OPTIMIZATION METHODS AND QUADRATIC PROGRAMMING

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DECLARATION

I declare that the topic " Optimization methods and Quadratic Programming " for my M.Sc. degree has not been submitted in any other institution or University for award of any other degree .

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CERTIFICATE

This is to certify that the project Thesis entitled "Optimization methods and Quadratic Programming" submitted by Ms. Richa Singh , Roll no: 410MA2112 for the partial fulfilment of the requirements of M.Sc. degree in Mathematics from National institute of Technology ,Rourkela is a authentic record of review work carried out by her under my supervision and guidance. The content of this dissertation has not been submitted to any other Institute or University for the award of any degree.

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ABSTRACT

Optimization is the process of maximizing or minimizing the objective function which satisfies the given constraints. There are two types of optimization problem linear and nonlinear. Linear optimization problem has wide range of applications, but all realistic problem cannot be modeled as linear program, so here non-linear programming gains its importance. In the present work I have tried to find the solution of non-linear programming Quadratic problem under different conditions such as when constraints are not present and when constraints are present in the form of equality and inequality sign. Graphical method is also highly efficient in solving problems in two dimensions. Wolfe's modified simplex method helps in solving the Quadratic programming problem by converting the quadratic problem in successive stages to linear programming which can be solved easily by applying two – phase simplex method. A variety of problems arising in the area of engineering, management etc. are modeled as optimization problem thus making optimization an important branch of modern applied mathematics.

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Chapter-1

INTRODUCTION TO OPTIMIZATION METHODS & QUADRATIC PROGRAMMING

Optimization constitutes a very important branch of modern applied mathematics. A variety of problems arising in the field of engineering design, operations research, management science, computer science, financial engineering and economics can be modeled as optimization is useful in real life.

It was the development of the simplex method for linear programming by G.B. Dantzig in the mid 40's which in the sense started the subject of mathematical optimization. Another major development was due to H.W. Kuhn and A.W. Tucker in 1951 who gave necessary/sufficient optimality conditions for non-linear programming problem, now known as Karush-Kuhn Tucker(KKT) conditions. In 1939 W. Karush had already developed conditions similar to those given by Kuhn Tucker.

The presence of linearity structure on the given optimization problem gave beautiful mathematical results and also helped greatly in its algorithmic development. However most of the real world applications lead to optimization problems which are inherently nonlinear and are void of linearity. Fortunately most often this nonlinearity is of 'parabola' type leading to the convexity structure which can be used to understand the convex optimization problems or Quadratic programming problem.

Chapter-2

PRE-REQUISITES TO QUADRATIC PROGRAMMING

Some definitions:

- Vector: A vector in n space is an ordered set of n real numbers.
  For e.g. $a=(a_1,a_2, \ldots,a_n)$ is a vector of elements or components.
- Null vector: The null vector is a vector whose elements are all zero.
  $0=(0,0,\ldots,0)$.The null vector corresponds to origin.
- Sum vector : The sum vector is a vector whose elements are all one.
  $1=(1,1,\ldots,1)$.
- Unit vector (e_i) :The unit vector(e_i) is a vector whose i^{th} element is one.
  $e_i=(1,0,\ldots,0)$.
  E^2, there are two unit vectors. E^n, there are n unit vectors.
- Orthogonal vectors: Two vectors a and b are said to be orthogonal if a.b=0
- Linear independence: A set of vectors $a_1,a_2, \ldots,a_k$ is linearly independent if the equation $\lambda_1 a_1 + \lambda_2 a_2 + \ldots\ldots + \lambda_k a_k = 0$ is satisfied only if $\lambda_1 = \lambda_2 = \ldots\ldots = \lambda_k = 0$.
- Linear dependence: A set of vectors which are not linearly independent are called linearly dependent.
- Spanning set: The set of vectors $a_1,a_2, \ldots,a_k$ in E^n is a spanning set in E^n if every vector in E^n can be expressed as a linear combination of vectors $a_1,a_2, \ldots,a_k$. where (k<n).
- Basis set: A set of vectors $a_1,a_2, \ldots,a_k$ in E^n is a basis set if
  i) it is linearly independent set
  ii) it is a spanning set of E^n , if it is a basis then k=n.
- Standard basis : The set of unit vectors $e_1,e_2,e_3,\ldots\ldots e_n$ is called the standard basis for E^n.
- Matrix : A matrix is a rectangular array of ordered numbers, arranged into rows and columns. $A= [a_{ij}]_{mxn}$. The elements a_{ij} for i=j i.e. a_{11}, a_{22}, a_{33} and so on are called principal diagonal elements ,others are called off diagonal elements.
- Square matrix: Any matrix in which number of rows is equal to number of columns is known as square matrix (m=n).
- Diagonal matrix: A square matrix in which all off diagonal elements are zero i.e, $a_{ij} =0$ for $(i\neq j)$ is called a diagonal matrix.
- Identity or Unit matrix : A diagonal matrix whose all principal elements are 1 is called an Identity or unit matrix denoted simply by I.

