“SOME STUDIES ON DYNKIN DIAGRAMS ASSOCIATED WITH KAC-MOODY ALGEBRA”

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DECLARATION

I hereby certify that the work which is being presented in the thesis entitled "**Some studies on Dynkin Diagram associated with Kac-Moody Algebra**" in partial fulfillment of the requirement for the award of the degree of Master of Science, submitted in the Department of Mathematics, National Institute of Technology, Rourkela is an authentic record of my own work carried out under the supervision of Prof. K.C. Pati. The matter embodied in this thesis has not been submitted by me for the award of any other degree.

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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Words at my command are inadequate to convey the profound to my parents whose love, affection and blessings has inspired me the most.

Amit Kumar Singh

ABSTRACT

In the present project report, a sincere report has been made to construct and study the basic information related to Simple Lie Algebras, Kac-Moody algebras and their corresponding Dynkin Diagrams.

In chapter-1, I have given the definitions of Lie Algebra and some of the terms related to Lie algebra, i.e. subalgebras, ideals, Abelian, solvability, nilpotency etc. Also, I have done the classifications of Classical Lie algebras.

In chapter-2, I addressed the basics of Representation Theory, i.e. structure constants, modules, reflections in a Euclidean space, root systems (simple roots) and their corresponding root diagrams. Then I have discussed the formation of Dynkin Diagrams and cartan matrices associated with the roots of the simple lie algebras.

In chapter-3, I have given the necessary theory based on Kac-Moody lie algebras and their classifications. Then the definition of the extended Dynkin diagrams for Affinization of Kac-Moody algebras and the Dynkin Diagrams associated with the affine Kac-Moody algebras are provided

Contents

1

Introduction

1.1 Basic Definitions, Examples:

Before going to my concerned topic **KAC-MOODY ALGEBRA**, we need to get a precise definition of **LIE ALGEBRA**, i.e. A Vector Space **L** over a field **F** with an operation **L** × **L** ⟶ **L** denoted by $(x\ ,\ y) \longmapsto [x\ ,\ y]$ and called the **bracket** or **commutator** of x and y is called a **Lie Algebra** if the following axioms are satisfied :

1. The bracket operation is **bilinear**, i.e.

   $[\alpha x + \beta y, z]$ = α $[x, z]$ + β $[y, z]$ for all scalars α, β in **F** and all elements x, y, z in **L**.

2. The bracket operation is **skew-symmetric**, i.e.
   $[x, x]$ = **0** for all x in **L**.

3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$ = **0** for all x, y, z in **L**.

The axiom is called **Jacobi's Identity**. The followings are some of the examples related to Lie algebra.[1]

Example 1 : Let A be an algebra over F (a vector space with an associative multiplication x · y). A is a Lie algebra A_L (also called A *as Lie algebra)* by defining [x , y] = x · y − y · x.

Example 2 : If A the algebra of all operators (endomorphisms) of a vector space V ; the corresponding A_L is called the *general Lie algebra of* V , gl(V). Concretely, taking number space R_n as V, this is the *general linear Lie algebra* gl(n,R) of all n × n real matrices, with [x , y] = x . y – y .x. Similarly gl(n,C).

Example 3 : The *special linear Lie algebra* sl(n,R) consists of all n × n real matrices with trace 0 (and has the same linear and bracket operations as gl(n,R)—it is a "sub Lie algebra"); similarly for C. For any vector space V we have sl(V), the *special linear Lie algebra* of V , consisting of the operators on V of trace 0.[2]

1.2 Subalgebras, Ideals:

- A subset **K** of a Lie Algebra **L** is called a **sub-algebra** of **L** if for all x, y in **K** and all α, β in **F**, one has $\alpha x + \beta y$ in **K** , $[x, y]$ in **K**.

- An **ideal I** of a Lie Algebra **L** is a sub-algebra of **L** with the property [**I**,**L**] is a subset of **I**, i.e. for all x in **I** and y in **L** one has $[x, y]$ in **I**. Every (non-zero) Lie Algebra has at least two

Ideals, namely the Lie Algebra **L** itself and the sub-algebra **0** consisting of the **zero element** only. Both these ideals are called **Trivial**. All non-trivial ideals are called **Proper ideals**.[1]

1.3 Abelian, Solvable & Nilpotent:

- The Lie Algebra **L** is called **abelian** or **commutative** if $[x, y]$ = **0** for all x, y in **L**.

