, the mode shapes are less sensitive to environmental effects, such as temperature, than natural frequencies. The disadvantages are also
apparent. First, measurement of the mode shapes requires a series of sensors; second, the measured mode shapes are more prone to noise contamination than natural frequencies. Shi et al. [65] extended the damage localization method based on multiple damage location assurance criterions (MDLAC) by using incomplete mode shape instead of modal frequency. The two-step damage detection procedure is to preliminarily localize the damage sites by using incomplete measured mode shapes and then to detect the damage site and its extent again by using measured natural frequencies. No expansion of the incomplete measured mode shapes or reduction of finite element model is required to match the finite-element model, and the measured information can be used directly to localize damage sites. The method was demonstrated in a simulated 2D planar truss model. Comparison showed that the proposed method is more accurate and robust in damage localization with or without noise effect than the original MDLAC method. In this method, the use of mode shape is only for preliminary damage localization, and the accurate localization and quantification of damage still rely on measured frequency changes.

Lee et al. [33] presented a neural network based technique for element-level damage assessments of structures using the mode shape differences or ratios of intact and damaged structures. The effectiveness and applicability of the proposed method using the mode shape differences or ratios were demonstrated by two numerical example analyses on a simple beam and a multi-girder bridge.

Hu and Afzal [28] proposed a statistical algorithm for damage detection in timber beam structures using difference of the mode shapes before and after damage. The different severities of damage, damage locations, and damage counts were simulated by removing mass from intact
beams to verify the algorithm. The results showed that the algorithm is reliable for the detection of local damage under different severities, locations, and counts.

Pawar et al. [54] investigated the effect of damage on beams with clamped boundary conditions using Fourier analysis of mode shapes in the spatial domain. The damaged mode shapes were expanded using a spatial Fourier series, and a damage index (DI) in the form of a vector of Fourier coefficients was formulated. A neural network (NN) was trained to detect the damage location and size using Fourier coefficients as input. Numerical studies showed that damage detection using Fourier coefficients and neural networks has the capability to detect the location and damage size accurately. However, the use of this method is limited to beams with clampedclamped boundary condition.

Abdo and Hori [2] suggested that the rotation (i.e., the first derivative of displacement) of mode shape is a sensitive indicator of damage. Based on a finite element analysis of a damaged cantilevered plate and a damaged simply-supported plate, the rotation of mode shape is shown to have better performance of multiple damage localization than the displacement mode shape itself.

Hadjileontiadis et al. [24] and Hadjileontiadis and Douka [25] proposed a response-based damage detection algorithm for beams and plates using Fractal Dimension (FD). This method calculates the localized FD of the fundamental mode shape directly. The damage features are established by employing a sliding window of length $M$ across the mode shape and estimating the FD at each position for the regional mode shape inside the window. Damage location and size are determined by a peak on the FD curve indicating the local irregularity of the fundamental mode shape introduced by the damage. If the higher mode shapes were considered, this method might give misleading information as demonstrated in their study.

Wang and Qiao [69] proposed a modified FD method termed 'generalized fractal dimension (GFD)' method by introducing a scale factor S in the FD algorithm, Instead of directly applying the algorithm to the fundamental mode shape, the GFD is applied to the 'uniform load surface' (ULS) to detect the damage. Three different types of damage in laminated composite beams have been successfully detected by the GFD. It should be pointed out that the GFD bears no conventional physical meaning as compared to the FD, and it only serves as an indicator of damage. A scale factor $S$ has to be carefully chosen in order to detect damage successfully.

Qiao and Cao [58] proposed a novel waveform fractal dimension-based damage identification algorithm. An approximate waveform capacity dimension (AWCD) was formulated first, from which an AWCD-based modal irregularity algorithm (AWCD-MAA) was systematically established. Then, the basic characteristics of AWCD-MAA on irregularity detection of mode shapes, e.g., crack localization, crack quantification, noise immunity, etc., were investigated based on an analytical crack model of cantilever beams using linear elastic fracture mechanics. In particular, from the perspective of isomorphism, a mathematical solution on the use of applying waveform FD to higher mode shapes for crack identification was originally proposed, from which the inherent deficiency of waveform FD to identify crack when implemented to higher mode shapes is overcome. The applicability and effectiveness of the AWCD-MAA was validated by an experimental program on damage identification of a cracked composite cantilever beam using directly measured strain mode shape from smart piezoelectric sensors.

Hong et al. [27] showed that the continuous wavelet transform (CWT) of mode shape using a Mexican hat wavelet is effective to estimate the Lipschitz exponent for damage detection of a damaged beam. The magnitude of the Lipschitz exponent can be used as a useful indicator of the
damage extent. It was also proved in their work that the number of the vanishing moments of wavelet should be at least 2 for crack detection in beams.

Douka et al. $[17,18]$ applied 1D symmetrical 4 wavelet transform on mode shape for crack identification in beam and plate structures. The position of the crack is determined by the sudden change in the wavelet coefficients. An intensity factor is also defined to estimate the depth of the crack from the coefficients of the wavelet transform.

Zhong and Oyadiji [72] proposed a crack detection algorithm in symmetric beam-like structures based on stationary wavelet transform (SWT) of mode shape data. Two sets of mode shape data, which constitute two new signal series, are, respectively, obtained from the left half and reconstructed right half of modal displacement data of a damaged simply supported beam. The difference of the detail coefficients of the two new signal series was used for damage detection. The method was verified using modal shape data generated by a finite element analysis of 36 damage cases of a simply supported beam with an artificial random noise of $5 \%$ SNR. The effects of crack size, depth and location as well as the effects of sampling interval were examined. The results show that all the cases can provide evidence of crack existence at the correct location of the beam and that the proposed method can be recommended for identification of small cracks as small as $4 \%$ crack ratio in real applications with measurement noise present. However, there are two main disadvantages of this method. First, the use of this method based on SWT requires fairly accurate estimates of the mode shapes. Second, the method cannot tell the crack location from its mirror image location due to its inherent limitation. Therefore, in applying the method, both the crack location predicted and its mirror image location should be checked for the presence of a crack.

Chang and Chen [12] presented a spatial Gabor wavelet-based technique for damage detection of a multiple cracked beam. Given natural frequencies and crack positions, the depths of the cracks are then solved by an optimization process based on traditional characteristic equation. Analysis and comparison showed that it can detect the cracks positions and depths and also has high sensitivity to the crack depth, and the accuracy of this method is good. The limitation of this method is very common in wavelet transform methods, that is, there are peaks near the boundaries in the wavelet plot caused by discontinuity and the crack cannot be detected when the crack is near the boundaries.

Cao and Qiao [15] proposed a novel wavelet transform technique (so called 'integrated wavelet transform'), which takes synergistic advantage of the SWT and the CWT, to improve the robustness of irregularity analysis of mode shapes in damage detection. Two progressive wavelet analysis steps are considered, in which the SWT-based multiresolution analysis (MRA) is first employed to refine the retrieved mode shapes, followed by CWT-based multiscale analysis (MSA) to magnify the effect of slight irregularity. The SWT-MRA is utilized to separate the multi-component modal signal, eliminate random noise and regular interferences, and thus extract purer damage information; while the CWT-MSA is employed to smoothen, differentiate or suppress polynomials of mode shapes to magnify the effect of irregularity. The choice of the optimal mother wavelet in damage detection is also elaborately discussed. The proposed methodology is evaluated using the mode shape data from the numerical finite element analysis and experimental testing of a cantilever beam with a through-width crack. The methodology presented provides a robust and viable technique to identify minor damage in a relatively lower signal-to-noise ratio environment.

### 2.3 CURVATURE/STRAIN MODE SHAPE-BASED METHODS:- It has been

shown by many researchers that the displacement mode shape itself is not very sensitive to small damage, even with high density mode shape measurement. As an effort to enhance the sensitivity of mode shape data to the damage, the mode shape curvature (MSC) is investigated as a promising feature for damage identification.

Pandey et al. [53] suggested for the first time that the MSC, that is, the second derivatives of mode shape, are highly sensitive to damage and can be used to localize it. The curvature mode shapes are derived using a central difference approximation. Result showed that the difference of curvature mode shapes from intact and damaged structure can be a good indicator of damage location. It is also pointed out that for the higher modes, the difference in modal curvature shows several peaks not only at the damage location but also at other positions, which may lead to a false indication of damage. Hence, in order to reduce the possibility of a false alarm, only first few low curvature mode shapes can be used for damage identification.

Abdel Wahab and De Roeck [1] investigated the accuracy of using the central difference approximation to compute the MSC based on finite element analysis. The authors suggested that a fine mesh is required to derive the modal curvature correctly for the higher modes and that the first mode will provide the most reliable curvature in practical application due to the limited number of sensors needed. Then, a damage indicator called 'curvature damage factor', which is the average absolute difference in intact and damaged curvature mode shapes of all modes, is introduced. The technique is further applied to a real structure, namely bridge Z 24 , to show its effectiveness in multiple damage location.

Swamidas and Chen [68] performed a finite element-based modal analysis on a cantilever plate with a small crack. It was found that the surface crack in the structure will affect most of the
modal parameters, such as the natural frequencies of the structure, amplitudes of the response and mode shapes. Some of the most sensitive parameters are the difference of the strain mode shapes and the local strain frequency response functions. By monitoring the changes in the local strain frequency response functions and the difference between the strain mode shapes, the location and severity of the crack that occurs in the structure can be determined.

Li et al. [35] presented a crack damage detection using a combination of natural frequencies and strain mode shapes as input in artificial neural networks (ANN) for location and severity prediction of crack damage in beam-like structures. In the experiment, several steel beams with six distributed surface-bonded strain gauges and an accelerometer mounted at the tip were used to obtain modal parameters such as resonant frequencies and strain mode shapes.

Amaravadi et al [4] proposed an orthogonal wavelet transform technique that operates on curvature mode shape for enhancing the sensitivity and accuracy in damage location. First, the curvature mode shape is calculated by central difference approximation from the displacement mode shapes experimentally obtained from SLV. Then, a threshold wavelet map is constructed for the curvature mode shape to detect the damage. The experimental results are reasonably accurate.

Kim et al. [32] proposed a curvature mode shape-based damage identification method for beamlike structures using wavelet transform. Using a small damage assumption and the Haar wavelet transformation, a set of linear algebraic equations is given by damage mechanics. With the aid of singular value decomposition, the singularities in the damage mechanism were discarded. Finally, the desired DI was reconstructed using the pseudo-inverse solution. The performance of the proposed method was compared with two existing NDE methods for an axially loaded beam without any special knowledge about mass density and an applied axial force. The effect of
random noise on the performance was examined. The proposed method was verified by a finite element model of a clamped-pinned pre-stressed concrete beam and by field test data on the I-40 Bridge over the RioGrande

Shi et al. [65] presented a damage localization method for beam, truss or frame type structures based on the modal strain energy change. The MSEC at the element level is suggested as an indicator for damage localization.

### 2.4 OTHER METHODS BASED ON MODAL PARAMETERS:-

Ren and De Roeck [60] proposed a damage identification technique from the finite element model using frequencies and mode shape change. The element damage equations have been established through the eigenvalue equations that characterize the dynamic behavior. Several solution techniques are discussed and compared. The results show that the SVD-R method based on the singular value decomposition (SVD) is most effective. The method has been verified by a simple beam and a continuous beam numerical model with numbers of simulated damage scenarios. The method is further verified by a laboratory experiment of a reinforced concrete beam.

Wang and Qiao [69] developed a general order perturbation method involving multiple perturbation parameters for eigenvalue problems with changes in the stiffness parameters. The perturbation method is then used iteratively with an optimization method to identify the stiffness parameters of structures. The generalized inverse method is used efficiently with the first order perturbations, and the gradient and quasi-Newton methods are used with the higher order perturbations.

Rahai et al. [59] presents a finite element-based approach for damage detection in structures utilizing incomplete measured mode shapes and natural frequencies. Mode shapes of a structure
are characterized as a function of structural stiffness parameters. More equations were obtained using elemental damage equation which requires complete mode shapes. This drawback is resolved by presenting the mode shape equations and dividing the structural DOFs to measured and unmeasured parts. The nonlinear optimization problem is then solved by the sequential quadratic programming (SQP) algorithm. Monte Carlo simulation is applied to study the sensitivity of this method to noise in measured modal displacements.

Hao and Xia [26] applied a genetic algorithm with real number encoding to minimize the objective function, in which three criteria are considered: the frequency changes, the mode shape changes, and a combination of the two. A laboratory tested cantilever beam and a frame structure were used to verify the proposed technique. The algorithm did not require an accurate analytical model and gave better damage detection results for the beam than the conventional optimization method.

Ruotolo and Surace [63] utilized genetic algorithm to solve the optimization problem. The objective function is formulated by introducing terms related to global damage and the dynamic behavior of the structure, i.e., natural frequencies, mode shapes and modal curvature. The damage assessment technique has been applied to both the simulated and experimental data related to cantilevered steel beams, each one with a different damage scenario. It is demonstrated that this method can detect the presence of damage and estimate both the crack positions and sizes with satisfactory precision. The problems related to the tuning of the genetic search and to the virgin state calibration of the model are also discussed.

## CHAPTER-3

## TRANSFORM METHODS

## CHAPTER-3

## TRANSFORM METHODS

### 3.1 LAPLACE TRANSFORMS

Laplace transforms provide a method for representing and analyzing linear systems using algebraic methods. The application of Laplace Transform methods is particularly effective for linear ODEs with constant coefficients, and for systems of such ODEs. To transform an ODE, we need the appropriate initial values of the function involved and initial values of its derivatives. In systems that begin undeflected and at rest the Laplace's' can directly replace the $\mathrm{d} / \mathrm{dt}$ operator in differential equations [70]. It is a superset of the phasor representation in that it has both a complex part, for the steady state response, but also a real part, representing the transient part. As with the other representations the Laplace " $s$ " is related to the rate of change in the system.

$$
\begin{equation*}
F(s)=\int_{0}^{\infty} f(t) e^{-s t} d t=\lim _{A \rightarrow \infty} \int_{0}^{A} e^{-s t} f(t) d t \tag{3.1}
\end{equation*}
$$

where,
$F(s)=$ the function in terms of the Laplace s
$F(t)=$ the function in terms of time t
The normal convention is to show the function of time with a lower case letter, while the same function in the s-domain is shown in upper case. Another useful observation is that the transform starts at $t=0 s$.The Laplace method is particularly advantageous for input terms that are piecewise defined, periodic or impulsive.

