LARGE AMPLITUDE FREE VIBRATION ANALYSIS OF COMPOSITE PLATES BY FINITE ELEMENT METHOD

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

Master of Technology
in
Structural Engineering

By

Anil Kumar Dash
ROLL NO. 208CE204

DEPARTMENT OF CIVIL ENGINEERING
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Anil Kumar Dash
ROLL NO. 208CE204

NATIONAL INSTITUTE OF TECHNOLOGY
ROURKELA

CERTIFICATE

This is to certify that the thesis entitled, ***"LARGE AMPLITUDE FREE VIBRATION ANALYSIS OF COMPOSITE PLATES BY FINITE ELEMENT METHOD"*** *submitted by* ***Anil Kumar Dash*** *in partial fulfillment of the requirements for the award of Master of Technology Degree in Civil Engineering with specialization in "****Structural Engineering****" at National Institute of Technology, Rourkela is an authentic work carried out by him under my supervision and guidance. To the best of my knowledge, the matter embodied in this thesis has not been submitted to any other university/ institute for award of any Degree or Diploma.*

Date: *25th MAY 2010*

Prof. Manoranjan Barik
Dept. of Civil Engineering
National Institute of Technology,
Rourkela-769008

CONTENTS

ABSTRACT

Most of the structural components are generally subjected to dynamic loadings in their working life. Very often these components may have to perform in severe dynamic environment where in the maximum damage results from the resonant vibrations. Susceptibility to fracture of materials due to vibration is determined from stress and frequency. Maximum amplitude of the vibration must be in the limited for the safety of the structure. Hence vibration analysis has become very important in designing a structure to know in advance its response and to take necessary steps to control the structural vibrations and its amplitudes.

The non-linear or large amplitude vibration of plates has received considerable attention in recent years because of the great importance and interest attached to the structures of low flexural rigidity. These easily deformable structures vibrate at large amplitudes. The solution obtained based on the lineage models provide no more than a first approximation to the actual solutions. The increasing demand for more realistic models to predict the responses of elastic bodies combined with the availability of super computational facilities have enabled researchers to abandon the linear theories in favor of non-linear methods of solutions.

In the present investigation, **large amplitude** free vibration analyses of composite **Mindlin's plates** have been carried out using a $\mathbf{C^0}$ eight noded Langragian element by finite element method. The formulation is based on "First order shear deformation theory". The large

deformation effect on plate structures has been taken care by the dynamic version of von Karman's field equation. The effects of variations in the Poisson's ratio, amplitude ratio, thickness parameter & plate aspect ratio on the non-linear frequency ratio has also been included in the research.

Chapter 1 includes the general introduction and the scope of present investigation. The review of literature confining to the scope of the study has been presented in the **Chapter 2**. The general methods of analysis of the laminated composite plates have been briefly addressed in this chapter. The **chapter 3** presents some information about the theoretical background of finite element method and composite materials. The **Chapter 4** comprises the mathematical formulation of the finite elements. The elastic stiffness and the mass matrices for the plate element have been formulated. The boundary conditions have been implemented by eliminating the constrained degrees of freedom from the global stiffness matrix. The **Chapter 5** briefly describes the computer program implementation of the theoretical formulation presented in Chapter 4. The different functions and the associated variables which have been used in writing the codes in **MATLAB** have been presented in brief. A few numbers of flow-chart of the computer program has been illustrated. Several numerical examples which include "large amplitude free vibration analysis" have been presented in the **Chapter 6** to validate the formulation of the proposed method. The **Chapter 7** sums up and concludes the present investigation. An account of possible scope of extension to the present study has been appended to the concluding remarks. Some important publications and books referred during the present investigation have been listed in the **References** section.

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2.1 Introduction

Fiber-reinforced composites, due to their high specific strength, and stiffness, which can be tailored depending on the design requirement, are fast replacing the traditional metallic structures in the weight sensitive aerospace and aircraft industries [**14**]. Indeed, composite materials present considerable potential for wide use in aircraft structures in the future, especially because of their advantages of improved toughness, reduction in structural weight, reduction in fatigue and corrosion problems [**33**]. Most of the structures experiences severe dynamic environment during their service life; thus the excited motions are likely to have large amplitudes. The **large amplitude** analysis of composite structures is far more complex due to anisotropy, material couplings, and transverse shear flexibility effects compared to their isotropic counterparts. The use of composite materials require complex analytical methods in order to predict accurately their response to external loading, especially in severe environments, which may induce **geometrically non-linear behaviour**. This requires appropriate design criteria and accurate estimation of the fatigue life [**60**]. In addition to the usual difficulties encountered generally in the non-linear analysis of structures, related to the fact that the theorem of superposition does not hold, existence and uniqueness of the solutions are generally not guaranteed [**25**].

2.1 Finite element non-linear analysis

In the finite element formulation, we assume that the displacements of the finite element assemblage are infinitesimally small and the material is linearly elastic. In addition we also assume that the nature of the boundary conditions remain unchanged during the application of the loads on the finite element assemblage. With these assumptions, the finite element equilibrium equations derived were for **static analysis** $KU = R$. These equations correspond to a linear analysis of a structural problem because the displacement response **U** is a linear function of applied load vector **R**. Now applying a load of $\boldsymbol{kR}$ instead of R, where k is a constant, if the corresponding displacements are **not kU**, **then we perform a nonlinear analysis [70]**. In **dynamic problem** if $(w_{max} > 0.2 \times h)$ or $(w_{max} > 0.02 \times b)$ then it is called "a problem of **large amplitude plate vibration**", where w_{max} is maximum amplitude of the plate under vibration, h is thickness, b is the breadth of the plate taken. Now we will perform non-linear analysis for dynamic problem.

The solution of non-linear problems by the finite element method is usually attempted by one of the **3** basic techniques: incremental or stepwise procedures, iterative or Newton methods and step-iterative or mixed procedures. In case of incremental procedures, load is subdivided into many small partial loads or increments usually equal in magnitude though generally they need not be equal. The load is applied one increment at a time, and during the application of each increment the equations are assumed to be linear. In other words a fixed value of stiffness matrix is assumed throughout each increment, but stiffness matrix may take different values during different load increments. The solution for each step of loading is obtained as an increment of the displacements. These displacement increments are accumulated to give the total displacement at any stage of loading and the incremental process is repeated until the total load is reached. The incremental method is analogous to the numerical method used for the integration of systems of linear or non-linear differential equations, such as the Euler method or Runge-Kutta method. The iterative procedure is a sequence of calculations in which the body or structure is fully loaded in each iteration. Because some approximate constant value of the stiffness matrix is used in each step, equilibrium is not necessarily satisfied. After each iteration, the portion of the total loading that is not balanced is calculated and used in the next step to compute an additional increment of displacements. The process is repeated until equilibrium is approximated to some acceptable degree. Some of the iterative methods are direct iteration technique and Newton-Raphson techniques. The mixed procedures utilize a combination of the incremental and iterative schemes. Here the load is applied incrementally, but after each increment successive iterations are performed.

Table 1.1 Classification of the non-linear analyses

Types of analysis	Description
Material non-linearity only	Infinitesimal displacement and strain; the stress-strain relation is non-linear.
Large displacement, large rotation, but small strains	Displacements and the rotations of fibers are large, but fiber extensions and angle changes between fibers are small. The stress-strain relationship may be linear or nonlinear.
Large displacement, large rotation large strains	Fiber extensions and angle changes between fibers are large, fiber displacement and rotations may also be large, the stress-strain relationship may be linear or non-linear.

The **geometrically non-linear analysis** of composite plates exhibits specific difficulties due to the anisotropic material behaviour, and to the higher non-linearity induced by a **higher stiffness**, inducing tensile mid-plane forces in plates higher, than that observed with conventional homogeneous materials. These structures with complex boundary conditions, loadings and shapes are not easily amenable to analytical solutions and hence one has to resort to numerical methods such as finite elements **[69]**. A considerable amount of effort has gone into the development of simple plate bending elements based on the Yang, Norris and Stavsky theory [**63**] which is a consistent extension of Mindlin's theory for homogeneous isotropic plates. The advantages of this approach are (i) it accounts for transverse shear deformation, (ii) it requires only $\mathbf{C^0}$ continuity of the field variables, and (iii) it is possible to develop finite elements based on 6 engineering degrees of freedom viz. 3 translations and 3 rotations [2]. However, the low-order elements, i.e. the 3-node triangular, 4-node and 8-node quadrilateral elements, locked and exhibited violent stress oscillations. Unfortunately, this element which is having the shear strain term based on the Mindlin's theory becomes very stiff when used to model thin structures, resulting inexact solutions. This effect is termed as **shear-locking** which makes this otherwise successful element unsuitable. Many techniques have been tried to overcome this, with varying degrees of success. The most prevalent technique to avoid **shear locking** for such elements is a **reduced or selective integration** scheme . In all these studies shear stresses at nodes are inaccurate and need to be sampled at certain optimal points derived from considerations based on the employed integration order .The use of the same interpolation functions for transverse displacement and section rotations in these elements results in a mismatch of the order of polynomial for the transverse shear strain field. This mismatch in the order of polynomials is responsible for shear locking [**20]**..

2.3 Present investigation

The aim of this thesis is to apply the theoretical finite element model developed to analyze the **large amplitude** (**geometrical non-linear)** free dynamic response of laminated composite plate (**Mindlin's plate,** $\frac{\boldsymbol{length}}{\boldsymbol{thickness}} = \mathbf{10} \ldots \ldots \mathbf{25}$) in order to investigate the effect of non-linearity on the non-linear resonance frequencies. Periodic displacement was assumed since the motion of plates vibrating freely with amplitude displacements of the order of their thickness is generally periodic [**45**]. An isoparametric quadratic plate bending element has been used. It also considers the shear deformation of the plate. Hence the formulation is applicable to both thin as well as thick plates. Consistent mass matrix has been used.

As the higher order terms in the strain-displacement relations are not known, in order to obtain the solution for non-linear free vibration problem an iterative procedure is adopted using linear strain-displacement relations for the first iteration. For the successive iterations the higher order terms of the strain-displacement relations have been evaluated from the scaled eigenvectors corresponding to given amplitude at a prescribed point of the previous iterations. The iteration process is continued until required convergence is reached. In the present investigation, non-linear free vibration analysis is done for several quadrangular plates. Various boundary conditions have been considered. The effect of variations in some material and/or geometric properties of the plate have also been studied.

CHAPTER-2 LITERATURE REVIEW

2.1 Introduction

The analysis of plate and shell structures has a long history starting with membrane theory and then the bending theories. Plate and shell analyses are mainly based on 3 theories:

1. The classical plate theory (CPT)
2. The first-order shear deformation theory (FSDT)
3. The higher-order shear deformation theory (HSDT)

The effect of transverse shear deformation, which may be essential in some cases, is included in FSDT and HSDT, whereas it is neglected in CPT due to the Kirchhoff hypothesis. The classical laminate plate theory is based on the Kirchhoff hypothesis that straight lines normal to the undeformed mid plane remain straight and normal to the deformed mid plane and do not undergo stretching in the thickness direction. These assumptions imply the vanishing of the transverse shear and transverse normal strains. The classical laminate theory has been used in the stress analysis of composite plates. However, it is only accurate for thin plates.

In FSDT, a first-order displacement field is assumed for transverse shear strain through the thickness. Appropriate shear correction factors are required in FSDT due to the assumption of constant transverse shear strain and shear stress through the plate thickness, which is contradictory to the zero shear stress condition on the bounding planes of the plate and actual stress states through the thickness. Higher-order polynomials are used to represent displacement components through the thickness of the plate in HSDT, and the actual transverse strain/stress through the thickness and the zero stress conditions on the top and bottom of a plate can be represented. A more accurate approximation of the transverse shear effect can thus be obtained with no shear correction factors. However, complexities in formulation and large computational effort make it economically unattractive. The free vibration of plates has been largely studied using the first order shear deformation theory (FSDT). The advent of digital computer along with its capability of exponentially increasing computing speed has made the analytically difficult problems amenable through the various numerical methods and thus making the literature rich in this area.

2.2 Review on laminated composite plate

Ganapati et al. [16] have studied nonlinear flexural vibrations of laminated orthotropic plate using C^0 shear flexible QUAD-8 plate element. The nonlinear governing equations are solved using the direct iteration technique. Numerical results are obtained for isotropic, orthotropic and cross-ply laminated plates with simply-supported boundary conditions on immovable edges. It is observed that hardening behaviour is increased for thick plates and orthotropic plates.

Bhimaraddi et al. [9] have presented a critical analysis on nonlinear vibrations of heated antisymmetric angle-ply laminated plates using the parabolic shear deformation theory. Strains due to initial imperfections have also been retained using the von Karman type large deflection model. Numerical results are obtained by using the single mode approach to simply-supported plates, thus reducing five governing equations to a single nonlinear time differential equation involving quadratic and cubic nonlinearities.