- More generally, a Lie Algebra **L** is said to be **solvable** if in the sequence of ideals of **L**(the derived series)
  $\mathbf{L}^{(0)} = \mathbf{L}$, $\mathbf{L}^{(1)} = [\mathbf{L},\mathbf{L}]$, $\mathbf{L}^{(2)}=[\mathbf{L}^{(1)},\mathbf{L}^{(1)}]$, $\mathbf{L}^{(3)}=[\mathbf{L}^{(2)},\mathbf{L}^{(2)}]$, …,$\mathbf{L}^{(i)} = [\mathbf{L}^{(i-1)},\mathbf{L}^{(i-1)}]$
  $\mathbf{L}^{(n)} = 0$ for some n.
  e.g. Abelian implies solvable whereas simple algebras are definitely non-solvable.
  A lie algebra $t_+(n)$ of the upper triangular matrices is a prototype of solvable algebras.

- A Lie Algebra L is said to be **nilpotent** if in the lower central series of **L**
  $\mathbf{L}^0 = \mathbf{L}$, $\mathbf{L}^1 = [\mathbf{L},\mathbf{L}]$, $\mathbf{L}^2 = [\mathbf{L},\mathbf{L}^1]$, … , $\mathbf{L}^i = [\mathbf{L},\mathbf{L}^{i-1}]$
  $\mathbf{L}^n = 0$ for some n.
  e.g. Any abelian algebra is nilpotent.
  A lie algebra $t_{++}(n)$ of strictly upper triangular matrices is the prototype of nilpotent algebras.

Clearly, $L^{(i)} \subset L^i$ for all I, so nilpotent algebras are solvable. But the converse is false.[1]

1.4 Simple & Semisimple Lie Algebra:

- A Lie algebra **L** is **simple** if it has no proper ideals and is not abelian.
- A Lie Algbera L is said to be semisimple if its radical is zero. Equivalently, **L** is semisimple if it does not contain any non-zero abelian ideals. In particular, a simple Lie algebra is semisimple.[1]

1.5 Classical Lie Algebras:

V is a finite-dimensional vector space over **F** and denotes **End V**, the set of linear transformations **V**→ **V** (**endomorphisms** of **V**).

- $\boldsymbol{A_l}$: **Special linear lie algebra**, denoted by $\mathfrak{sl}$**(V)** or $\mathfrak{sl}(\boldsymbol{l+1}, \mathbf{F})$. It is the **End V** having trace 0.
  The dimension of $\boldsymbol{A_l}$ is at most $\boldsymbol{(l+1)^2 - 1}$.

- $\boldsymbol{B_l}$: **Orthogonal lie algebra**, denoted by $\mathfrak{o}$**(V)** or $\mathfrak{o}(\boldsymbol{2l+1}, \mathbf{F})$. It consists of **End V** satisfying the following property :

$$f\,(x(v), w) = -\,f\,(v, x(w))$$

or, $$sx = -\,x^t s$$

where $s = = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$ and $x = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$ The dimension of B_l is $2\,l^2$ + l.

- $\boldsymbol{C_l}$: **Sympletic Lie algebra**, denoted by $\mathfrak{sp}$**(V)** or $\mathfrak{sp}$**(2$\boldsymbol{l}$, F)**. The **End V** satisfies the following property :

  $f\ (x(v), w)$ = - $f\ (v,, x(w))$

  or, sx = - $x^t s$

  where $s = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ and $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$

  The dimension of $\boldsymbol{C_l}$ is $2l^2$ + l.

- $\boldsymbol{D_l}(\boldsymbol{l} \geq \mathbf{2})$: **Orthogonal lie algebra**, denoted by $\mathfrak{o}$**(V)** or $\mathfrak{o}$**(2l,F)**. The construction of $\boldsymbol{D_l}$ is identical that for $\boldsymbol{B_l}$, except that dim **V** = 2$\boldsymbol{l}$ is even and s has the simpler form $\begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$.[1]

2

Representations

In representation theory, a **Lie algebra representation** or **representation of a Lie algebra** is a way of writing a Lie algebra as a set of matrices (or endomorphisms of a vector space) in such a way that the Lie bracket is given by the commutator.

Structure Constants:

Let $\mathfrak{g}$ be a lie algebra and take a basis $\{X_1, X_2, \ldots, X_n\}$ for (the vector space) $\mathfrak{g}$. By bilinearity [,] – operation in $\mathfrak{g}$ is completely determined once the values $[X_i, X_j]$ are known. We know them by writing them as linear combinations of X_i. The coefficients c_{ij}^{k} in the relations $[X_i, X_j] = c_{ij}^{k} X_k$ (sum over repeated indices !) are called the **structure constants** of $\mathfrak{g}$ (relative to the given basis).[1]

Modules:

Let $\mathfrak{g}$ be a lie algebra. It is often convenient to use the language of modules along with the (equivalent) language of representations.

