Numerical solutions of integral equations by using CAS wavelets

A THESIS

Submitted **by partial fulfillment of the**

requirement of the award of the degree of

Master of Science

In

Mathematics

By

Deo kumari sahu

Under the supervision of

Prof. S. Saha Ray

May, 2012

DEPARTMENT OF MATHEMATICS

NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA-769008

ODISHA, INDIA

NATIONAL INSTITUTE OF TECHNOLOGY, ROURKELA

CERTIFICATE

This is to certify that the project Thesis entitled “Numerical solutions of integral equations by using CAS wavelets” submitted by Deo kumari sahu, Roll no: 410ma2105 for the partial fulfillment of the requirements of M.Sc. degree in Mathematics from National institute of Technology, Rourkela is a authentic record of review work carried out by her under my supervision and guidance. The content of this dissertation has not been submitted to any other Institute or University for the award of any degree.

Dr. S. Saha Ray

Associate Professor

Department of Mathematics

National Institute of Technology Rourkela,

Rourkela-769008

Odisha, India

DECLARATION

I declare that the topic " Numerical solutions of integral equations by using CAS wavelets" for my M.Sc. degree has not been submitted in any other institution or University for award of any other degree .

Place:

Date:

Deo kumari sahu

Roll no. 410ma2105

Department of Mathematics

National Institute of Technology

Rourkela-769008

Odisha, India

ACKNOWLEDGEMENT

I would like to warmly acknowledge and express my deep sense of gratitude and indebtedness to my supervisor Dr. S. Saha Ray, Associate Professor, Department of Mathematics, National Institute of Technology, Rourkela, Orissa, for his keen guidance, constant encouragement and prudent suggestions during the course of my study and preparation of the final manuscript of this project .

Thanks to all my classmates and my friends for their support any encourage.

Also thanks, to all my family members who have all supported me on my journey.

Finally, thanks to all my good wisher. It has been a pleasure to work on Mathematics in National Institute of Technology, Rourkela.

ABSTRACT

Wavelets are mathematical functions that cut up data into different frequency components and study each component with a resolution matched to its scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. Wavelets are developed independently in the field of mathematics, quantum physics, electrical engineering, and seismic geology. Interchange between this field during the last ten years have led to many new wavelet application such as image compression, turbulence, human vision, radar and earthquake prediction. In this we introduce a numerical method of solving integral equation by using CAS wavelets. This method is method upon CAS wavelet approximations. The properties of CAS wavelets are first presented. CAS wavelet approximations methods are then utilized to reduce the integral equations to the solution of algebraic equations.

KEYWORDS

Kernel

Fredholm integral equations

Volterra integral equations

Integro-differential equations

Contents **Page no.**

1. Introduction 8
2. Properties of CAS wavelets 10
3. Method of solving fredholm integral equations 13
4. Example of fredholm integral equation solved using CAS wavelets 14
5. Integro-differential equation-introduction 22
6. Properties of CAS wavelets 23
7. Method of solving integro-differential equations by using CAS wavelets 28
8. Example of integro-differential equation solved using CAS wavelets 29
9. Conclusion 39
10. References 40

INTRODUCTION

Wavelets theory is a relatively new and emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation [1]. Wavelet algorithms process data at different scales or resolutions. If we look at a signal at a large “window” we would notice gross features. Similarly, if we look at a signal with the small “window” we would notice small “window”. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [2].

For Fredholm-Hammerstein integral equations, the classical method of successive approximations was introduced in [3]. A variation of the Nystrom method was presented in [4]. A collocation type method was developed in [5]. In [6], Yalcinbas applied a Taylor polynomial solution to Volterra-Fredholm integral equations.

LINEAR FREDHOLM INTEGRAL EQUATIONS

In this thesis, we are concerned with the application of CAS wavelets to the numerical solution of a Fredholm integral equation of the form

$$y(x) = f(x) + \int_0^1 K(x,t)y(t)dt \qquad 0 \le x,t \le 1 \tag{1}$$

The general linear Fredholm integral equations of the second kind for a function $\varphi(x)$ is an equation of the type

$$\Phi(x) - \lambda\int_0^1 K(x,y)\Phi(y)dy = f(x) \qquad (0 \le x \le 1) \tag{2}$$

While the linear Fredholm integral equation of the first kind is given by

$$\int_0^1 K(x,y)\Phi(y)dy = f(x) \qquad (0 \le x \le 1) \tag{3}$$

In eq. (1) , where $f(x)$ and the kernel $K(x,t)$ are assumed to be in $L^2(R)$ on the interval $0 \le x,t \le 1$ and eq. (1) has a unique solution y to be determined.

In this thesis, a new numerical method to solve Fredholm integral equations is introduced. The method consists of reducing the integral equation to a set of algebraic equations by expanding the solution as CAS wavelets with unknown coefficients. The CAS wavelets are first given. The product operational matrix and orthonormality property of CAS wavelets basis are then utilized to evaluate the coefficients of CAS wavelets expansion.

Properties of CAS wavelets

Wavelets and CAS wavelet

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the family of continuous wavelets as [7]

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-a}{b}\right), \qquad a, b \in \mathrm{R}, a \neq 0 \ ,$$

If we restrict the parameters a and b to discrete values as

$a = a_0^{-k}, b = nb_0 a_0^{-k}, a_0 > 1, b_0 > 0$ and n and k are positive integers

We have the following family of discrete wavelets

$$\Psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \Psi\left(a_0^{k} t - nb_0\right),$$

where $\Psi_{nm}(t)$ form a wavelet basis for $L^2(R)$.In particular when $a_0 = 2$

and $b_{0=}1$ then $\Psi_{k,n}(t)$ forms an orthonormal basis [7].

CAS wavelets $\Psi_{nm}(t) = \Psi(k, n, m, t)$ have four arguments

n = 0, 1, 2….2^k-1, k can be assume any nonnegative integer, m is any

Integer and t is the normalized time. They are defined on the interval [0, 1) as

$$\Psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} CAS_m\left(2^k t - n\right), & \text{for } \frac{\mathrm{n}}{2^{\mathrm{k}}} \leq t < \frac{n+1}{2^k} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

where $CAS_m(t) = \cos(2m\pi t) + \sin(2m\pi t)$ (5)

The dilation parameter is $a = 2^{-k}$ and translation parameter is $b = n2^{-k}$.