Srinivas [55] has developed a sufficiently accurate refined analysis of composite laminates, which is much simpler than exact 3D analysis, for static and dynamic of composite laminates. He applied variational approach and considered transverse shear and inertia.

Chandrashekhara et al. [12] have investigated non-linear static and dynamic analysis heated laminated plates: a finite element approach by the use of a shear flexible finite element model. A

wide variety of results are presented for the nonlinear response of rectangular and circular plates under thermal and thermo-mechanical loads. The influences of anisotropy, boundary conditions, aspect ratio, rotary inertia and stacking sequence on the thermally induced response are studied.

Liew [28] solved the vibration of thick symmetric laminates by Reissner/Mindlin plate theory and the *p*-ritz method with various combinations of boundary conditions. to incorporate the effects of transverse shear deformation and rotary inertia, first-order Reissner/Mindlin plate theory is employed. Finally results in terms of non-dimensional frequency parameters for various boundary conditions, aspect ratios and relative thickness ratios are presented.

Large amplitude free flexural vibration analysis of composite stiffened plates have been carried out by **Kant**[14] using a nine-noded Lagrangian element. The element is based on the first order shear deformation theory. The large deformation effect of the stiffened plated structures has been taken care by the dynamic version of von Karman's field equations. The non-linear equations obtained have been solved by the direct iteration technique using the linear modeshapes as the starting vectors.

Singh et al. [56] investigated the large amplitude vibratory behaviour of unsymmetrically laminated plates. For this purpose, an efficient and accurate four-node shear flexible rectangular material with six degrees of freedom per node. The element assumes bi-cubic polynomial distribution with sixteen generalized undetermined coefficients for the transverse displacement.

The element stiffness and mass matrices are computed numerically by employing 3 × 3 Gauss-Legendre product rules. The element is found to be free of shear locking and does mot exhibit any spurious modes. In order to compute the nonlinear frequencies, linear mode shape corresponding to the fundamental frequency is assumed as the spatial distribution and nonlinear finite element equations are reduced to a single nonlinear second-order differential equation.

The geometrically non-linear free vibration of thin composite laminated plates is investigated by **Harras et al.** [19] using a theoretical model based on Hamilton's principle and spectral analysis previously applied to obtain the non-linear mode shapes and resonance frequencies of thin straight structures, such as beams, plates and shells. The **Von Karman's non-linear strain displacement** relationships have been employed. In the formulation, the transverse displacement **w** of the plate mid-plane has been taken into account and the in-plane displacements **u** and **v** have been neglected in the non-linear strain energy expressions.

A large amplitude vibration analysis of pre-stressed functionally graded material (FGM) laminated plates that are composed of a shear deformable functionally graded layer and two surface-mounted piezoelectric actuator layers has been carried out by **Yang et al.** [63]. Nonlinear governing equations of motion are derived within the context of Reddy's higher-order shear deformation plate theory to account for transverse shear strain and rotary inertia. A semi-analytical method that is based on one dimensional differential quadrature and Galerkin technique is proposed to predict the large amplitude vibration behavior.

Singha et al. [53] have presented the large amplitude free flexural vibration behaviors of thin laminated composite skew plates are investigated using finite element approach. The formulation includes the effects of shear deformation, in-plane and rotary inertia. The geometric non-linearity based on **Von Karman's** assumptions is introduced. The non-linear governing equations obtained employing Lagrange's equations of motion are solved using the direct iteration technique. The study reveals the redistribution of vibrating mode shape at certain amplitude of vibration depending on geometric and lamination parameters of the plate. Also, the degree of hardening behavior increases with the skew angle and its rate of change depends on the level of amplitude of vibration.

Amabili [5] has worked on theory and experiments for large-amplitude vibrations of rectangular plates with geometric imperfections. The von Karman nonlinear strain–displacement relationships are used to describe the geometric nonlinearity. A specific boundary condition, with restrained normal displacement at the plate edges and fully free in-plane displacements, not previously considered, has been introduced as a consequence that it is very close to the experimental boundary condition. The nonlinear equations of motion are studied by using a code based on pseudo-arclength continuation method. A thin rectangular stainless-steel plate has been inserted in a metal frame; this constraint is approximated with good accuracy by the newly introduced boundary condition. The plate inserted into the frame has been measured with a 3D laser system in order to reconstruct the actual geometry and identify geometric imperfections (out-of-planarity). The plate has been experimentally tested in laboratory for both the first and second vibration modes for several excitation magnitudes in order to characterize the nonlinearity of the plate with imperfections. Numerical results are able to follow experimental results with good accuracy for both vibration modes and for different excitation.

Li and **Cheng** [27] have proposed Differential quadrature method for nonlinear vibration of Orthotropic plates with finite deformation and transverse shear effect. Based on the Reddy's theory of plates with the effect of higher-order shear deformations, the governing equations for nonlinear vibration of orthotropic plate s with finite deformations are presented. The nonlinear free vibration is analyzed by the differential quadrature method. The differential quadrature approach suggested by Wang and Bert is extended to handle the multiple boundary conditions of the plate. The results show that the presented differential quadrature method is fairly reliable and valid. Influences of geometric and material parameters, transverse shear deformations and rotation inertia, as well as vibration amplitudes, on the nonlinear free vibration characteristics of orthotropic plates are studied. **Sundararajan et al.** [57] have developed the non-linear formulation of free flexural vibrations of functionally graded rectangular and skew plates under thermal environments, based on von Karman's assumptions. The nonlinear governing equations obtained using Lagrange's equations of motion are solved using finite element procedure. The results obtained here reveal that the temperature field and gradient index have significant effect on the nonlinear vibration of the functionally graded plate.

The large amplitude, geometrically non-linear periodic vibrations of shear deformable composite laminated plates, a **p**-version, hierarchical finite element is employed to define the model, taking into account the effects of the rotary inertia, transverse shear and geometrical non-linearity **Ribeiro** [45]. Harmonic forces are applied transversely to the plates and the steady-state periodic solutions are sought in the time domain by the shooting method. Fixing the amplitude of excitation and varying its frequency, response curves are derived. Several cases of modal coupling are found and the ensuing motions are analysed. The influences that the fibers orientations have on the forced vibrations are investigated. The efficiency and accuracy of the methods employed are discussed.

Malekzadeh [31] used a differential quadrature (DQ) method, to present large amplitude free vibration analysis of laminated composite skew thin plates. The governing equations are based on the thin plate theory (TPT) and the geometrical nonlinearity is modeled using Green's strain in conjunction with von Karman assumptions. Some new results for laminated composite skew plates with different mixed boundary conditions are presented and are compared with those obtained using the first order shear deformation theory based DQ(FSDT-DQ) method. Excellent agreements exist between the solutions of the two approaches but with much lower computational efforts of the present DQ methodology with respect to FSDT-DQ method.

A nine-noded isoparametric plate-bending element has been used for the analysis of free undamped vibration of isotropic and fiber reinforced laminated composite plates **Pandit et al.[**39]. The effect of shear deformation has been incorporated in the formulation by considering the first-order shear deformation theory. An effective mass lumping scheme with rotary inertia has been recommended. Two types of mass lumping schemes have been formed. In one lumping scheme rotary inertia has also been introduced. Numerical examples of isotropic and composite rectangular plates having different fiber orientations angles, thickness ratios, and aspect ratio have been solved. The present results are very close to the analytical solutions. Few examples have been presented as new results.

Yongsheng et al. [65] have developed large Amplitude Flexural Vibration of the Orthotropic Composite Plate Embedded with Shape Memory Alloy Fibers. Based on the nonlinear theory of symmetrically laminated anisotropic plates, the governing equations of flexural vibration in terms of displacement and stress functions are derived. The numerical results show that the relationship between nonlinear natural frequency ratio and temperature for the nonlinear plate has similar characteristics compared with that of the linear one, and the effects of temperature on forced response behavior during phase transformation from Martensite to Austenite are significant. The effects of the volume fraction of the SMA fiber, aspect ratio and free vibration amplitude on the dynamical behavior of the plate are also discussed.

Allahverdizadeh et al. [3] Vibration amplitude and thermal effects on the nonlinear behavior of thin circular functionally graded plates, formulated in terms of von-Karman's dynamic equations, and a semi-analytical approach is developed. The plate thickness is constant and the material properties of the functionally graded plate are assumed to vary continuously through the thickness, according to a power-law distribution of the volume fraction of the constituents. For harmonic vibrations, by using assumed-time-mode method and Kantorovich time averaging technique, governing equations are solved. The nonlinear frequencies and associated stresses are determined at large amplitudes of vibration. Effects of material compositions and thermal loads on the vibration characteristics and stresses are examined. The numerical results obtained here are compared with available published results, based on various approaches. **Houmat** [43] has proposed large amplitude free vibration of shear deformable laminated composite annular sector plates by a sector *p*-element. The effects of out-of-plane shear deformations, rotary inertia, and geometric non- linearity are taken into account. The shape functions are derived from the shifted Legendre orthogonal polynomials. The accuracy of the solution is improved simply by increasing the polynomial order. The time-dependent coefficients are described by a truncated Fourier series. The equations of free motion are obtained using the harmonic balance method and solved by the linearized updated mode method. The linear frequencies are found to converge rapidly down-wards as the polynomial order is increased.

Lal et al. [25] have dealt with nonlinear free vibration of laminated composite plates on elastic foundation with random system properties. The basic formulation of the problem is based on higher-order shear displacement theory including rotatory inertia effects and von Karman-type Non-linear strain displacement relations. A C^0 finite element is used for descretization of the laminate. A direct iterative method in conjunction with first-order Taylor series based perturbation technique procedure is developed to solve random nonlinear generalized eigenvalue problem. The developed probabilistic procedure is successfully used for the nonlinear free vibration problem with a reasonable accuracy.

Malekzadeh [32] has developed differential quadrature large amplitude free vibration analysis of laminated skew plates based on FSDT based on the first order shear deformation theory (FSDT) using differential quadrature method (DQM). The geometrical nonlinearity is modeled using Green's strain and von Karman assumptions in conjunction with the FSDT of plates. After transforming and discretizing the governing equations, which includes the effects of rotary inertia, direct iteration technique as well as harmonic balance method is used to solve the resulting discretized system of equations. The effects of skew angle, thickness-to-length ratio, aspect ratio and also the impact due to different types of boundary conditions on the convergence and accuracy of the method are studied.

A mesh-free least-squares-based finite difference (LSFD) method is applied for solving large-amplitude free vibration problem of arbitrarily shaped thin plates by **Wu et al.** [62]. In this approximate numerical method, the spatial derivatives of a function at a point are expressed as weighted sums of the function values of a group of supporting points. This method can be used to solve strong form of partial differential equations (PDEs), and it is especially useful in solving problems with complex domain geometries due to its mesh-free and local approximation characteristics. In this study, the displacement components of thin plates are constructed from the product of a spatial function and a periodic temporal function. Consequently, the nonlinear PDE is reduced to an ordinary differential equation (ODE) in terms of the temporal function.

Gajbir et al. [15] have studied Nonlinear vibration analysis of composite laminated and sandwich plates with random material properties Nonlinear vibration analysis is performed using a C^0 assumed strain interpolated finite element plate model based on Reddy's third order theory.

An earlier model is modified to include the effect of transverse shear variationalongtheplatethicknessandVon-Karmannonlinearstrainterms.Monte Carlo Simulation with Latin Hypercube Sampling technique is used to obtain the variance of linear and nonlinear natural frequencies of the plate due to randomness in its material properties. This chaotic nature of the dispersion of nonlinear eigenvalues is also revealed in eigenvalue sensitivity analysis.

Jayakumar et al. [23] have studied on nonlinear free vibrations of simply supported piezo-laminated rectangular plates with immovable edges utilizing Kirchhoff's hypothesis and von Karman strain–displacement relations. The effect of random material properties of the base structure and actuation electric potential difference on the nonlinear free vibration of the plate is examined. The study is confined to linear-induced strain in the piezoelectric layer applicable to low electric fields. The von Karman's large deflection equations for generally laminated elastic plates are derived in terms of stress function and transverse deflection function.

A review of the recent development of the finite element analysis for laminated composite plates from 1990 is presented by **Zhang et al.** [68]. The literature review is devoted to the recently developed finite elements based on the various laminated plate theories for the free vibration and dynamics, buckling and post-buckling analysis, geometric nonlinearity and large deformation analysis, and failure and damage analysis of composite laminated plates. The material nonlinearity effects and thermal effects on the buckling and post-buckling analysis, the first-ply failure analysis and the failure and damage analysis were emphasized specially. The future research is summarized finally.