- Transpose matrix: The transpose of a matrix $A=[a_{ij}]$ denoted by $A^T$is a matrix obtained by interchanging the rows and columns of A.
- Symmetric matrix: A square matrix A is said to be symmetric if the matrix A remains the same by interchanging the rows and columns of A(i.e, $a_{ij}= a_{ji}$ or $A^T=A$)
- Row matrix: A matrix having only a single row is called a row matrix .It is an 1xn matrix.
- Column matrix: A matrix having only a single column is called a column matrix .It is an mx1 matrix.
- Null matrix: A matrix whose all elements are zero is called a null matrix.
- Rank of a matrix: A positive integer r is said to be the rank of a matrix A denoted by $\rho(A)$ if
  - *i)* Matrix A possess at least one r-rowed minor which is not zero.
  - *ii)* Matrix A does not possess any nonzero (r+1)-rowed minor.
- Equivalent matrices: Two matrices A and B are said to be equivalent, if and only if $\rho(A)=\rho(B)$ denoted by $A\sim B$.
- Quadratic forms: Let $x=(x_1,x_2, \ldots,x_n)$ and nxn matrix $A=[a_{ij}]$ then a function of n variables denoted by $f(x_1,x_2, \ldots,x_n)$ or Q(x) is called a quadratic forms in n space if $Q(x)=x^TA\,x = \sum \sum a_{ij}x_ix_j$

  Properties of Quadratic forms:
  - i) Positive definite : A quadratic form Q(x) is positive definite iff Q(x) is positive (>0) for all $x\neq 0$.
  - ii) Positive semi-definite: A quadratic form Q(x) is positive semi definite iff, Q(x) is non-negative (≥ 0) for all x and there exists an $x\neq 0$ for which Q(x)=0 .
  - iii) Negative definite: A quadratic form Q(x) is negative definite iff, -Q(x) is positive definite.
  - iv) Negative semi-definite: A quadratic form Q(x) is negative semi-definite iff, -Q(x) is positive semi- definite.
  - v) Indefinite: A quadratic form Q(x) is indefinite if Q(x)is positive for some x and negative for some other.
- Difference equation: An equation relating the values of a function y and one or more of its differences Δy ,$\Delta^2 y$,… for each value of a set of numbers is called a difference equation.
- Order of difference equation: The difference between the highest and the lowest suffix of the equation is called the order of difference equation.

  $Y_{k+1} + 3y_k=0$

  Here, k+1-k=1 is the order of the difference equation.

- Feasible solution: Solution values of the decision variables x_j(j=1,2,3,……,n) which satisfy the constraints and non-negativity conditions is known as feasible solution.
- Basic feasible solution: Collection to all feasible solutions to a problem constitutes a convex set whose extreme points correspond to the basic feasible solution.
- Extreme points to a convex set: A point x in a convex set c is called an extreme point if x cannot be expressed as a convex combination of any two distinct points $x^{(1)}$ and $x^{(2)}$ in c.

Chapter-3

CONVEX FUNCTIONS AND THEIR PROPERTIES

Definition

- <u>Convex functions</u>: Let $S \subseteq R^n$ be a convex set and f: S →R. Then f is called a convex function if for all x, u ∈ S and for all $0 \le \lambda \le 1$,we have $f(\lambda x + (1-\lambda)u) \le \lambda f(x) + (1-\lambda) f(u)$

  Some examples of convex functions are:

  $i) f(x) = x^2, x \in R$

  $ii) f(x) = |x|, x \in R$

  $iii) f(x) = e^x, x \in R$

  $iv) f(x) = e^{-x}, x \in R$

  $v) f(x) = -\sqrt{1-x^2}, -1 \le x \le 1$

- <u>Concave functions</u>: Let $S \subseteq R^n$ be a convex set and f: S →R. Then f is called a concave function if for all x, u ∈ S and for all $0 \le \lambda \le 1$,we have $f(\lambda x + (1-\lambda)u) \ge \lambda f(x) + (1-\lambda) f(u)$.

  Some examples of convex functions are:

  $i) f(x) = \ln x$, x>0

  $ii) f(x) = -|x|, x \in R$

  $iii) f(x) = +\sqrt{1-x^2}, -1 \le x \le 1$

- <u>Properties</u>

  1. If a function is both convex and concave, then it has to be a linear function.
  2. A function may be neither convex nor concave.

  e.g. $f(x) = \sin x, -\frac{\pi}{2} \le \frac{\pi}{2}$ or $f(x) = x^3, x \in R$

  3. The domain of a convex function has to be a convex set.
  4. A convex/concave function need not be differentiable.

e.g. $f(x) = |x|, x \in R$ is convex but not differentiable at x=0.

5. Convex functions need not even be continuous.

e.g. f(x)={x^2 if -1≤x≤1} and {2 if x=1}

It is not continuous at x=2. However, convex functions are always continuous in the interior of its domain.

6. If f and g are two convex functions defined over a convex set $S \subseteq R^n$ then

i) f+g

ii) αf(α≥0)

iii) h(x)=Max $x \in S$ (f(x),g(x)) are convex functions.

7. If f and g are two concave functions defined over a convex set $S \subseteq R^n$ then

i) f+g

ii) αf(α≥0)

iii) h(x)=Min $x \in S$ (f(x),g(x)) are concave functions.

Chapter-4

UNCONSTRAINED PROBLEMS OF OPTIMIZATION

Some important results:

- A necessary condition for a continuous function $f(x)$ with continuous first and second partial derivatives to have an extreme point at $x_0$is that each first partial derivative of f(x),evaluated at $x_0$vanish
  i.e.
  $$\nabla f(x_0) = 0 \; where \; \nabla \equiv \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right]$$
  is the gradient vector.
- A sufficient condition for a stationary point x_0 to be an extreme point is that the Hessian matrix H evaluated at x_0 is
  i) Negative definite when x_0 is a maximum point and
  ii) Positive definite when x_0 is minimum point.

Example: Find the maximum or minimum of the function

$$f(x) = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 8x_2 - 12x_3 + 56$$

Applying the necessary condition

$$\Delta f(x_0) = 0$$

where $\nabla \equiv \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right] = (0,0,0)$

$$\frac{\partial f}{\partial x_1} = 2x_1 - 4 = 0,$$
$$\frac{\partial f}{\partial x_2} = 2x_2 - 8 = 0,$$
$$\frac{\partial f}{\partial x_3} = 2x_3 - 12 = 0$$

The solution of these simultaneous equations is given by

$x_0 = (2,4,6)$ is the only point that satisfies the necessary condition.