The set of CAS wavelets are an orthonormal basis.

FUNCTION APPROXIMATION

A function $f(t)$ defined over [0, 1) may be expanded as

$$f(t)=\sum_{n=1}^{\infty}\sum_{m\in}c_{nm}\Psi_{nm}(t) \tag{6}$$

where $c_{nm}=(f(t),\Psi_{nm}(t))$. If the infinite series in eq. (6) is truncated, then eq. (6) can be written as

$$f(t)\approx\sum_{n=1}^{2^k}\sum_{m=-M}^{M}c_{nm}\Psi_{nm(}(t)=C^T\Psi(t) \tag{7}$$

where C and $\Psi(t)$ are $2^k(2M+1) \times 1$ matrices given by

$$C=\left[c_{1(-M)},c_{1(-M+1)},.....,c_{1(M)},c_{2(-M)},...,c_{2(M)},...,c_{2^k(-M)},.....,c_{2^k(M)}\right]^T \tag{8}$$

$$\Psi(t)=\left[\Psi_{1(-M)}(t),\Psi_{1(-M+1)}(t),.....,\Psi_{1M}(t),\Psi_{2(-M)}(t),.....,\Psi_{2M}(t),.....,\Psi_{2^k(-M)}(t),........,\Psi_{2^kM}(t)\right]^T \tag{9}$$

The product operational matrix of the CAS wavelet

Let $\Psi(t)\Psi^T(t)\approx\widehat{C}\Psi(t)$ (10)

Where $\hat{C}$ is a $2^k(2M+1) \times 2^k(2M+1)$ product operational matrix.

Let $M = 1$ and $k = 1$, thus we have

$$C = [c_{1(-1)},c_{10},c_{11},c_{2(-1)},c_{20},c_{21}]^T \tag{11}$$

$$\Psi(t)=\left[\Psi_{1(-1)}(t),\Psi_{10}(t),\Psi_{11}(t),\Psi_{2(-1)}(t),\Psi_{20}(t),\Psi_{21}(t)\right]^T \tag{12}$$

In eq. (12) we have

$$\left.\begin{aligned}\Psi_{1(-1)}(t)&=2^{1/2}(\cos(4\pi t)-\sin(4\pi t))\\ \Psi_{10}(t)&=2^{1/2}\\ \Psi_{11}(t)&=2^{1/2}(\cos(4\pi t)+\sin(4\pi t))\end{aligned}\right\}\qquad 0\le t<\frac{1}{2} \tag{13}$$

$$\left.\begin{array}{l}\Psi_{2(-1)}(t) = 2^{1/2}(\cos(4\pi t) - \sin(4\pi t)) \\ \Psi_{20}(t) = 2^{1/2} \\ \Psi_{21}(t) = 2^{1/2}(\cos(4\pi t) + \sin(4\pi t))\end{array}\right\} \qquad \frac{1}{2} \le t < 1 \tag{14}$$

Method of solving fredholm integral equations

Consider the fredholm integral equations which are given eq. (1). To use CAS wavelets, we have to approximate $y(x)$ as

$$y(x) = C^T \Psi(x) \tag{15}$$

and $$f(x) = d^T \Psi(x) \tag{16}$$

and $$K(x,t) = \Psi(x)^T K \Psi(t) \tag{17}$$

where C and $\Psi(x)$ are defined as eqn. (6) and (7).

Also where K is $2^k(2M+1) \times 2^k(2M+1)$ matrices and elements of K are calculated as

$$\int_0^1 \int_0^1 \Psi_{ni}(x) \Psi_{lj}(t) K(x,t)\, dt\, dx$$

Where n=1,2…,2^k, i=$-M, \dots\dots.., M$, l=1,….,2^k,, j=$-M, \dots. M$

Then we have

$$C^T \Psi(x) = d^T \Psi(x) + \lambda \int_0^1 \Psi(x)^T K \Psi(t)^T C\, dt \tag{18}$$

Thus with the orthonormality of CAS Wavelets we have

$$\Psi(x)^T C = \Psi(x)^T d + \lambda \Psi(x)^T KC \tag{19}$$

Eq. (17) is a linear systems of C and thus

C = $(I-K)^{-1}$d, where I is identity matrix.

Example-1

$$y(x)=\cos(4\pi x)+\int_0^1 ty(t)dt$$

Let us approximate

$y(x)=C^T\Psi(x)$ and $f(x)=d^T\Psi(x)$

Take k = 1 and M=1

For $x=0$

$$\Psi_{0(-1)}(x)=\sqrt{2}(\cos(4\pi x)-\sin(4\pi x))=\sqrt{2}$$

$$\Psi_{00}(x)=\sqrt{2}$$

$$\Psi_{01}(x)=\sqrt{2}(\cos(4\pi x)+\sin(4\pi x))=\sqrt{2}$$

$$\Psi_{1(-1)}(x)=0$$

$$\Psi_{10}(x)=0$$

$$\Psi_{11}(x)=0$$

So, $\Psi(x)=\left[\sqrt{2},\ \sqrt{2},\ \sqrt{2},\ 0,\ 0,\ 0\right]^T$

and $f(x)=d^T\Psi(x)$

$$\Rightarrow \cos(4\pi x)=\left[d_1\ \ d_2\ \ d_3\ \ d_4\ \ \mathrm{d}_5\ \ \mathrm{d}_6\right]\begin{bmatrix}\sqrt{2}\\ \sqrt{2}\\ \sqrt{2}\\ 0\\ 0\\ 0\end{bmatrix}$$

=> $\cos(4\pi x) = \sqrt{2}d_1 + \sqrt{2}d_2 + \sqrt{2}d_3$

=> $1 = \sqrt{2}d_1 + \sqrt{2}d_2 + \sqrt{2}d_3$ (20)

$for \quad x = \frac{1}{4}$

$\Psi_{0(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = -\sqrt{2}$

$\Psi_{00}(x) = \sqrt{2}$

$\Psi_{01}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = -\sqrt{2}$

$\Psi_{1(-1)}(x) = 0$

$\Psi_{10}(x) = 0$

$\Psi_{11}(x) = 0$

So, $\Psi(x) = \left[-\sqrt{2}, \sqrt{2}, -\sqrt{2}, 0, 0, 0\right]^T$

So, $f(x) = d^T \Psi(x)$

$$\Rightarrow \cos(4\pi x) = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & \mathrm{d}_5 & \mathrm{d}_6 \end{bmatrix} \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ -\sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -1 = -\sqrt{2}d_1 + \sqrt{2}d_2 - \sqrt{2}d_3$ (21)