CHAPTER-3 THEORETICAL BACKGROUND

3.1 Basic concept

In order to analyze an engineering system, a mathematical model is developed to describe the system. While developing the mathematical model, some assumptions are made for simplification. Finally, the governing mathematical expression is developed to describe the behavior of the system. The mathematical expression usually consists of differential equations and given conditions. These differential equations are usually very difficult to obtain solutions which explain the behavior of the given engineering system. With the advent of high performance computers, it has become possible to solve such differential equations. Various numerical solution techniques have been developed and applied to solve numerous engineering problems in order to find their approximate solutions. Especially, the finite element method has been one of the major numerical solution techniques. One of the major advantages of the finite element method is that a general purpose computer program can be developed easily to analyze the various kinds of problems.

The finite element method requires division of problem domain into many subdomains and each subdomain is called a *finite element*. Therefore, the problem domain consists of many finite element patches. The *finite element method* (FEM), or *finite element analysis* (FEA), it based on a idea of building a complicated object with simple blocks, or dividing a complicated object into small and manageable pieces. Application of this simple idea can be found everywhere in life, as well as in engineering.

The advent of the digital computer along with its exponentially increasing computational speed as well as core memory capacity has given the investigators a new direction to the analysis of the complicated structures thereby evolving simpler and more efficient methodologies. The widely used numerical methods to solve PDEs are the

- Finite element method (FEM)
- Finite volume methods (FVM)
- Finite difference methods (FDM)
- Exceptionally efficient higher-order version hp-FEM
- Generalized finite element method (GFEM)
- Extended finite element method (XFEM)
- Spectral finite element method (SFEM)
- Mesh-free finite element method
- Discontinuous Galerkin finite element method (DGFEM), etc

Among all the existing numerical methods, the finite element method is undoubtedly the most versatile and accurate one specially for structures having irregular geometry, material anisotropy, non-homogeneity and any type of loading and boundary conditions.

3.1.1 Why finite element method?

- Design analysis: hand calculations, experiments, and computer simulations.
- FEM/FEA is the most widely applied computer simulation method in engineering.
- Closely integrated with CAD/CAM applications.

3.1.2 Applications of FEM in engineering

- Mechanical/Aerospace/Civil/Automobile Engineering
- Structure analysis(static/dynamics, linear/nonlinear)
- Thermal/fluid flows
- Electromagnetic
- Geomechanics
- Biomechanics

3.1.3 A brief history of FEM

- 1943--------Courant (Variational methods)
- 1956--------Turner, Clough, Martin and Topp (Stiffness)
- 1960--------Clough("Finite element", plane problem)
- 1970--------application of mainframe computers
- 1980--------Microcomputers, pre and postprocessors
- 1990--------Analysis of large structural systems

3.1.4 Computer implementations

- Preprocessing (build FE model, loads and constraints)
- FEA solver (assemble and solve system of equations)
- Postprocessing (sort and display the results)

3.1.5 Available Commercial FEM Software Packages

- *ANSYS* (General purpose PC and workstations)
- *SDRC/I-DEAS* (Complete CAD/CAM/CAE package)
- *NASTRAN* (General purpose FEA on mainframe)

- *ABAQUS* (Nonlinear and dynamic analyses)
- *COSMOS* (General purpose FEA)
- *ALGOR* (PC and workstations)
- *PATRAN* (Pre /Post Processor)
- *Hyper-Mesh* (Pre /Post Processor)
- *Dyna-3D* (crash/impact analysis)

3.2 Composites

Generally speaking any material consisting of two or more components with different properties and distinct boundaries between the components can be referred to as a composite material. Moreover, the idea of combining several components to produce a material with properties that are not attainable with the individual components has been used by man for thousands of years. Correspondingly, the majority of natural materials that have emerged as a result of a prolonged evolution process can be treated as composite materials. Composite materials can be classified into two groups such as ‘filled materials’ and ‘Reinforced materials’.

Table 3.1 Difference between filled and reinforced composite materials

Filled materials	**Reinforced materials**
The main feature of these materials is the existence of some basic or matrix material whose properties are improved by filling it with some particles. Usually the matrix volume fraction is more than 50% in such materials.	The basic components of these materials (sometimes referred to as ‘advanced composites’) are long and thin fibers possessing high strength and stiffness. The fibers are bound with a matrix material whose volume fraction less than 50%.

Fibers used in advanced composite are two types are natural fibers (i.e. carbon, boron, steel, glass, aramid, polyethylene fibers) and natural fibers (i.e. wood, coir, bamboo, wool, cotton, rice, natural silk, asbestos). Fig.3.1 shows different types of fabrications in composites.

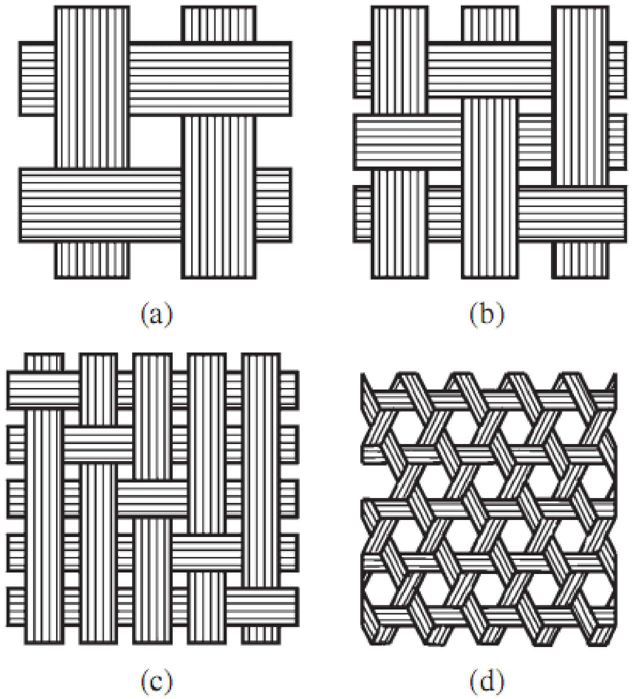


Fig.3.1 Plain (a), twill (b), and (c) biaxial woven, (d) triaxial woven fabrics

3.3 Stress-strain relationships

In a composite material the fibers may be oriented in a arbitrary manner. Depending upon the arrangements of fibers, the material may behave differently in different directions. According to their behaviour, composites may be characterized as generally:

Anisotropic (There are no symmetric planes w.r.t the alignment of fibers. Fibers are arranged in three- non mutual perpendicular direction**)**

Monoclinic (There are one symmetric planes w.r.t the alignment of fibers.**)**

Orthotropic (There are 3 mutually perpendicular symmetric planes w.r.t the alignment of fiber**)**

Transversely isotropic (there are three-planes of symmetry and, as such, it is orthotropic. In one of the planes of symmetry the material is treated as isotropic. An example of transversely isotropic material is a composite reinforced with continuous unidirectional fibers with all the fibers aligned in x_1 direction. In this case the material in the plane perpendicular to fibers(x_2-x_3 plane**)** is treated as isotropic.

Isotropic (every plane is a plane of symmetry. For example a composite containing a large no. of randomly oriented fibers behaves in an isotropic manner**)**

3.3.1 Generalized Hooke's law

Cauchy generalized Hooke's law to 3D elastic bodies and stated that the 6 components of stress are linearly related to the **six** components of strain. The stress-strain relationship written in matrix form, where the **six** components of stress and strain are organized into column vectors, is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{3.3.1}$$

In general, stress-strain relationships such as these are known as constitutive relations. There are 36 stiffness matrix components. However, it can be shown that conservative materials possess a strain energy density function and as a result, the stiffness and compliance matrices are symmetric. Therefore, only **21** stiffness components are actually independent in Hooke's law. The vast majority of engineering materials are conservative. Please note that the stiffness matrix is traditionally represented by the symbol **C**, while **S** is reserved for the compliance matrix. This convention may seem backwards, but perception is not always reality.

$$\{\sigma\} = [C]\{\varepsilon\} \tag{3.3.2}$$

$$\{\varepsilon\} = [S]\{\sigma\} \tag{3.3.3}$$

$$[C] = [S]^{-1} \tag{3.3.4}$$

3.3.2 Compliance matrix for different materials

$$[S] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & \frac{\nu_{61}}{G_{12}} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & \frac{\nu_{62}}{G_{12}} \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & \frac{\nu_{63}}{G_{12}} \\ 0 & 0 & 0 & \frac{1}{G_{23}} & \frac{\nu_{54}}{G_{13}} & 0 \\ 0 & 0 & 0 & \frac{\nu_{45}}{G_{23}} & \frac{1}{G_{13}} & 0 \\ \frac{\nu_{16}}{E_1} & \frac{\nu_{26}}{E_2} & \frac{\nu_{36}}{E_3} & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \quad MONOCLINIC \tag{3.3.5}$$

$$[S] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \quad ORTHOTROPIC \tag{3.3.6}$$

$$[S] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{21}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu_{23})}{E_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix} \quad TRANSVERSELY\ \text{ISOTROPIC} \tag{3.3.7}$$

$$[S] = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \quad ISOTROPIC \qquad (3.3.8)$$

3.3.3 Stiffness matrix [C] for different materials

$$[C] = \begin{bmatrix} [L] & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & [M] \end{bmatrix} \qquad (3.3.9)$$

Value of [L] and [M] for **orthotropic** material

$$[L]_O = \frac{1}{D_O} \begin{bmatrix} E_1\left(1 - \frac{E_3}{E_2}\nu_{23}^2\right) & E_2\left(\nu_{12} + \frac{E_3}{E_2}\nu_{13}\nu_{23}\right) & E_3(\nu_{13} + \nu_{12}\nu_{23}) \\ E_2\left(\nu_{12} + \frac{E_3}{E_2}\nu_{13}\nu_{23}\right) & E_2\left(1 - \frac{E_3}{E_1}\nu_{13}^2\right) & E_3\left(\nu_{23} + \frac{E_2}{E_1}\nu_{12}\nu_{13}\right) \\ E_3(\nu_{13} + \nu_{12}\nu_{23}) & E_3\left(\nu_{23} + \frac{E_2}{E_1}\nu_{12}\nu_{13}\right) & E_3\left(1 - \frac{E_2}{E_1}\nu_{12}^2\right) \end{bmatrix}$$

$$D_O = \frac{(E_1E_2E_3 - \nu_{23}^2 E_1 E_3^2 - \nu_{12}^2 E_2^2 E_3 - 2\nu_{12}\nu_{13}\nu_{23}E_2E_3^2 - \nu_{13}^2 E_2 E_3^2)}{E_1E_2E_3}$$

$$[M]_O = \begin{bmatrix} G_{23} & 0 & 0 \\ 0 & G_{13} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

Value of [L] and [M] for **Transversely Isotropic** material

$$[L]_{TI} = \frac{1}{D_{TI}} \begin{bmatrix} E_1(1-\nu_{23}^2) & E_2\nu_{12}(1+\nu_{23}) & E_2\nu_{12}(1+\nu_{23}) \\ E_2\nu_{12}(1+\nu_{23}) & E_2\left(1-\frac{E_2}{E_1}\nu_{12}^2\right) & E_2\left(\nu_{23}+\frac{E_2}{E_1}\ \nu_{12}^2\right) \\ E_3\nu_{12}(1+\nu_{23}) & E_3\left(\nu_{23}+\frac{E_2}{E_1}\nu_{12}^2\right) & E_2\left(1-\frac{E_2}{E_1}\nu_{12}^2\right) \end{bmatrix}$$

$$D_{TI} = \left(1-\nu_{23}^2 - 2(1+\nu_{23})\frac{E_2}{E_1}\nu_{12}^2\right)$$

$$[M]_{TI} = \begin{bmatrix} \frac{E_2}{2(1+\nu_{23})} & 0 & 0 \\ 0 & G_{12} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

Value of [L] and [M] for **Isotropic** material

$$[L]_I = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 0 & 0 \\ 0 & 1-\nu & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}$$

$$[M]_I = \begin{bmatrix} \frac{E}{2(1+\nu)} & 0 & 0 \\ 0 & \frac{E}{2(1+\nu)} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}$$

Table 3.2 Engineering material constants

MATERIAL	INDEPENDENT	DEPENDENT
MONOCLINIC	E_1, E_2, E_3 G_{23}, G_{13}, G_{12} $\nu_{12}, \nu_{13}, \nu_{23}$ $\nu_{16}, \nu_{26}, \nu_{45}, \nu_{36}$	
ORTHOTROPIC	E_1, E_2, E_3 G_{23}, G_{13}, G_{12} $\nu_{12}, \nu_{13}, \nu_{23}$	
TRANSVERSELY ISOTROPIC	E_1, E_2 G_{12} ν_{12}, ν_{23}	$E_3 = E_2, G_{13} = G_{12}$ $G_{23} = \frac{E_2}{2(1+\nu_{23})}$ $\nu_{13} = \nu_{12}$
ISOTROPIC	$E_1 (= E)$ $\nu_{12} (= \nu)$	$E_2 = E_3 = E, \nu_{13} = \nu_{23} = \nu$ $G_{23} = G_{13} = G_{12} = \frac{E}{2(1+\nu)}$

3.4 Plane-strain condition

There are circumstances when the stresses and strains do not vary in a certain direction. This direction is designated by z-axis. Although the stresses and strains do not vary along z-axis, they may vary in planes perpendicular to z-axis. This condition is referred to as plane-strain condition. When the plane-strain condition exists, the three dimension analysis simplifies considerably. For an isotropic material, the normal strain ε_z and the out-of-plane shear strain γ_{xz} *and* γ_{yz} are zero. For fiber reinforced composites these strains are not necessarily zero.