## Properties of Laplace transform:

1. Linearity:

$$
\begin{equation*}
\mathcal{L}\left\{c_{1} f(t)+c_{2} g(t)\right\}=c_{1} \mathcal{L}\{f(t)\}+c_{2} \mathcal{L}\{g(t)\} . \tag{3.2}
\end{equation*}
$$

2. First derivative:

$$
\begin{equation*}
\mathcal{L}\left\{f^{\prime}(t)\right\}=s \mathcal{L}\{f(t)\}-f^{\prime}(0) \tag{3.3}
\end{equation*}
$$

3. Second derivative:

$$
\begin{equation*}
\mathcal{L}\left\{f^{\prime \prime}(t)\right\}=s^{2} \mathcal{L}\{f(t)\}-s f(0)-f^{\prime}(0) \tag{3.4}
\end{equation*}
$$

4. Higher order derivative:

$$
\begin{equation*}
\mathcal{L}\left\{f^{(n)}(t)\right\}=s^{n} \mathcal{L}\{f(t)\}-s^{n-1} f(0)-s^{n-2} f^{\prime}(0)-\cdots-s f^{n-2}(0)-f^{n-1}(0) . \tag{3.5}
\end{equation*}
$$

5. $\mathcal{L}\{-t f(t)\}=F^{\prime \prime}(s)$
where $F(s)=\mathcal{L}\{f(t)\}$. this also implies $\mathcal{L}\{t f(t)\}=-F^{\prime \prime}(s)$.
6. $\mathcal{L}\left\{e^{a t} f(t)\right\}=F(s-a)$
where $F(s)=\mathcal{L}\{f(t)\}$. this implies $e^{a t} f(t)=\mathcal{L}^{-1}\{F(s-a)\}$.
The original time signal $f(t)$ can be obtained through an inverse transform of $F(s)$ and is written as

$$
\mathcal{L}^{-1}\{F(s)\}=f((t), \quad \text { if } F(s)=\mathcal{L}\{f(t)\} .
$$

Let us consider $f(t)=1$ for $t>0$. the Laplace transform is

$$
\begin{equation*}
F(s)=\int_{0}^{\infty} f(t) e^{-s t} d t=\int_{0}^{\infty} 1 e^{-s t} d t=-\left.\frac{1}{s} e^{-s t}\right|_{0} ^{\infty}=\left[-\frac{1}{s} e^{-s \infty}\right]-\left[-\frac{1}{s} e^{-s 0}\right]=\frac{1}{s} \tag{3.8}
\end{equation*}
$$

This transform can be performed only analytically and there is no numerical implementation. This restricts the use of Laplace transform for analysis of problems with higher complexities, which need to be solved numerically.

### 3.2 FOURIER TRANSFORMS

The Fourier transform is a tool, this can adopt easily and spread throughout system and it is used in many fields of science as a mathematical or physical tool to alter a problem into one that can be more easily solved. The Fourier transform, in essence, decomposes or separates a waveform or function into sinusoids of different frequency which sum to the original waveform. It identifies or distinguishes the different frequency sinusoids and their respective amplitudes. The main advantage of Fourier transform in structural dynamics and wave propagation problems is that several important characteristics of system can be obtained directly from the transformed frequency domain method. Fourier transform can be implemented analytically, semi analytically and numerically in the form of Continuous Fourier Transform (CFT), Fourier Series (FS) and Discrete Fourier Transform (DST) respectively.

In Fourier analysis the complex exponential function $e^{i x}$ is often used. We have by the Euler formula

$$
\begin{equation*}
e^{i x}=\cos x+i \sin x \tag{3.9}
\end{equation*}
$$

Hence, $e^{i x}$ can be considered as complex sinusoid. In general a complex valued function has the form of

$$
\begin{equation*}
f(x)=a(x)+i b(x) \quad x \in \mathbb{R} \tag{3.10}
\end{equation*}
$$

here $a(x)$ is the real part of the $f(x)$ and $b(x)$ is the imaginary part of the $f(x)$, and both $a$ and bare real valued functions.

### 3.2.1 CONTINUOUS FOURIER TRANSFORMS

The both forward and inverse continuous Fourier transform of any time signal $f(t)$, generally shown as transform pair, which are

$$
\begin{equation*}
F(t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \hat{F}(\omega) e^{i \omega t} d \omega, \quad \hat{F}(\omega)=\int_{-\infty}^{\infty} F(t) e^{-i \omega t} d t \tag{3.11}
\end{equation*}
$$

where $\hat{F}(\omega)$ is the CFT of $F(t), \omega$ is the angular frequency, $i$ is the complex number $(\sqrt{-1})$. $\widehat{F}(\omega)$ should be complex and a plot of amplitude of this function against frequency will give the spectral density of the time signal $F(t)$. As an example, consider a rectangular pulse width'd'. this function can be written as

$$
F(t)=\left\{\begin{array}{cc}
F_{0} & -\frac{d}{2} \leq t \leq \frac{d}{2}  \tag{3.12}\\
0 & \text { otherwise }
\end{array}\right.
$$

Substitute the equation (3.12) in the equation (3.11) of $\hat{F}(\omega)$, we get

$$
\begin{equation*}
\hat{F}(\omega)=F_{0} d\left\{\frac{\sin \frac{\omega d}{2}}{\frac{\omega d}{2}}\right\} \tag{3.13}
\end{equation*}
$$

Now, allow the pulse to propagate in the time domain by an amount of $t_{0}$ seconds, and this can be written in mathematical form

$$
F(t)=\left\{\begin{array}{cc}
F_{0} & -t_{0} \leq t \leq t_{0}+d  \tag{3.14}\\
0 & \text { otherwise }
\end{array}\right.
$$

Substitute the above equation (3.14) in equation (3.13), we get

$$
\begin{equation*}
\hat{F}(\omega)=F_{0} d\left\{\frac{\sin \frac{\omega d}{2}}{\frac{\omega d}{2}}\right\} e^{-i \omega\left(t_{0}+\frac{d}{2}\right)} \tag{3.15}
\end{equation*}
$$

The magnitude of the equations (3.13) and (3.15) are same. The second transform exists phase information. The change of phase in frequency domain refers the propagation of signal in the time domain. By using CFT, spread of the signal can be calculated in the time and frequency domain.

## Properties of CFT

(a) Linearity: Consider, two functions $F(t)$ and $G(t)$ for the incident and reflected waves respectively in time and $a$ and $b$ are the constants. The Fourier Transform of a function $a F(t)+b G(t)$ can be obtained as $a \hat{F}(\omega)+b \hat{G}(\omega)$. This expression can also be written as $a F(t)+b G(t) \Leftrightarrow a \widehat{F}(\omega)+b \widehat{G}(\omega)$. The symbol $\Leftrightarrow$ represents the Fourier Transform.
(b) Scaling: If multiply with some non zero constant $k$ to the time signal $F(t)$ will become $F(k t)$. The Continuous Fourier Transform of the signal $F(k t)$ can be written as $F(k t) \Leftrightarrow \frac{1}{k} \widehat{F}\left(\frac{\omega}{k}\right)$. This represents that compression occurs in time domain results and dilation occurs in frequency domain results. The amplitude however decreases to keep the energy constant.
(c) Time shifting: A shift in the time signal by $t_{s}$ is manifested as a phase change in the transformed frequency domain obtained through Continuous Fourier Transform. After shifting the time the transform pair can be written as $F\left(t-t_{s}\right) \Leftrightarrow \widehat{F}(\omega) e^{-i \omega t_{s}}$.
(d) Symmetric property of the CFT: The CFT of the time signal $F(t)$ is in complex form, split this complex form into real and imaginary parts by using equation (3.9). The real and imaginary part of the CFT can be written as

$$
\begin{gather*}
\hat{F}_{R}(\omega)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} F(t) \cos (\omega t) d t  \tag{3.16}\\
\widehat{F}_{I}(\omega)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} F(t) \sin (\omega t) d t \tag{3.17}
\end{gather*}
$$

The first integral is an even function and second is an odd function.
That is $\widehat{F}_{R}(\omega)=\widehat{F}_{R}(-\omega)$ and $\widehat{F}_{I}(\omega)=\widehat{F}_{I}(\omega)$.

Consider a point on CFT say origin $(\omega=0)$, the transform on the right side of the origin can be written as $\hat{F}(\omega)=\hat{F}_{R}(\omega)+i \widehat{F}_{I}(\omega)$. Similarly on the left side can be written as $\widehat{F}(-\omega)=\widehat{F}_{R}(-\omega)+i \widehat{F}_{I}(-\omega)=\widehat{F}_{R}(-\omega)-i \widehat{F}_{I}(-\omega)$. This origin point is called Nyquist frequency and it is very important to determine the half of the total frequency rang.
(e) Convolution: This property of the CFT is very useful for understanding the signal processing aspects and it has great importance in wave propagation analysis. This property can be obtained by multiplying two time signals $F_{1}(t)$ and $F_{2}(t)$ each other.

$$
\begin{equation*}
\hat{F}_{12}(\omega)=\int_{-\infty}^{\infty} F_{1}(t) F_{2}(t) e^{-i \omega t} d t \tag{3.18}
\end{equation*}
$$

Substitute the equation (3.11) in equation (3.18) for both these functions written as

$$
\begin{align*}
& \hat{F}_{12}(\omega)=\int_{-\infty}^{\infty} \hat{F}_{1}(\bar{\omega}) \int_{-\infty}^{\infty} F_{2}(t) e^{-i(\omega-\bar{\omega}) t} d t d \bar{\omega} \\
& \hat{F}_{12}(\omega)=\int_{-\infty}^{\infty} \hat{F}_{1}(\bar{\omega}) \hat{F}_{2}(\omega-\bar{\omega}) d \bar{\omega} \tag{3.19}
\end{align*}
$$

The above equation (3.19) can also be written as

$$
\begin{equation*}
F_{1}(t) F_{2}(t) \Leftrightarrow \int_{-\infty}^{\infty} \hat{F}_{1}(\bar{\omega}) \hat{F}_{2}(\omega-\bar{\omega}) d \bar{\omega} \tag{3.20}
\end{equation*}
$$

Conversely we can also write as

$$
\begin{equation*}
\hat{F}_{1}(\omega) \hat{F}_{2}(\omega) \Leftrightarrow \int_{-\infty}^{\infty} F_{1}(\tau) F_{2}(t-\tau) d \tau \tag{3.21}
\end{equation*}
$$

### 3.2.2 FOURIER SERIES

Fourier series is in between the Continuous Fourier Transform (CFT) and Discrete Fourier Transform (DFT). The inverse transform is represented by a series while forward
transform is still in the integral form as in the CFT. i.e. out of both equations, one still needs the mathematical description of the time signal to get the transforms.

The Fourier Series (FS) can be written as

$$
\begin{equation*}
F(t)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left[a_{n} \cos \left(2 \pi n \frac{t}{T}\right)+b_{n} \sin \left(2 \pi n \frac{t}{T}\right)\right] \tag{3.22}
\end{equation*}
$$

where $n=0,1,2,3, \cdots$

$$
\begin{equation*}
a_{n}=\frac{2}{T} \int_{0}^{T} F(t) \cos \left(\frac{2 \pi n t}{T}\right) d t, \quad b_{n}=\frac{2}{T} \int_{0}^{T} F(t) \sin \left(\frac{2 \pi n t}{T}\right) d t \tag{3.23}
\end{equation*}
$$

The equation (3.22) and equation (3.23) are corresponding to the inverse transform and forward transform of the CFT respectively. Where $T$ is the discrete representation of time signal, and it introduces the periodicity of time signal. The equations (3.22) and (3.23) can be writing in terms of complex exponentials as

$$
\begin{gather*}
F(t)=\frac{1}{2} \sum_{-\infty}^{\infty}\left(a_{n}-b_{n}\right) e^{i \omega_{n} t}=\sum_{-\infty}^{\infty} \hat{F}_{n} e^{i \omega_{n} t} \quad n=0, \pm 1, \pm 2, \cdots  \tag{3.24}\\
\hat{F}_{n}=\frac{1}{2}\left(a_{n}-b_{n}\right)=\frac{1}{T} \int_{0}^{T} F(t) e^{-i \omega_{n} t} d t \tag{3.25}
\end{gather*}
$$

where $\omega_{n}=\frac{2 \pi n}{T}$, and the signal repeats after $T$ seconds due to enforced periodicity.
Now this time signal, can also be write in terms of fundamental frequency as

$$
\begin{equation*}
F(t)=\sum_{-\infty}^{\infty} \hat{F}_{n} e^{i 2 \pi n f_{0} t}=\sum_{-\infty}^{\infty} \hat{F}_{n} e^{i n \omega_{0} t} \tag{3.26}
\end{equation*}
$$

where the fundamental frequency $f_{0}=\frac{\omega_{0}}{2 \pi}=\frac{1}{T}$ in radians per second or Hz .
Consider the rectangular time signal and substitute in equation (3.25), we get

$$
\begin{equation*}
\widehat{F}_{n}=\frac{F_{0}}{T}\left\{\frac{\sin \frac{n \pi d}{T}}{\frac{n \pi d}{T}}\right\} e^{-i\left(\frac{2 \pi n}{T}\right)\left(t_{0}+\frac{d}{2}\right)} \tag{3.27}
\end{equation*}
$$

### 3.2.3 DISCRETE FOURIER TRANSFORM

The transform pair of CFT requires mathematical description of the time signal and their integration. Data of the time signals are obtained from experiments. Hence, what we require is the numerical representation for the transform pair, which is called the Discrete Fourier Transform (DFT). It is an alternative way of mathematical representation of CFT in terms of summations. Here the ultimate aim is to replace the integral involved in computation of the Fourier coefficient by summation for numerical implementation. Divide the time signal into $M$ number of constant triangles. The height of the triangle is $F_{m}$ and width of the triangle is $\Delta t=\frac{T}{M}$. From equation (3.15) we can observe that the continuous transform of the rectangle is a sinc function. Similarly in discrete system, idealizing the signal as rectangle. Therefore, the DFT of the signal will be the summation of $M$ sinc functions. Hence the second integral of equation (3.25) can be written as

$$
\begin{equation*}
\hat{F}_{n}=\Delta T\left[\frac{\sin \left(\frac{\omega_{n} \Delta T}{2}\right)}{\left(\frac{\omega_{n} \Delta T}{2}\right)}\right] \sum_{m=0}^{M} F_{m} e^{-i \omega_{n} t_{m}} \tag{3.28}
\end{equation*}
$$

If $\Delta T$ is very small, the value of above equation (3.28) nearly equal to unity. Hence the forward and inverse Discrete Fourier Transform (DFT) can be written as

$$
\begin{gather*}
\hat{F}_{n}=\hat{F}\left(\omega_{n}\right)=\Delta T \sum_{m=0}^{N-1} F_{m} e^{-i \omega_{n} t_{m}}=\Delta T \sum_{m=0}^{N-1} F_{m} e^{-i 2 \pi n m / N}  \tag{3.29}\\
F_{m}=F\left(t_{m}\right)=\frac{1}{T} \sum_{n=0}^{N-1} \hat{F}_{n} e^{i \omega_{n} t_{m}}=\frac{1}{T} \sum_{n=0}^{N-1} \hat{F}_{n} e^{i 2 \pi n m / N} \tag{3.30}
\end{gather*}
$$

Here, $n$ and $m$ varies from 0 to $N-1$.
The DFT of a real function is symmetric about nyquist frequency as like in the case of CFT. One side of DFT coefficients about nyquist frequency is the complex conjugate of the coefficients on other side of it. Thus $N$ real points are transformed to $\frac{N}{2}$ complex points. Nyquist frequency can be calculated by using the following expression

$$
\begin{equation*}
f_{n y q}=\frac{1}{2 \Delta T} \tag{3.31}
\end{equation*}
$$

### 3.3 WAVELET TRANSFORMS

The word wavelet has been derived from the French word. The equivalent French word ondelette meaning "small wave" was used by Morlet and Grossmann [23, 48, 49] in the early 1980s. A wavelet is a wave-like oscillation with an amplitude that starts out at zero, increases, and then decreases back to zero. It can typically be visualized as a "brief oscillation" like one might see recorded by a seismograph or heart monitor. Generally, wavelets are purposefully crafted to have specific properties that make them useful for signal processing. Wavelets can be combined, using a "shift, multiply and sum" technique called convolution, with portions of an unknown signal to extract information from the unknown signal. Wavelets are defined by the wavelet function $\psi(t)$ (i.e. the mother wavelet) and scaling function $\varphi(t)$ (also called father wavelet) in the time domain. The wavelet function is in effect a band-pass filter and scaling it for each level halves its bandwidth. This creates the problem that in order to cover the entire spectrum, an infinite number of levels would be required. The scaling function filters the lowest level of the transform and ensures all the spectrum is covered.