$for \ \mathrm{x} = \frac{1}{8}$

$\Psi_{0(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = -\sqrt{2}$

$\Psi_{00}(x) = \sqrt{2}$

$\Psi_{01}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = \sqrt{2}$

$\Psi_{1(-1)}(x) = 0$

$\Psi_{10}(x) = 0$

$\Psi_{11}(x) = 0$

So, $\Psi(x) = \left[-\sqrt{2},\ \sqrt{2},\ \sqrt{2},\ 0,\ 0,\ 0\right]^T$

and $f(x) = d^T \Psi(x)$

$$\Rightarrow \cos(4\pi x) = \left[d_1 \quad d_2 \quad d_3 \quad d_4 \quad \mathrm{d}_5 \quad \mathrm{d}_6\right] \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 0 = -\sqrt{2}d_1 + \sqrt{2}d_2 + \sqrt{2}d_3 \tag{22}$$

for $\mathrm{x} = \dfrac{1}{2}$

$\Psi_{0(-1)}(x) = 0$

$\Psi_{00}(x) = 0$

$\Psi_{01}(x) = 0$

$\Psi_{1(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = \sqrt{2}$

$\Psi_{10}(x) = \sqrt{2}$

$\Psi_{11}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = \sqrt{2}$

and $f(x) = d^T \Psi(x)$

$$\Rightarrow \cos(4\pi x) = \left[d_1 \quad d_2 \quad d_3 \quad d_4 \quad \mathrm{d}_5 \quad \mathrm{d}_6\right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$\Rightarrow 0 = \sqrt{2}d_4 + \sqrt{2}d_5 + \sqrt{2}d_6 \tag{23}$$

for $x = \dfrac{3}{4}$

$$\Psi_{0(-1)}(x) = 0$$
$$\Psi_{00}(x) = 0$$
$$\Psi_{01}(x) = 0$$
$$\Psi_{1(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = -\sqrt{2}$$
$$\Psi_{10}(x) = \sqrt{2}$$
$$\Psi_{11}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = -\sqrt{2}$$
$$f(x) = d^T\Psi(x)$$

$$\Rightarrow \cos(4\pi x) = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & \mathrm{d}_5 & \mathrm{d}_6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$\Rightarrow -1 = -\sqrt{2}d_4 + \sqrt{2}d_5 - \sqrt{2}d_6 \tag{24}$$

for $x = \dfrac{3}{8}$

$$\Psi_{0(-1)}(x) = 0$$
$$\Psi_{00}(t) = 0$$
$$\Psi_{01}(x) = 0$$
$$\Psi_{1(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = \sqrt{2}$$
$$\Psi_{10}(x) = \sqrt{2}$$
$$\Psi_{11}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = -\sqrt{2}$$

So, $\Psi(x) = \left[0,\ 0,\ 0, \sqrt{2},\ \sqrt{2},\ \sqrt{2}\right]^T$

$$f(x) = d^T\Psi(x)$$

$$\Rightarrow \cos(4\pi x) = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & \mathrm{d}_5 & \mathrm{d}_6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$\Rightarrow 0 = \sqrt{2}d_4 + \sqrt{2}d_5 - \sqrt{2}d_6 \qquad (25)$$

Solving equation (20), (21) and (22) we get

$d_1 = \frac{1}{2\sqrt{2}}$, $d_6 = 0$, $d_3 = \frac{1}{2\sqrt{2}}$

and on solving equation (23), (24) and (25) we get

$d_4 = \frac{1}{2\sqrt{2}}$, $d_5 = 0$, $d_6 = \frac{1}{2\sqrt{2}}$

So, $d = [d_1, d_2, d_3, d_4, d_5, d_6]^T$

$$\Rightarrow d = [\frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}},]^T$$

Now, K is $2^k(2M+1) \times 2^k(2M+1)$ matrices where elements of K can be calculate as

$$\int_0^1 \int_0^1 \Psi_{ni}(x)\Psi_{lj}(t)k(x,t)dtdx$$

where $n = 1, 2....2^k$, $i = -M,....., M$, $l = 1,...2^k$, $j = -M,..., M$

Then we have

$$C^T\Psi(x) = d^T\Psi(x) + \lambda\int_0^1 \Psi(x)^T K\Psi(t)\Psi(t)^T C dt$$

Thus with the orthonormality of CAS wavelets we have

$$\Psi(x)^T C = \Psi(x)^T d + \lambda\Psi(x)^T KC$$

And answer is $C = (I-K)^{-1}d$

where I is the identity matrix.

Here in this example $K(x,t)$ is t

Since k = 1 and M = 1

So, n = l = 1, 2 and i = j = -1, 0, 1

So, k will be 6 x 6 matrixes

And since

$$\int_0^{1/2} \Psi_{1(-1)}(x)dx = 0 \qquad \int_0^{1/2} \Psi_{1(-1)}(t)tdt = \frac{\sqrt{2}}{8\pi}$$

$$\int_0^{1/2} \Psi_{10}(x)dx = \frac{\sqrt{2}}{2} \qquad \int_0^{1/2} \Psi_{10}(t)tdt = \frac{\sqrt{2}}{8}$$

$$\int_0^{1/2} \Psi_{11}(x)dx = 0 \qquad \int_0^{1/2} \Psi_{11}(t)tdt = \frac{-\sqrt{2}}{8\pi}$$

and

$$\int_{1/2}^{1} \Psi_{2(-1)}(x)dx = 0 \qquad \int_{1/2}^{1} \Psi_{2(-1)}(t)tdt = \frac{\sqrt{2}}{8\pi}$$

$$\int_{1/2}^{1} \Psi_{20}(x)dx = \frac{\sqrt{2}}{2} \qquad \int_{1/2}^{1} \Psi_{20}(t)tdt = \frac{3\sqrt{2}}{8}$$

$$\int_{1/2}^{1} \Psi_{21}(x)dx = 0 \qquad \int_{1/2}^{1} \Psi_{21}(t)tdt = \frac{-\sqrt{2}}{8\pi}$$