3.5 Plane-stress condition

Under the plane-stress condition one of the normal stresses and both out-of-plane shear stresses are zero. Plane stress condition may approximate the stresses in a thin-reinforced composite plate when the fibers are parallel to the x-y plane and the plate is loaded by forces along the edges such that the forces are parallel to the plane of the plate and are distributed uniformly over the thickness. The plane-stress condition does not provide the stresses exactly, not even for this thin-plate problem. Nevertheless, for many thin wall structures it is a useful approximation, yielding answers within reasonable accuracy.

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \qquad (3.5.1)$$

$$\{\sigma\} = [Q]\{\varepsilon\} \qquad (3.5.2)$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \qquad (3.5.3)$$

$$[Q] = \begin{bmatrix} \frac{E_1}{(1-\nu_{12}\nu_{21})} & \frac{\nu_{12}E_2}{(1-\nu_{12}\nu_{21})} & 0 \\ \frac{\nu_{12}E_2}{(1-\nu_{12}\nu_{21})} & \frac{E_2}{(1-\nu_{12}\nu_{21})} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \quad ORTHOTROPIC \qquad (3.5.4)$$

$$[Q] = [S]^{-1} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{13}} \end{bmatrix}^{-1} \quad TRANSVERSELY\ ISPTROPIC \qquad (3.5.5)$$

$$[Q] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad ISOTROPIC \qquad (3.5.6)$$

$$[Q]_\theta = [T]^T [Q][T] \qquad (3.5.7)$$

$$[T] = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \qquad (3.5.8)$$

$c = cos\theta, \qquad s = sin\theta$

[Q] and **[S]** are the stiffness and compliance matrices of different '**single layer composite plate**' under plane-stress condition respectively. ***[Q]*****$_\theta$** is the stiffness matrix of '**single layer composite plate'** under **plane-stress** condition when fibers are arranged at an angle θ with the x-axis on x-y plane. [T] is the transformation matrix.

3.6 Laminated composite

Composites are frequently made up of layers (plies) bonded together to form a laminate. A layer may consist of short fibers, unidirectional continuous fibers, or woven or braided fibers embedded in a matrix. A layer containing woven or braided fibers is referred to as fabric. Adjacent plies having the same material and same orientation are referred to as a ply group. Since the properties and the orientations are the across the ply group may be treated as one layer.

3.6.1 Basic assumptions

The following basic assumption are made

- Each lamina or ply of the laminate is quasi-homogeneous and orthotropic, but the orientation of the fiber may change from lamina to lamina.
- All displacements are continuous throughout the lamina.
- All deformations in the laminate are considered to be small.
- The laminate is thin and loaded in its plane only. The laminate and its layers are assumed to be in a plane stress condition expect the edges $(\sigma_z = \tau_{xz} = \tau_{yz} = 0)$.
- Transverse shear strains $\tau_{xz}\ and\ \tau_{yz}$ are negligible. This implies that a line originally straight and perpendicular to laminate remain straight and perpendicular to the deformed state.
- The bonds between plies in a laminae are perfect, that is plies will not slip over each other, and displacements and strains are continuous across interface of plies.
- Stress-strain and strain-displacement relations are linear.

3.6.2 Laminate code

The orientations of unidirectional continuous plies are specified by the angle θ w.r.t x-axis. The angle θ is positive in the counter clockwise direction. Example:-

[$45_3/0_4/90_2/60$] This laminate contains four ply groups, the first contain thre plies in the 45-degree direction, the second containing four plies in 0-degree direction, the third containing two plies in the 90-degree direction, the fourth containing one ply in the 60-degree direction. Different types of

Laminates on the basis of ply orientation are:-

- **Symmetrical laminates -** Laminate is symmetrical w.r.t midplane

  ex: $[45_2/0_2]_s$ or [45/45/0/0/0/0/45/45]

- **Balanced laminates** – For every ply in +θ direction there is an identical ply in –θ direction. Ex. $[45_2/0_2/-45_2/-0_2]$

- **Cross-ply laminates -** Ex. [0/90/90/0/0]

- **Angle-ply laminates** – Ex. [45/45/-45/-45/45]

- ***π*/4 laminates** - it consists plies in 0, 45, 90 and -45 degree directions. The no. of plies in each direction is same (balanced laminate). In addition, the layup is also symmetrical.

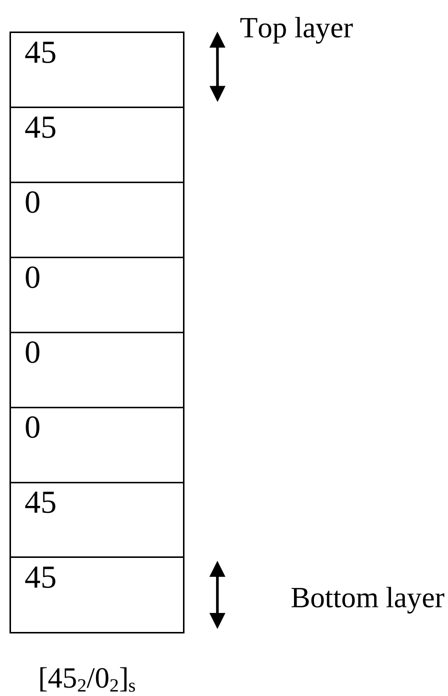


Fig.3.2 Different layers of a laminate

3.6.3 Stiffness matrices of thin laminates

Thin laminates are characterized by 3 stiffness matrices denoted [a], [b] and [d]. In this section we determine these matrices for thin, flat laminate undergoing small deformations. The analyses are based on the laminate plate theory and are formulated using the approximations that the strains vary linearly across the laminate, (out of plane) shear deformations are negligible and the out-of-plane normal stress σ_z and the shear stresses τ_{xz}, τ_{yz} are small compared with the in-plane σ_x, σ_y *and* σ_{xy} stresses. These approximations imply that the stress-strain relationship under plane-stress conditions may be applied.

$\{\sigma\} = [Q]\{\varepsilon\}$ (3.6.1)

$$\{\varepsilon\} = \{\varepsilon^0\} + z\{\kappa\} \qquad (3.6.2)$$

$$\{\sigma\} = [Q]\{\varepsilon^0\} + [Q]\, z\{\kappa\} \qquad (3.6.3)$$

$$\{N\} = \int_{-h_b}^{h_t} \sigma\, dz = \int_{-h_b}^{h_t} ([Q]\{\varepsilon^0\} + [Q]\, z\{\kappa\})\ dz \qquad (3.6.4)$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h_b}^{h_t} [\bar{Q}] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} dz + \int_{-h_b}^{h_t} z\ [\bar{Q}]\, dz \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \qquad (3.6.5)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h_b}^{h_t} z\, [\bar{Q}] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} dz + \int_{-h_b}^{h_t} z^2\ [\bar{Q}]\, dz \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \qquad (3.6.6)$$

$$\kappa_x = -\frac{\partial^2 w^0}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w^0}{\partial y^2}, \ \kappa_{xy} = -\frac{2\, \partial^2 w^0}{\partial x\, \partial y} \qquad (3.6.7)$$

$$[a] = \int_{-h_b}^{h_t} [\bar{Q}]\ dz \qquad (3.6.8)$$

$$[b] = \int_{-h_b}^{h_t} z\, [\bar{Q}]\ dz \qquad (3.6.9)$$

$$[d] = \int_{-h_b}^{h_t} z^2 \, [\bar{Q}] \ dz \qquad (3.6.10)$$

$$a_{ij} = \int_{-h_b}^{h_t} \bar{Q}_{ij} \ dz \ , \quad b_{ij} = \int_{-h_b}^{h_t} z \, \bar{Q}_{ij} \ dz \ , \ d_{ij} = \int_{-h_b}^{h_t} z^2 \, \bar{Q}_{ij} \ dz \qquad (3.6.11)$$

$$[\bar{Q}] = [Q]_{\theta} \qquad (3.6.12)$$

$$a_{ij} = \sum_{k=1}^{K} \left(\bar{Q}_{ij}\right)_k (z_k - z_{k-1}) \qquad (3.6.12)$$

$$b_{ij} = \frac{1}{2} \sum_{k=1}^{K} \left(\bar{Q}_{ij}\right)_k ({z_k}^2 - {z_{k-1}}^2) \qquad (3.6.13)$$

$$d_{ij} = \frac{1}{3} \sum_{k=1}^{K} \left(\bar{Q}_{ij}\right)_k ({z_k}^3 - {z_{k-1}}^3) \qquad (3.6.14)$$

Where, $\boldsymbol{\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0}$ are the strains in the reference plane (in-plane deformations) and $\boldsymbol{\kappa_x, \kappa_y, \kappa_{xy}}$ are the curvatures of the reference plane of the laminate. $\boldsymbol{[Q]}$ stiffness matrix of a ply (single layer) when fibers are along x-axis. $[\boldsymbol{\bar{Q}}]$ is the stiffness of a ply when fibers are aligned at an angle with x-axis. $\boldsymbol{Nx}$**:** Stress resultant in the x direction over a unit width along the y direction. $\boldsymbol{Nxy}$ $(or\ \boldsymbol{Nyx})$: Membrane shear stress resultant over a unit width along the y direction (or respectively along the x direction): $\boldsymbol{Mx}$ is Moment resultant along the x axis, due to the stresses σ_y over a unit width along the x direction. $\boldsymbol{Mxy}$ $(or - \boldsymbol{Myx})$: Twisting moment along the x axis (or y axis), due to the shear stress τ_{xy} over a unit width along the y direction (or x direction): $\boldsymbol{K}$ is the total number of plies (or ply groups) in the laminate. $\boldsymbol{z_k, z_{k-1}}$ are the distances from the reference plane to the two surfaces of the $\boldsymbol{k}^{\text{th}}$ ply. $[\overline{\boldsymbol{Q}}_{\boldsymbol{ij}}]_{\boldsymbol{k}}$ is the element stiffness matrix of the $\boldsymbol{k}^{\text{th}}$ ply, $\mathbf{h_t}$ and $\mathbf{h_b}$ are the height of the topmost and bottom surfaces from the reference plane. **[a], [b], [d]** are the stiffness matrix of the laminate. Fig 3.3 shows different layers in a laminate with reference plane.