Now by checking the sufficiency condition we have to determine whether this point is maxima or minima.

Hessian matrix, evaluated at (2, 4, 6) is given by

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The principal minor determinants of H:

$$|2|, \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

have the values 2, 4, 8 respectively. Thus, each principal minor determinant is positive. Hence, this is positive definite and the point (2, 4, 6) yields a minimum of f(x).

Chapter-5

CONSTRAINED OPTIMIZATION WITH EQUALITY CONSTRAINTS

Lagrangian method

In non-linear programming problem if objective function is differentiable and has equality constraints optimization can be achieved by the use of Lagrange multipliers.

Formulation

Consider the problem of maximizing or minimizing $z = f(x_1,x_2)$ subject to the constraints $g(x_1,x_2) = c$ and $x_1,x_2 \geq 0$ where c is a constant. We assume that $f(x_1,x_2)$ and $g(x_1,x_2)$ are differentiable w.r.t x_1 and x_2. Let us introduce a differentiable function $h(x_1,x_2)$ differentiable w.r.t x_1 and x_2 and defined by $h(x_1,x_2) \equiv g(x_1,x_2) - c$.

The problem is restated as maximize $z = f(x_1,x_2)$ subject to the constraints $h(x_1,x_2)=0$ and $x_1,x_2 \geq 0$.

To find the necessary conditions for a maximum (or minimum) value of z ,a new function is formed by introducing a Lagrange multiplier λ , as $L(x_1,x_2,\lambda) = f(x_1,x_2) - \lambda h.(x_1,x_2)$.

The number λ is an unknown constant and the function $L(x_1,x_2,\lambda)$ is called the Lagrangian function with Lagrange multiplier λ. The necessary conditions for a maximum or minimum of $f(x_1,x_2)$ subject to $h(x_1,x_2)=0$ are thus given by

Necessary condition

$$\frac{\partial L(x_1,x_2,\lambda)}{\partial x_1} = 0$$

$$\frac{\partial L(x_1,x_2,\lambda)}{\partial x_2} = 0$$

$$\frac{\partial L(x_1,x_2,\lambda)}{\partial \lambda} = 0$$

Their partial derivatives are given by

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2}$$

$\frac{\partial L}{\partial \lambda} = -h$ where L ,f and h stand for the functions.

The necessary conditions for maximum or minimum of $f(x_1,x_2)$ are given by $f_1 = \lambda h_1$; $f_2 = \lambda h_2$ and $-h(x_1,x_2)=0$.

Sufficient condition

Let the Lagrangian function for n variables and one constraint be $L(x,\lambda) = f(x) - \lambda h.(x)$.

The necessary conditions for a stationary point to be a maximum or minimum are

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \qquad (j=1,2,\ldots,n) \text{ and}$$

$$\frac{\partial L}{\partial \lambda} = -h(x) = 0$$

The value of λ is obtained by $\lambda = \frac{\partial f / \partial x_j}{\partial h / \partial x_j}$ (for j=1,2,….,n)

The sufficient conditions for a maximum or minimum require the evaluation at each stationary point of n-1 principal minors of the determinant given:

$$\begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x2} & \ldots\ldots\ldots & \frac{\partial h}{\partial xn} \\ \frac{\partial h}{\partial x1} & \frac{\partial^2 f}{\partial x1^2} - \lambda \frac{\partial^2 h}{\partial x1^2} & \frac{\partial^2 f}{\partial x1 \partial x2} - \lambda \frac{\partial^2 h}{\partial x1 \partial x2} & \ldots\ldots\ldots & \frac{\partial^2 f}{\partial x1 \partial xn} - \lambda \frac{\partial^2 h}{\partial x1 \partial xn} \\ \frac{\partial h}{\partial x2} & \frac{\partial^2 f}{\partial x2 \partial x1} - \lambda \frac{\partial^2 h}{\partial x2 \partial x1} & \frac{\partial^2 f}{\partial x2^2} - \lambda \frac{\partial^2 h}{\partial x2^2} & \ldots\ldots\ldots & \frac{\partial^2 f}{\partial x2 \partial xn} - \lambda \frac{\partial^2 h}{\partial x2 \partial xn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h}{\partial xn} & \frac{\partial^2 f}{\partial xn \partial x1} - \lambda \frac{\partial^2 h}{\partial xn \partial x1} & \frac{\partial^2 f}{\partial xn \partial x2} - \lambda \frac{\partial^2 h}{\partial xn \partial x2} & \ldots\ldots\ldots & \frac{\partial^2 f}{\partial xn^2} - \lambda \frac{\partial^2 h}{\partial xn^2} \end{vmatrix} = \Delta_{n+1}$$

If $\Delta_3 > 0$, $\Delta_4 < 0$, $\Delta_5 > 0$, the signs pattern being alternate ,the stationary point is a local maximum .If $\Delta_3 < 0$, $\Delta_4 < 0$,…….. $\Delta_{n+1} < 0$, the sign being always negative, the stationary point is a local minimum.

Example:

Obtain the set of necessary and sufficient conditions for the following NLP *Minimize* $z = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$ subject to the constraints: $x_1 + x_2 + x_3 = 11; \quad x_1, x_2, x_3 \geq 0$

Solution: We formulate the Lagrangian function as

$$L(x_1, x_2, \lambda) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200 - \lambda(x_1 + x_2 + x_3 - 11).$$

The necessary conditions for the stationary point are

$$\frac{\partial L}{\partial x_1} = 4x_1 - 24 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 8 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 11) = 0$$

The solution to the simultaneous equations yields the stationary point $x_0 = (x_1, x_2, x_3) = (6,2,3)\,; \lambda = 0$ .The sufficient condition for stationary point to minimum is that both the minors Δ_3 and Δ_4 should be negative.