So, K = $\begin{bmatrix} 0 & \frac{1}{8\pi} & 0 & 0 & \frac{1}{8\pi} & 0 \\ 0 & 0.125 & 0 & 0 & 0.125 & 0 \\ 0 & \frac{-1}{8\pi} & 0 & 0 & \frac{-1}{8\pi} & 0 \\ 0 & \frac{1}{8\pi} & 0 & 0 & \frac{1}{8\pi} & 0 \\ 0 & 0.375 & 0 & 0 & 0.375 & 0 \\ 0 & \frac{-1}{8\pi} & 0 & 0 & \frac{-1}{8\pi} & 0 \end{bmatrix}$

Then C = $(I-K)^{-1}d$

Now $(I-k)^{-1}$ = $\begin{bmatrix} 1 & -0.0796 & 0 & 0 & -0.0796 & 0 \\ 0 & 1.2500 & 0 & 0 & 0.2500 & 0 \\ 0 & -0.0796 & 1 & 0 & -0.0796 & 0 \\ 0 & 0.0796 & 0 & 1 & 0.0796 & 0 \\ 0 & 0.7500 & 0 & 0 & 1.7500 & 0 \\ 0 & -0.0796 & 0 & 0 & -0.0796 & 1 \end{bmatrix}$

Then $(I-k)^{-1}d = \begin{bmatrix} 1 & -0.0796 & 0 & 0 & -0.0796 & 0 \\ 0 & 1.2500 & 0 & 0 & 0.2500 & 0 \\ 0 & -0.0796 & 1 & 0 & -0.0796 & 0 \\ 0 & 0.0796 & 0 & 1 & 0.0796 & 0 \\ 0 & 0.7500 & 0 & 0 & 1.7500 & 0 \\ 0 & -0.0796 & 0 & 0 & -0.0796 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ 0 \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ 0 \\ \frac{1}{2\sqrt{2}} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ 0 \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ 0 \\ \frac{1}{2\sqrt{2}} \end{bmatrix}$$

$C = (I - K)^{-1}d$

$$\Rightarrow C = \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \end{bmatrix}$$

Now $y(x) = C^T \Psi(x)$

$$= \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2^{1/2}(\cos(4\pi x) - \sin(4\pi x)) \\ 2^{1/2} \\ 2^{1/2}(\cos(4\pi x) + \sin(4\pi x)) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y(x) = \cos(4\pi x)$$

Therefore, $y(x) = \cos(4\pi x)$ is the exact solution of the given integral equation.

Integro-differential equations

Introduction

wavelets $\Psi_{nm}(t)=\Psi(k,n,m,t)$ involve four argument $n,\ m,\ k$ and t, where n=0,1,….,2^k- 1, k is assumed any nonnegative integer , m is any integer and t is the normalized time. Recently, Yousefi and Banifatemi [11] introduced the CAS wavelets which are defined by In this thesis , we introduce a new numerical method to solve the linear Fredholm integro-differential equation

$$\begin{cases} y'(t) = f(t) + \int_0^1 K(t,s)y(s)ds, \quad 0 \le \mathrm{t} \le 1 \\ y(0) = y_0, \end{cases} \tag{26}$$

Where the function $f(t)\in \mathrm{L}^2([0,1])$, **t**he kernel $K(t,s)\in \mathrm{L}^2([0,1]\,\mathrm{x}\,[0,1])$ are known and $y(t)$ is the unknown function to be determined . This method reduces the integral equation to set of algebraic equations by expanding $y(t)$ as CAS wavelets with unknown coefficients.

Properties of CAS wavelets

Wavelets and CAS wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\Phi(t)$ called the mother wavelets. When the dilation is 2 and the translation parameter is 1 we have the following family of discrete wavelets [12].

$$\Psi_{kn}(t) = 2^{\frac{k}{2}}\Phi(2^k t - n),$$

where Ψ_{kn} form a wavelet orthonormal basis for $L^2(R)$

$$\psi_{nm}(t) = \begin{cases} 2^{k} CAS_m\left(2^k t - n\right), & \text{for } \frac{n}{2^k} \le t \le \frac{n+1}{2^k} \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

where $CAS_m(t) = \cos(2m\pi t) + \sin(2m\pi t)$

The set of CAS wavelets also forms an orthonormal basis for $L^2([0,1])$. Here we use CAS wavelets to solve integro-differential equations by introducing the operational matrix of integration.

Function approximation

A function f(t) defined over [0,1] may be expanded as

$$f(t)=\sum_{n=0}^{\infty}\sum_{m\in Z}c_{nm}\Psi_{nm}(t) \tag{28}$$

Where $c_{nm}=(f(t),\Psi_{nm}(t)),$ (29)

If the infinite series in equation (28) is truncated, then equation (28) can be written as

$$f(t)\approx\sum_{n=0}^{2^k-1}\sum_{m=-M}^{M}c_{nm}\Psi_{nm}(t)\equiv C^T\Psi(t) \tag{30}$$

Where C and Ψ(t) are $2^k(2M+1)$ x 1 matrices given by

$$C=\left[c_{1(-M)},c_{1(-M+1)},.....,c_{1(M)},c_{2(-M)},...,c_{2(M)},...,c_{2^k(-M)},....,c_{2^k(M)}\right]^T \tag{31}$$

$$\Psi(t)=\left[\Psi_{1(-M)}(t),\Psi_{1(-M+1)}(t),.....,\Psi_{1M}(t),\Psi_{2(-M)}(t),.....,\Psi_{2M}(t),.....,\Psi_{2^k(-M)}(t),........,\Psi_{2^kM}(t)\right]^T \tag{32}$$

CAS wavelet operational matrix of integration

We have to construct the 6 x 6 matrix P for M=1 and k=1.