Since from the last few decades many researchers showing their interest for the development and application wavelets in many fields. Some of the researchers include Morlet and Grossmann [23]
developed formulation for the Continuous Wavelet Transform, Meyer [41] and Mallt [38] developed multi resolution analysis using wavelets, Daubechies [16] for proposal of orthogonal compactly supported wavelets and Stromberg [67] for early works on discrete wavelet transform. As a mathematical tool, wavelets can be used to extract information from many different kinds of data of signals and images. Sets of wavelets are generally needed to analyze data fully. A set of "complementary" wavelets will deconstruct data without gaps or overlap so that the deconstruction process is mathematically reversible. Thus, sets of complementary wavelets are useful in wavelet based compression/decompression algorithms where it is desirable to recover the original information with minimal loss.

### 3.3.1 TYPES OF WAVELETS

We have several wavelet functions such as Morlet wavelets, Mexican hat wavelets, Mayer wavelets, Shanon wavelets, Daubechies wavelets e.t.c., out of these wavelet functions Morlet wavelets and Mexican hat wavelets are explained here, Daubechies wavelet function explained in chapter 4.

### 3.3.1.1 MORLET'S WAVELET

The wavelet defined by Morlet is:

$$
\begin{equation*}
\widehat{g}(\omega)=e^{-2 \pi^{2\left(v-v_{0}\right)^{2}}} \tag{3.32}
\end{equation*}
$$

It is a complex wavelet which can be decomposed in two parts, one for the real part, and the other for the imaginary part.

$$
\begin{align*}
& g_{r}(x)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{x^{2}}{2}} \cos \left(2 \pi v_{0} x\right)  \tag{3.33}\\
& g_{i}(x)=\frac{1}{\sqrt{2 \pi}} e^{-\frac{x^{2}}{2}} \sin \left(2 \pi v_{0} x\right) \tag{3.34}
\end{align*}
$$

where $v_{o}$ is a constant. The admissibility condition is verified only if $v_{o}>0.8$. Figure 1 shows the real part of the Morlet wavelet function and figure 2 shows the imaginary part of the Morlet wavelet function.


Figure 3.1: (a) Morlet's wavelet: real part


Figure 3.1: (b) Morlet's wavelet Imaginary part

### 3.3.1.2 MEXICAN HAT WAVELET

The Mexican hat defined by Murenzi is:

$$
\begin{equation*}
g(x)=\left(1-x^{2}\right) e^{-\frac{1}{2} x^{2}} \tag{3.35}
\end{equation*}
$$

It is the second derivative of a Gaussian (see figure 2).


Figure 3.2: Mexican Hat wavelet.

### 3.3.1.3 THE CONTINUOUS WAVELET TRANSFORM

The Morlet-Grossmann definition of the continuous wavelet transform for a $1 D$ signal $f(x) \in$ $L^{2}(\mathbb{R})$ is:

$$
\begin{equation*}
W(a, b)=\frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x) \psi^{*}\left(\frac{x-b}{a}\right) d x \tag{3.36}
\end{equation*}
$$

where $\psi^{*}(x)$ is the mother wavelet, $a(>0)$ is the scaling parameter, it represents the frequency content of the wavelet and $b$ is the translation parameter, it represents the location of wavelet in time. The basis function of the wavelet transform retains the time locality and frequency locality. The transform is characterized by the following three properties:

1. It is a linear transformation,
2. It is covariant under translations:

$$
\begin{equation*}
f(x) \rightarrow f(x-u) \quad W(a, b) \rightarrow W(a, b-u) \tag{3.37}
\end{equation*}
$$

3. It is covariant under dilations:

$$
\begin{equation*}
f(x) \rightarrow f(s x) \quad W(a, b) \rightarrow s^{-\frac{1}{2}} W(s a, s b) \tag{3.38}
\end{equation*}
$$

The last property makes the wavelet transform very suitable for analyzing hierarchical structures. It is like a mathematical microscope with properties that do not depend on the magnification.

### 3.3.1.4 DISCRETE WAVELET TRANSFORM

The discrete wavelet transform (DWT) can be derived from this theorem if we process a signal which has a cut-off frequency. A digital analysis is provided by the discretisation of equation (3.34) with some simple considerations on the modification of the wavelet pattern by dilation. Usually the wavelet function $\psi^{*}(x)$ has no cut-off frequency and it is necessary to suppress the values outside the frequency band in order to avoid aliasing effects. We can work in Fourier space, computing the transform scale by scale. The number of elements for a scale can be reduced, if the frequency bandwidth is also reduced. This is possible only for a wavelet which also has a cut-off frequency. The decomposition proposed by Littlewood and Paley provides a very nice illustration of the reduction of elements scale by scale. This decomposition is based on an iterative dichotomy of the frequency band. The associated wavelet is well localized in Fourier space where it allows a reasonable analysis to be made although not in the original space. The
search for a discrete transform which is well localized in both spaces leads to multiresolution analysis.

### 3.3.2 MULTI-RESOLUTION ANALYSIS

In any discretised wavelet transform, there are only a finite number of wavelet coefficients for each bounded rectangular region in the upper half plane. Still, each coefficient requires the evaluation of an integral. To avoid this numerical complexity, one needs one auxiliary function, the father wavelet $\boldsymbol{\varphi} \boldsymbol{\epsilon} \boldsymbol{L}^{2}(\mathbb{R})$. The sequence of nested subspaces $V_{j}$ plays an important role in the construction of wavelet functions. These nested subspaces represent the multi resolution of wavelet function in $L^{2} \mathbb{R}$. The closed subspaces $V_{j}$ for $j \in \mathbb{Z}$ with the following properties,

1. $\{0\} \cdots \subset V_{-2} \subset V_{-1} \subset V_{0} \subset V_{1} \subset V_{2} \subset \cdots L^{2} \mathbb{R}$
2. $\quad \cup_{j \in \mathbb{Z}} V_{j}=L^{2} \mathbb{R}$.
3. $\bigcap_{j \in \mathbb{Z}} V_{j}=\{0\}$
4. $\quad F(t) \in V_{j}$ if and only if $F(2 t) \in V_{j+1}$

These properties of subspaces are related with scaling relation. Thus, the embedded subspaces $V_{j}$ essentially reduces the problem of obtaining $V_{0}$.
5. Each subspace is spanned by integers translates of a single function,

$$
F(t) \in V_{j} \Leftrightarrow F(t+1) \in V_{j} \quad \forall_{j} \in \mathbb{Z}
$$

Based on the above properties multi resolution analysis, we can conclude that, first we need to find the scaling function $\varphi(t) \in V_{0}$. Such that its integer translates $\{\varphi(t-k), k \in \mathbb{Z}\}$ are the Riesz bases for the space $V_{0}$. Similarly, $\varphi(2 t-k)$ will form a basis for the space $V_{1}$. Thus,

$$
\begin{align*}
V_{0} & =\operatorname{span} \overline{\{\varphi(t-k), k \in \mathbb{Z}\}}  \tag{3.39}\\
V_{1} & =\operatorname{span} \overline{\{\varphi(2 t-k), k \in \mathbb{Z}\}} \tag{3.40}
\end{align*}
$$

From equation (3.37) and (3.38), we can conclude that $V_{0} \in V_{1}$. Therefore the basis function of space $V_{0}$ can be write in terms of basis function $V_{1}$ as

$$
\begin{equation*}
\varphi(t)=\sum_{-\infty}^{\infty} a_{k} \varphi(2 t-k) \tag{3.41}
\end{equation*}
$$

where $a_{k}, k \in \mathbb{Z}$ are filter coefficients. The equation (3.39) gives the scaling function $\varphi(t)$ for the space $V_{0}$. The basis function $V_{j}$ can be defined as

$$
\begin{equation*}
\varphi_{j, k}(t)=2^{\frac{j}{2}} \varphi\left(2^{j} t-k\right) \tag{3.42}
\end{equation*}
$$

where $j$ and $k$ are the dilation and translation indices respectively. In the above equation $j$ represents the frequency content and $k$ represents the time in the analysis of time signal. Let us consider the approximation of the function $F(t)$ into the subspace $V_{j}$ by the scaling function $\varphi_{j, k}(t)$ as $P_{j} F$,

$$
\begin{equation*}
P_{j} F=\sum_{-\infty}^{\infty} c_{j, k} \varphi_{j, k}(t) \tag{3.43}
\end{equation*}
$$

In the above equation (3.41), $c_{j, k}$ are the approximation coefficients and $j \rightarrow \infty, P_{j} F \rightarrow F$.
Similarly the wavelet function $\psi(t)$ and its translates $\psi(t-k), k \in \mathbb{Z}$ are the Riesz basis for the subspace $w_{0}$, can be expressed in terms of basis functions for $V_{1}$ as

$$
\begin{equation*}
\psi(t)=\sum_{-\infty}^{\infty} b_{k} \psi(2 t-k) \tag{3.44}
\end{equation*}
$$

Where $b_{k}=(-1)^{k} a_{1-k}$ and form the Riesz basis for subspace $w_{j}$ can be written as

$$
\begin{equation*}
\psi_{j, k}(t)=2^{\frac{j}{2}} \psi\left(2^{j} t-k\right) \tag{3.45}
\end{equation*}
$$

Let us consider the approximation of the function $F(t)$ into the subspace $W_{j}$ by the wavelet function $\psi_{j, k}(t)$ as $Q_{j} F$,

$$
\begin{equation*}
Q_{j} F=\sum_{-\infty}^{\infty} d_{j, k} \psi_{j, k}(t) \tag{3.46}
\end{equation*}
$$

Here, $d_{j, k}$ are called as detailed coefficients.
The next step is to obtain the closure subspace $w_{j,} j \in \mathbb{Z}$ for the subspace $V_{j}$ and its orthogonal compliment such that

$$
\begin{equation*}
V_{j+1}=V_{j} \oplus W_{j} \text { and } V_{j} \perp W_{j} \tag{3.47}
\end{equation*}
$$

where the symbol $\oplus$ represents the direct sum. The above equation (3.45) shows that the subspaces $V_{j}$ and $W_{j}$ are orthogonal to each other and we can write expression for $P_{j+1} F$.

$$
\begin{equation*}
P_{j+1}=P_{j} F+Q_{j} F \tag{3.48}
\end{equation*}
$$

Using the above procedure, calculate the approximations of the $F(t)$ at higher scale from the approximations of lower scale.

## CHAPTER-4

## WAVELET INTEGRALS

## CHAPTER-4

## WAVELET INTEGRALS

### 4.1 EVALUATION OF WAVELET INTEGRALS

Many wavelet algorithms require the evaluation of integrals which involve combination of wavelets, scaling functions and their derivatives. We collectively refer to such integrals as wavelet integrals. There are a few instances in which wavelet integrals can be evaluated analytically. The accuracy and efficiency of the technique used to compute the integrals will have a significant impact on the wavelet performance of the wavelet algorithm. For instance wavelet-based algorithms for solving differential equations typically lead to integrals involving wavelets and their derivatives. Wavelet integrals[8] which are commonly encountered in construction of Daubechies compactly supported wavelet algorithms:
(a) Filter coefficients,
(b) Wavelet and Scaling function coefficients,
(c) Moments of Wavelet and Scaling functions and
(d) Connection coefficients.

### 4.2 CONSTRUCTION OF DAUBECHIES COMPACTLY SUPPORTED WAVELETS

For a wavelet with compact support, $\varphi(t)$ can be considered finite in length and is equivalent to the scaling filter $g$. An orthogonal wavelet is entirely defined by the scaling filter - a low-pass
finite impulse responce (FIR) filter of length $2 N$ and sum 1. In biorthogonal wavelets, separate decomposition and reconstruction filters are defined. For analysis with orthogonal wavelets the high pass filter is calculated as the quadrature mirror filter of the low pass, and reconstruction filters are the time reverse of the decomposition filters. Daubechies and Symlet wavelets can be defined by the filter coefficients.