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 4 & 0 \end{vmatrix} = -8 \quad \Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = -48$$

Since Δ_3 and Δ_4 both are negative , $x_0 = (6,2,3)$ provides the solution to the NLPP. The stationary point is local minimum .Thus, $x_0 = (6,2,3)$ provides the solution to NLPP.

Sufficient conditions for a NLPP with more than one equality constraints

Optimize $z = f(x), x \in R^n$ subject to the constraints $g_i(x) = 0, i = 1,2,....,m$ *and* $x \geq 0$ $L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$ $(m < n)$

where m= number of equality constraints = number of Lagrangian multipliers

n = number of unknowns

$$\frac{\partial L}{\partial x_j} = 0 \text{ for } j = 1,2,...,n$$

$$\frac{\partial L}{\partial \lambda_j} = 0 \text{ for } j = 1,2,....,m$$

Provide the necessary conditions for stationary points of $f(x)$. The function $L(x,\lambda)$, $f(x)$ and $g(x)$ all possess partial derivatives of first and second order with respect to the decision variables.

$M = \left| \frac{\partial^2 L(x,\lambda)}{\partial x_i \partial x_j} \right|_{nxn}$ for all i and j be the matrix of second order partial derivatives of $L(x,\lambda)$ w.r.t decision variables.

$V = \left| \frac{\partial \ g_i(x)}{\partial x_j} \right|_{mxn}$ where i=1,2,…..,m; j=1,2,……,n

Define the square matrix $H_B = \begin{vmatrix} 0 & V \\ V^T & M \end{vmatrix}_{(m+n)x(m+n)}$ where O is an mxm null matrix. The matrix H_B is called the bordered Hessian matrix .Then the sufficient conditions for maximum and minimum is: $(x^*, \boldsymbol{\lambda}^*)$ be the stationary point for the Lagrangian function L (x,λ) and H_B^* be the value of corresponding bordered Hessian matrix

i) X * is a maximum point , if starting with principal minor of order (m+1), the last (n-m) principal minors of H_B^* form an alternating sign pattern starting with $(-1)^{m+n}$

ii) X * is a minimum point , if starting with principal minor of order (2m+1), the last (n-m) principal minors of H_B^* have the sign of $(-1)^m$

<u>Example:</u> Optimize $z = 4x_1^{\ 2} + 2x_2^{\ 2} + x_3^{\ 2} - 4x_1x_2$ such that

$x_1 + x_2 + x_3 = 15; 2x_1 - x_2 + 2x_3 = 20; \ x_1, x_2, x_3 \geq 0$

Solution: $z = f(x) = 4x_1^{\ 2} + 2x_2^{\ 2} + x_3^{\ 2} - 4x_1x_2$ such that

$g_1(x) = x_1 + x_2 + x_3 - 15;$
$g_2(x) = 2x_1 - x_2 + 2x_3 - 20; \ x_1, x_2, x_3 \geq 0$

The Lagrangian function is given by

$$L(x,\lambda) = f(x) - \lambda_1 g_1(x) - \lambda_2 g_2(x)$$
$$= 4x_1^{\,2} + 2x_2^{\,2} + x_3^{\,2} - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

The stationary point (x^*, $\boldsymbol{\lambda}^*$) can be obtained by the following necessary conditions

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$
$$\frac{\partial L}{\partial x_2} = 4x_2 - \lambda_1 + \lambda_2 - 4x_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$
$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$
$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$$
$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

Solving equation (i) and (v) we get

$$x_1 = \frac{2\lambda_1 + \lambda_2}{4}$$
$$x_2 = \frac{2\lambda_1}{4}$$
$$x_3 = \frac{\lambda_1 + 2\lambda_2}{2}$$

Substituting the values of x_1, x_2, x_3 in equation (iv) and (v) we get

$$7\lambda_1 + 5\lambda_2 = 60 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (vi)$$
$$5\lambda_1 + 10\lambda_2 = 80 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (vii)$$

Solving equation (vi) and (vii) we get

$$\lambda_1 = \frac{40}{9}; \lambda_2 = \frac{52}{9}; x_1 = \frac{33}{9}; x_2 = \frac{10}{3}; x_3 = 8$$

$$x^* = (x_1, x_2, x_3) = (\frac{33}{9}, \frac{10}{3}, 8)$$
$$\lambda^* = (\lambda_1, \lambda_2) = (\frac{40}{9}, \frac{52}{9}))$$

For this stationary point (x^*, λ^*) the bordered Hessian matrix is given by

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \text{.............................}(i)$$
$$\frac{\partial L}{\partial x_2} = 4x_2 - \lambda_1 + \lambda_2 - 4x_1 = 0 \text{.............................}(ii)$$
$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 \text{.......................................}(iii)$$
$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15) = 0 \text{..............................}(iv)$$
$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20) = 0 \text{............................}(v)$$

Solving equation (i) and (v) we get

$$x_1 = \frac{2\lambda_1 + \lambda_2}{4}$$
$$x_2 = \frac{2\lambda_1}{4}$$
$$x_3 = \frac{\lambda_1 + 2\lambda_2}{2}$$

Substituting the values of x_1, x_2, x_3 in equation (iv) and (v) we get

$$7\lambda_1 + 5\lambda_2 = 60 \text{...}(vi)$$
$$5\lambda_1 + 10\lambda_2 = 80 \text{..}(vii)$$