The six basis function are given by

$$\left.\begin{aligned}\Psi_{0(-1)}(t)&=2^{1/2}(\cos(4\pi t)-\sin(4\pi t))\\ \Psi_{00}(t)&=2^{1/2}\\ \Psi_{01}(t)&=2^{1/2}(\cos(4\pi t)+\sin(4\pi t))\end{aligned}\right\}\qquad 0\le t<\frac{1}{2} \tag{33}$$

$$\left.\begin{aligned}\Psi_{1(-1)}(t)&=2^{1/2}(\cos(4\pi t)-\sin(4\pi t))\\ \Psi_{10}(t)&=2^{1/2}\\ \Psi_{11}(t)&=2^{1/2}(\cos(4\pi t)+\sin(4\pi t))\end{aligned}\right\}\qquad \frac{1}{2}\le t<1 \tag{34}$$

By integrating (33) and (34) from 0 to t and representing it to the matrix form, we obtain

$$\int_0^1 \Psi_{0(-1)}(t')dt' = \begin{cases} \frac{\sqrt{2}}{4\pi}(\cos(4\pi t)+\sin(4\pi t)-1) & 0 \le \text{t} < \frac{1}{2} \\ 0 & \frac{1}{2} \le t < 1 \end{cases}$$

$$= \frac{1}{4\pi}(-\Psi_{00}(t)+\Psi_{01}(t)$$

$$= \left[0, -\frac{1}{4\pi}, \frac{1}{4\pi}, 0,0,0\right]\Psi_6(t)$$

$$\int_0^1 \Psi_{00}(t')dt' = \begin{cases} \sqrt{2t} & 0 \le \text{t} < \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{1}{2} \le \text{t} < 1 \end{cases}$$

$$= \frac{1}{4\pi}\Psi_{0(-1)}(t) + \frac{1}{4}\Psi_{00}(t) - \frac{1}{4\pi}\psi_{01}(t) + \frac{1}{2}\Psi_{10}(t)$$

$$= \left[\frac{1}{4\pi}, \frac{1}{4}, -\frac{1}{4\pi}, 0, \frac{1}{2}, 0\right]\Psi_6(t)$$

Similarly we have

$$\int_0^1 \Psi_{01}(t')dt' = \frac{1}{4\pi}(\Psi_{0(-1)}(t)+\Psi_{00}(t))$$

$$= \left[\frac{1}{4\pi}, \frac{1}{4\pi}, 0,0,0,0\right]\Psi_6(t),$$

$$\int_0^1 \Psi_{1(-1)}(t')dt' = \frac{1}{4\pi}(-\Psi_{10}(t)+\Psi_{11}(t))$$

$$= \left[0,0,0,0, -\frac{1}{4\pi}, \frac{1}{4\pi}\right]\Psi_6(t)$$

$$\int_0^1 \Psi_{10}(t')dt' = \frac{1}{4\pi}\Psi_{1(-1)}(t) + \frac{1}{4}\Psi_{10}(t) - \frac{1}{4\pi}\Psi_{11}(t)$$

$$= \left[0,0,0,\frac{1}{4\pi},\frac{1}{4},\frac{1}{4\pi}\right]\Psi_6(t)$$

$$\int_0^1 \Psi_{11}(t')dt' = \frac{1}{4\pi}(\Psi_{1(-1)}(t) + \Psi_{10}(t))$$

$$= \left[0,0,0,0,\frac{1}{4\pi},\frac{1}{4\pi}\right]\Psi_6(t).$$

Thus $\int_0^1 \Psi(t')dt' = P_{6x6}\Psi_6(t)$

Where, $\Psi_6(t) = \left[\Psi_{0(-1)}, \Psi_{00}, \Psi_{01}, \Psi_{1(-1)}, \Psi_{10}, \Psi_{11}\right]^T$

and

$$P_{6x6} = \frac{1}{4}\begin{bmatrix} 0 & -\frac{1}{\pi} & \frac{1}{\pi} & 0 & 0 & 0 \\ \frac{1}{\pi} & 1 & -\frac{1}{\pi} & 0 & 2 & 0 \\ \frac{1}{\pi} & \frac{1}{\pi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\pi} & \frac{1}{\pi} \\ 0 & 0 & 0 & \frac{1}{\pi} & 1 & -\frac{1}{\pi} \\ 0 & 0 & 0 & 0 & \frac{1}{\pi} & \frac{1}{\pi} \end{bmatrix}$$

Hence $P_{6x6} = \begin{bmatrix} L_{3x3} & F_{3x3} \\ 0_{3x3} & L_{3x3} \end{bmatrix}$

$$where \quad L_{3x3} = \begin{bmatrix} 0 & -\frac{1}{\pi} & \frac{1}{\pi} \\ \frac{1}{\pi} & 1 & -\frac{1}{\pi} \\ \frac{1}{\pi} & \frac{1}{\pi} & 0 \end{bmatrix}$$

and $F_{3x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

In general, we have

$$\int_0^1 \Psi(t')dt' = P\Psi(t)$$

Where $\Psi(t)$ is given as in eq. (32) and P is a $2^k(2M+1)$ x $2^k(2M+1)$ matrix given by

$$P = \frac{1}{2^{k+1}} \begin{bmatrix} L & F & F & .. & .. & F \\ 0 & L & F & .. & .. & F \\ .. & 0 & \ddots & \ddots & .. & .. \\ .. & .. & \ddots & \ddots & .. & .. \\ .. & .. & .. & .. & .. & F \\ 0 & 0 & .. & .. & 0 & L \end{bmatrix}$$

Where F and L are (2M+1) x (2M+1) matrices

Method to solve linear Fredholm integro-differential equations.

Consider the linear fredholm integro-differential equation in equation (26).

We approximate

$$y(t) \approx Y^T \Psi(t), \quad \mathrm{y}(0) = \mathrm{Y}_0{}^T \Psi(t), \quad \mathrm{f(t)} \approx \mathrm{X}^\mathrm{T} \Psi(t)$$

and $K(t,s) \approx \Psi(t)^T K \Psi((s)$

Where Y', Y_0 and X are the coefficients. Also K is a $2k(2M+1)\, x\, 2k(2M+1)$ matrix and the elements of K can be calculated as

$$\int_0^1 \int_0^1 \Psi_{ni}(t) \Psi_{lj}(s) K(t,s) dtds, \qquad n = l = 0,....,2k-1, \qquad i, j = -M,...,M$$

Then we have

$$y(t) = \int_0^1 y'(s)ds + y(0) \approx \int_0^1 Y'^T \ \Psi(s)ds + Y_0{}^T \Psi(t)$$

$$= Y'^T \ P\Psi(t) + Y_0{}^T \Psi(t) = (Y'^T \ P + Y_0{}^T)\Psi(t)$$

Substituting in equation (26), we have

$$\Psi(t)^T Y' = \Psi(t)^T X + \Psi(t)^T (P^T Y' + Y_0) + \int_0^1 \Psi(t)^T K \Psi(s) \Psi(s)^T (P^T Y' + Y_0) ds$$

$$\Rightarrow \Psi(t)^T Y' = \Psi(t)^T X + \Psi(t)^T P^T Y' + Y_0) + \Psi(t)^T K(P^T Y' + Y_{0)})$$

$$\Rightarrow (\mathrm{I - KP^T} - P^T) Y' = KY_0 + Y_0 + X$$

Thus solving this linear system we can get the vector Y'.