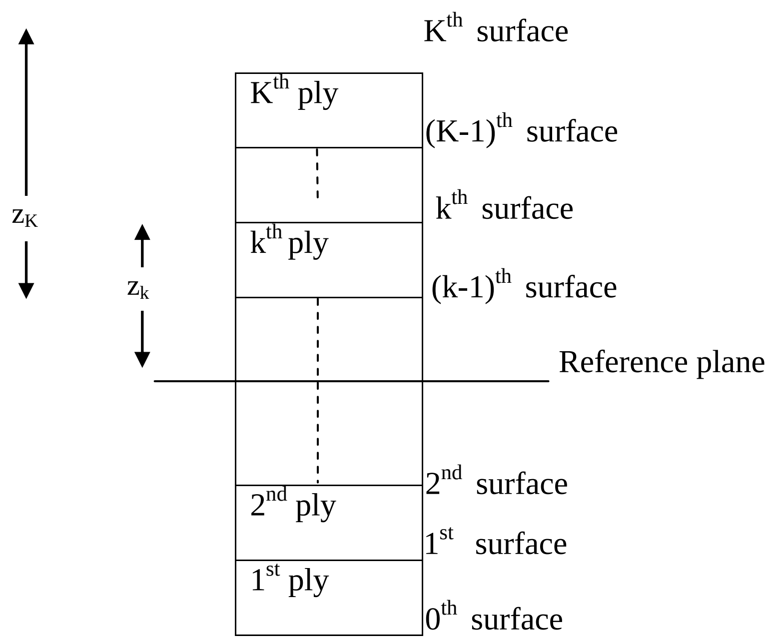


Fig.3.3 Different notation on a laminate

In the presence of shear deformations the force-strain relations are, as below:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ V_x \\ V_y \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & b_{11} & b_{12} & b_{16} & 0 & 0 \\ a_{12} & a_{22} & a_{26} & b_{12} & b_{22} & b_{26} & 0 & 0 \\ a_{16} & a_{26} & a_{66} & b_{16} & b_{26} & b_{66} & 0 & 0 \\ b_{11} & b_{12} & b_{16} & d_{11} & d_{12} & d_{16} & 0 & 0 \\ b_{12} & b_{22} & b_{26} & d_{12} & d_{22} & d_{26} & 0 & 0 \\ b_{16} & b_{26} & b_{66} & d_{16} & d_{26} & d_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{S}_{11} & \bar{S}_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{S}_{12} & \bar{S}_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \qquad (3.6.15)$$

Constitutive matrix or rigidity matrix of laminate $[D] = \begin{bmatrix} [a] & [b] & 0 \\ [b] & [d] & 0 \\ 0 & 0 & [V] \end{bmatrix}$ *(3.6.16)*

The significance of [a],[b],[d] matrices

a_{ij} are the in-plane stiffnesses that relate the in plane forces $\boldsymbol{N_x, N_y, N_{xy}}$ to the in-plane deformations $\boldsymbol{\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0}$. **$d_{ij}$** are the bending stiffnesses that relate the moments $\boldsymbol{Mx, My, Mxy}$ to the in-plane curvatures $\boldsymbol{\kappa_x, \kappa_y, \kappa_{xy}}$. **$b_{ij}$** are the in-plane-out-plane coupling stiffnesses that relate the in-plane forces $\boldsymbol{N_x, N_y, N_{xy}}$ to curvatures $\boldsymbol{\kappa_x, \kappa_y, \kappa_{xy}}$. and the moments $\boldsymbol{Mx, My, Mxy}$ to the in-plane deformations $\boldsymbol{\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0}$.

Table 3.3 [a],[b],[d] matrices show different types of couplings as follows:

Coupling types	Description
Extension-shear	When the elements a_{16}, a_{26} are not zero, in-plane normal forces N_x, N_y cause shear deformation γ_{xy}^0 and a twist force N_{xy} causes elongation in the x and y direction.
Bending-twist	When the elements d_{16}, d_{26} are not zero, bending moment M_x, M_y cause twist of the laminate and a twist moment M_{xy} causes curvature in the x-z and y-z plane.
Extension-twist and bending-shear	When the elements $b_{16}, b_{26}$are not zero, in-plane normal forces N_x, N_y cause twist and bending moments M_x, M_y results in shear deformationγ_{xy}^0.
In-plane-out-of-plane	When the elements b_{ij} are not zero, in-plane forces cause curvature and moments cause in-plane deformations.
Extension-extension	When a_{12} is not zero, N_x cause elongation in y-direction and N_y cause elongation in x-direction.
Bending-bending	When d_{12} is not zero, M_x causes κ_y in y-z plane and M_y causes κ_x in x-z plane.

CHAPTER-4 FINITE ELEMENT FORMULATION

4.1 Introduction

In the finite element analysis, the continuum is divided into a finite number of elements having finite dimensions and reducing the continuum having infinite degrees of freedom to finite number of unknowns. The formulation presented here is based on assumed displacement pattern within the element and can be applied to linear, quadratic, cubic or any other higher order element by incorporating appropriate shape functions. In the following the element mass and stiffness matrices of the plate are derived. The element mass and stiffness matrices are then assembled to form the overall mass and stiffness matrices. Necessary boundary conditions are then incorporated. Reduced integration technique has been used to obtain the element mass and stiffness matrices. A **non-linear finite element model of an isoparametric plate** element is developed of the governing equations. A composite plate (**Mindlin's plate**) is chosen for the present analysis.

4.2 Assumptions

The formulation is based on the following assumptions:

- The material of the plate obeys Hooke's law.

- The bending deformations follow Mindlin's hypothesis therefore the normal perpendicular to the middle plane of the plate before bending remains straight, but not necessarily normal to the middle plane of the plate after bending.
- The deflection in the z-direction is a function of x and y only.
- The transverse normal stresses are neglected.

4.3 Equilibrium equations

The equations of motion for free un-damped vibration of an elastic system undergoing large displacements can be expressed in the following matrix form.

- Free vibration analysis $[K]\{\delta\} + [M]\{\ddot{\delta}\} = \{0\}$ *(4.3.2)*

[K] and [M] are overall stiffness & mass matrix and {δ} is displacement vector.

4.4 The transformation of co-ordinates

The arbitrary shape of the whole plate is mapped into a Master Plate of square region [-1, +1] in the s-t plane with the help of the relationship given by [**69**].

$x = \sum_{i=1}^{8} N_i(s,t)x_i \qquad y = \sum_{i=1}^{8} N_i(s,t)y_i$ *(4.4.1)*

Where $(x_{i,}y_i)$ are the coordinates of the i^{th} node on the boundary of the plate in the x-y plane and $N_i(s,t)$ are the corresponding cubic serendipity shape functions presented below.

$$N_1 = ¼\,(\eta - 1)(1 - \xi)(\eta + \xi + 1)$$

$$N_2 = ½\,(1 - \eta)(1 - \xi^2)$$

$$N_3 = ¼\,(\eta - 1)(1 + \xi)(\eta - \xi + 1)$$

$$N_4 = ½\,(1 - \eta^2)(1 + \xi)$$

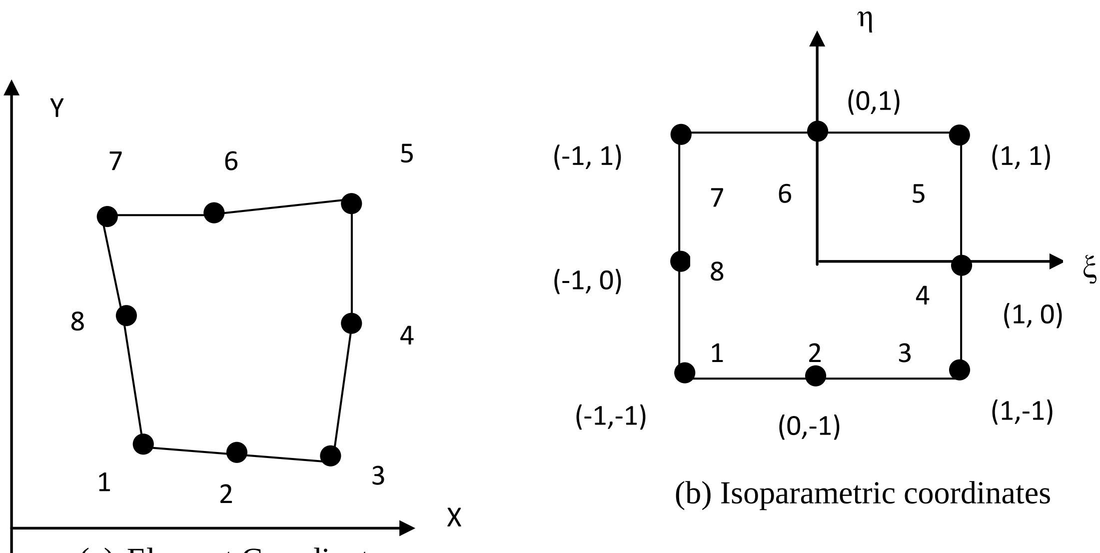
$$N_5 = ¼\,(1 + \eta)(1 + \xi)(\eta + \xi - 1)$$

$$N_6 = ½\,(1 + \eta)(1 - \xi^2)$$

$$N_7 = \frac{1}{4}(1+\eta)(1-\xi)(\eta-\xi-1)$$

$$N_8 = \frac{1}{2}(1-\eta^2)(1-\xi)$$

$$[N] = [N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6 \ N_7 \ N_8 \ ] \qquad (4.4.2)$$



(a) Element Coordinates

(b) Isoparametric coordinates

Figure 4.1 Quadratic Isoparametric Plate Element

$\sum_{i=1}^{8} N_i = 1$ at any point inside the element.

Quadratic elements are preferred for stress analysis, because of their high accuracy and flexibility in modeling complex geometry, such as curved boundaries. The displacement functions of the plate element are expressed in terms of the local (ξ-η) coordinate system whereas the strains are in terms of the derivatives of the displacements with respect to the x and y coordinates. Hence before establishing the relationship between the strain and the displacement the first and second order derivatives of the displacement w with respect to the x-y coordinates are expressed in terms of those of the (ξ-η) coordinates using the chain rule of differentiation and are obtained as below.

$$\begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \end{Bmatrix} \qquad (4.4.3)$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \; and \; [J]^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \qquad (4.4.4)$$

$[J]$ is the Jacobian matrix, $[J]^{-1}$ is the inverse jacobian matrix. V is the volume.

$$\partial x \, \partial y = |J| \; \partial \xi \, \partial \eta \qquad (4.4.5)$$

$$\partial \mathrm{V} = |J| \, h \, \partial \xi \, \partial \eta \qquad (4.4.6)$$

4.5 Plate element formulation

The displacement field at any point within the element is given by

$$\{f\} = \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \begin{Bmatrix} u - z\theta_x \\ v - z\theta_y \\ w \end{Bmatrix} \qquad (4.5.1)$$

Owing to the shear deformations, certain warping in the section occurs as shown in Fig. 3.3. However, considering the rotations θ_x and θy as the average and linear variation along the thickness of the plate, the angles ϕ_x and ϕ_y denoting the average shear deformation in and x-y directions respectively are given by:

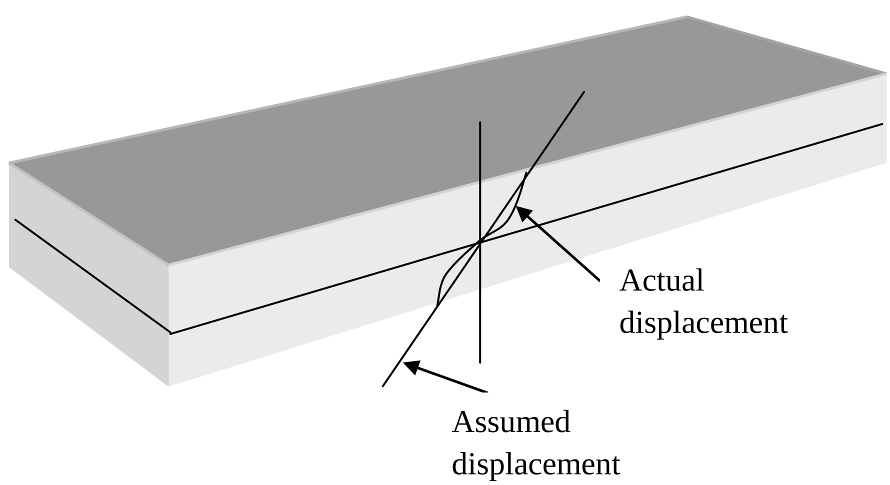


Fig 4.3 Deformation of plate cross-section

$$\begin{Bmatrix} \theta_x \\ \theta_y \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w}{\partial x} + \emptyset_x \\ \frac{\partial w}{\partial y} + \emptyset_y \end{Bmatrix} \qquad (4.5.2)$$