Solving equation (vi) and (vii) we get

$$\lambda_1 = \frac{40}{9}; \lambda_2 = \frac{52}{9}; x_1 = \frac{33}{9}; x_2 = \frac{10}{3}; x_3 = 8$$

$$x^* = (x_1, x_2, x_3) = (\frac{33}{9}, \frac{10}{3}, 8)$$
$$\lambda^* = (\lambda_1, \lambda_2) = (\frac{40}{9}, \frac{52}{9}))$$

For this stationary point (x^*, λ^*) the bordered Hessian matrix is given by the following necessary conditions

$$\frac{\partial L}{\partial x_1} = 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - \lambda_1 + \lambda_2 - 4x_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - 2\lambda_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 15) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

$$\frac{\partial L}{\partial \lambda_2} = -(2x_1 - x_2 + 2x_3 - 20) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (v)$$

Solving equation (i) and (v) we get

$$x_1 = \frac{2\lambda_1 + \lambda_2}{4}$$

$$x_2 = \frac{2\lambda_1}{4}$$

$$x_3 = \frac{\lambda_1 + 2\lambda_2}{2}$$

Substituting the values of x_1, x_2, x_3 in equation (iv) and (v) we get

$$7\lambda_1 + 5\lambda_2 = 60 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (vi)$$

$$5\lambda_1 + 10\lambda_2 = 80 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (vii)$$

Solving equation (vi) and (vii) we get

$$\lambda_1 = \frac{40}{9}; \lambda_2 = \frac{52}{9}; x_1 = \frac{33}{9}; x_2 = \frac{10}{3}; x_3 = 8$$

$$x^* = (x_1, x_2, x_3) = (\frac{33}{9}, \frac{10}{3}, 8)$$

$$\lambda^* = (\lambda_1, \lambda_2) = (\frac{40}{9}, \frac{52}{9}))$$

For this stationary point (x^*, λ^*) the bordered Hessian matrix is given by

$$H_B = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix} = 72$$

n-m=3-2=1

2m+1=2x2+1=5

The determinant$|H*_B|$ has sign of $(-1)^2$ i.e. positive.Therefore x* is a minimum point.

Chapter-6

CONSTRAINTS IN THE FORM OF INEQUALITIES

(Kuhn – Tucker Necessary conditions)

Maximize f(x), x =($x_1,x_2,\ldots\ldots,x_n$) subject to m number of inequalities constraints

$g_i(x) \le b_i$,i=1,2,…..,m. including the non-negativity constraints $x \ge 0$ which are written as $-x \le 0$, the necessary conditions for a local maxima or stationary point(s) at $\bar{x}$ are

i) $\frac{\partial L}{\partial x_j}(\bar{x},\bar{\boldsymbol{\lambda}},\bar{S})=0$ j=1,2,…..,n

ii) $\bar{\lambda}_i\,[g_i(\bar{x}) - b_i]=0$

iii) $g_i(\bar{x}) \le b_i$

iv) $\bar{\boldsymbol{\lambda}}_i \ge 0$ i=1,2,……..,m.

(Kuhn-Tucker Sufficient conditions)

The Kuhn-Tucker conditions which are necessary are also sufficient if f(x) is concave and the feasible space is convex , i.e. if f(x) is strictly concave and $g_i(x)$, i=1,2,….m are convex.

Example:

Determine x_1, x_2, x_3 so as to maximize $z=-x_1^2-x_2^2-x_3^2+4x_1+6x_2$ subject to constraints $x_1+x_2 \le 2; 2x_1+3x_2 \le 12;\ x_1, x_2 \ge 0$

Solution: $f(x)=-x_1^2-x_2^2-x_3^2+4x_1+6x_2$ $x \in R^n$

$g_1(x)=x_1+x_2-2;$
$g_2(x)=2x_1+3x_2-12; x_1, x_2, x_3 \ge 0$

First we decide about the concavity and convexity of f(x)

$$H_B=\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} \text{n=3, m=2,n-m=1}$$

Therefore $|H_B| = -8 < 0$.Thus, f(x) is concave. Clearly $g_1(x)$ and $g_2(x)$ are convex in x .Thus the Kuhn-Tucker conditions will be the necessary and sufficient conditions for a maximum. These conditions are obtained by partial derivatives of Lagrangian function.

$L(x,\lambda,s) = f(x) - \lambda_1[g_1(x)+s_1^2] - \lambda_2[g_2(x)+s_2^2]$ where $s=(s_1,s_2)$, $\boldsymbol{\lambda}=(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$ and s_1,s_2 being slack variables and $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2$ are Lagrangian multipliers. The Kuhn-Tucker conditions are given by

$$a) L(x,\lambda,s) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2 - \lambda_1(x_1 + x_2 - 2 + s_1^2) - \lambda_2(2x_1 + 3x_2 - 12 + s_2^2)$$

$$i) \frac{\partial L}{\partial x_1} = 2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0.$$

$$ii) \frac{\partial L}{\partial x_2} = -2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0$$

$$iii) \frac{\partial L}{\partial x_3} = -2x_3 = 0$$

$$b) i) \lambda_1(x_1 + x_2 - 2) = 0.$$

$$ii) \lambda_2(2x_1 + 3x_2 - 12) = 0.$$

$$c) i) x_1 + x_2 - 2 \le 0$$

$$ii) 2x_1 + 3x_2 - 12 \le 0$$

$$d) \lambda_1 \ge 0, \lambda_2 \ge 0$$

Now, four different cases may arise:

Case 1: ($\boldsymbol{\lambda}_1$=0, $\boldsymbol{\lambda}_2$=0)

In this case, the system (a) of equations give: $x_1=2$, $x_2=3$,$x_3=0$. However, this solution violates both the inequalities of (c) given above.