A vector space **V**, endowed with an operation $\mathfrak{g}\times\mathbf{V}\rightarrow\mathbf{V}$(denoted by $(\mathbf{x},\mathbf{v}) \rightarrow \mathbf{x.v}$) is called an $\mathfrak{g}$-module if the following conditions are satisfied:

- $(ax + by).v = a(x.v) + b(y.v)$
- $x.(av + bw) = a(x.v) + b(x.w)$
- $[x, y].v = x.y.v - y.x.v$, where $x, y \in \mathfrak{g}$; $v, w \in \mathbf{V}$; $a, b \in \mathbf{F}$.[1]

Representations:

A representation of a lie algebra $\mathfrak{g}$ on a vector space **V** is a homomorphism (say φ) of $\mathfrak{g}$ into the general linear lie algebra $\mathfrak{gl}(\mathbf{V})$ of **V**. φ assigns to each X in $\mathfrak{g}$ an operator $\varphi(X)$:$\mathbf{V} \rightarrow \mathbf{V}$ depending linearly on X (thus, $\varphi(aX + bY) = a\varphi(X) + b\varphi(Y)$) and satisfying $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$. A vector space **V**, together with the representation φ, is called an $\mathfrak{g}$**-space** or $\mathfrak{g}$**-module**.

For example, the adjoint representation ad : $\mathfrak{g} \rightarrow \mathfrak{gl}(\mathbf{V})$, where ad x $(y) = [x , y]$. It preserves the bracket as follows :

$$[\text{ad} \ , \text{ad}\ y]\ (z) = \text{ad}\ x\ \text{ad}\ y\ (z) - \text{ad}\ y\ \text{ad}\ x\ (z)$$

$$= \text{ad}\ x\ ([y , z]) - \text{ad}\ y\ ([x , z]$$

$$= [x , [y , z]] - [y , [x , z]]$$

$$= [x , [y , z]] + [[x , z] , y]$$

$$= [[x , y] , z]$$

$$= \text{ad}\ [x , y]\ (z).$$ [1]

Reflections in a Euclidean Space:

We are here concerned with a fixed Euclidean space **E**, i.e. a finite dimensional vector space over **R** equipped with a positive definite symmetric bilinear form (α , β). Geometrically, a reflection in **E** is an invertible linear transformation leaving pointwise fixed some hyperplane (subspace of codimension one) and sending any vector orthogonal to that hyperplane into its negative.

Ultimately, a reflection is orthogonal, i.e. preserves the inner product on **E**. Any non-zero vector $\boldsymbol{\alpha}$ determines a reflection σ_α, with **reflecting hyperplane** $P_\alpha = \{\beta \in E : (\beta , \alpha) = 0\}$. An explicit formula for $\sigma_\alpha(\beta) = \beta - 2\frac{(\beta , \boldsymbol{\alpha})}{(\boldsymbol{\alpha} , \boldsymbol{\alpha})} \boldsymbol{\alpha}$. (This is because it sends $\boldsymbol{\alpha}$ to $-\boldsymbol{\alpha}$ and fixes all points in P_α).

We replace $2\frac{(\beta , \boldsymbol{\alpha})}{(\boldsymbol{\alpha} , \boldsymbol{\alpha})}$ by $< \beta, \alpha >$ and is linear only in the first variable β.[1]

Root Systems:

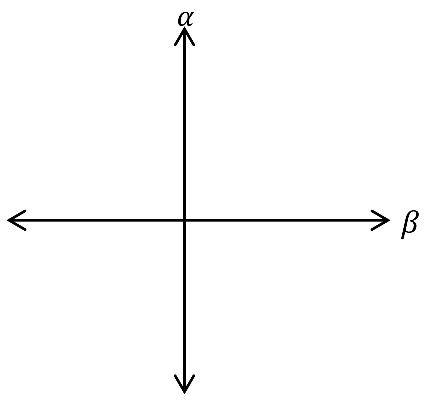
A subset Φ of the Euclidean space E is called a root system in E if the following axioms are satisfied:

- Φ is finite, spans E, and doesn't contain 0.
- If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.
- If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant .
- If $\alpha, \beta \in \Phi$, then $< \beta, \alpha > \in \mathbf{Z}$.[1]

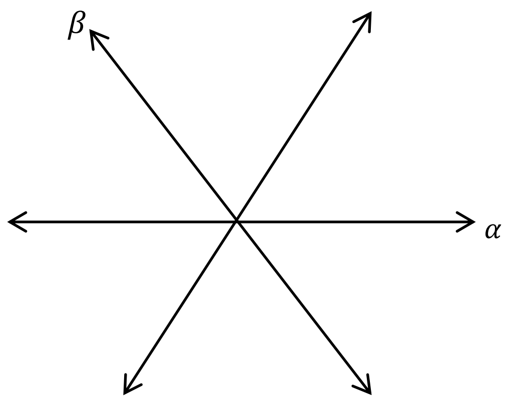
The ***simple roots*** can be defined as are the positive roots that cannot be written as the linear combination of other positive roots. If there are $\boldsymbol{r}$ simple roots for an algebra L of rank $\boldsymbol{r}$ and they form a basis of the root system.[7]