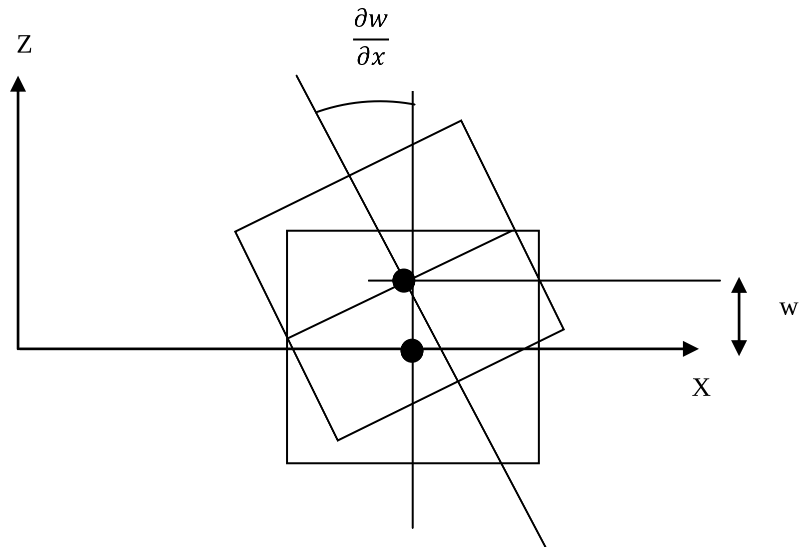


Fig.4.4 Kirchhoff theory

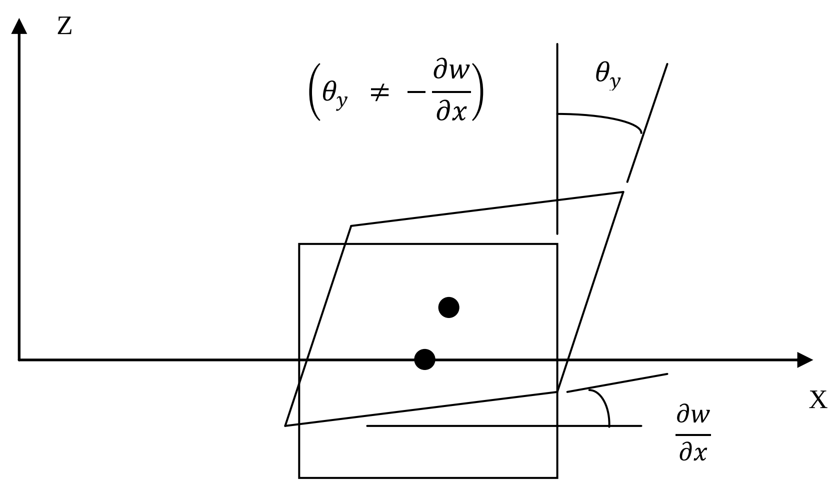


Fig.4.5 Mindlin theory

4.5.1 Linear and non-linear stiffness matrix

The plate strains are described in terms of middle surface displacements i. e. x-y plane coincides with the middle surface as shown in Fig.4.6. The strain matrix is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \\ -\emptyset_x \\ -\emptyset_y \end{Bmatrix} \tag{4.5.3}$$

The stress matrix is given by

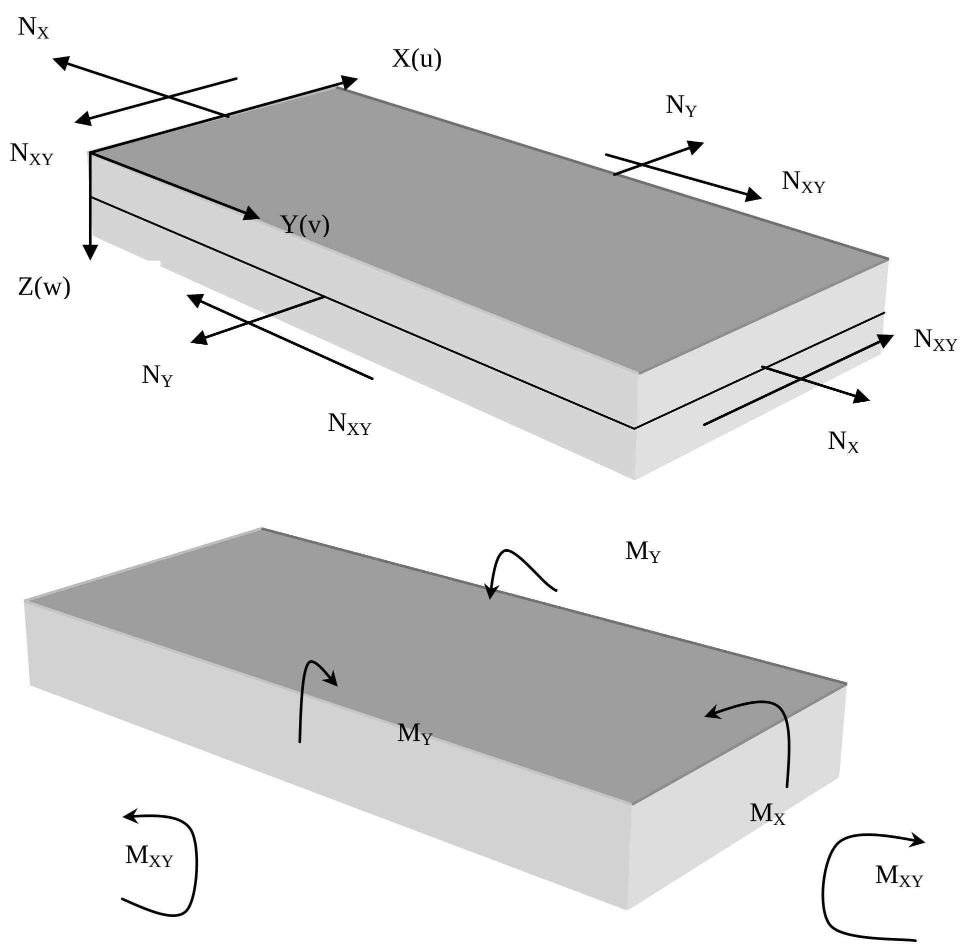


Fig.4.6 In-plane and bending resultants for a flat plate

$$\{\sigma\} = \begin{Bmatrix} N_X \\ N_Y \\ N_{XY} \\ M_X \\ M_Y \\ M_{XY} \\ V_X \\ V_Y \end{Bmatrix} \qquad (4.5.4)$$

The stresses are defined in terms of usual stress resultants: $N_x = \sigma_x h$ where σ_x is the average membrane stress etc. Now if the deformed shape is considered as shown in the

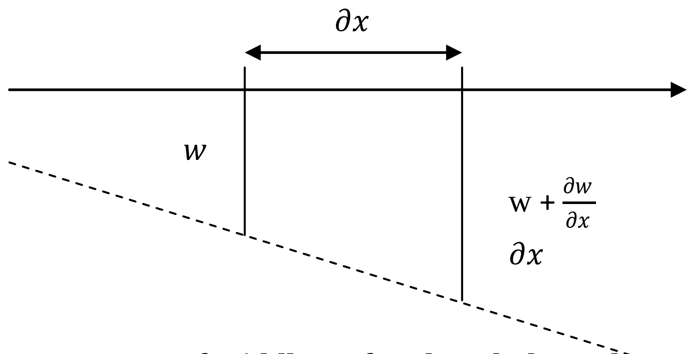


Fig.4.7 Increase of middle surface length due to lateral displacement

Fig.4.7 it is seen that displacement w produces some additional extension in the x and y directions of the middle surface and the length dx stretches to

$$dx' = dx\sqrt{1 + (\partial w/\partial x)^2} = dx\{1 + ½(\partial w/\partial x)^2 + \cdots\cdots\cdots\cdots\} \qquad (4.5.5)$$

In defining the x- elongation the following relationship can be written (to second approximation)

$$\varepsilon_x = \left(\frac{\partial u}{\partial x}\right) + ½ \left(\frac{\partial w}{\partial x}\right)^2 \qquad (4.5.6)$$

In the similar manner considering the other components the strain is given by

$$\{\varepsilon\} = \{\varepsilon_L\} + \{\varepsilon_{NL}\} \qquad (4.5.7)$$

Where $$\{\varepsilon_L\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \\ -\emptyset_x \\ -\emptyset_y \end{Bmatrix} \quad and \quad \{\{\varepsilon_{NL}\}\} = \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (4.5.8)$$

In which the first term is the linear expression and the second term gives non-linear terms. If linear elastic behavior is considered, the stress-strain relationship is written as

$$\{\sigma\} = [D]\{\varepsilon\} = [D]\{\varepsilon_L\}\{\varepsilon_{NL}\}\{\sigma_L\}\{\sigma_{NL}\} \qquad (4.5.9)$$

Where **[*D*]** the rigidity matrix

For **composite** material $$[D] = \begin{bmatrix} [a] & [b] & 0 \\ [b] & [d] & 0 \\ 0 & 0 & [s] \end{bmatrix} \qquad (4.5.10)$$

[a], [b], [d], [s]are the stiffness matrix of the laminates of composite plate. Description of these stiffness matrices presented in **chapter-3**.

For **isotropic** material

$$[D] = \begin{bmatrix} DXA & DIA & 0 & 0 & 0 & 0 & 0 & 0 \\ DIA & DYA & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & DXYA & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & DXF & DIF & 0 & 0 & 0 \\ 0 & 0 & 0 & DIF & DYF & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & DXYF & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & S_X & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_Y \end{bmatrix} \qquad (4.5.11)$$

$DXA = \frac{Eh}{(1-\nu^2)}$, $DIA = \upsilon \times DXA$, $DYA = DXA$, $DXYA = \frac{(1-\nu)}{2} \times DXA$,

$$DXF = \frac{Eh^3}{12 \times (1-\nu^2)}, DIF = \nu \times DXF, \qquad DYF = DXF, \qquad DXYF = \frac{(1-\nu)}{2} \times DXF$$

$$S_X = S_Y = \alpha \times Gh$$

E= modulus of elasticity, G=Shear modulus, ν= Poisson's ratio, h=Thickness of plate

The non-linear strain component can be conveniently written as

$$\{\varepsilon_{NL}\} = ½ \begin{bmatrix} \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} = ½[A]\{\theta\} \qquad (4.5.12)$$

The slope can be related to nodal parameters as follows

$$\{\theta\} = \begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} = \sum_{i=1}^{8} \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial y} & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} = [G]\{\delta\} \qquad (4.5.13)$$

$$[G] = \sum_{i=1}^{8} \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial y} & 0 & 0 \end{bmatrix} \qquad (4.5.14)$$

Then $$d\{\varepsilon_{NL}\} = ½d[A]\{\theta\} + ½[A]d\{\theta\} = [A]\{G\}d\{\delta\} \qquad (4.5.15)$$

This above equation is due to an interesting property of matrix $[A]$ and $\{\theta\}$, $\{\varepsilon\}$ is approximated as

$$d\{\varepsilon\} = [\bar{B}]d\{\varepsilon\} \qquad (4.5.16)$$

Where

$$[\bar{B}] = [B_L] + [B_{NL}(\delta)] \qquad (4.5.17)$$

In which $[B_L]$, is the same matrix as in linear infinitesimal strain analysis and is given by

$$[B_L] = \sum_{i=1}^{8} \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & -\frac{\partial N_i}{\partial y} \\ 0 & 0 & 0 & \frac{\partial N_i}{\partial y} & -\frac{\partial N_i}{\partial x} \\ 0 & 0 & \frac{\partial N_i}{\partial x} & -N_i & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial y} & 0 & -N_i \end{bmatrix} \qquad (4.5.18)$$

Only $[B_{NL}]$ depends on the displacements. In general, $[B_{NL}]$ is found to be a linear function of such displacements. Therefore,

$$d\{\varepsilon\} = d\{\varepsilon_L\} + d\{\varepsilon_{NL}\} = ([B_L] + [B_{NL}])\, d\{\delta\} \qquad (4.5.19)$$

$$[B_{NL}] = [A]\{G\} \qquad (4.5.20)$$

$$\{\varepsilon_{NL}\} = ½[B_{NL}]\{\delta\} \qquad (4.5.21)$$

The virtual work equation in Langrangian coordinate system is

$$\int_V d\{\varepsilon\}^T \ \{\sigma\}dv - d\{\sigma\}^T \{R\} = \{0\} \qquad (4.5.22)$$

Where $d\{\varepsilon\}$ is the variation in the Green's strain vector $\{\varepsilon\}$, associated with the displacement $d\{\varepsilon\}$, $\{\sigma\}$ is the Polio-Kirchhoff's stress and $\{R\}$ is a vector of generalized forces associated with the displacements $d\{\varepsilon\}$.