Conditions to calculate the filter coefficients:

1. For uniqueness, normalization is done by considering the area under the scaling function to be unity,

$$
\begin{equation*}
\int_{-\infty}^{\infty} \varphi(t) d t=1 \tag{4.1}
\end{equation*}
$$

The above equation leads to the following condition on the filter coefficients,

$$
\begin{equation*}
\sum_{-\infty}^{\infty} a_{k}=2 \tag{4.2}
\end{equation*}
$$

2. For Daubechies wavelets, the integer translates of scaling functions are orthogonal, i.e.,

$$
\begin{equation*}
\int_{-\infty}^{\infty} \varphi(t) \varphi(t+1) d t=\delta_{0, l} \quad l \in Z \tag{4.3}
\end{equation*}
$$

where $\delta_{0, l}=\left\{\begin{array}{lr}1, & l=0 \\ 0, & \text { otherwise }\end{array}\right.$

This gives the condition on the filer coefficients as

$$
\begin{equation*}
\sum_{k=-\infty}^{\infty} a_{k} a_{k+2 l}=2 \delta_{0, l,} \quad l \in Z \tag{4.4}
\end{equation*}
$$

The conditions given by equations (4.2) and (4.4) are not sufficient to get unique set of filter coefficients. For an $N$ coefficient system, $\frac{N}{2}+1$ equations can get from equations (4.2) and (4.4), remaining $\frac{N}{2}-1$ equations can get by imposing some conditions on the wavelet functions. For Daubechies wavelets, assumed that the scaling functions represents exactly the polynomials order $M$. Where $M=\frac{N}{2}$. Therefore the polynomial order $M$ as

$$
\begin{equation*}
f(t)=a_{0}+a_{1} t+a_{2} t^{2}+\cdots+a_{M-1} t^{M-1} \tag{4.5}
\end{equation*}
$$

The above polynomial is in expanded form, similar to the equation (3.42) for $j=0$ and can be written as

$$
\begin{equation*}
f(t)=\sum_{k=-\infty}^{\infty} c_{k} \varphi(t-k) \tag{4.6}
\end{equation*}
$$

Since $\psi(t)$ are orthogonal to the translates of $\varphi(t)$, taking inner product of equation (4.6) with $\psi(t)$ gives

$$
\begin{equation*}
\langle f(t), \psi(t)\rangle=\sum_{k=-\infty}^{\infty} c_{k}\langle\varphi(t-k), \psi(t)\rangle \equiv 0 \tag{4.7}
\end{equation*}
$$

Substitute the equation (4.5) in equation (4.7), we get

$$
\begin{equation*}
a_{0} \int_{-\infty}^{\infty} \psi(t) d t+a_{0} \int_{-\infty}^{\infty} \psi(t) t d t+\cdots+a_{M-1} \int_{-\infty}^{\infty} \psi(t) t^{M} d t \equiv 0 \tag{4.8}
\end{equation*}
$$

The above expression is valid for all values of $a_{j}$. Where $j=0,1,2,3, \cdots, M-1$. Consider $a_{j}=1$ and all other $a_{j}=0$ gives

$$
\begin{equation*}
\int_{-\infty}^{\infty} \psi(t) t^{l} d t \quad l=0,1,2, \ldots \ldots \ldots, M-1 \tag{4.9}
\end{equation*}
$$

The above equation can be written in term of filter coefficients, we get

$$
\begin{equation*}
\sum_{k=-\infty}^{\infty}(-1)^{k} a_{k} k^{l}=0 \quad l=0,1,2, \ldots \ldots \ldots, M-1 \tag{4.10}
\end{equation*}
$$

where M is the moment of wavelet function. The scaling functions $\varphi(t)$ are obtained by solving recursively the dilation equation (4.2), (4.4) and (4.5). Which can be expanded for DN as,

$$
\begin{equation*}
\varphi(t)=a_{0} \varphi(2 t)+a_{1} \varphi(2 t-1)+\cdots+a_{N-1} \varphi(2 t-N+1) \tag{4.11}
\end{equation*}
$$

Above equation can be written as the following equations,

$$
\begin{aligned}
& \varphi(0)=\mathrm{a}_{0} \varphi(0) \\
& \varphi(1)=\mathrm{a}_{0} \varphi(2)+\mathrm{a}_{1} \varphi(1)+\mathrm{a}_{2} \varphi(0) \\
& \varphi(2)=\mathrm{a}_{0} \varphi(4)+\mathrm{a}_{1} \varphi(3)+\mathrm{a}_{2} \varphi(2)+\mathrm{a}_{3} \varphi(1)+\mathrm{a}_{4} \varphi(0) \\
& \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
& \varphi(\mathrm{N}-2)=\mathrm{a}_{\mathrm{N}-3} \varphi(\mathrm{~N}-1)+\mathrm{a}_{\mathrm{N}-2} \varphi(\mathrm{~N}-2)+\mathrm{a}_{\mathrm{N}-1} \varphi(\mathrm{~N}-3) \\
& \varphi(\mathrm{N}-1)=\mathrm{a}_{\mathrm{N}-1} \varphi(\mathrm{~N}-1)
\end{aligned}
$$

This can also be written in the matrix form,

$$
\left[\begin{array}{ccccccc}
a_{0} & 0 & 0 & \cdots & 0 & 0 & 0  \tag{4.12}\\
a_{2} & a_{1} & a_{0} & \cdots & 0 & 0 & 0 \\
a_{4} & a_{3} & a_{2} & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & & \cdots & a_{N-1} & a_{N-2} \\
0 & 0 & 0 & & a_{N-3}
\end{array}\right]\left[\begin{array}{c}
\varphi(0) \\
\varphi(1) \\
\varphi(2) \\
\cdots(N-2) \\
\varphi(N-1)
\end{array}\right]=\left[\begin{array}{c}
\varphi(0) \\
\varphi(1) \\
\varphi(2) \\
\cdots \\
\varphi(N-2) \\
\varphi(N-1)
\end{array}\right]
$$

The above equation possesses eigenvalue problem and can be solved to obtain $\varphi$ as the eigenvector. The matrix A is known as the filter coefficients $a_{k}$ and can be solved from equations. Filter coefficients are given table for $\mathrm{N}=22$. Where N is the order of Daubechies wavelet. The function "dbwavf" in the MATLAB wavelet toolbox gives the filter coefficients. Table 1 shows the values of filter coefficients for $\mathrm{N}=22$.

### 4.3 SCALING FUNCTION AND WAVELET FUNCTION

Scaling functions $\varphi(t)$ and wavelet functions $\psi(t)$ can be calculate by using the following equations,

$$
\begin{align*}
& \varphi(t)=\sum_{-\infty}^{\infty} a_{k} \varphi(2 t-k)  \tag{4.13}\\
& \psi(t)=\sum_{-\infty}^{\infty} b_{k} \varphi(2 t-k) \tag{4.14}
\end{align*}
$$

where $b_{k}=(-1)^{k} a_{1-k}$

The following figures are shows the scaling functions and wavelet functions for the $D 4, D 6, D 12$ and $D 22$. By observing these figures we can say that smoothness increases by increasing order of Debauchies.

Table 4.1: Filter Coefficients for $\mathrm{N}=22$

| Filter Coefficients for N=22 |  |  |  |
| :---: | :---: | :---: | :---: |
| $a_{0}$ | 0.0264377294333137 | $a_{11}$ | 0.0443145095659574 |
| $a_{1}$ | 0.203741535201907 | $a_{12}$ | 0.0294734895982882 |
| $a_{2}$ | 0.636254348460787 | $a_{13}$ | -0.0217291381089843 |
| $a_{3}$ | 0.969707536626371 | $a_{14}$ | -0.00472468792819149 |
| $a_{4}$ | 0.582605597780612 | $a_{15}$ | 0.00696983509023781 |
| $a_{5}$ | -0.229491852355296 | $a_{16}$ | -0.000436416206187889 |
| $a_{6}$ | -0.387820982791004 | $a_{17}$ | -0.00126292559260676 |
| $a_{7}$ | 0.0933997381355307 | $a_{18}$ | 0.000352354877907889 |
| $a_{8}$ | 0.211866179836356 | $a_{19}$ | 0.0000769884777628895 |
| $a_{9}$ | -0.0657325829045156 | $a_{20}$ | -0.0000489812643698126 |
| $a_{10}$ | -0.0939586317975106 | $a_{21}$ | $6.35586363589251 \mathrm{E}-06$ |



Figure 4.1(a): Scaling function for $\mathrm{N}=2$.


Figure 4.1(b): Wavelet function for $\mathrm{N}=2$.


Figure 4.2(a): Scaling function for $\mathrm{N}=4$.


Figure 4.2(b): Wavelet function for $\mathrm{N}=4$.


Figure 4.3(a): Scaling function for $\mathrm{N}=12$.


Figure 4.3(b): Wavelet function for $\mathrm{N}=12$.


Figure 4.4(a): Scaling Function for $\mathrm{N}=22$


Figure 4.4(b): Wavelet Function for $\mathrm{N}=22$

### 4.4 CONNECTION COEFFICIENTS

Connection coefficients ( $\Omega_{n}^{r}$ ) are integrals involving combination of wavelets and scaling functions, their translates and their derivatives. They are frequently encountered during the discretization of ordinary and partial differential equations. The classes of wavelets for which connection coefficients are usually desired are those wavelets which are orthogonal, biorthogonal and compact support. The evaluation of connection coefficients for orthogonal wavelets:

$$
\begin{align*}
& \Omega_{j-k}^{l}=\int \varphi^{\prime}(\tau-k) \varphi(\tau-j) d \tau  \tag{4.15}\\
& \Omega_{j-k}^{2}=\int \varphi^{\prime \prime}(\tau-k) \varphi(\tau-j) d \tau \tag{4.16}
\end{align*}
$$

### 4.5 MOMENT OF SCALING FUNCTIONS

Let $\mu_{k}$ be the $k^{\text {th }}$ moment of scaling function $\varphi(x)$

$$
\begin{equation*}
\mu_{k}=\int \varphi(x) x^{k} d x \quad k=0,1,2, \ldots \frac{N}{2} \tag{4.17}
\end{equation*}
$$

The moment of scaling functions are easily calculated by using the following three recursive equations given in Amaratunga and Williams (1997) [3],

Always the zero-th moment, $M_{0}$ of $\varphi(x)$ is 1by the normalization of $\varphi(x)$,

$$
\begin{gather*}
\mu_{0}^{0}=\int \varphi(x) d x=1  \tag{4.18}\\
\mu_{0}^{l}=\frac{1}{2^{k+1}-2} \sum_{i=0}^{N-1} \sum_{l=1}^{k}\binom{k}{l} p_{i} i^{l} \mu_{k-l} \tag{4.19}
\end{gather*}
$$

$$
\begin{equation*}
\mu_{k}^{l}=\sum_{i=0}^{l}\binom{l}{i} k^{l-i} \mu_{0}^{i} \tag{4.20}
\end{equation*}
$$

Table4. 2: First order $\left(\Omega^{1}\right)$ and Second order $\left(\Omega^{2}\right)$ connection coefficients for $N=22$.

| Connection Coefficients for N=22 |  |  |  |
| :--- | :--- | :--- | :--- |
| $\Omega_{1}^{l}$ | 0 | $\Omega_{1}^{2}$ | -3.47337840801057 |
| $\Omega_{2}^{l}$ | -0.913209538941252 | $\Omega_{2}^{2}$ | 2.16026933978281 |
| $\Omega_{3}^{l}$ | 0.347183551085084 | $\Omega_{3}^{2}$ | -0.602653205128255 |
| $\Omega_{4}^{l}$ | -0.145808621125452 | $\Omega_{4}^{2}$ | 0.259980735100996 |
| $\Omega_{5}^{l}$ | 0.0565725329081428 | $\Omega_{5}^{2}$ | -0.113990617900921 |
| $\Omega_{6}^{l}$ | -0.0189618931390823 | $\Omega_{6}^{2}$ | 0.0444431406647273 |
| $\Omega_{7}^{l}$ | 0.00525147114923078 | $\Omega_{7}^{2}$ | -0.0144806539860068 |
| $\Omega_{8}^{l}$ | -0.00115026373575154 | $\Omega_{8}^{2}$ | 0.00377470685481452 |
| $\Omega_{9}^{l}$ | 0.000188643492331173 | $\Omega_{9}^{2}$ | -0.000752954288200255 |
| $\Omega_{10}^{l}$ | -0.0000214714239949848 | $\Omega_{10}^{2}$ | 0.000108557139788138 |
| $\Omega_{11}^{l}$ | $1.56173825099191 \mathrm{E}-06$ | $\Omega_{11}^{2}$ | -0.0000103988485588956 |
| $\Omega_{12}^{l}$ | $-8.57426004680047 \mathrm{E}-08$ | $\Omega_{12}^{2}$ | $5.55791376315321 \mathrm{E}-07$ |
| $\Omega_{13}^{l}$ | $5.88963918586183 \mathrm{E}-09$ | $\Omega_{13}^{2}$ | $3.72688117300783 \mathrm{E}-09$ |
| $\Omega_{14}^{l}$ | $3.91513557517705 \mathrm{E}-10$ | $\Omega_{14}^{2}$ | $-5.40977631679387 \mathrm{E}-09$ |
| $\Omega_{15}^{l}$ | $-1.13407211122615 \mathrm{E}-10$ | $\Omega_{15}^{2}$ | $4.73721978304495 \mathrm{E}-10$ |
| $\Omega_{16}^{l}$ | $-3.25544726094433 \mathrm{E}-12$ | $\Omega_{16}^{2}$ | $3.16562432692258 \mathrm{E}-11$ |
| $\Omega_{17}^{l}$ | $-1.60928747640703 \mathrm{E}-14$ | $\Omega_{17}^{2}$ | $2.31293236538858 \mathrm{E}-13$ |
| $\Omega_{18}^{l}$ | $-1.06221294110902 \mathrm{E}-17$ | $\Omega_{18}^{2}$ | $-1.81854918345231 \mathrm{E}-15$ |
| $\Omega_{19}^{l}$ | $2.4074144309226 \mathrm{E}-18$ | $\Omega_{19}^{2}$ | $-1.72101206062974 \mathrm{E}-16$ |
| $\Omega_{20}^{l}$ | $1.95845511016976 \mathrm{E}-17$ | $\Omega_{20}^{2}$ | $1.70753154840549 \mathrm{E}-16$ |
| $\Omega_{21}^{l}$ | $-1.16229834607148 \mathrm{E}-17$ | $\Omega_{21}^{2}$ | $-2.32763511072116 \mathrm{E}-16$ |

Table 4.3: Moment of Scaling Functions for $\mathrm{N}=22$

| Moment of Scaling Functions for N=22 |  |
| :---: | :---: |
| $\mu_{0}^{0}$ | 1 |
| $\mu_{0}^{1}$ | 2.31726465941445 |
| $\mu_{0}^{2}$ | 5.36971550177098 |
| $\mu_{0}^{3}$ | 12.2319705607159 |
| $\mu_{0}^{4}$ | 26.8773186714991 |
| $\mu_{0}^{5}$ | 55.6323219714068 |
| $\mu_{0}^{6}$ | 104.846598989408 |
| $\mu_{0}^{7}$ | 169.689702637117 |
| $\mu_{0}^{8}$ | 205.772899131726 |
| $\mu_{0}^{9}$ | 92.0630949558087 |
| $\mu_{0}^{10}$ | -326.991286254096 |

### 4.6 IMPACT LOAD

A discretized impulse time signal $F(t)$ has shown in figure(4.5). The time window varies from $0 \mu s$ to $500 \mu s$. The duration of time signal is $50 \mu \mathrm{~s}$, and it starts at $100 \mu \mathrm{~s}$ and ends at $150 \mu \mathrm{~s}$. This load signal generated by using Gaussian function in the MATLAB.