Algebra	Root system Δ[4]
$\boldsymbol{A_l}$	$e_i - e_j$
$\boldsymbol{B_l}$	$\pm e_i \pm e_j , \pm e_i$
$\boldsymbol{C_l}$	$\pm e_i \pm e_j , \pm 2e_i$
$\boldsymbol{D_l}$	$\pm e_i \pm e_j$
$\boldsymbol{E_6}$	$\pm e_i \pm e_j , \pm\frac{1}{2}(\pm e_1 \pm e_2 \pm \ldots \pm e_5 - e_6 - e_7 + e_8)$
$\boldsymbol{E_7}$	$\pm e_i \pm e_j , \pm (e_8 - e_7), \pm\frac{1}{2}(\pm e_1 \pm e_2 \pm \ldots \pm e_6 - e_7 + e_8)$
$\boldsymbol{E_8}$	$\pm e_i \pm e_j , \pm\frac{1}{2}(\pm e_1 \pm \ldots \pm e_8)$
$\boldsymbol{F_4}$	$\pm e_i \pm e_j, \pm e_i , \pm\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$
$\boldsymbol{G_2}$	$e_i - e_j , \pm(e_i \pm e_j) \mp e_k$

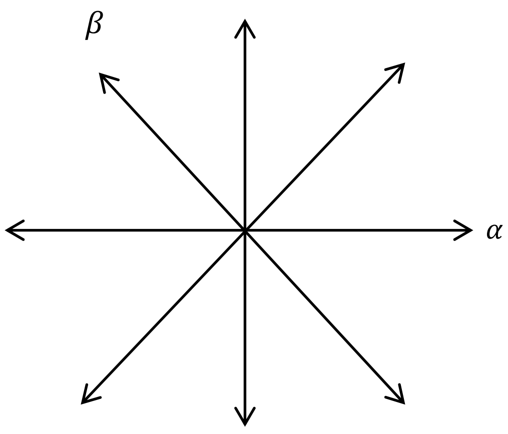
Root Diagrams of Lie Algebras[1]



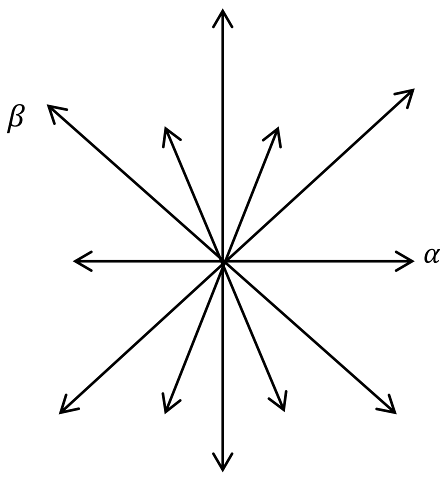
Root Diagram of $A_1 \times A_1$



Root Diagram of A_2



Root Diagram of B_2



Root Diagram of G_2

Cartan Matrix:

A **generalised Cartan matrix** is a square matrix A = (a_{ij}) with integer entries such that

- For diagonal entries, a_{ii} = 2.
- For non-diagonal entries, $a_{ij} \leq 0$.
- a_{ij} = 0 if and only if a_{ji} = 0.
- A can be written as DS, where D is a diagonal matrix, and S is a symmetric matrix.

We can always choose D with positive diagonal entries and if S is positive definite, then A is said to be a **Cartan Matrix**.

The Cartan Matrix of a simple lie algebra is the matrix whose elements are the scalar products

$$a_{ij} = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}.$$ [4]

Killing Form:

Let $\mathfrak{g}$ be any lie algebra. If x , y ∈ $\mathfrak{g}$, define $\kappa(x\,,y) = Tr(\text{ad}\ x\,,ad\ y)$. Then κ is a symmetric bilinear form on $\mathfrak{g}$, called the **Killing form**.

κ is also **associative**, i.e. $\kappa([x\,,y],z) = \kappa(x\,,[y\,,z])$.

It follows from $Tr([x\,,y],z) = Tr(x\,,[y\,,z])$ for endomorphisms $x\,,y\,,z$ of a finite dimensional vector space.[1]

Coxeter Graphs and Dynkin Diagrams:

If $\alpha\,,\beta$ are distinct positive roots, then we know that $<\alpha,\beta><\beta,\alpha>$ = 0, 1, 2 or 3. Define the **Coxeter graph** of $\aleph$ to be a graph having l vertices, the i^{th} joined to the $j^{th}(i \neq j)$ by $<\alpha_i\,,\alpha_j><\alpha_j,\alpha_i>$ edges.