The finite element approximation to the equation (4.5.22) in the B-notation is

$$d\{\delta\}^T \ \int_V [\bar{B}]^T\{\sigma\}dv - d\{\sigma\}^T\{R\} = \{0\} \qquad (4.5.23)$$

Which because $d\{\delta\}$ is arbitrary, gives the equilibrium equation as

$$\int_V [\bar{B}]^T [D]\{\varepsilon\}dv - \{R\} = \{0\} \qquad (4.5.24)$$

or

$$\int_V ([B_L] + [B_{NL}])^T [D](B_L + ½B_{NL})\,\{\delta\}\, dv - \{R\} = \{0\}$$

or $$[K_S]\{\delta\} - \{R\} = \{0\} \quad \textit{(4.5.25)}$$

Where $[K_S] =$

$$\int_V([B_L]^T[D][B_L] + [B_{NL}]^T[D][B_L] + ½[B_L]^T[D][B_{NL}] + ½[B_{NL}]^T[D][B_{NL}])\ \mathrm{dv} \quad \textit{(4.5.26)}$$

If N-notation is followed then $[K_S]$ would be obtained as

$$[K_S] = \left\{[K_0] + {}^1\!/_2\,[N_1] + {}^1\!/_3\,[N_2]\right\} \quad \textit{(4.5.27)}$$

The first term in the curly brackets of the above equation *i. e*. $[K_0]$ is independent of the displacements {δ}. $[N_1]$ is linearly dependent upon {δ} and $[N_2]$ is quadratically dependent upon {δ}. The matrix $[K_S]$ is known as secant stiffness matrix. The secant stiffness matrix obtained with B-notation is un-symmetric and that obtained with N-notation is symmetric. The correlation between the two notations is expressed as

$$[K_0] = \int_V [B_L]^T[D][B_L]dv \quad \textit{(4.5.28)}$$

$$[N_1] = \int_V([B_L]^T[D][B_{NL}] + [B_{NL}]^T[D][B_L] + [G]^T[S_L][G])\ dv \quad \textit{(4.5.29)}$$

$$[N_2] = \int_V([B_{NL}][D][B_{NL}] + [G][S_{NL}][G])dv \quad \textit{(4.5.30)}$$

The matrices $[S_L]$ and $[S_{NL}]$ together give symmetric stress matrix. The symmetric stress matrix is introduced as

$$[S] = [S_L] + [S_{NL}] \quad \textit{(4.5.30)}$$

Where

$$[S] = \begin{bmatrix} N_X & N_{XY} \\ N_{XY} & N_Y \end{bmatrix} \quad \textit{(4.5.31)}$$

$$[S_L] = \begin{bmatrix} DXA\left(\frac{\partial u}{\partial x}\right) & DXYA\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ DXYA\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & DYA\left(\frac{\partial v}{\partial y}\right) \end{bmatrix} \quad \textit{(4.5.32)}$$

$$[S_{NL}] = \begin{bmatrix} ½\left\{DXA\left(\frac{\partial w}{\partial x}\right)^2 + DIA\left(\frac{\partial w}{\partial y}\right)^2\right\} & DXYA\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \\ DXYA\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) & ½\left\{DIA\left(\frac{\partial w}{\partial x}\right)^2 + DYA\left(\frac{\partial w}{\partial y}\right)^2\right\} \end{bmatrix} \quad \textit{(4.5.33)}$$

From the above relation, it can be deducted that

$$[K_L] = [K_0] \qquad (4.5.33)$$

and non-linear stiffness matrix

$$[K_{NL}] = \frac{1}{2}[N_1] + \frac{1}{3}[N_2] \qquad \text{in N-notation}$$

$$= \int_V ([B_{NL}][D][B_L] + [B_L][D][B_{NL}] + [B_{NL}][D][B_{NL}])\, dv \qquad (4.5.34)$$

in B-notation

4.5.2 Mass matrix

The acceleration field at any point is given by

$$\{\ddot{f}\} = \begin{Bmatrix} \ddot{u} - z\ddot{\theta}_x \\ \ddot{v} - z\ddot{\theta}_y \\ \ddot{w} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -z & 0 \\ 0 & 1 & 0 & 0 & -z \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \\ \ddot{\theta}_x \\ \ddot{\theta}_y \end{Bmatrix} = [G_p] \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \\ \ddot{\theta}_x \\ \ddot{\theta}_y \end{Bmatrix} \qquad (4.5.35)$$

Where $[G_p] = \begin{bmatrix} 1 & 0 & 0 & -z & 0 \\ 0 & 1 & 0 & 0 & -z \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ (4.5.36)

Expressing acceleration field in terms of nodal parameters,

$$\{\ddot{f}\} = [G_p][N]\{\ddot{\delta}\} \qquad (4.5.37)$$

in which

$$[N] = \sum_{i=1}^{8} \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 \\ 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix} \qquad (4.5.38)$$

$$\{\ddot{\delta}\} = \sum_{i=1}^{8} \begin{Bmatrix} \ddot{u}_i \\ \ddot{v}_i \\ \ddot{w}_i \\ \ddot{\theta}_{xi} \\ \ddot{\theta}_{yi} \end{Bmatrix} \qquad (4.5.39)$$

Applying D'Alembert's principle for any infinitesimal element within the element, the inertia force resisting the acceleration can be written as

$$\{F_p\} = [\rho_p]\{\ddot{f}\}\, dx\, dy\, dz \qquad (4.5.40)$$

In which

$$[\rho_p] = \begin{bmatrix} \rho_p & 0 & 0 \\ 0 & \rho_p & 0 \\ 0 & 0 & \rho_p \end{bmatrix} \qquad (4.5.41)$$

Where δp is the density of plate material

The work done by the acceleration forces for an infinitesimal element is

$$\delta W = \{f\}^T \{F_p\} = \{\delta\}^T [N]^T [G_p]^T [\rho_p][G_p][N]\{\ddot{\delta}\}\, dx\, dy\, dz \qquad (4.5.42)$$

The internal work done by the distributed acceleration forces for an entire element is

$$W = \int_V \partial W = \int_V \{\delta\}^T [N]^T [G_p]^T [\rho_p][G_p][N]\{\ddot{\delta}\}\, dx\, dy\, dz \qquad (4.5.43)$$

From the above relation, the mass matrix is obtained as

Mass matrix is obtained as

$$[M] = \int_V [N]^T [G_p]^T [\rho_p][G_p][N]\, dx\, dy\, dz \qquad (4.5.44)$$

$$[M] = \iint [N]^T [m_p][N]\, dx\, dy$$

Where $$[m_p] = \int_{-½}^{+½} [G_p]^T [\rho_p][G_p]\, dz = \int_{-½}^{+½} \begin{bmatrix} 1 & 0 & 0 & -z & 0 \\ 0 & 1 & 0 & 0 & -z \\ 0 & 0 & 1 & 0 & 0 \\ -z & 0 & 0 & z^2 & 0 \\ 0 & -z & 0 & 0 & z^2 \end{bmatrix} dz \qquad (4.5.45)$$

Where $[M]$ is the consistent mass matrix $[m_p]$ the lumped mass matrix

$$[m_p] = \begin{bmatrix} \rho h & 0 & 0 & 0 & 0 \\ 0 & \rho h & 0 & 0 & 0 \\ 0 & 0 & \rho h & 0 & 0 \\ 0 & 0 & 0 & \frac{\rho h^3}{12} & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho h^3}{12} \end{bmatrix} \qquad (4.5.46)$$

4.6 Solution procedure

By assembling the finite elements and applying the kinematic boundary conditions, the equations of motion for the linear free vibration of a given plate may be written as

$$[M]\{\emptyset\}_0 - \lambda[K_L]\{\emptyset\}_0 = \{0\} \qquad (4.5.47)$$

or $$w_L^2[M]\{\emptyset\}_0 = [K_L]\{\emptyset\}_0$$

where, [M] and $[K_L]$ denote the system mass and linear stiffness matrices respectively, $\overline{\omega}_L$ the fundamental linear frequency and $\{\phi\}_0$ the corresponding linear mode shape normalized with the maximum component to unity. The plate deflection $w_{max}\{\emptyset\}_0$ is then used to obtain the non-linear stiffness matrix $[K_{NL}]$. The equation of motion for non-linear free vibration is

$$\overline{\omega}_{NL}^2 [M]\{\emptyset\}_i = ([K_L] + [K_{NL}])\{\emptyset\}_i \qquad (4.5.48)$$

where, $\overline{\omega}_{NL}$ is the fundamental non-linear frequency associated with the amplitude ratio (=w/h) and $\{\phi\}_I$, the corresponding normalized mode shape of i^{th} iteration. The solution of above equation can be obtained using the direct iteration method. The steps involved are:

***Step* 1**: The fundamental linear frequency and corresponding linear mode shape is calculated by solving above equation with all the terms in $[K_{NL}]$ being set to zero.

***Step* 2**: The mode shape is normalized by appropriately scaling the eigenvector ensuring that the maximum displacement is equal to the desired amplitude W_{max}/h.

***Step* 3**: The terms in the stiffness matrix $[K_{NL}]$ are computed using the normalized mode shape.

$\boldsymbol{Step\ 4}$: The equations are then solved to obtain new eigenvalues and corresponding eigenvectors.

$\boldsymbol{Step\ 5}$: Steps 2 to 4 are now repeated with $w_{max}\,\{\phi\}_i$ until convergence criterion is satisfied.

The convergence criteria used in the present study are

- Displacement norm defined as $\sum \Delta w_i^2 / \sum w_i^2$
- Frequency norm defined as $|\Delta w_i| / w_i$

Where w_i and Δw_i are the change in displacement and change in non-linear frequency during the i^{th} iteration cycle.

CHAPTER-5 COMPUTER IMPLEMENTATION

5.1 Introduction

The finite element method has been established as a powerful numerical tool because of its broad spectrum of generality and its ease of applicability to rather more complex and difficult problems showing greater efficacy in its solution than that of any other existing similar techniques. This advantage of the method over others has led various research organizations and modern industries to endeavour the development of general purpose software packages and other in-house codes for solving practical problems of more complex nature. In an effort to make the method more powerful and to address more complicated problems, the finite element analysis programmes themselves become extremely complex and computationally involved. These programmes are available as black box modules which are to be used with the help of CAD programs. These conventional programmes cannot easily be modified to perform a desired task necessitating redesign and rebuild of finite element libraries to suit one's need. Hence there is a requirement for finite element analysis programmes to be easily modifiable to introduce new analysis procedures, and new kinds of design of structural components or even emerging technology of new materials whenever needed. In the present investigation, the computer codes have been generated with such modularity which is amenable to easy modification whenever the need arises. Throughout all these years the finite element codes have been developed employing procedural language such as FORTRAN which is unstructured in its nature. Now there is a trend to pay attention to the verification, portability and reusability of the computer programmes during the process of their development and to the possibility of the use of other software products. However, FORTRAN does not have the provision to meet all these requirements. The **MATLAB** coding are efficient for the finite element problems. A computer programme based on the formulation given in the previous chapter is developed in **MATLAB7.2b** for large amplitude free vibration analysis of composite plates. There are 5 degrees of freedom per node (viz. $u, v, w, \theta x, \theta y$). Composite plates of rectangular as well as square shape have been analyzed by this program.

5.2 Application Domain

The Computer Programmes have been developed in the present investigation by making use of the MATLAB code to include a wide spectrum of application domain. Computer programme codes have been written to incorporate various boundary conditions of the structures. They have the analytical modules to solve the following types of problems:

- Large amplitude free vibration analysis of isotropic plate.
- Large amplitude free vibration analysis of laminated composite plate.

5.3 Description of the Programme

The finite element procedure involves 3 basic steps in terms of the computation carried out which may be termed as:

1. Preprocessor
2. Processor
3. Post processor

5.3.1 Preprocessor

This module of the programme reads the necessary information about the geometry and boundary conditions of the plate, material properties, loading configuration and its magnitude, and its properties etc. Also in this module, all the **nodal coordinates** and the **nodal connectivity** are generated.

5.3.1.1 Automatic Mesh Generation

The mesh division for the structures analyzed is generated automatically. The algorithm for this purpose is provided by a function **rectangularmesh()** . The plate structure is divided into a number of elements by assigning the number of divisions in each direction. This information is given as input to the problem under consideration. The elements are numbered automatically moving from left to right and top to bottom as shown in the Fig.5.1