Case2: ($\boldsymbol{\lambda}_1$=0, $\boldsymbol{\lambda}_2\neq0$)

In this case, (b) gives $2x_1+3x_2=12$ and a(i) and (ii) give $-2x_1+4=2\boldsymbol{\lambda}_2$, $-2x_2+6=3\boldsymbol{\lambda}_2$

Solution of these simultaneous equations gives $x_1=24/13$,$x_2=36/13$,$\boldsymbol{\lambda}_2=2/13>0$ and also equation (a) (iii) gives $x_3=0$.This solution violates c (i). So, this solution is discarded.

Case 3: ($\boldsymbol{\lambda}_1\neq0$, $\boldsymbol{\lambda}_2\neq0$)

In this case, (b) (i) and (ii) gives $x_1+x_2=2$ and $2x_1+3x_2=12$. These equations give $x_1=-6$ and $x_2=8$. Thus (a)(i), (ii), (iii) yield $x_3=0$,$\boldsymbol{\lambda}_1=68$,$\boldsymbol{\lambda}_2=-26$ .Since $\boldsymbol{\lambda}_2=-26$ violates the condition (d). So, this solution is discarded.

Case 4: ($\lambda_1 \neq 0$, $\lambda_2 = 0$)

In this case (b) (i) gives $x_1+x_2=2$ .This together with (a) (i) and (ii) gives $x_1=1/2$,$x_2=3/2$,$\lambda_1=3>0$. Further from a (iii) $x_3=0$.This solution does not violate any of the Kuhn-Tucker conditions. Hence, the optimum (maximum) solution to the given problem is :

$x_1=1/2$, $x_2=3/2$,$x_3=0$; with $\lambda_1=3$, $\lambda_2=0$ the maximum value of the objective function is $z=17/2$.

Chapter-7

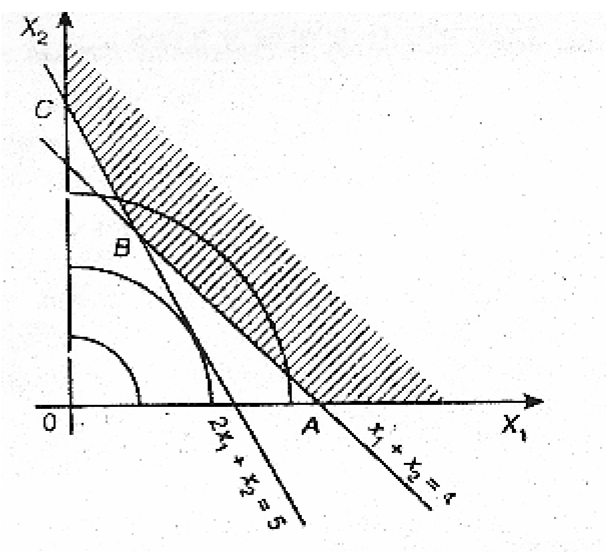
GRAPHICAL METHOD (Non-linear objective function and linear constraints)

Example: Minimize the distance of the origin from the convex region bounded by the constraints $x_1 + x_2 \geq 4; 2x_1 + x_2 \geq 5$; and $x_1, x_2 \geq 0$. Verify that Kuhn-Tucker necessary conditions hold at the point of minimum distance.

Solution: Minimizing the distance of the origin from the convex region is equivalent to finding the length of radius i.e. minimum distance from origin to the tangent which just touches the convex region and is bounded by the given constraints

i.e. $\min(r^2 = z) = x_1^{\ 2} + x_2^{\ 2}$ such that $x_1 + x_2 \geq 4; 2x_1 + x_2 \geq 5$; and $x_1, x_2 \geq 0$

The feasible region will lie in the first quadrant as $x_1, x_2 \geq 0$ .We plot the lines $x_1 + x_2 = 4; 2x_1 + x_2 = 5$ .The region shaded by the lines is the unbounded convex feasible region. We have to search for a point (x_1, x_2) which gives a minimum value of $x_1^{\ 2} + x_2^{\ 2}$ and lies in the feasible region



The (slope)gradient of the tangent to the tangent to the circle $x_1^{\ 2} + x_2^{\ 2} = k$

$$2x_1 + 2x_2 \frac{dx_2}{dx_1} = 0$$

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$

Slope of the line $x_1 + x_2 = 4;$ is -1 and slope of the line $2x_1 + x_2 = 5$ is -2 .

Case 1: If the line $x_1 + x_2=4$ is tangent to the circle $x_1^2 + x_2^2 = k$ then $\frac{dx_2}{dx_1} = -\frac{x_1}{x_2} = -1$ then $x_1=x_2$. On solving $x_1 + x_2=4$ and $x_1=x_2$ we get $x_1=2$ and $x_2=2$.The line touches the circle at point (2,2).

Case2:IF the line $2x_1 + x_2 = 5$ is tangent to the circle $x_1^2 + x_2^2 = k$ $\frac{dx_2}{dx_1} = -\frac{x_1}{x_2} = -2$ then $x_1=2x_2$. On solving $2x_1 + x_2 = 5$ and $x_1=2x_2$ we get $x_1=2, x_2=1$.The line touches the circle at the point (2,1).