Then $Y^T = Y' P + Y_0{}^T$

or $y(t) \approx Y^T \Psi(t)$

Example -2

Consider the integro-differential equation

$$\begin{cases} y'(x) = xe^{x} + e^{x} - x + \int_{0}^{1} xy(t)dt \\ \\ y(0) = 0 \end{cases}$$

The exact solution is xe^{x}

We first approximate $y(0) = Y_0^{\,T}\Psi(x)$

$$0 == Y_0^{\,T}\begin{bmatrix} \Psi_{0(-1)}(x) \\ \Psi_{00}(x) \\ \Psi_{01}(x) \\ \Psi_{1(-1)}(x) \\ \Psi_{10}(x) \\ \Psi_{11}(x) \end{bmatrix}$$

For $x = 0$

$$\Psi_{0(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = \sqrt{2}$$

$$\Psi_{00}(x) = \sqrt{2}$$

$$\Psi_{01}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = \sqrt{2}$$

$$\Psi_{1(-1)}(x) = 0$$

$$\Psi_{10}(x) = 0$$

$$\Psi_{11}(x) = 0$$

So, $\Psi(x) = \left[\sqrt{2},\ \sqrt{2},\ \ \sqrt{2},\ 0,\ 0,\ 0\right]^{T}$

So, $y(x)=\begin{bmatrix}y_1, & y_2, & y_3, & y_4, & y_5, & y_6\end{bmatrix}\begin{bmatrix}\sqrt{2}\\ \sqrt{2}\\ \sqrt{2}\\ 0\\ 0\\ 0\end{bmatrix}$

$$\Rightarrow 0=\sqrt{2}y_1+\sqrt{2}y_2+\sqrt{2}y_3 \tag{35}$$

$for\ \ x=\frac{1}{4}$

$\Psi_{0(-1)}(x)=\sqrt{2}(\cos(4\pi x)-\sin(4\pi x))=-\sqrt{2}$

$\Psi_{00}(x)=\sqrt{2}$

$\Psi_{01}(x)=\sqrt{2}(\cos(4\pi x)+\sin(4\pi x))=-\sqrt{2}$

$\Psi_{1(-1)}(x)=0$

$\Psi_{10}(x)=0$

$\Psi_{11}(x)=0$

So, $\Psi(x)=\left[-\sqrt{2},\sqrt{2},-\sqrt{2},0,0,0\right]^T$

So, $y(x)=\begin{bmatrix}y_1, & y_2, & y_3, & y_4, & y_5, & y_6\end{bmatrix}\begin{bmatrix}-\sqrt{2}\\ \sqrt{2}\\ -\sqrt{2}\\ 0\\ 0\\ 0\end{bmatrix}$

$$\Rightarrow 0=-\sqrt{2}y_1+\sqrt{2}y_2-\sqrt{2}y_3 \tag{36}$$

$x=\frac{1}{8}$

$\Psi_{0(-1)}(x)=\sqrt{2}(\cos(4\pi x)-\sin(4\pi x))=-\sqrt{2}$

$\Psi_{00}(x)=\sqrt{2}$

$\Psi_{01}(x)=\sqrt{2}(\cos(4\pi x)+\sin(4\pi x))=\sqrt{2}$

$\Psi_{1(-1)}(x)=0$

$\Psi_{10}(x)=0$

$\Psi_{11}(x)=0$

So, $\Psi(x)=\left[-\sqrt{2},\ \sqrt{2},\ \sqrt{2},\ \ 0,\ 0,\ 0\right]^T$

So, $y(x)=\left[y_1,\ y_2\ \ ,y_3\ \ ,y_4,\ y_5,\ y_6\right]\begin{bmatrix}-\sqrt{2}\\ \sqrt{2}\\ \sqrt{2}\\ 0\\ 0\\ 0\end{bmatrix}$

$$\Rightarrow 0=-\sqrt{2}y_1+\sqrt{2}y_2+\sqrt{2}y_3 \qquad (37)$$

$x=\dfrac{1}{2}$

$\Psi_{0(-1)}(x)=0$

$\Psi_{00}(x)=0$

$\Psi_{01}(x)=0$

$\Psi_{1(-1)}(x)=\sqrt{2}(\cos(4\pi x)-\sin(4\pi x))=\sqrt{2}$

$\Psi_{10}(x)=\sqrt{2}$

$\Psi_{11}(x)=\sqrt{2}(\cos(4\pi x)+\sin(4\pi x))=\sqrt{2}$

So, $\Psi(x)=\left[0,\ 0,\ 0,\sqrt{2},\ \sqrt{2},\ \sqrt{2}\right]^T$

So, $y(x)=\left[y_1, y_2,\ y_3,\ y_4,\ y_5,\ y_{61}\right]\begin{bmatrix}0\\ 0\\ 0\\ \sqrt{2}\\ \sqrt{2}\\ \sqrt{2}\end{bmatrix}$

$$\Rightarrow 0=\sqrt{2}y_4+\sqrt{2}y_5+\sqrt{2}y_6 \qquad (38)$$

$x=\dfrac{3}{4}$

$\Psi_{0(-1)}(x)=0$

$\Psi_{00}(x)=0$

$\Psi_{01}(x)=0$

$\Psi_{1(-1)}(x)=\sqrt{2}(\cos(4\pi x)-\sin(4\pi x))=-\sqrt{2}$

$\Psi_{10}(x)=\sqrt{2}$

$\Psi_{11}(x)=\sqrt{2}(\cos(4\pi x)+\sin(4\pi x))=-\sqrt{2}$

So, $y(x) = [y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6] \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\sqrt{2} \\ \sqrt{2} \\ -\sqrt{2} \end{bmatrix}$

$$\Rightarrow 0 = -\sqrt{2}y_4 + \sqrt{2}y_5 - \sqrt{2}y_6 \qquad (39)$$

$x = \dfrac{3}{8}$

$\Psi_{0(-1)}(x) = 0$

$\Psi_{00}(t) = 0$

$\Psi_{01}(x) = 0$

$\Psi_{1(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = \sqrt{2}$

$\Psi_{10}(x) = \sqrt{2}$

$\Psi_{11}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = -\sqrt{2}$

So, $\Psi(x) = [0,\ 0,\ 0, \sqrt{2},\ \sqrt{2},\ \sqrt{2}]^T$

So, $y(x) = [y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6] \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ \sqrt{2} \\ -\sqrt{2} \end{bmatrix}$

$$\Rightarrow 0 = \sqrt{2}y_4 + \sqrt{2}y_5 + \sqrt{2}y_6 \qquad (40)$$

On solving equation (35), (36), (37), (38), (39) and (40)