Figure 4.5: Discredited impulse time signal

## CHAPTER-5

# WAYELET BASED SPECTRAL FINITE 

 ELEMENT FORMULATION
## CHAPTER 5

## WAVELET BASED SPECTRAL FINITE ELEMENT FORMULATION

### 5.1 REDUCTION OF WAVE EQUATIONS TO ORDINARY DIFFERENTIAL

## EQUATIONS

Governing wave equations for higher order composite beam [37] are,

$$
\begin{gather*}
I_{0} \frac{\partial^{2} u}{\partial t^{2}}-I_{1} \frac{\partial^{2} \emptyset}{\partial t^{2}}-A_{11} \frac{\partial^{2} u}{\partial x^{2}}+B_{11} \frac{\partial^{2} \emptyset}{\partial x^{2}}=0  \tag{5.1}\\
I_{0} \frac{\partial^{2} w}{\partial t^{2}}-A_{55}\left(\frac{\partial^{2} w}{\partial x^{2}}-\frac{\partial \emptyset}{\partial x}\right)=0  \tag{5.2}\\
I_{2} \frac{\partial^{2} \emptyset}{\partial t^{2}}-I_{1} \frac{\partial^{2} u}{\partial t^{2}}-A_{55}\left(\frac{\partial w}{\partial x}-\emptyset\right)+B_{11} \frac{\partial^{2} u}{\partial x^{2}}-D_{11} \frac{\partial^{2} \emptyset}{\partial x^{2}}=0 \tag{5.3}
\end{gather*}
$$

where $u(x, t), w(x, t)$ and $\emptyset(x, t)$ are the axial, transverses and shear displacements respectively.

$$
\begin{align*}
& {\left[A_{i j}, B_{i j}, D_{i j}\right]=\sum \int_{Z_{i}}^{Z_{i+1}} Q_{i j}\left[1, z, z^{2}\right] b d z}  \tag{5.4}\\
& {\left[I_{0}, I_{1}, I_{2}\right]=\sum \int_{Z_{i}}^{Z_{i+1}} \rho\left[1, z, z^{2}\right] b d z} \tag{5.5}
\end{align*}
$$

where $A_{i j}, B_{i j}$ and $D_{i j}$ are the in-plane laminate maduli coefficients, in-plane/ flexure coupling laminate moduli coefficients and flexural laminate stiffness coefficient. $I_{0}, I_{1}$, and $I_{2}$ are the inertial constants. These constants can be calculated by the following expression. $\rho$ is the mass density, $b$ is the width of the beam and $z$ is the thickness of each ply of laminate. The force boundary conditions associated with the governing differential equations are

$$
\begin{align*}
& A_{11} \frac{\partial u}{\partial x}-B_{11} \frac{\partial \phi}{\partial x}=P  \tag{5.6}\\
& A_{55} \frac{\partial w}{\partial x}-A_{55} \phi=V \tag{5.7}
\end{align*}
$$

$$
\begin{equation*}
-B_{11} \frac{\partial u}{\partial x}+D_{11} \frac{\partial \phi}{\partial x}=M \tag{5.8}
\end{equation*}
$$

where $P(x, t), V(x, t)$ and $M(x, t)$ are the axial, transverse force and moment respectively. In the formulation WSFE, reduce the governing differential equations (5.1)-(5.3) to ODE using Daubechies scaling functions for approximation in time. The displacements $u(x, t), w(x, t)$ and $\phi(x, t)$ discredited at $n$ points in the time window. Therefore the number of sampling points $\tau=0,1,2, \cdots, n-1$. the time at a particular instant can be obtained by

$$
\begin{equation*}
t=\Delta t \tau \tag{5.9}
\end{equation*}
$$

where $\Delta t$ is the time interval between two sampling points. Approximate the functions $u(x, t), w(x, t)$ and $\phi(x, t)$ by scaling functions $\varphi(\tau)$ at an arbitrary scale as

$$
\begin{array}{ll}
u(x, t)=u(x, \tau)=\sum_{k} u_{k} \varphi(\tau-k), & k \in Z \\
w(x, t)=w(x, \tau)=\sum_{k} w_{k} \varphi(\tau-k), & k \in Z \\
\emptyset(x, t)=\emptyset(x, \tau)=\sum_{k} \emptyset_{k} \varphi(\tau-k), & k \in Z \tag{5.12}
\end{array}
$$

Substitute above expressions in the governing wave equations we get

$$
\begin{gather*}
\frac{I_{0}}{\Delta t^{2}} \sum_{k} u_{k} \varphi^{\prime \prime}(\tau-k)-\frac{I_{1}}{\Delta t^{2}} \sum_{k} \emptyset_{k} \varphi^{\prime \prime}(\tau-k)+\sum_{k}\left(-A_{11} \frac{d^{2} u_{k}}{d x^{2}}+B_{11} \frac{d^{2} \emptyset_{k}}{d x^{2}}\right) \varphi(\tau-k)=0  \tag{5.13}\\
\frac{I_{0}}{\Delta t^{2}} \sum_{k} w_{k} \varphi^{\prime \prime}(\tau-k)-A_{55} \sum_{k}\left(\frac{d^{2} w_{k}}{d x^{2}}-\frac{\emptyset_{k}}{d x}\right) \varphi(\tau-k)=0  \tag{5.14}\\
\frac{I_{2}}{\Delta t^{2}} \sum_{k} \emptyset_{k} \varphi^{\prime \prime}(\tau-k)-\frac{I_{1}}{\Delta t^{2}} \sum_{k} u_{k} \varphi^{\prime \prime}(\tau-k)+\sum_{k}\left[\left(-A_{55} \frac{d w_{k}}{d x}-\emptyset_{k}\right)+B_{11} \frac{d^{2} u_{k}}{d x^{2}}-D_{11} \frac{d^{2} \emptyset_{k}}{d x^{2}}\right] \varphi(\tau-k) \\
=0 \tag{5.15}
\end{gather*}
$$

Taking inner product of on both sides of above equations with the translates of scaling functions $\varphi(\tau-j) d \tau$, and using their orthogonal properties, we get n simultaneous equations ordinary differential equations are

$$
\begin{gather*}
\frac{1}{\Delta t^{2}} \sum_{k=j-N+2}^{j+N-2} \Omega_{j-k}^{2}\left(I_{0} u_{k}-I_{1} \emptyset_{k}\right)-A_{11} \frac{d^{2} u_{j}}{d x^{2}}+B_{11} \frac{d^{2} \emptyset_{j}}{d x^{2}}=0  \tag{5.16}\\
\frac{1}{\Delta t^{2}} \sum_{k=j-N+2}^{j+N-2} \Omega_{j-k}^{2} I_{0} w_{k}-A_{55}\left(\frac{d^{2} w_{j}}{d x^{2}}-\frac{d \emptyset_{j}}{d x}\right)=0  \tag{5.17}\\
\frac{1}{\Delta t^{2}} \sum_{k=j-N+2}^{j+N-2} \Omega_{j-k}^{2}\left(I_{2} \emptyset_{k}-I_{1} u_{k}\right)-A_{55}\left(\frac{d w_{j}}{d x}-\emptyset_{j}\right)+B_{11} \frac{d^{2} u_{j}}{d x^{2}}-D_{11} \frac{d^{2} \emptyset_{j}}{d x^{2}}=0 \tag{5.18}
\end{gather*}
$$

where $j=0,1,2,3, \ldots, n-1, \quad N$ is the order of the Daubechies wavelet and $\Omega_{j-k}^{1}, \Omega_{j-k}^{2}$ are the first and second order connection coefficients respectively.

$$
\begin{aligned}
& \Omega_{j-k}^{1}=\int \varphi^{\prime}(\tau-k) \varphi(\tau-j) d \tau \\
& \Omega_{j-k}^{2}=\int \varphi^{\prime \prime}(\tau-k) \varphi(\tau-j) d \tau
\end{aligned}
$$

For compactly supported wavelets these first and second order connection coefficients [8] are non zeros at an interval of $k=j-N+2$ to $k=j+N-2$. By observing the equations(ODEs), the connection coefficients near the boundaries at $j=0$ and $j=n-1$ lie outside the time window $\left[0, t_{f}\right]$. The falling of these connections coefficients outside the time window will create problems with finite length data sequence. Therefore the coefficients at the boundary need treatment. This treatment can be done using capacitance matrix and penalty function methods $[56,57]$. To solve this boundary value problem, a wavelet based extrapolation scheme proposed by Amaratunga and Williams(1997)[3,71] is used. This method is particularly suitable for approximation time for the ease of imposition of initial conditions. The treatment of
boundaries for finite domain analysis can be done by using wavelet extrapolation technique. The equation (5.11) can be written in matrix algebraic equations and can be solved by conventional techniques. The equation (5.11) gives the $n$ algebraic coupled equations. $u_{j}$ are known in the interval of time 0 to $t_{f}$. Where $j$ varies from 0 to $n-1$. Out of $n$ algebraic coupled equations, some equations contain coefficients $u_{j}$ corresponding to $j=0$ to $j=N-2$. This condition shows that, the coefficients $u_{j}$ lie outside of the time window [ $0-t_{f}$ ]. Similarly the same problem exist on other boundary for $j=(n-1)-N+2$ to $j=n-1$.

### 5.1.1 WAVELET EXTRAPOLATION TECHNIQUE

The $N$ coefficients of Daubechies scaling function has $p=\frac{N}{2}$ vanishing moments, and that its translates can be combined to give exact representation of polynomial order $p-1$. Assume that $u_{j}$ has a polynomial representation of order $p-1$ in the vicinity of the left boundary $t=0$, we have

$$
\begin{equation*}
u(\tau)=\sum_{k} u_{k} \varphi(\tau-k),=\sum_{l=0}^{p-1} c_{l} \tau^{l} \tag{5.19}
\end{equation*}
$$

where $c_{l}$ are constant coefficients. By taking inner product of equation (5.12) with $\varphi(\tau-j)$ on both sides, we get

$$
\begin{equation*}
u_{j}=\sum_{l=0}^{p-1} c_{l} \mu_{j}^{l} \quad j=-1,-2,-3, \cdots,-N+2 . \tag{5.20}
\end{equation*}
$$

where $\mu_{j}^{l}$ are the moment of scaling functions defined in equation (4.17) and solving the recursive equations in 4.4. Solve the equation (5.12) using finite difference scheme, we get $p-1$ initial values of $u(\tau)$. Substitute these $u(\tau)$ values in equation (5.13) and get the $c_{l}$ values in
terms of initial values. Like that the unknown coefficients $u_{j}$ on LHS can calculate by using the following relation

$$
\left[\begin{array}{c}
u_{-1}  \tag{521}\\
u_{-2} \\
\vdots \\
u_{-N+2}
\end{array}\right]=\left[\begin{array}{cccc}
\mu_{-1}^{0} & \mu_{-1}^{1} & \cdots & \mu_{-1}^{p-1} \\
\mu_{-2}^{0} & \mu_{-2}^{1} & \cdots & \mu_{-2}^{p-1} \\
\vdots & \vdots & \cdots & \vdots \\
\mu_{-N+2}^{0} & \mu_{-N+2}^{1} & \cdots & \mu_{-N+2}^{p-1}
\end{array}\right]\left[\begin{array}{c}
c_{0} \\
c_{1} \\
\vdots \\
c_{p-1}
\end{array}\right]
$$

Similarly assume the polynomial representation to obtain the unknown coefficients $u_{j}$ at the RHS as

$$
\begin{equation*}
u_{j}=\sum_{l=0}^{p-1} c_{l} \mu_{j-n}^{l} \quad j=(n-1)-p+1,(n-1)-p+2, \cdots, n-1 \tag{5.22}
\end{equation*}
$$

The above equation (5.15) can be written in matrix form as

$$
\left[\begin{array}{cccc}
\mu_{-p}^{0} & \mu_{-p}^{1} & \cdots & \mu_{-p}^{p-1}  \tag{5.23}\\
\mu_{-p+1}^{0} & \mu_{-p+1}^{1} & \cdots & \mu_{-p+1}^{p-1} \\
\vdots & \vdots & \cdots & \vdots \\
\mu_{-1}^{0} & \mu_{-1}^{1} & \cdots & \mu_{-1}^{p-1}
\end{array}\right]\left[\begin{array}{c}
c_{0} \\
c_{1} \\
\vdots \\
c_{p-1}
\end{array}\right]=\left[\begin{array}{c}
u_{(n-1)-p+1} \\
u_{(n-1)-p+2} \\
\vdots \\
u_{(n-1)}
\end{array}\right]
$$

calculate the $c_{l}$ values from the above equation (5.16), and substitute back in equation (5.15) for $j=n, n+1, n+2, \cdots,(n-1)+N-2$, we get

$$
\left[\begin{array}{c}
u_{n}  \tag{5.24}\\
u_{n+1} \\
\vdots \\
u_{n-1+N-2}
\end{array}\right]=\left[\begin{array}{cccc}
\mu_{0}^{0} & \mu_{0}^{1} & \cdots & \mu_{0}^{p-1} \\
\mu_{1}^{0} & \mu_{1}^{1} & \cdots & \mu_{1}^{p-1} \\
\vdots & \vdots & \cdots & \vdots \\
\mu_{-N+2}^{0} & \mu_{-N+2}^{1} & \cdots & \mu_{-N+2}^{p-1}
\end{array}\right]\left[\begin{array}{c}
c_{0} \\
c_{1} \\
\vdots \\
c_{p-1}
\end{array}\right]
$$

Finally, after getting the unknown coefficients on LHS and RHS, substitute these unknown
coefficients in equation (5.9), (5.10) and (5.11), and the system of coupled equations can be written in matrix form as

$$
\begin{gather*}
{\left[\Gamma^{1}\right]^{2}\left(I_{0}\left\{u_{j}\right\}-I_{1}\left\{\emptyset_{j}\right\}\right)-A_{11}\left\{\frac{d^{2} u_{j}}{d x^{2}}\right\}+B_{11}\left\{\frac{d^{2} \emptyset_{j}}{d x^{2}}\right\}=0}  \tag{5.25}\\
{\left[\Gamma^{1}\right]^{2} I_{0}\left\{w_{j}\right\}-A_{55}\left\{\frac{d^{2} w_{j}}{d x^{2}}-\frac{d \emptyset_{j}}{d x}\right\}=0}  \tag{5.26}\\
{\left[\Gamma^{1}\right]^{2}\left(I_{2}\left\{\emptyset_{j}\right\}-I_{1}\left\{u_{j}\right\}\right)-A_{55}\left\{\frac{d w_{j}}{d x}-\emptyset_{j}\right\}+B_{11}\left\{\frac{d^{2} u_{j}}{d x^{2}}\right\}-D_{11}\left\{\frac{d^{2} \emptyset_{j}}{d x^{2}}\right\}=0} \tag{5.27}
\end{gather*}
$$

where $\Gamma^{1}$ is the first order connection coefficient matrix obtain after using the wavelet extrapolation technique. $\Gamma^{2}$ is the second order connection coefficient matrix[8] evaluated independently, and this can also be written as $\left[\Gamma^{1}\right]^{2}$. This modification of second order connection coefficient matrix will helps in the imposition of initial conditions for non periodic solutions[43]. The reduced ordinary differential equations (ODE) are coupled in wavelet spectral finite element (WSFE). Decoupling of these equations can be done by eigenvalue analysis of connection coefficient matrix $\Gamma^{1}$ and can be written as

$$
\begin{equation*}
\Gamma^{1}=Ф \Pi \Phi^{-1} \tag{5.28}
\end{equation*}
$$

Here $\Phi$ is the eigen vector matrix of $\Gamma^{1}$ and $\Pi$ is the diagonal matrix containing corresponding eigenvalues $-i \omega_{n}$. Similarly the eigenvalue analysis of second order connection coefficient $\Gamma^{2}$ can be written as

$$
\begin{equation*}
\Gamma^{2}=\Phi \Pi^{2} \Phi^{-1} \tag{5.29}
\end{equation*}
$$