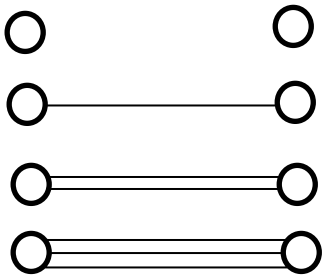
Examples[1]:

$A_1 \times A_1$

A_2

B_2

G_2



The coxeter graph determines the numbers $< \alpha_i , \alpha_j >$ in case all roots have equal lengths, since then $< \alpha_i , \alpha_j > = < \alpha_j , \alpha_i >$. In case more than one root length occurs(Ex. B_2 or G_2), the graph fails to tell us which of a pair of vertices should correspond to a short simple root, which to alone.

When a double or triple edge occurs in the coxeter graph, we can add an arrow pointing to the shorter of the two roots. This additional information gives us the cartan integers, and resulting figure **Dynkin Diagram**.[1]

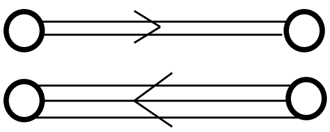
To a cartan matrix is associated a Dynkin Diagram, consisting of vertices representing the simple roots and (oriented) lines connecting them. The Dynkin Diagram an algebra L of rank $\boldsymbol{r}$ is constructed using the following rules:

1. Draw $\boldsymbol{r}$ vertices, one for each simple root α_i.
2. Connect the vertices i and j with number of lines equal to max$\{|A_{ij}| , |A_{ji}|\}$, or equivalently to the product $A_{ij}A_{ji}$.
3. If $|A_{ij}| > |A_{ji}|$, then draw an arrow pointing towards j from i, i.e. from the biggest to the smallest root.[7]

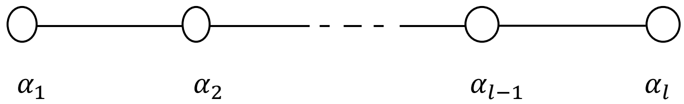
Examples[1]:

B_2

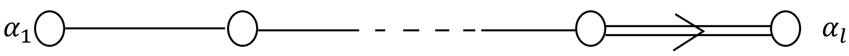
G_2



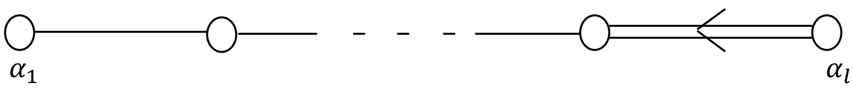
Dynkin Diagrams of Simple Lie Algebra[1]