We get $Y_0 = [0,0,0,0,0,0]^T$

For $k = 1, M = 1$, We get $\mathrm{K} = 6 \times 6$ matrice and here $k(x,t) = x$

so, K $= \begin{bmatrix} 0 & \frac{1}{8\pi} & 0 & 0 & \frac{1}{8\pi} & 0 \\ 0 & 0.125 & 0 & 0 & 0.125 & 0 \\ 0 & \frac{-1}{8\pi} & 0 & 0 & \frac{-1}{8\pi} & 0 \\ 0 & \frac{1}{8\pi} & 0 & 0 & \frac{1}{8\pi} & 0 \\ 0 & 0.375 & 0 & 0 & 0.375 & 0 \\ 0 & \frac{-1}{8\pi} & 0 & 0 & \frac{-1}{8\pi} & 0 \end{bmatrix}$

Now we approximate $f(x) = X^T \Psi(t)$

For $x = 0$

$\Psi_{0(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = \sqrt{2}$

$\Psi_{00}(x) = \sqrt{2}$

$\Psi_{01}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = \sqrt{2}$

$\Psi_{1(-1)}(x) = 0$

$\Psi_{10}(x) = 0$

$\Psi_{11}(x) = 0$

So, $\Psi(x) = \left[\sqrt{2},\ \sqrt{2},\ \sqrt{2},\ 0,\ 0,\ 0\right]^T$

So, $f(x) = [x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6] \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow e^0 - 0 = \sqrt{2}x_1 + \sqrt{2}x_2 + \sqrt{2}x_3$$
$$\Rightarrow 1 = \sqrt{2}x_1 + \sqrt{2}x_2 + \sqrt{2}x_3 \qquad (42)$$

$$x = \frac{1}{4}$$

$$\Psi_{0(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = -\sqrt{2}$$

$$\Psi_{00}(x) = \sqrt{2}$$

$$\Psi_{01}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = -\sqrt{2}$$

$$\Psi_{1(-1)}(x) = 0$$

$$\Psi_{10}(x) = 0$$

$$\Psi_{11}(x) = 0$$

So, $\Psi(x) = \left[-\sqrt{2},\ \sqrt{2},\ -\sqrt{2}\ ,0,\ 0,\ 0\right]^T$

So, $f(x) = [x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6] \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ -\sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow e^{\frac{1}{4}} - \frac{1}{4} = -\sqrt{2}x_1 + \sqrt{2}x_2 - \sqrt{2}x_3$$

$$\Rightarrow 1.034 = -\sqrt{2}x_1 + \sqrt{2}x_2 - \sqrt{2}x_3 \qquad (43)$$

$$x = \frac{1}{8}$$

$$\Psi_{0(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = -\sqrt{2}$$

$$\Psi_{00}(x) = \sqrt{2}$$

$$\Psi_{01}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = \sqrt{2}$$

$$\Psi_{1(-1)}(x) = 0$$

$$\Psi_{10}(x) = 0$$

$$\Psi_{11}(x) = 0$$

So, $\Psi(x) = \left[-\sqrt{2},\ \sqrt{2},\ \sqrt{2},\ 0,\ 0,\ 0\right]^T$

So, $f(x)=[x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6]\begin{bmatrix}-\sqrt{2}\\ \sqrt{2}\\ \sqrt{2}\\ 0\\ 0\\ 0\end{bmatrix}$

$$\Rightarrow e^{\frac{1}{8}}-\frac{1}{8}=-\sqrt{2}x_1+\sqrt{2}x_2+\sqrt{2}x_3$$

$$\Rightarrow 1.008=-\sqrt{2}x_1+\sqrt{2}x_2+\sqrt{2}x_3 \qquad (44)$$

$$x=\frac{1}{2}$$

$$\Psi_{0(-1)}(x)=0$$

$$\Psi_{00}(x)=0$$

$$\Psi_{01}(x)=0$$

$$\Psi_{1(-1)}(x)=\sqrt{2}(\cos(4\pi x)-\sin(4\pi x))=\sqrt{2}$$

$$\Psi_{10}(x)=\sqrt{2}$$

$$\Psi_{11}(x)=\sqrt{2}(\cos(4\pi x)+\sin(4\pi x))=\sqrt{2}$$

So, $\Psi(x)=\left[0,\ 0,\ 0,\ \sqrt{2},\ \sqrt{2},\ \sqrt{2}\right]^T$

So, $f(x)=[x_1,\ x_2,\ x_3,\ x_4,\ x_5, x_{61}]\begin{bmatrix}0\\ 0\\ 0\\ \sqrt{2}\\ \sqrt{2}\\ \sqrt{2}\end{bmatrix}$

$$\Rightarrow e^{\frac{1}{2}}-\frac{1}{2}=\sqrt{2}x_4+\sqrt{2}x_5+\sqrt{2}x_6$$

$$\Rightarrow 1.1487=\sqrt{2}x_4+\sqrt{2}x_5+\sqrt{2}x_6 \qquad (45)$$

$$x=\frac{3}{4}\in[\frac{1}{2},1]$$

$$\Psi_{0(-1)}(x)=0$$

$$\Psi_{00}(x)=0$$

$$\Psi_{01}(x)=0$$

$$\Psi_{1(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = -\sqrt{2}$$
$$\Psi_{10}(x) = \sqrt{2}$$
$$\Psi_{11}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = -\sqrt{2}$$