Here, $\Pi^{2}$ is a diagonal matrix with diagonal terms $-\omega_{n}{ }^{2}$. These matrices $\Gamma^{1}$ and $\Gamma^{2}$ are independents of the problem and depends only on the order of wavelet $N$. After decoupling of the equations (5.18),(5.19) and (5.20) can be written as

$$
\begin{equation*}
-I_{0} \omega_{n}^{2} \hat{u}_{j}+I_{1} \omega_{n}^{2} \widehat{\emptyset}_{j}-A_{11} \frac{d^{2} \hat{u}_{j}}{d x^{2}}+B_{11} \frac{d^{2} \widehat{\emptyset}_{j}}{d x^{2}}=0 \tag{5.30}
\end{equation*}
$$

$$
\begin{gather*}
-I_{0} \omega_{n}^{2} \widehat{w}_{j}-A_{55}\left(\frac{d^{2} \widehat{w}_{j}}{d x^{2}}-\frac{d \widehat{\emptyset}_{j}}{d x}\right)=0  \tag{5.31}\\
-I_{2} \omega_{n}^{2} \widehat{\emptyset}_{j}+I_{1} \omega_{n}^{2} \widehat{u}_{j}-A_{55}\left(\frac{d \widehat{w}_{j}}{d x}-\widehat{\emptyset}_{j}\right)+B_{11} \frac{d^{2} \widehat{u}_{j}}{d x^{2}}-D_{11} \frac{d^{2} \widehat{\emptyset}_{j}}{d x^{2}}=0 \tag{5.32}
\end{gather*}
$$

where $\widehat{u}_{j}=\Phi^{-1} u_{j}, \widehat{w}_{j}=\Phi^{-1} w_{j}$ and $\widehat{\emptyset}_{j}=\Phi^{-1} \emptyset_{j}$. Similarly the force boundary conditions given by (5.6)-(5.8) can be written as

$$
\begin{array}{r}
A_{11} \frac{\partial \hat{u}_{j}}{\partial x}-B_{11} \frac{\partial \widehat{\emptyset}_{j}}{\partial x}=\hat{P}_{j} \\
A_{55} \frac{\partial \widehat{w}_{j}}{\partial x}-A_{55} \widehat{\varnothing}_{j}=\widehat{V}_{j} \\
-B_{11} \frac{\partial \hat{u}_{j}}{\partial x}+D_{11} \frac{\partial \widehat{\emptyset}_{j}}{\partial x}=\widehat{M}_{j} \tag{5.35}
\end{array}
$$

where $j=0,1,2, \cdots, n-1$ and $P_{j}, V_{j}$ and $M_{j}$ are the transformed force boundary conditions of $P(x, t), V(x, t)$ and $M(x, t)$ respectively.

### 5.1.2 CALCULATION OF WAVE NUMBERS AND WAVE AMPLITUDES

The parameters such as wave numbers and wave speeds are required to understand the wave mechanics in the wave guide and these are required in SFE formulation to know the wave characteristics like the wave mode is in propagating mode or damping mode or combination of both. This spectral analysis starts with the partial differential equations (5.1)-(5.3) governing the waveguide. $u(x, t), w(x, t)$ and $\emptyset(x, t)$ are the field variables in spatial and temporal dimensions. These field variables are transformed to frequency domain by using DFT as

$$
\begin{align*}
u & =\sum_{n=0}^{N-1} \widehat{u}_{n}\left(x, \omega_{n}\right) e^{i \omega_{n} t}  \tag{5.36}\\
w & =\sum_{n=0}^{N-1} \widehat{w}_{n}\left(x, \omega_{n}\right) e^{i \omega_{n} t} \tag{5.37}
\end{align*}
$$

$$
\begin{equation*}
\emptyset=\sum_{n=0}^{N-1} \widehat{\emptyset}_{n}\left(x, \omega_{n}\right) e^{i \omega_{n} t} \tag{5.38}
\end{equation*}
$$

Here $\omega_{n}$ is the discrete circular frequency in $\mathrm{rad} / \mathrm{sec}, N$ is the total number of frequency points used in the transformation and $\hat{u}_{n}, \widehat{w}_{n}$ and $\widehat{\emptyset}_{n}$ are the DFT coefficients, varies with $x$ only. The discrete circular frequency $\omega_{n}$ related with the time window as

$$
\omega_{n}=n \Delta \omega=\frac{n \omega_{f}}{N}=\frac{n}{N \Delta t}=\frac{n}{T}
$$

where $\Delta t$ is the time sampling rate and $\omega_{f}$ is the highest frequency captured by $\Delta t . N$ decides the frequency content of the load and considering the wrap around problem and $\Delta \omega$ decides the aliasing problem [18]. Substitute the equations (5.36)-(5.38) in equations (5.1)-(5.3), we get

$$
\begin{gather*}
-I_{0} \omega_{n}^{2} \widehat{u}_{n}+I_{1} \omega_{n}^{2} \widehat{\emptyset}_{n}-A_{11} \frac{d^{2} \widehat{u}_{n}}{d x^{2}}+B_{11} \frac{d^{2} \widehat{\emptyset}_{n}}{d x^{2}}=0  \tag{5.39}\\
-I_{0} \omega_{n}^{2} \widehat{w}_{n}-A_{55}\left(\frac{d^{2} \widehat{w}_{n}}{d x^{2}}-\frac{d \widehat{\emptyset}_{n}}{d x}\right)=0  \tag{5.40}\\
-I_{2} \omega_{n}^{2} \widehat{\emptyset}_{n}+I_{1} \omega_{n}^{2} \widehat{u}_{n}-A_{55}\left(\frac{d \widehat{w}_{n}}{d x}-\widehat{\emptyset}_{n}\right)+B_{11} \frac{d^{2} \widehat{u}_{n}}{d x^{2}}-D_{11} \frac{d^{2} \widehat{\emptyset}_{n}}{d x^{2}}=0 \tag{5.41}
\end{gather*}
$$

Assuming the solutions of above equations as $\widehat{u}_{n}=c_{u} e^{-i k x}, \widehat{w}_{n}=c_{w} e^{-i k x}$ and $\widehat{\emptyset}_{n}=c_{\emptyset} e^{-i k x}$. Here $c_{u}, c_{w}$ and $c_{\emptyset}$ are the constants, later these can be derived from boundary values and $k$ is the wave number. Substituting these solutions in equations (5.39)-(5.41), we get

$$
\begin{gather*}
-I_{0} \omega_{n}^{2} c_{u}+I_{1} \omega_{n}^{2} c_{\emptyset}+A_{11} k^{2} c_{u}-B_{11} k^{2} c_{\emptyset}=0  \tag{5.42}\\
-I_{0} \omega_{n}^{2} c_{w}+A_{55} k^{2} c_{w}-A_{55} i k c_{\emptyset}=0  \tag{5.43}\\
-I_{2} \omega_{n}^{2} c_{\emptyset}+I_{1} \omega_{n}^{2} c_{u}+A_{55} i k c_{w}+A_{55} c_{\emptyset}-B_{11} k^{2} c_{u}+D_{11} k^{2} c_{\emptyset}=0 \tag{5.44}
\end{gather*}
$$

The above equations can also be write in matrix form, we get

$$
\left[\begin{array}{ccc}
-I_{0} \omega_{n}^{2}+A_{11} k^{2} & 0 & I_{1} \omega_{n}^{2}-B_{11} k^{2}  \tag{5.45}\\
0 & -I_{0} \omega_{n}^{2}+A_{55} k^{2} & -A_{55} i k \\
I_{1} \omega_{n}^{2}-B_{11} k^{2} & A_{55} i k & -I_{2} \omega_{n}^{2}+D_{11} k^{2}+A_{55}
\end{array}\right]\left[\begin{array}{l}
c_{u} \\
c_{w} \\
c_{\varnothing}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]
$$

These wave numbers $k$ were obtained as a function of circular frequency $\omega$, and computation of wave numbers are very difficult for higher composite structures due to its coupling of transverse and shear displacements and elastic coupling. In order to calculate the wave numbers and associated wave amplitudes for such problems have been proposed by S. Gopalakrishnan [21]. Singular value decomposition (SVD) and polynomial evaluation problem (PEP) methods are used to solve these problems. In this project, PEP method is used to solve the equation (5.42)(5.44) and those equations are written in the form PEP in $k$ as

$$
\begin{equation*}
A_{0} k^{m}+A_{1} k^{m-1}+\cdots+A_{m}=0 \tag{5.46}
\end{equation*}
$$

Here $A_{i}, i=0$ to $m$ are the $p \times p$ matrices, where $p$ is the number of independent variables in the governing equations. Thus, the PEP is of the order $p \times m$. In this project, $p=3$ and $m=2$. The PEP for equation(5.45) is given as

$$
\begin{equation*}
A_{0} k^{2}+A_{1} k+A_{2}=0 \tag{5.47}
\end{equation*}
$$

where

$$
A_{0}=\left[\begin{array}{ccc}
A_{11} & 0 & -B_{11} \\
0 & A_{55} & 0 \\
-B_{11} & 0 & D_{11}
\end{array}\right] \quad A_{1}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -A_{55} \\
0 & A_{55} & 0
\end{array}\right] \quad A_{2}=\left[\begin{array}{ccc}
-I_{0} \omega_{n}^{2} & 0 & I_{1} \omega_{n}^{2} \\
0 & -I_{0} \omega_{n}^{2} & 0 \\
I_{1} \omega_{n}^{2} & 0 & -I_{2} \omega_{n}^{2}+A_{55}
\end{array}\right]
$$

By solving the equation (5.47)'using "polyeig" in the MATLAB, gives the eigenvalues and eigenvectors. These eigenvalues represents the wavenumbers and eigenvectors represents the corresponding wave amplitudes.

### 5.2 SPECTRAL FINITE ELEMENT FORMULATION

The equations (5.39) to (5.41) are solved exactly to get the shape functions in terms of some unknown constants. These unknown constants are again solved in terms of boundary values and these are useful in the formulation of elemental dynamic stiffness matrix of the transformed nodal displacements with transformed nodal forces.


Figure 5.1: Composite beam element with nodal forces and nodal displacements The above figure shows the beam element with three degrees of freedom such as $u_{1,2}, w_{1,2}$ and $\emptyset_{1,2}$ represents the axial, transverse and shear displacements at nodel and node 2 respectively and $P_{1,2}, V_{1,2}$ and $M_{1,2}$ are the axial, transverse forces and moments at nodel and node2 respectively. The transformed ODEs (5.30) to (5.32) are need to be solved for the $\widehat{u}_{j}, \widehat{w}_{j}$ and $\widehat{\emptyset}_{j}$ and the actual solutions $u(x, t), w(x, t)$ and $\emptyset(x, t)$ are obtained from inverse wavelet transform and solve these equations exactly, gives the exact shape functions (interpolating functions) which are

$$
\begin{align*}
& \widehat{u}(x)=c_{1} R_{11} e^{-i k_{1} x}+c_{2} R_{12} e^{-i k_{1}(L-x)}+c_{3} R_{13} e^{-i k_{2} x}+c_{4} R_{14} e^{-i k_{2}(L-x)}+c_{5} R_{15} e^{-i k_{3} x} \\
& +c_{6} R_{16} e^{-i k_{3}(L-x)}  \tag{5.48}\\
& \widehat{w}(x)=c_{1} R_{21} e^{-i k_{1} x}+c_{2} R_{22} e^{-i k_{1}(L-x)}+c_{3} R_{23} e^{-i k_{2} x}+c_{4} R_{24} e^{-i k_{2}(L-x)}+c_{5} R_{25} e^{-i k_{3} x} \\
& +c_{6} R_{26} e^{-i k_{3}(L-x)}  \tag{5.49}\\
& \widehat{\emptyset}(x)=c_{1} R_{31} e^{-i k_{1} x}+c_{2} R_{32} e^{-i k_{1}(L-x)}+c_{3} R_{33} e^{-i k_{2} x}+c_{4} R_{34} e^{-i k_{2}(L-x)}+c_{5} R_{35} e^{-i k_{3} x} \\
& +c_{6} R_{36} e^{-i k_{3}(L-x)} \tag{5.50}
\end{align*}
$$

where $L$ is the length of the element and $k_{1}, k_{2}$ and $k_{3}$ are the wave numbers corresponding to the axial, transverse and shear modes of the element and explained in Mitra \& Gopalakrishnan (in press)

The above equations (5.48)-(5.50) can be written in matrix form as

$$
\begin{align*}
& \left\{\hat{u}_{(x)}^{e}\right\}=\left[R_{1}\right][\Theta]\{C\} \tag{5.51}
\end{align*}
$$

Here $\left\{\hat{u}_{(x)}^{e}\right\}$ is the nodal displacement vector of element at each node. $\left[R_{1}\right]$ is the $3 \times 6$ amplitude ratio matrix and this matrix can be obtained by solving the PEP equation (5.47). The solution of equation (5.47) gives the eigenvalues and eigenvectors. [ $R_{1}$ ] is the eigenvector matrix and eigenvalues are the wave numbers $\pm k_{1}, \pm k_{2}$ and $\pm k_{3}$. [ $\left.\Theta\right]$ is the diagonal matrix with the diagonal elements $\quad\left[e^{-i k_{1} x}, \mathrm{e}^{-\mathrm{i} \mathrm{k}_{1}(L-x)}, \mathrm{e}^{-\mathrm{i} \mathrm{k}_{2} x}, \mathrm{e}^{-\mathrm{i} \mathrm{k}_{2}(L-x)}, \mathrm{e}^{-\mathrm{i} \mathrm{k}_{3} x}, \mathrm{e}^{-\mathrm{i} \mathrm{k}_{3}(L-x)}\right] \quad$ and $\{C\}=\left\{\begin{array}{lllll}c_{1} & c_{2} & c_{3} & c_{4} & c_{5} \\ c_{6}\end{array}\right\}^{T}$ are the unknown coefficients and these are obtained by applying the boundary conditions at node 1 and node 2 as shown in figure 5.1.

Applying the boundary conditions at two nodes of element as shown in figure 5.1 to get unknown constants $\{C\}$ in terms of nodal displacements. Put $x=0$ in equation (5.51), gives the nodal displacements at node1, and the equation becomes

$$
\begin{equation*}
\left\{\hat{u}_{1}^{e}\right\}=\left[R_{1}\right]\left[\Theta_{1}\right]\{C\} \tag{5.52}
\end{equation*}
$$

Put $x=L$ in equation (5.51), gives the nodal displacements at node 2 , and the equation becomes

$$
\begin{equation*}
\left\{\hat{u}_{2}^{e}\right\}=\left[R_{1}\right]\left[\Theta_{2}\right]\{C\} \tag{5.53}
\end{equation*}
$$

Here $\left\{\hat{u}_{1}^{e}\right\}=\left\{\hat{u}_{1} \widehat{w}_{1} \widehat{\emptyset}_{1}\right\}^{T}$ is the nodal displacement vector at node 1 , similarly $\left\{\hat{u}_{2}^{e}\right\}=$ $\left\{\hat{u}_{2} \widehat{w}_{2} \widehat{\emptyset}_{2}\right\}^{T}$ is the nodal displacement vector at node 2. $\left[\Theta_{1}\right]$ and $\left[\Theta_{2}\right]$ are the diagonal matrices and these are shown in the next page.