Out of these two points (2,1) lies outside the feasible region ,but point (2,2) lies in the feasible region. So , $\min z = x_1^2 + x_2^2 = 2^2 + 2^2 = 8$, $x_1 = 2, x_2 = 2$

Verification of Kuhn-Tucker condition : To verify (2,2) satisfies Kuhn-Tucker conditions

$f(x) = x_1^2 + x_2^2;$

$g_1(x) = x_1 + x_2 - 4;$
$g_2(x) = 2x_1 + x_2 - 5; x_1, x_2 \geq 0$

$L(x,\lambda,s) = f(x) - \lambda_1[g_1(x) + s_1^2] - \lambda_2[g_2(x) + s_2^2]$ where $s=(s_1,s_2)$, $\boldsymbol{\lambda}=(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$ and s_1 , s_2

Being slack variables and $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ are Lagrangian multipliers .The Kuhn-Tucker conditions are given by $L(x,\lambda,s) = x_1^2 + x_2^2 - \lambda_1(x_1 + x_2 - 4) - \lambda_2(2x_1 + x_2 - 5)$

$a) i) \frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 2\lambda_2 = 0.$

$ii) \frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - \lambda_2 = 0$

at (2,2) solving we get $\lambda_1 = 4, \lambda_2 = 0$

$b) i) \lambda_1(x_1 + x_2 - 4) = 0.$

$ii) \lambda_2(2x_1 + x_2 - 5) = 0.$

$c) i) x_1 + x_2 - 4 \geq 0$

$ii) 2x_1 + x_2 - 5 \geq 0$

$d) \lambda_1 \geq 0, \lambda_2 \geq 0$

(2,2) satisfies a), b), c) conditions of the Kuhn-Tucker for minima.

Hence, Min z= 8, $x_1=2$, $x_2=2$ is the solution and it satisfies Kuhn- Tucker conditions.

Chapter-8

QUADRATIC PROGRAMMING (Wolfe's method)

$Max\ z = f(x) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k$ subject to the constraints :

$\sum_{j=1}^{n} \mathrm{a}_{ij} x_j \le b_j, x_j \ge 0 (i = 1,2,............, m;\ j = 1,2,............, n)$ where $c_{jk} = c_{kj}$

For all j and k , $b_i \geq 0$ for all i=1,2,………,m. Also assume the quadratic form $\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k$ be negative semi-definite.

Outline of the iterative procedure is

Step 1: First we convert the inequality constraints into equations by introducing slack variables q_1^2 in the i^{th} constraint (i=1,2,3,………,m) and the slack variables r_j^2 in the j^{th} non-negativity constraint (j=1,2,……,n).

Step 2: Then, we construct the Lagrangian function

$$L(x,q,r,\lambda,\mu) = f(x) - \sum_{i=1}^{m} \lambda_i [\sum_{i=1}^{n} a_{ij} x_j - b_i + q_i^2] - \sum_{j=1}^{n} \mu_j [-x_j + r_j^2]$$

$$where\ x = (x_1, x_2, ..., x_n),$$

$$q = (q_1^2, ..., q_m^2)$$

$$r = (r_1^2, ..., r_n^2)$$

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$$

$$x = (\mu_1, \mu_2, ..., \mu_n),$$

Differentiating L partially w.r.t. the components of x , q , r, $\boldsymbol{\lambda}$, μ and equating the first order partial derivatives to zero , Kuhn-Tucker conditions are obtained.

Step 3: We introduce the non-negative artificial variable v_j , j =1,2,…..,n in the Kuhn-Tucker conditions

$c_j + \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} a_{ij} \lambda_i + \mu_j = 0$ for $= 1,2,....., \mathrm{n}$ and to construct an objective function

$$z_v = v_1 + v_2 + ... + v_n.$$

Step 4: We obtain the initial basic feasible solution to the following linear programming problem min $z_v = v_1 + v_2 + ... + v_n$ subject to the constraints

$$\sum_{k=1}^{n} x_k c_{jk} - \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} a_{ij}\lambda_i + \mu_j + v_j = -c_j \quad \text{for} \quad (\text{j} = 1,2,.....,\text{n})$$

$$\sum_{j=1}^{n} a_{ij} x_j + q_i^{\ 2} = b_i$$

$$where \ i = (1,2,..........., m), v_j, \lambda_i, \mu_j, x_j \geq 0 (i = 1,2,........, m; j = 1,2,.........., n)$$

and satisfying the complementary slackness condition :

$$\sum_{j=1}^{n} \mu_j \text{x}_j + \sum_{i=1}^{m} \lambda_i s_i = 0 \ (where \ s_i = q_i^{\ 2}) \ \lambda_i s_i = 0 \ and \ \mu_j \text{x}_j = 0$$

$$(i = 1,2,........, m; j = 1,2,.........., n)$$

Step 5: Now we apply 2- phase simplex method to find and optimum solution of Linear Programming problem in step 4 .The solution must satisfy the above complementary slackness condition.

Step 6 : Thus the optimum solution obtained in step 5 is the optimal solution of the given Quadratic programming problem (QPP).

Example:

$$Max\ z = 8x_1 + 10x_2 - 2x_1^{\ 2} - x_2^{\ 2} \text{ subject to}$$

$$3\text{x}_1 + 2\text{x}_2 \leq 6 \text{ and } \text{x}_1, \text{x}_2 \geq 0$$

Solution: We convert all the inequality constraints to $\leq$

$$3\text{x}_1 + 2\text{x}_2 \leq 6 \text{ and } -\text{x}_1 \leq 0, -\text{x}_2 \leq 0$$

Now we introduce slack variables

$$3\text{x}_1 + 2\text{x}_2 + q_1^{\ 2} = 6 \text{ and}$$

$$-\text{x}_1 + r_1^{\ 2} = 0,$$

$$-\text{x}_2 + r_2^{\ 2} = 0$$

So the problem now becomes

$$Max\ z = 8x_1 + 10x_2 - 2x_1^{\ 2} - x_2^{\ 2}$$

$3x_1 + 2x_2 + q_1^{\ 2} = 6$ and
$-x_1 + r_1^{\ 2} = 0,$
$-x_2 + r_2^{\ 2} = 0$

To obtain the Kuhn-Tucker condition we construct Lagrange function

$$L(x_1, x_2, \lambda_1, \mu_1, \mu_2, q_1, r_1, r_2) = (8x_1 + 10x_2 - 2x_1^{\ 2} - x_2^{\ 2}) - \lambda_1(3x_1 + 2x_2 + q_1^{\ 2} - 6)$$
$$- \mu_1(-x_1 + r_1^{\ 2}) - \mu_2(-x_2 + r_2^{\ 2})$$