So, $f(x) = \left[x_1,\ x_2,\ x_3,\ x_4,\ x_x,\ x_6\right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$$\Rightarrow e^{\frac{3}{4}} - \frac{3}{4} = -\sqrt{2}x_4 + \sqrt{2}x_5 - \sqrt{2}x_6$$

$$\Rightarrow 1.3670 = -\sqrt{2}x_4 + \sqrt{2}x_5 - \sqrt{2}x_6 \tag{46}$$

$$x = \frac{3}{8}$$
$$\Psi_{0(-1)}(x) = 0$$
$$\Psi_{00}(x) = 0$$
$$\Psi_{01}(x) = 0$$
$$\Psi_{1(-1)}(x) = \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) = \sqrt{2}$$
$$\Psi_{10}(x) = \sqrt{2}$$
$$\Psi_{11}(x) = \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) = -\sqrt{2}$$

So, $\Psi(x) = \left[0,\ 0,\ 0,\ \sqrt{2}, \sqrt{2},\ \sqrt{2}\right]^T$

So, $f(x) = \left[x_1\ , x_2,\ x_3\ , x_4\ , x_5,\ x_6\right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ \sqrt{2} \\ -\sqrt{2} \end{bmatrix}$

$$\Rightarrow e^{\frac{3}{8}} - \frac{3}{8} = \sqrt{2}x_4 + \sqrt{2}x_5 - \sqrt{2}x_6$$

$$\Rightarrow 1.0799 = \sqrt{2}x_4 + \sqrt{2}x_5 - \sqrt{2}x_6 \qquad (47)$$

On solving equation (42), (43), (44), (45), (46) and (47)

We get

$$X = [-0.002,\ 0.179,\ -0.009,\ -0.101,\ 0.889,\ 0.024]$$

$$Now, \left(I - KP^T - P^T\right)Y' = KY_0 + Y_0 + X$$

$$Y' = \begin{bmatrix} -0.123579 \\ 0.337076 \\ -0.0590784 \\ -0.22257 \\ -0.255339 \\ 0.14557 \end{bmatrix}$$

Now, $Y^T = Y'^T P + Y_0^{\ T}$

$$\Rightarrow Y^T = [0.0221223 \quad 0.342209 \quad 0.0366578 \quad 0.0203233 \quad 0.448058 \quad 0.0141958]$$

$$Then, \quad \mathrm{y(t)} = Y^{\mathrm{T}}\Psi(t)$$

$$\Rightarrow y(t) = 0.083127613\cos(4\pi t) + 0.020556301\sin(4\pi t) + 0.483956609$$

Table

This table shows that difference of exact solution and numerical solution.

where y is the exact solution and $\hat{y}$ is the numerical solution.

Absolute errors in CAS wavelet method (M=1, k=1)

x	y	$\hat{y}$	$\lvert y-\hat{y}\rvert$
0.1	0.110517091	0.46201024	0.351493149
0.2	0.244280551	0.440063871	0.19578332
0.3	0.404957642	0.418117503	0.013159861
0.4	0.596729879	0.39728942	0.199440459
0.5	0.824360635	0.374224765	0.45013587
0.6	1.09327128	0.352278397	0.740992883
0.7	1.409626895	0.330332028	1.079294867
0.8	1.780432743	0.308385659	1.472047084
0.9	2.2136428	0.286439293	1.927203507

1. CONCLUSION

CAS wavelets gives an efficient and accurate method for solving the Fredholm integral equations. This method reduces an integral equations into set of algebraic equations. The integration of the product of two CAS wavelets function vectors is an identity matrix, which makes computation of integral equation attractive. It is also shown that the CAS wavelets provide an exact solution.

CAS wavelet also provide an efficient method for solving integro-differential equations by reducing an integral equations into a set of algebraic equations.

10.REFERENCE:-

[1] C.K. Chui, Wavelets: A mathematical Tool for signal Analysis, SIAM, Philadelphia PA, 1997.

[2] G. Beylkin, R. Coifman, V. Rokhlin, Fast wavelet transforms and numerical algorithms, I, Commun. Pure Appl. Math. 44 (1991) 141-183.

[3] F.G. Tricomi, Integral Equations, Dover Publications, 1982.

[4] L.J. Lardy, A variation of Nystrom's method for Hammerstein equations, J. Integral Equat. 3 (1981) 43-60.

[5] S. Kumar, I.H. Sloan, A new collocation-type method for Hammerstein integral equations, J. Math. Comp. 48 (1987) 123-129.

[6] S. Yalcinbas, Taylor polynomial solution of nonlinear Volterra–Fredholm integral equations, Appl. Math. Comp. 127 (2002) 195-206

[7] J.S. Gu, W.S. Jiang, The Haar wavelets operational matrix of integration, Int. J. Sys. Sci. 27 (1996) 623-628.

[8] A.M. Wazwaz, A First Course in Integral Equations, World scientific Publishing Company, New Jersey, 1997.

[9] M. Razzaghi, S. Yousefi, The Legendre wavelets operational matrix of integration, Int. J. Syst. Sci. 32 (4) (2001) 495-502

[10] F.Khellat, S.A.Yousefi, The Linear Legendre mother wavelets operational matrix of integration and its application, J. Franklin Inst. 343 (2006) 181-190.

[11] S. Yousefi, A. Banifatemi, Numerical Solution of Fredholm integral equations by using CAS wavelets, Appl. Math. Comput. (2006), doi:10.1016/j.amc.2006.05.081.

[12] J.S. Gu, W.S. Jiang, The Haar wavelets operational matrix of integration, Int. J. Sys.Sci. 27 (1996) 623628.

[13] M. Tavassoli Kajani, A. Hadi Venchch, Solving linear integro-differential equation with Legendre wavelets, Int. J. Comp. Math. 81 (6) (2004) 719726.

[14] P. Darania Ali Ebadian, A method for the numerical solution of the integro-differential equations, Appl. Math. Comput. (2006), doi:10.1016/j.amc.2006.10.046.

[15] H. Danfu, S. Xufeng, Numerical solution of integro-differential equations by using CAS wavelet operational matrix of integration, Applied Mathematics and computation 194 (2007) 460-466.

[16] IEEE Computational Science and Engineering summer 1995, vol.2, num.2, Published by IEEE Computer society, 10662 Los vaqueros Circle, Los Alamitos, CA 90720, USA