From equations (5.52) and (5.53), the nodal displacement vector $\left\{\hat{u}^{e}\right\}=\left\{\hat{u}_{1}^{e} \hat{u}_{2}^{e}\right\}^{T}$ of an element can be written as

$$
\left\{\hat{u}^{e}\right\}=\left[\begin{array}{l}
{\left[T_{11}\right]}  \tag{5.54}\\
{\left[T_{12}\right]}
\end{array}\right]\{C\}=\left[T_{1}\right]\{C\}
$$

where $\left\{\hat{u}^{e}\right\}=\left\{\hat{u}_{1}^{e} \hat{u}_{2}^{e}\right\}=\left\{\hat{u}_{1} \widehat{w}_{1} \widehat{\emptyset}_{1} \hat{u}_{2} \widehat{w}_{2} \widehat{\emptyset}_{2}\right\}^{T}$ and $\left[T_{11}\right]_{3 \times 6}=\left[R_{1}\right]\left[\Theta_{1}\right]$ and similarly $\left[T_{12}\right]_{3 \times 6}=\left[R_{1}\right]\left[\Theta_{2}\right]$.

The derivatives of the interpolating functions are given as

$$
\begin{align*}
\frac{\partial \widehat{u}}{\partial x}= & -i k_{1} c_{1} R_{11} e^{-i k_{1} x}+i k_{1} c_{2} R_{12} e^{-i k_{1}(L-x)}-i k_{2} c_{3} R_{13} e^{-i k_{2} x}+i k_{2} c_{4} R_{14} e^{-i k_{2}(L-x)} \\
& -i k_{3} c_{5} R_{15} e^{-i k_{3} x}+i k_{3} c_{6} R_{16} e^{-i k_{3}(L-x)}  \tag{5.55}\\
\frac{\partial \widehat{w}}{\partial x}= & -i k_{1} c_{1} R_{21} e^{-i k_{1} x}+i k_{1} c_{2} R_{22} e^{-i k_{1}(L-x)}-i k_{2} R_{23} e^{-i k_{2} x}+i k_{2} c_{4} R_{24} e^{-i k_{2}(L-x)} \\
& -i k_{3} c_{5} R_{25} e^{-i k_{3} x}+i k_{3} c_{6} R_{26} e^{-i k_{3}(L-x)}  \tag{5.56}\\
\frac{\partial \widehat{\emptyset}}{\partial x}= & -i k_{1} c_{1} R_{31} e^{-i k_{1} x}+i k_{1} c_{2} R_{32} e^{-i k_{1}(L-x)}-i k_{2} c_{3} R_{33} e^{-i k_{2} x}+i k_{2} c_{4} R_{34} e^{-i k_{2}(L-x)} \\
& -i k_{3} c_{5} R_{35} e^{-i k_{3} x}+i k_{3} c_{6} R_{36} e^{-i k_{3}(L-x)} \tag{5.57}
\end{align*}
$$

The above equations (5.55)-(5.57) can be written in matrix form as

$$
\begin{align*}
& \left\{\begin{array}{l}
\hat{P} \\
\hat{V} \\
\hat{M}
\end{array}\right\}=\left[\begin{array}{cccccc}
-i k_{1} R_{11} & i k_{1} R_{12} & -i k_{2} R_{13} & i k_{2} R_{14} & -i k_{3} R_{15} & i k_{3} R_{16} \\
-i k_{1} R_{21} & i k_{1} R_{22} & -i k_{2} R_{23} & i k_{2} R_{24} & -i k_{3} R_{25} & i k_{3} R_{26} \\
-i k_{1} R_{31} & i k_{1} R_{32} & -i k_{2} R_{33} & i k_{2} R_{34} & -i k_{3} R_{35} & i k_{3} R_{36}
\end{array}\right]\left[\begin{array}{ccc}
e^{-i k_{1} x} & & \\
\mathrm{e}^{-\mathrm{ik}(L-x)} & & \\
& \mathrm{e}^{-\mathrm{ik} k_{2} x} & \\
& \mathrm{e}^{-i \mathrm{k}_{2}(L-x)} & \\
& & \mathrm{e}^{-i \mathrm{k}_{3} x} \\
\\
& & \mathrm{e}^{-\mathrm{ik}(L-x)}
\end{array}\right]\left\{\begin{array}{l}
c_{1} \\
c_{2} \\
c_{3} \\
c_{4} \\
c_{5} \\
c_{6}
\end{array}\right\} \\
& \left\{\hat{F}_{(x)}^{e}\right\}=\left[R_{2}\right][\Theta]\{C\} \tag{5.58}
\end{align*}
$$

Here $\left\{\hat{F}_{(x)}^{e}\right\}$ is the nodal force vector of element at each node. $[\Theta]$ is the diagonal matrix with the diagonal elements $\quad\left[e^{-i k_{1} x}, \mathrm{e}^{-i k_{1}(L-x)}, \mathrm{e}^{-i \mathrm{k}_{2} x}, \mathrm{e}^{-i \mathrm{k}_{2}(L-x)}, \mathrm{e}^{-i \mathrm{k}_{3} x}, \mathrm{e}^{-\mathrm{i} \mathrm{k}_{3}(L-x)}\right] \quad$ and $\{C\}=\left\{\begin{array}{lllll}c_{1} & c_{2} & c_{3} & c_{4} & c_{5}\end{array} c_{6}\right\}^{T}$ are the unknown coefficients and these are obtained by applying the boundary conditions at node 1 and node 2 as shown in figure 5.1.

Applying the boundary conditions at two nodes of element as shown in figure 5.1 to get unknown constants $\{C\}$ in terms of nodal forces. Put $x=0$ in equation (5.58), gives the nodal forces at nodel, and the equation becomes

$$
\begin{equation*}
\left\{\hat{F}_{1}^{e}\right\}=\left[R_{2}\right]\left[\Theta_{1}\right]\{C\} \tag{5.59}
\end{equation*}
$$

Put $x=L$ in equation (5.58), gives the nodal displacements at node 2 , and the equation becomes

$$
\begin{equation*}
\left\{\hat{F}_{2}^{e}\right\}=\left[R_{2}\right]\left[\Theta_{2}\right]\{C\} \tag{5.60}
\end{equation*}
$$

Here $\left\{\widehat{F}_{1}^{e}\right\}=\left\{\widehat{P}_{1} \widehat{V}_{1} \widehat{M}_{1}\right\}^{T}$ is the nodal force vector at node 1 , similarly $\left\{\widehat{F}_{2}^{e}\right\}=\left\{\hat{P}_{2} \widehat{V}_{2} \widehat{M}_{2}\right\}^{T}$ is the nodal force vector at node 2. $\left[\Theta_{1}\right]$ and $\left[\Theta_{2}\right]$ are the diagonal matrices as given above.

From equations (5.59) and (5.60), the nodal force vector $\left\{\hat{F}^{e}\right\}=\left\{\hat{F}_{1}^{e} \hat{F}_{2}^{e}\right\}^{T}$ of an element can be written as

$$
\left\{\hat{F}^{e}\right\}=\left[\begin{array}{l}
{\left[T_{21}\right]}  \tag{5.61}\\
{\left[T_{22}\right]}
\end{array}\right]\{C\}=\left[T_{2}\right]\{C\}
$$

where $\left\{\hat{F}^{e}\right\}=\left\{\widehat{F}_{1}^{e} \widehat{F}_{2}^{e}\right\}=\left\{\hat{P}_{1} \widehat{V}_{1} \widehat{M}_{1} \hat{P}_{2} \widehat{V}_{2} \widehat{M}_{2}\right\}^{T}$ and $\left[T_{21}\right]_{3 \times 6}=\left[R_{2}\right]\left[\Theta_{1}\right]$ and similarly $\left[T_{22}\right]_{3 \times 6}=\left[R_{2}\right]\left[\Theta_{2}\right]$.

From equation (5.54),

$$
\begin{equation*}
\{C\}=\left[T_{1}\right]^{-1}\left\{\hat{u}^{e}\right\} \tag{5.62}
\end{equation*}
$$

Substitute the equation (5.62) in equation (5.61) we get

$$
\begin{equation*}
\left\{\hat{F}^{e}\right\}=\left[T_{2}\right]\left[T_{1}\right]^{-1}\left\{\hat{u}^{e}\right\} \tag{5.63}
\end{equation*}
$$

The above equation can also be written as

$$
\begin{equation*}
\left\{\hat{F}^{e}\right\}=\left[K_{D}^{e}\right]\left\{\hat{u}^{e}\right\} \tag{5.64}
\end{equation*}
$$

where $\left[K_{D}^{e}\right]$ is the elemental dynamic stiffness matrix of size $6 \times 6$ and this matrix is divided into four sub matrices of size $3 \times 3$, and rewrite the equation (5.64) as

$$
\left\{\begin{array}{l}
\hat{F}_{1}^{e}  \tag{5.65}\\
\hat{F}_{2}^{e}
\end{array}\right\}=\left[\begin{array}{ll}
K_{11}^{e} & K_{12}^{e} \\
K_{21}^{e} & K_{22}^{e}
\end{array}\right]\left\{\begin{array}{l}
\hat{u}_{1}^{e} \\
\hat{u}_{2}^{e}
\end{array}\right\}
$$

where $\hat{F}_{j}^{i}$ and $\hat{u}_{j}^{i}$ are the nodal force and displacement vectors of the $i^{\text {th }}$ element at $j^{t h}$ node.

### 5.3 MODELING OF DE-LAMINATION IN COMPOSITE BEAM

The modeling of embedded de-lamination of the composite beam is done according to the method presented by Nag et al.[51]. The following figure 5.2 shows the embedded de-lamination of the graphite - epoxy cantilevered beam.


Figure 5.2: Graphite-Epoxy [0] 8 layered cantilever beam
where $L$ is the length of the beam, $L_{1}$ is the distance between the free end and edge of the delamination, $L_{d}$ is the de-lamination length, 2 h is the depth of the beam and b is the width of the beam. Figure 5.3(a) and 5.3(b) are shows the cross sections of the beam at the end and at the delaminated portion of the composite beam


Figure 5.3(a): Cross section at the end


Figure 5.3(b): Cross section at the de-lamination


Figure 5.4: Modeling of an embedded de-lamination with base and sub laminates.


Figure 5.5: Representation of the base and sub base laminates by spectral elements.
Figure 5.2 shows the delaminated composite beam divided into four elements as shown in Figure5.4, out of four elements element 1 and element 2 are called as base laminates and element 3 and element 4 are called as sub laminates. Every element of the beam is considered as individual structural wave guides and modeled as coupled composite Timoshenko beam by using WSFE method as discussed in previous section.

From equation (5.65), the nodal force vector for element 1 is written as

$$
\left\{\begin{array}{l}
\hat{F}_{1}^{1}  \tag{5.66}\\
\hat{F}_{4}^{1}
\end{array}\right\}=\left[\begin{array}{ll}
K_{11}^{1} & K_{12}^{1} \\
K_{21}^{1} & K_{22}^{1}
\end{array}\right]\left\{\begin{array}{l}
\hat{u}_{1}^{1} \\
\hat{u}_{4}^{1}
\end{array}\right\}
$$

Similarly, the nodal force vector for element 2 is written as

$$
\left\{\begin{array}{l}
\hat{F}_{7}^{2}  \tag{5.67}\\
\hat{F}_{2}^{2}
\end{array}\right\}=\left[\begin{array}{ll}
K_{11}^{2} & K_{12}^{2} \\
K_{21}^{2} & K_{22}^{2}
\end{array}\right]\left\{\begin{array}{l}
\hat{u}_{7}^{2} \\
\hat{u}_{2}^{2}
\end{array}\right\}
$$

The kinematic assumption at the interfaces of the base and sub laminates is that the constant cross sectional slope. From this assumption, the constant and continuous slope at the two interfaces between base laminates and sub base laminates gives relation of nodal displacement vector of sub laminates in term of base laminates.


Figure 5.6: Force balance at the interface between base and sub laminate elements.
From the above Figure, the nodal displacement vector at node 3 and node 5 is written in terms of node 4 as

$$
\widehat{u}_{3}^{4}=\left\{\begin{array}{l}
\hat{u}_{3} \\
\widehat{w}_{3} \\
\widehat{\emptyset}_{3}
\end{array}\right\}=\left\{\begin{array}{c}
\hat{u}_{4}+h_{1} \widehat{\emptyset}_{4} \\
\widehat{w}_{4} \\
\widehat{\emptyset}_{4}
\end{array}\right\} \quad \text { and } \quad \hat{u}_{5}^{3}=\left\{\begin{array}{l}
\hat{u}_{5} \\
\widehat{w}_{5} \\
\widehat{\emptyset}_{5}
\end{array}\right\}=\left\{\begin{array}{c}
\hat{u}_{4}-h_{2} \widehat{\emptyset}_{4} \\
\widehat{w}_{4} \\
\widehat{\emptyset}_{4}
\end{array}\right\}
$$

From the above expression $\hat{u}_{3}^{4}$ and $\hat{u}_{5}^{3}$ can be written as

$$
\begin{equation*}
\hat{u}_{3}^{4}=S_{1} \hat{u}_{4} \quad \hat{u}_{5}^{3}=S_{2} \hat{u}_{4} \tag{5.68}
\end{equation*}
$$

Similarly, the nodal displacement vector at node 6 and node 8 is written in terms of node 7 as

$$
\hat{u}_{6}^{4}=\left\{\begin{array}{l}
\hat{u}_{6} \\
\widehat{w}_{6} \\
\widehat{\emptyset}_{6}
\end{array}\right\}=\left\{\begin{array}{c}
\hat{u}_{7}+h_{1} \widehat{\emptyset}_{7} \\
\widehat{w}_{7} \\
\widehat{\emptyset}_{7}
\end{array}\right\} \quad \text { and } \quad \hat{u}_{8}^{3}=\left\{\begin{array}{l}
\hat{u}_{8} \\
\widehat{w}_{8} \\
\widehat{\emptyset}_{8}
\end{array}\right\}=\left\{\begin{array}{c}
\hat{u}_{7}-h_{2} \widehat{\emptyset}_{7} \\
\widehat{w}_{7} \\
\widehat{\emptyset}_{7}
\end{array}\right\}
$$

From the above expression $\hat{u}_{6}^{4}$ and $\hat{u}_{8}^{3}$ can be written as

$$
\begin{equation*}
\hat{u}_{6}^{4}=S_{1} \hat{u}_{7} \quad \hat{u}_{8}^{3}=S_{2} \hat{u}_{7} \tag{5.69}
\end{equation*}
$$

where $\widehat{u}_{j}^{i}$ represents the nodal displacement vector of $i^{t h}$ element at $j^{t h}$ node. $S_{1}$ and $S_{2}$ are the transformation matrices in terms of top and bottom sub laminate thicknesses $2 h_{1}$ and $2 h_{2}$.

$$
S_{1}=\left[\begin{array}{ccc}
1 & 0 & h_{1} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] \quad \text { and } S_{2}=\left[\begin{array}{ccc}
1 & 0 & -h_{2} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