The necessary and sufficient conditions are

$$\frac{\partial L}{\partial x_1} = 8 - 4x_1 - 3\lambda_1 + \mu_1 = 0$$
$$\frac{\partial L}{\partial x_2} = 10 - 2x_2 - 2\lambda_1 + \mu_2 = 0$$

Defining $s_1 = q_1^{\ 2}$ we have $\boldsymbol{\lambda}_1 s_1 = 0$ $, \mu_1 x_1 = 0$

$3x_1 + 2x_2 + s_1 = 6 \ x_1, x_2, \lambda_1, \mu_1, \mu_2, s_1 \geq 0$

Modified linear programming

Now introducing artificial variables v1 and v2 we have

$$Max \ \ v_z = -v_1 - v_2$$
$$s.t \ \ 4x_1 + 3\lambda_1 - \mu_1 + v_1 = 8$$
$$2x_2 + 2\lambda_1 - \mu_2 + v_2 = 10$$
$$3x_1 + 2x_2 + s_1 = 6$$

Table 1

BV	C_B	X_B	$X_1(0)$	$X_2(0)$	$\boldsymbol{\lambda}_1(0)$	$\mu_1(0)$	$\mu_2(0)$	$V_1(-1)$	$V_2(-1)$	$S_1(0)$
V_1	-1	8	4	0	3	-1	0	1	0	0
V_2	-1	10	0	2	2	0	-1	0	1	0
S_1	0	6	3	2	0	0	0	0	0	1
	Zv=-18		-4	-2	-5	1	1	0	0	0

$\boldsymbol{\lambda}_1$ cannot be the entering variable, since s_1 is basic variable $\boldsymbol{\lambda}_1 s_1=0$. So x_1 is the entering variable, since μ_1 is not basic variable. (x_2 can also be the entering variable as μ_2 is not basic variable) Min ratio (8/4,6/3) there is a tie. So we take $x_2$as entering variable. Min ratio(10/2,6/2)

Table 2

BV	C_B	X_B	$X_1(0)$	$X_2(0)$	$\boldsymbol{\lambda}_1(0)$	$\mu_1(0)$	$\mu_2(0)$	$V_1(-1)$	$V_2(-1)$	$S_1(0)$
V_1	-1	8	4	0	3	-1	0	1	0	0
V_2	-1	4	-3	0	2	0	-1	0	1	0
X_2	0	3	3/2	1	0	0	0	0	0	½
	Zv=-12		-1	0	-5	1	1	0	0	0

Now $\boldsymbol{\lambda}_1$ can enter as s_1 is not basic variable. Leaving variable (8/3,4/2) is v_2.

Table 3

BV	C_B	X_B	$X_1(0)$	$X_2(0)$	$\boldsymbol{\lambda}_1(0)$	$\mu_1(0)$	$\mu_2(0)$	$V_1(-1)$	$V_2(-1)$	$S_1(0)$
V_1	-1	2	17/2	0	0	-1	3	1	-3	0
$\boldsymbol{\lambda}_1$	0	2	-3/2	0	1	0	-1	0	1	0
X_2	0	3	3/2	1	0	0	0	0	0	½
	Zv=-2		-17/2	0	0	1	-3	0	4	0

Min. ratio (4/17,2)

Table 4

BV	C_B	X_B	$X_1(0)$	$X_2(0)$	$\lambda_1(0)$	$\mu_1(0)$	$\mu_2(0)$	$V_1(-1)$	$V_2(-1)$	$S_1(0)$
x_1	0	4/17	1	0	0	-2/17	6/17	2/17	-6/17	0
λ_1	0	40/17	0	0	1	-3/17	8/17	20/17	-6/17	0
X_2	0	45/17	0	1	0	3/17	-9/17	-3/17	9/17	35/34
	Zv=0		0	0	0	0	0	1	1	0

The optimum solution is x_1=4/17, x_2=45/17, λ_1=40/17.

$$v_2 = v_1 = \mu_1 = \mu_2 = s_1 = 0$$

This satisfies the condition $\lambda_1 s_1=0$, $\mu_1 x_1=0$, $\mu_2 x_2=0$ and the restriction sign of the Lagrangian multipliers. So, the maximum value of z is max(z) = 6137/289.

Conclusion

Present work demonstrates methods to solve the optimization problems which are of Quadratic in nature .As discussed earlier concept of convex functions have been used to solve the optimization problems.

Three different cases are considered when the problems are :

i) Unconstrained
ii) Constrained in form of equality
iii) Constrained in form of inequalities

Graphical method has proved very efficient in solving problems in two dimensions.

Wolfe's method converts the Quadratic programming to linear programming in successive steps which can be solved easily by two phase simplex method. Thus, the Quadratic Programming Problems can be handled easily.

References

(1) Mokhter S Bazaraa , Hanif .D. Sherali, C. N. Shetty, ***Nonlinear Programming Theory and Algorithms*** , John Wiley and Sons Third edition.

(2) Suresh Chandra ; Jayadeva, Aparna Mehera, ***Numerical Optimization with Application***, Narosa Publishing House,Reprint 2011

(3) S.D Sharma , Himanshu Sharma, ***Operations Research*** Kedar Nath Ram Nath,1972

(4) Kanti Swarup, P.K.Gupta, Man Mohan,***Operations Research*** ,Sultan Chand& Sons,1978.