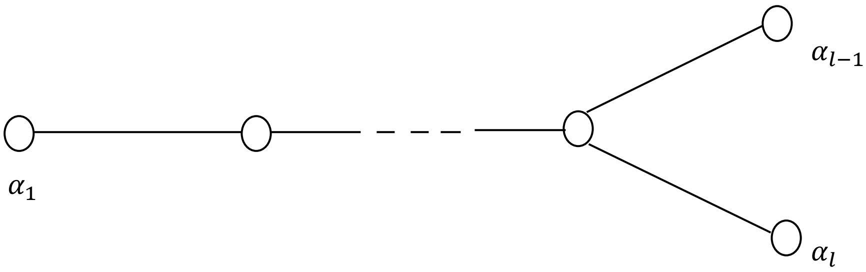
$\boldsymbol{A_l (l \geq 1)}$



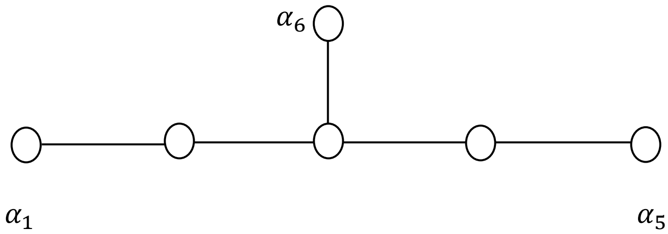
$\boldsymbol{B_l (l \geq 2)}$



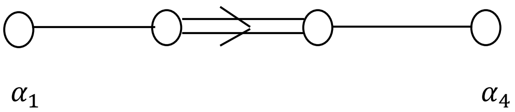
$\boldsymbol{C_l (l \geq 3)}$



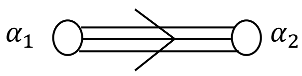
$\boldsymbol{D_l (l \geq 4)}$



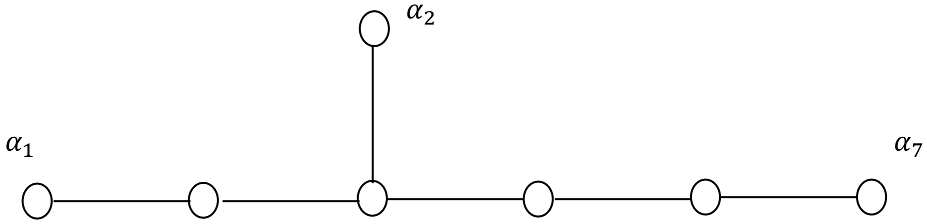
$\boldsymbol{E_6}$



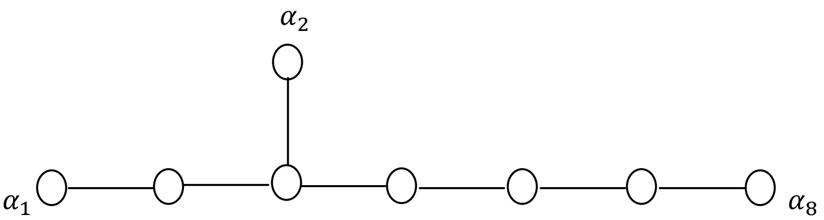
$\boldsymbol{F_4}$



$\boldsymbol{G_2}$



$\boldsymbol{E_7}$



$\boldsymbol{E_8}$

Cartan matrices[1]

$\boldsymbol{A_l (l \leq 1)}$

$$\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} & & & & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & & & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

$\boldsymbol{B_l (l \geq 2)}$

$$\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} & & & & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} & \mathbf{0} & & & \mathbf{0} \\ & & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{-1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & & \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

$\boldsymbol{C_l (l \geq 3)}$

$$\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} & & & & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} & & & & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} & \mathbf{-1} & & & \mathbf{0} \\ & & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & & & \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

$D_l(l \geq 4)$

$$\begin{pmatrix} 2 & -1 & 0 & & & & & 0 \\ -1 & 2 & -1 & & & & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & & & -1 & 2 & -1 & -1 \\ 0 & 0 & & & 0 & -1 & 2 & 0 \\ 0 & 0 & & & 0 & -1 & 0 & 2 \end{pmatrix}$$

F_4

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

G_2

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

E_6

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

E_7

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$\boldsymbol{E_8}$

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

3

Kac-Moody Algebra

3.1 Basic Definition

A Kac-Moody algebra can be defined as follows:

- A generalised cartan matrix C = (c_{ij}) of rank r.
- A vector space W over the complex numbers of dimension $2n - r$.
- A set of n linearly independent elements α_i of W and a set of n linearly independent elments $\alpha_i^*$of the dual space, such that $\alpha_i^*(\alpha_i)$ = c_{ij}. The α_i are known as coroots and α_i^* are known as roots.

The Kac-Moody algebra is the lie algebra $\mathfrak{g}$ defined by the generators e_i and f_i and the elements of W and the relations

- $[e_i\ , f_i]$ = α_i
- $[e_i\ , f_j]$ = 0 for $i \neq j$
- $[e_i\ , x]$ = $\alpha_i^*(x)e_i$ for $x \in$ W
- $[\ f_i\ , x]$ =$-\ \alpha_i^*(x)f_i$ for $x \in$ W
- $[\ x, x']$ = 0 for $x, x' \in$ W
- ad$(e_i)^{1-c_{ij}}$ (e_j) = 0
- ad$(f_i)^{1-c_{ij}}$ (f_j) = 0

where ad : $\mathfrak{g}$ → End(V), ad $x(y)$= $[x\ , y]$ is the adjoint representation of $\mathfrak{g}$.

A real (possibly infinite dimensional) Lie algebra is also considered as a Kac-Moody algebra if its complexification is a Kac-Moody algebra.[3]

3.2 Types of Kac-Moody Algebra

Properties of Kac-Moody algebra depend on the algebraic properties of its generalised cartan matrix C. If C is indecomposable, i.e. assume that there is no decomposition of the set of indices I into a disjoint union of non-empty subsets $I_1$and I_2 such that c_{ij} = 0 for all $i\ \in\ I_1$ and $j\ \in I_2$.

An important subclass of Kac-Moody algebras corresponds to symmetrizable generalised cartan matrices C, which can be decomposed in DS, where D is a diagonal matrix with positive integer entries and S is the symmetric matrix.

The Kac-Moody algebras are broadly divided into three classes

- A positive definite matrix S gives a finite-dimensional simple lie algebra.
- A positive semidefinite matrix S gives an infinite-dimensional Kac-Moody algebra of **affine lie algebra**.

- An indefinite matrix S gives rise to a Kac-moody algebra of **indefinite type**.
- Since the diagonal entries of C and S are positive, S can't be negative definite or negative semidefinite.
- An indefinite matrix S, but for each proper subset of I, the corresponding submatrix is positive definite or positive semidefinite gives rise to a Kac-moody algebra of **hyperbolic type**.[3]

3.3 Cartan Matrix of Kac-Moody Algebra

A generalised cartan matrix C = (c_{ij}) is defined as follows:

- The diagonal entries are all 2.
- The off-diagonal entries are all either non-positive, with c_{ij} = 0 if and only if c_{ji} = 0.
- The Cartan matrix is indecomposable.
- The Cartan matrix is symmetrizable, and the symmetrized matrix is positive definite.