From the force equilibrium equation at interface $A B$ as shown in figure (5.5)

$$
\left\{\begin{array}{c}
\hat{P}_{4}  \tag{5.70}\\
\widehat{V}_{4} \\
\hat{M}_{4}
\end{array}\right\}+\left\{\begin{array}{c}
\hat{P}_{3} \\
\widehat{V}_{3} \\
\widehat{M}_{3}
\end{array}\right\}+\left\{\begin{array}{c}
0 \\
0 \\
h_{2} \hat{P}_{3}
\end{array}\right\}+\left\{\begin{array}{c}
\hat{P}_{5} \\
\widehat{V}_{5} \\
\widehat{M}_{5}
\end{array}\right\}+\left\{\begin{array}{c}
0 \\
0 \\
-h_{1} \hat{P}_{5}
\end{array}\right\}=\left\{\begin{array}{l}
0 \\
0 \\
0
\end{array}\right\}
$$

The equation (5.70) can be written as

$$
\begin{equation*}
\hat{F}_{4}+S_{1}^{T} \hat{F}_{3}+S_{2}^{T} \hat{F}_{5}=0 \tag{5.71}
\end{equation*}
$$

Similarly, the force equilibrium equation at interface CD as shown in figure (5.5)

$$
\left\{\begin{array}{c}
\hat{P}_{7}  \tag{5.72}\\
\widehat{V}_{7} \\
\widehat{M}_{7}
\end{array}\right\}+\left\{\begin{array}{c}
\hat{P}_{6} \\
\hat{V}_{6} \\
\widehat{M}_{6}
\end{array}\right\}+\left\{\begin{array}{c}
0 \\
0 \\
h_{2} \hat{P}_{6}
\end{array}\right\}+\left\{\begin{array}{c}
\hat{P}_{8} \\
\hat{V}_{8} \\
\widehat{M}_{8}
\end{array}\right\}+\left\{\begin{array}{c}
0 \\
0 \\
-h_{1} \hat{P}_{8}
\end{array}\right\}=\left\{\begin{array}{l}
0 \\
0 \\
0
\end{array}\right\}
$$

The equation (5.72) can be written as

$$
\begin{equation*}
\hat{F}_{7}+S_{1}^{T} \widehat{F}_{6}+S_{2}^{T} \widehat{F}_{8}=0 \tag{5.73}
\end{equation*}
$$

The nodal force vector for element 3 is written as

$$
\left\{\begin{array}{l}
\hat{F}_{5}^{3}  \tag{5.74}\\
\hat{F}_{8}^{3}
\end{array}\right\}=\left[\begin{array}{ll}
K_{11}^{3} & K_{12}^{3} \\
K_{21}^{3} & K_{22}^{3}
\end{array}\right]\left\{\begin{array}{l}
\hat{u}_{5}^{3} \\
\hat{u}_{8}^{3}
\end{array}\right\}
$$

Substitute $\hat{u}_{5}^{3}$ and $\hat{u}_{8}^{3}$ values in equation (5.74) and pre-multiplying with $S_{2}^{T}$ on both sides, we get

$$
\left\{\begin{array}{l}
S_{2}^{T} \hat{F}_{5}^{3}  \tag{5.75}\\
S_{2}^{T} \hat{F}_{8}^{3}
\end{array}\right\}=\left[\begin{array}{ll}
S_{2}^{T} K_{11}^{3} S_{2} & S_{2}^{T} K_{12}^{3} S_{2} \\
S_{2}^{T} K_{21}^{3} S_{2} & S_{2}^{T} K_{22}^{3} S_{2}
\end{array}\right]\left\{\begin{array}{l}
\hat{u}_{4}^{3} \\
\hat{u}_{7}^{3}
\end{array}\right\}
$$

Similarly, The nodal force vector for element 4 is written as

$$
\left\{\begin{array}{l}
\hat{F}_{3}^{4}  \tag{5.76}\\
\hat{F}_{6}^{4}
\end{array}\right\}=\left[\begin{array}{ll}
K_{11}^{4} & K_{12}^{4} \\
K_{21}^{4} & K_{22}^{4}
\end{array}\right]\left\{\begin{array}{l}
\hat{u}_{3}^{4} \\
\hat{u}_{6}^{4}
\end{array}\right\}
$$

Substitute $\hat{u}_{3}^{4}$ and $\hat{u}_{6}^{4}$ values in equation (5.76) and pre-multiplying with $S_{1}^{T}$ on both sides, we get

$$
\left\{\begin{array}{l}
S_{1}^{T} \hat{F}_{5}^{3}  \tag{5.77}\\
S_{1}^{T} \hat{F}_{8}^{3}
\end{array}\right\}=\left[\begin{array}{ll}
S_{1}^{T} K_{11}^{4} S_{1} & S_{1}^{T} K_{12}^{4} S_{1} \\
S_{1}^{T} K_{21}^{4} S_{1} & S_{1}^{T} K_{22}^{4} S_{1}
\end{array}\right]\left\{\begin{array}{l}
\hat{u}_{4}^{4} \\
\hat{u}_{7}^{4}
\end{array}\right\}
$$

After calculation of elemental nodal force vectors, assemble the four spectral elements, we get

$$
\left\{\begin{array}{l}
\widehat{F}_{1}  \tag{5.78}\\
\widehat{F}_{4} \\
\widehat{F}_{7} \\
\widehat{F}_{2}
\end{array}\right\}=\left[\begin{array}{cccr}
K_{11}^{1} & K_{12}^{1} & 0 & 0 \\
K_{21}^{1} & K_{22}^{1}+S_{1}^{T} K_{11}^{4} S_{1}+S_{2}^{T} K_{11}^{3} S_{2} & S_{1}^{T} K_{12}^{4} S_{1}+S_{2}^{T} K_{12}^{3} S_{2} & 0 \\
0 & S_{1}^{T} K_{21}^{4} S_{1}+S_{2}^{T} K_{21}^{3} S_{2} & K_{11}^{2}+S_{1}^{T} K_{22}^{4} S_{1}+S_{2}^{T} K_{22}^{3} S_{2} & K_{12}^{2} \\
0 & 0 & K_{21}^{2} & K_{22}^{2}
\end{array}\right]\left\{\begin{array}{l}
\hat{u}_{1} \\
\hat{u}_{4} \\
\hat{u}_{7} \\
\hat{u}_{2}
\end{array}\right\}
$$

Nodal displacements at fixed end are zero i.e. at node $1, \hat{u}_{1}=0$ and nodal forces vectors are zero at node 4 and node 7. Using these conditions, the equation (5.78) can be written as

$$
\begin{gather*}
\left\{\begin{array}{c}
\widehat{F}_{1} \\
0 \\
0 \\
\hat{F}_{2}
\end{array}\right\}=\left[\begin{array}{cccc}
K_{11}^{1} & K_{12}^{1} & 0 & 0 \\
K_{21}^{1} & K_{22}^{1}+S_{1}^{T} K_{11}^{4} S_{1}+S_{2}^{T} K_{11}^{3} S_{2} & S_{1}^{T} K_{12}^{4} S_{1}+S_{2}^{T} K_{12}^{3} S_{2} & 0 \\
0 & S_{1}^{T} K_{21}^{4} S_{1}+S_{2}^{T} K_{21}^{3} S_{2} & K_{11}^{2}+S_{1}^{T} K_{22}^{4} S_{1}+S_{2}^{T} K_{22}^{3} S_{2} & K_{12}^{2} \\
0 & 0 & K_{21}^{2} & K_{22}^{2}
\end{array}\right]\left\{\begin{array}{c}
0 \\
\hat{u}_{4} \\
\hat{u}_{7} \\
\hat{u}_{2}
\end{array}\right\}  \tag{5.79}\\
\{\hat{F}\}=[\overline{\bar{K}}]\{\hat{u}\} \tag{5.80}
\end{gather*}
$$

where $[\overline{\bar{K}}]$ is the reconstructed dynamic stiffness matrix for the spectral element with embedded de-lamination of cantilever composite beam.

## CHAPTER 6

# NUMERICAL EXAMPLES AND 

## CONCLUSIONS

## CHAPTER 6

## NUMERICAL EXAMPLES AND CONCLUSIONS

The wave propagation of the composite beam is obtained by using WSFE as discussed in the previous chapter for broad band impulse load as shown in Figure (4.5). These numerical experiments are done over an eight layered AS4/3501-6 graphite-epoxy composite beam. The properties of AS4/3501-6 graphite-epoxy composite beam are given in the following table 6.1.

Table 6.1: Material properties of the Graphite- Epoxy composite beam:

| Material properties |  |
| :---: | :---: |
| $E_{11}$ | 141.90 Gpa. |
| $E_{22}$ | 9.78 Gpa. |
| $G_{12}=G_{13}$ | 6.13 Gpa. |
| $G_{23}$ | 0.42 Gpa |
| $v_{12}$ | $1449 \mathrm{~kg} / \mathrm{m}^{3}$ |
| Mass density $(\rho)$ |  |

The length $(L)$, breadth $(2 b)$ and depth $(2 h)$ of the beam are $0.5 \mathrm{~m}, 0.01 \mathrm{~m}$ and 0.01 m respectively as shown in Figure (5.2). The de-lamination is provided at a distance of $0.25 \mathrm{~m}\left(L_{1}\right)$ from free end of the cantilever beam and extends towards the fixed end at mid depth and at various depths above the center line of the beam.

### 6.1 WAVE RESPONSES TO THE IMPULSE LOAD

Wave propagation is studied in a AS4/3501-6 graphite-epoxy cantilevered composite beam due to the impulse load applied axially and transversely at the free end. The Figure (6.1) and (6.2), shows the axial and transverse velocities in a undamped $\left[0_{8}\right]$ cantilever composite beam respectively. These velocities are simulated using WSFE with $N=22, \Delta t=2 \mu s$ with total time window $T_{w}=1024$ and for the length, width and depth of beam are $0.5 \mathrm{~m}, 0.01 \mathrm{~m}$ and 0.01 m respectively.


Figure 6.1: Axial tip velocity of undamaged $\left[0_{8}\right]$ composite beam


Figure 6.2: Transverse tip velocity of undamaged $\left[0_{8}\right]$ composite beam

The figure (6.3) and (6.4) shows the axial and transverse tip velocities of the beam of same configuration as mentioned in the above plot, but here different length of de-lamination $\left(L_{d}\right)$ is embedded in the beam and compared with undamaged response. The de-laminations are embedded along the centerline of the beam and the de-lamination starts at a distance of 0.25 m from free end and extends towards fixed end with different length of 10 mm and 20 mm . From figure (6.3), it is clearly observed that, the axial velocity of undamaged beam and de-lamination of 10 mm and 20 mm produced the same response. In case of transverse velocity, the damaged responses compared with undamaged response shows the early reflections from the fixed end due to its de-lamination and amplitude of these reflected waves increases with increase in the length of de-lamination which can be observed from figure (6.4). Figures (6.5) and (6.6) shows the axial and transverse tip velocities for undamaged and different length of de-lamination ( 30 mm and 40 mm ) along the centerline of the eight layered composite beam.


Figure 6.3: Axial tip velocities for undamaged and different de-lamination lengths ( $L_{d}=10 \mathrm{~mm}$ and 20 mm ) along the centerline of $\left[0_{8}\right]$ layered composite beam


Figure 6.4: Transverse tip velocities for undamaged and different de-lamination lengths ( $L_{d}=10 \mathrm{~mm}$ and 20 mm ) along the centerline of $\left[0_{8}\right]$ composite beam


Figure 6.5: Axial tip velocities for undamaged and different de-lamination lengths ( $L_{d}=30 \mathrm{~mm}$ and 40 mm ) along the centerline of $\left[0_{8}\right]$ layered composite beam


Figure 6.6: Transverse tip velocities for undamaged and different de-lamination lengths $\left(L_{d}=30 \mathrm{~mm}\right.$ and 40 mm$)$ along the centerline of $\left[0_{8}\right]$ composite beam


Figure 6.7: Axial tip velocities for undamaged and different position ( 0.25 m and 0.35 m ) of 20 mm de-lamination from free end and along the centerline of $\left[0_{8}\right]$ layered composite beam


Figure 6.8: Transverse tip velocities for undamaged and different position ( 0.25 m and 0.35 m ) of 20 mm de-lamination from free end and along the centerline of $\left[0_{8}\right]$ composite beam

The figure (6.7) and (6.8) shows the axial and transverse tip velocities of undamaged beam and 20 mm de-lamination at different positions from the free end and along the centerline of the $\left[0_{8}\right]$ layered composite beam. From figure (6.7), it is clearly observed that, the axial velocity of undamaged beam and 20 mm de-lamination at different positions from the free end and along the centerline of the composite beam produce the same response. From figure (6.8), it can be observed that the position of reflections due to the de-lamination changes by changing the delamination position.


Figure 6.9: Axial tip velocities for undamaged and different depths above the centerline (h1=h, $\mathrm{h} 1=\mathrm{h} / 2$ and $\mathrm{h} 1=\mathrm{h} / 4$ ) of 20 mm de-lamination of $\left[0_{8}\right]$ layered composite beam

From figure (6.9) it can be observed that the response of undamaged and 20 mm de-lamination at $h_{1}=\mathrm{h}$ i.e. at the centerline of the beam is same in the axial tip velocity. The response, for $\mathrm{h} 1=\mathrm{h} / 2$ and $\mathrm{h} 1=\mathrm{h} / 4$ gives undulations in the axial tip velocity. Where $\mathbf{h} 1$ is the depth of de-lamination from the top surface of the beam.


Figure 6.10: Transverse tip velocities for undamaged and different depths above the centerline (h1 $=\mathrm{h}, \mathrm{h} 1=\mathrm{h} / 2$ and $\mathrm{h} 1=\mathrm{h} / 4$ ) of 20 mm de-lamination of $\left[\mathrm{O}_{8}\right.$ ] layered composite beam


Figure 6.11: Axial tip velocities for different orientation $\left([0]_{8},[30]_{8},[45]_{8}\right.$, and $\left.[60]_{8}\right)$ of 20 mm de-lamination along the centerline of composite beam

Figure (6.10) shows the transverse tip velocity of undamaged $[0]_{8}$ composite beam compared with different depths above the centerline ( $h_{1}=\mathrm{h}, h_{1}=\mathrm{h} / 2$ and $h_{1}=\mathrm{h} / 4$ ) of 20 mm de-lamination of beam. Undulations can be observed from figure (6.10) by reducing the $h_{1}$ by which number of undulations increases and the amplitude of these undulations initially decreases and later on increases. In figure (6.11) and (6.12), comparing the axial and transverse tip velocities for different ply orientation of $[0]_{8},[30]_{8},[45]_{8}$, and $[60]_{8}$ layered composite beam with 20 mm de-lamination along the centerline. From these figures, it can be observed that the amplitude of the undulations increases and shifting of these undulations occurs by increasing the ply orientation.


Figure 6.12: Transverse tip velocities for different orientation $\left([0]_{8},[30]_{8},[45]_{8}\right.$, and $\left.[60]_{8}\right)$ of 20 mm de-lamination along the centerline of composite beam

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