For any column matrix $a > 0$ if all the entries are positive, and $a < 0$ if all the entries are negative. We now define an $r \times r$ cartan matrix C as

- If $\mathrm{C}b > 0$. C is **finite** if and only if C is symmetric and the symmetrized matrix has signature $(+ + \cdots +)$,
- If $\mathrm{C}b = 0$. C is **affine** if and only if C is symmetric and the symmetrized matrix has signature $(+ + \cdots + 0)$,
- If $\mathrm{C}b < 0$. C is **hyperbolic** if and only if det C < 0 and deletion of any row and the corresponding column gives a direct sum of affine or finite matrices.

for some $r \times 1$ matrix $b > 0$.[7]

3.4 Affine Lie Algebra

An affine Lie Algebra is constructed out of a affine cartan matrix C, that has the following conditions

- c_{ii}= 2,
- The off diagonal elements are non-positive integers and c_{ij} =0 if and only if c_{ji} =0,
- det C=0 and deletion of any row corresponding column gives the direct sum of finite cartan matrices.

The matrix C is thus positive semidefinite.[3]

Dynkin diagrams of affine Kac-Moody algebra:

Consider a generalised cartan matrix C = $c_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$.

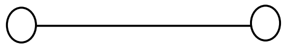
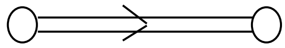
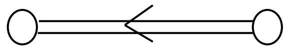
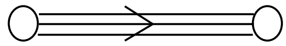
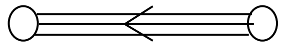
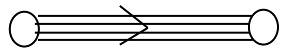
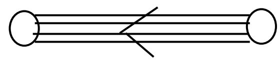
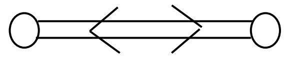
Here ($\alpha_1, \alpha_2, . . ., \alpha_l$) are independent vectors in l –dimensional Euclidean space.

The Dynkin diagram associated with the $l \times l$ cartan matrix C is obtained by the following rules:

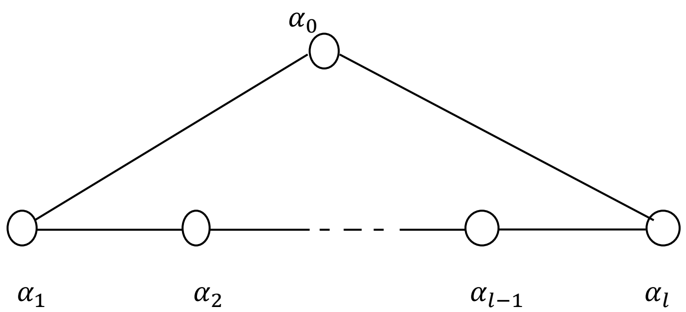
- The diagram has l vertices , which correspond to the l simple roots $\alpha_1, \alpha_2, . . ., \alpha_l$.

- When o $\leq c_{ij} \,.\, c_{ji} \leq 4$ the vertices i and j are connected by $\eta_{ij} = \max(|A_{ij}|,|A_{ji}|)$ lines with $A_{ij} = 0, -1, -2, \ldots$. If $|A_{ij}| > |A_{ji}|$ and $|A_{ij}| > 1$, the η_{ij} lines are equipped with an arrow pointing from j to i.[3]

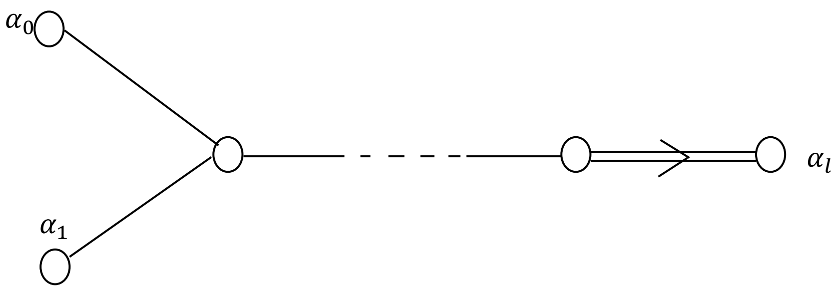
The possible links between i and $j (i \neq j)$ are restricted by the above rules. The vertices for generalised cartan matrix(GCM) of finite or affine type are given in the following table[3]:

$\lvert A_{ij} \rvert$	$\lvert A_{ji} \rvert$	i j
0	**0**	
1	**1**	
1	**2**	
2	**1**	
1	**3**	
3	**1**	
1	**4**	
4	**1**	
2	**2**	

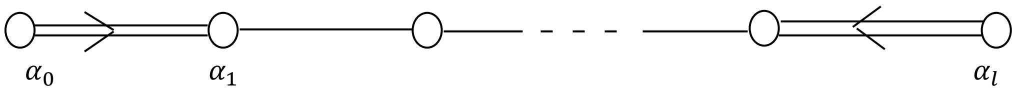
Dynkin diagram[4]



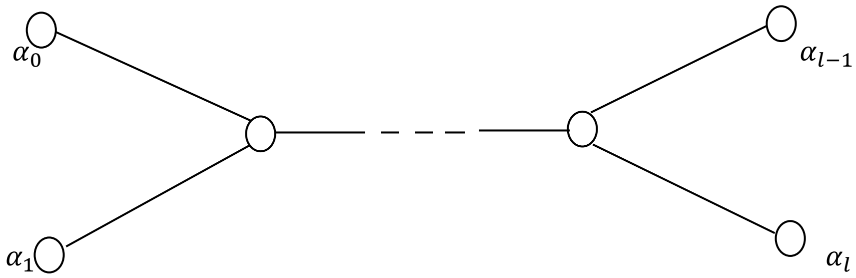
$\boldsymbol{A_l^{(1)}(l \leq 1)}$



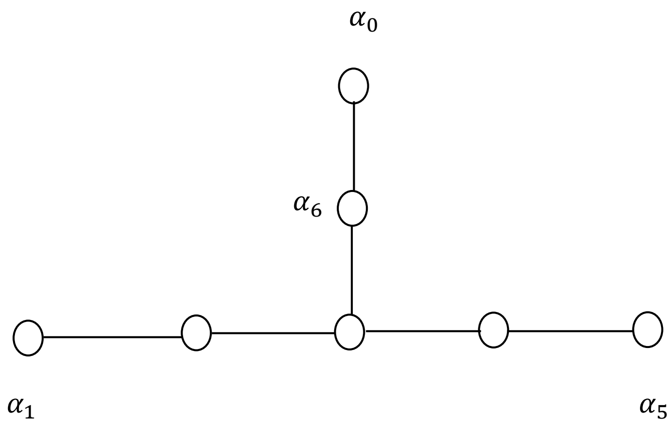
$\boldsymbol{B_l^{(1)}(l \geq 2)}$



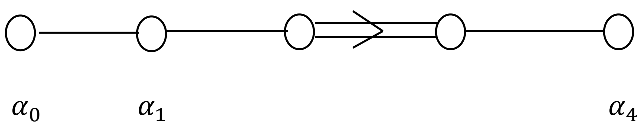
$\boldsymbol{C_l^{(1)} (l \geq 3)}$



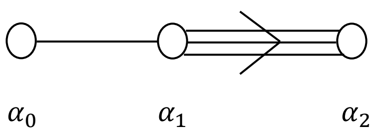
$\boldsymbol{D_l^{(1)} \ (l \geq 4)}$



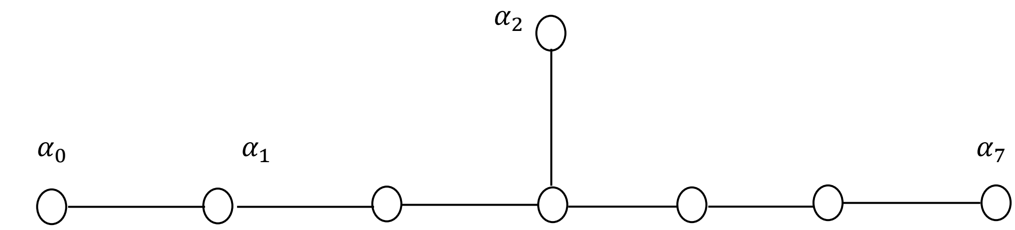
$\boldsymbol{E_6^{(1)}}$



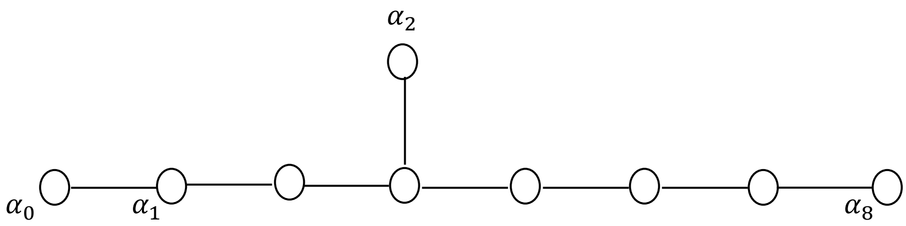
$\boldsymbol{F_4^{(1)}}$



$\boldsymbol{G_2^{(1)}}$



$\boldsymbol{E_7}^{(\mathbf{1})}$



$\boldsymbol{E_8}^{(\mathbf{1})}$

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