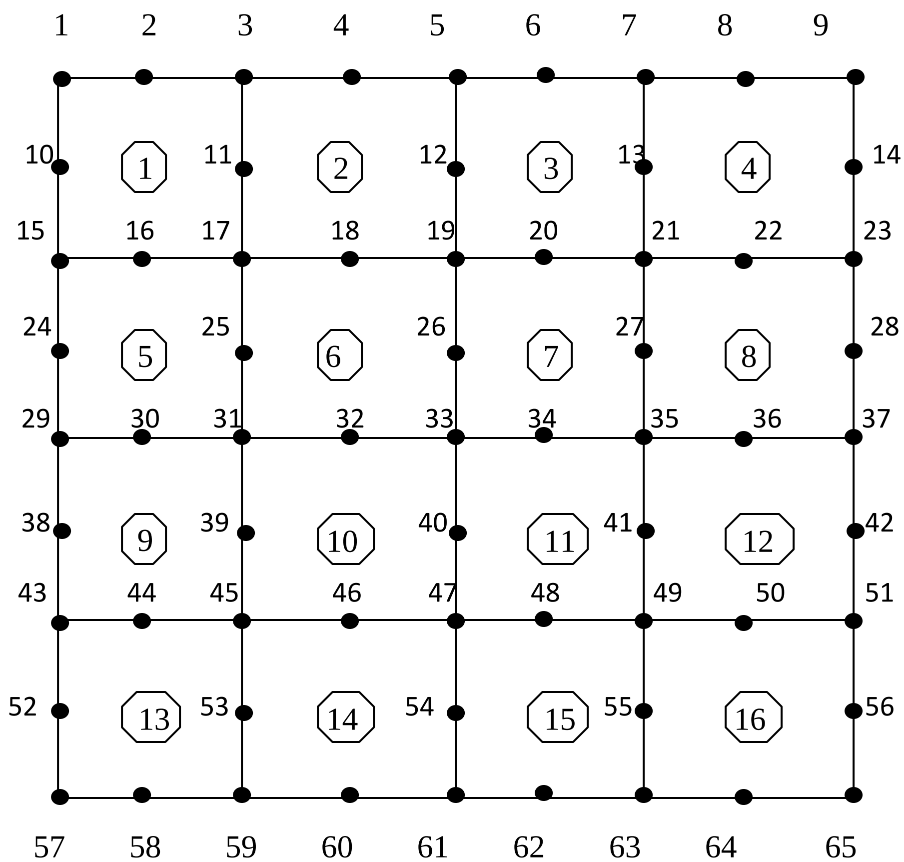


Fig.5.1 Finite element plate meshing

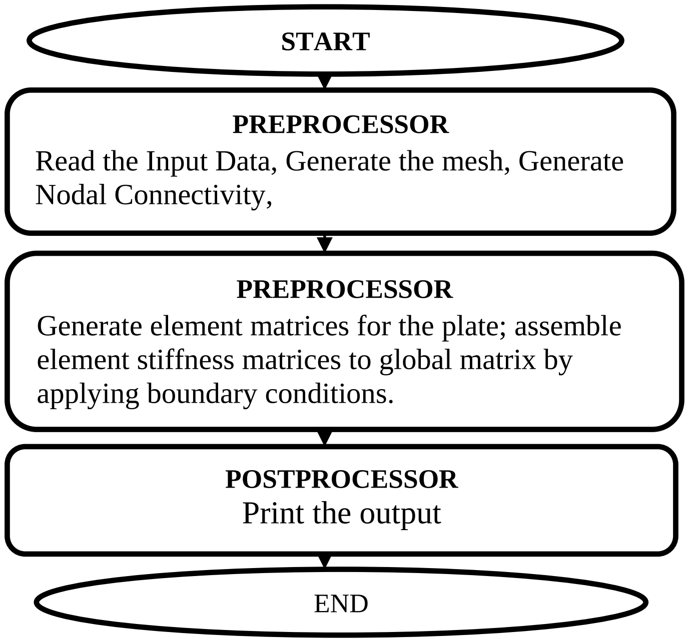


Fig.5.2: **Basic Elements of the Computer Programmes**

5.3.2 Processor

This module of the programmes performs the following tasks:

1. Generation of the element matrices.
2. Assembly of the element matrices into global matrices.
3. Imposition of the boundary conditions.
4. Determination of eigenvalues and eigenvectors for the free vibration analysis using simultaneous vector iteration technique.

5.3.3 Postprocessor

In this part of the programme, all the input data are echoed to check for their accuracy. The function **print-disp()** is used to print the output data in terms of **non-linear frequency ratio, eigenvalue** etc. The results are stored in a series of separate output files for each category of problems analyzed and those values are used to prepare tables and graphs etc.

5.4 different functions and variables used

5.4.1 input variables

length : length of the composite plate

breadth : breath of the composite plate

thichness : thickness of plate

$E1$: young's modulus in the longitudinal direction

$E2$: young's modulus in the transverse direction

$G12$: shear modulus on plane 1 along direction 2

$G13$: shear modulus on plane 1 along direction 3

$G23$: shear modulus on plane 2 along direction 3

ν_{12} : poisson's ration on plane 1 along direction 2

ν_{13} : poisson's ration on plane 1 along direction 3

ν_{23} : poisson's ration on plane 2 along direction 3

theta : vector of ply orientation, angle is taken $+ve$ in counter clockwise direction.

nop : no. of plies present in the laminate.

z : $[-\frac{h}{2}:\frac{h}{nop}:\frac{h}{2}]$ vector of different "ply thickness" bottom to top.

rho : density of plate

5.4.2 function **retangularmesh()**

This function generates the rectangular mesh divisions and calculates the nodal coordinates

Variables used in this function are

x : length of the plate in x-direction

y : length of the plate in y-direction

numberelementsx : number of elements in x-direction

numberelementsy : number of elements in y-direction

lengthx : length of an element in x-direction

lengthy : length of an element in y-direction

numberelements : total no. of elements present in the meshing

totnode : total no. of nodes present

nodecoordinates : matrix of x and y coordinate value of all nodes

nodenum : index no. of nodes

5.4.3 function **linearglobalstiff()**

This function compute the linear global stiffness matrix for the Mindlin plate. At first element stiffness is formed then globalization all element stiffnesses taken place using function **assemble().**

Variables used in this function are:

Astiff : in-plane stiffness of the laminate

Bstiff : in-plane-out-of-plane coupling stiffness of the laminate

Dstiff : bending stiffness of the laminate

Sstiff : shear stiffness matrix

Dmatrix : stress-strain constituent matrix

K : global stiffness matrix

indice : nodal conectivities for each element

elementdof : element degrees of freedom

ndof : no. of degree of freedoms per node

gdof : global no. of degree of freedoms

b_m : membranestrain-displacement matrix

b_b : bending strain-displacement matrix

b_s : shearstrain-displacement matrix

e : element counter

q : gauss point counter

xi , eta : natural coordinate

5.4.4 function **massComposite()**

This function compute the mass matrix for the mindlin plate.

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This function solves the integrations. For bending part, ‘complete’integration method used and for shear part ‘reduced’ integration method applied.

Locations : gauss point locations

Weight : gauss point weights

5.4.6 function **jacobian()**

This function finds the jacobian matrix, inverse jacobian matrix, and x and y-derivatives.

jacobianmatrix : jacobian matrix

invjacobian : inverse of jacobian matrix

xyderivatives : derivatives w.r.t. x and y

naturalderivatives : derivatives w.r.t. xi and eta

nodecoordinates : nodal coordinates at element level

5.4.7 function **essentialBc()**

Essential boundary condition for the mindlin plate.

ssss : all edges are simply supported

cccc : all edges are fixed

scsc : ‘s’ for simply supported, ‘c’ for clamped

5.4.8 function **shapefunction()**

This function evaluate the shape functions and its derivatives for the plate elements

Shape : Shape functions

naturalDerivatives : derivatives w.r.t. xi and eta

xi, eta : natural coordinates (-1 ... +1)

5.4.9 function **nlstifVonKrmn()**

This function computes the nonlinear stiffness matrix as per Von-Karman’s non-linear strain displacement theory.

iter : counter for iteration

wbyh : amplitude ratio $\frac{w_{max}}{h}$

toll : tolerance = abs((lamda(iter)-lamda(iter-1))/lamda(iter))

ratio : frequency ratio $\frac{w_{nl}}{w_l}$

5.4.10 function **eig()**

This function solves the eigen value problems. Find eigen vector and the eigen value (frequencies).

v : eigen vector

d : eigen value

5.4.11 function **mshape()**

This function finds the non-linear strain and modeshapes of u, w . Variables used are:

u	: displacement along x-axis (in-plane displacement)
v	: displacement along y-axis (in-plane displacement)
w	: transverse displacement of the plate
activedofw	: index of node numbering having non-zero value of w

5.4.12 function **linspace()**

This function is used to draw the grid of graph.

5.4.13 function **contour()**

This function plot the contour map of the mode shapes.

5.5 Input data

- The input data required for the analysis are as follows:
- Plate dimensions
- Mesh division, number of nodes of element, degrees of freedom at each node, number of Gauss points, number of loading conditions, degree of freedom of the amplitude taken as reference for normalizing
- Boundary conditions of the structure
- Material properties of the plate
- Number of amplitude levels and tolerance for convergence and
- Amplitude data

5.6 Output data

- The input data,
- Ratio of Non-linear frequency to Linear frequency,

- mode-shapes in each iteration, and number of iterations taken for convergence

5.7 Program Flow Chart

A flow chart illustrating the complete program has been presented in fig.5.3.

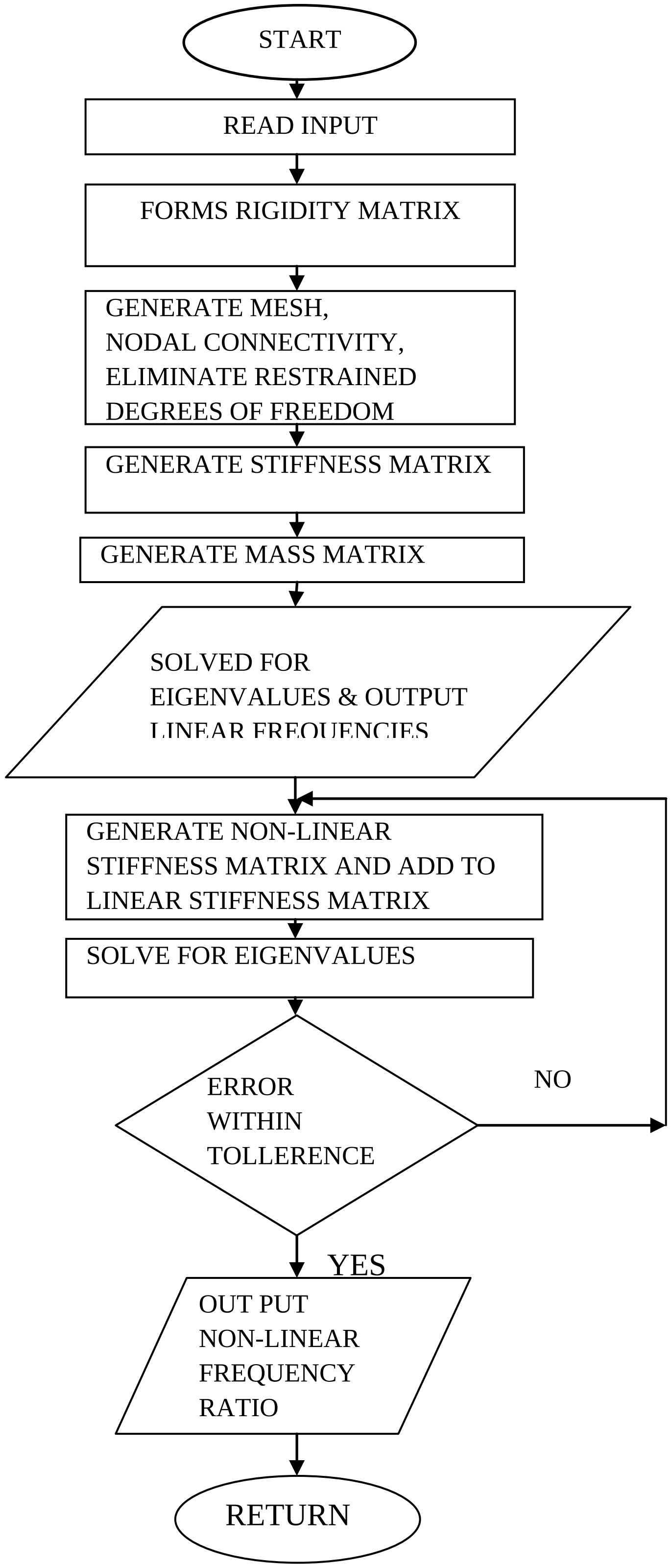

Fig 5.3 Program Flow Chart

CHAPTER -6 RESULTS AND DISCUSSION

The finite element formulation of large amplitude free vibrations of isotropic and composite plates has been presented in chapter-4. The computer programming based on this method has been put forward in chapter-5.Examples have been worked out to validate the proposed approach. A number of examples have been presented and comparisons have been made with the results of earlier investigators wherever possible. The examples include square and rectangular plates with various boundary conditions. Eight- noded plate element is considered with five degrees of freedom per node.

6.1 LARGE AMPLITUDE FREE VIBRATION ANALYSIS OF ISOTROPIC PLATE

Table 6.1

Non-linear frequency ratios $\left({}^{\omega_{NL}}/_{\omega_L}\right)$ for the square isotropic plate with various boundary conditions at different amplitude ratios.

$$(E = 10.92 \times 10^6\ {}^{N}/_{m^2},\ h = 0.01\ m, \rho = 100\ {}^{kg}/_{m^3}, \nu = 0.3\ )$$

	Amplitude Ratio (w/h)					Boundary conditions
	0.2	0.4	0.6	0.8	1.0	
present	1.0093	1.0320	1.0590	1.1223	1.1650	CCCC
Rao[42]	1.007	1.0276	1.0608	1.1047	1.1578	
Mei[33]	1.0062	1.0256	1.0564	1.0969	1.1429	
present	1.0256	1.1007	1.2110	1.3733	1.5299	SSSS
Goswami[18]	1.0263	1.1012	1.2165	1.3629	1.5325	
Ganapati[16]	1.0250	1.10021	1.20803	1.35074	1.51347	
present	1.0145	1.0560	1.1230	1.2067	1.2870	SSCC
Mei[33]	1.0097	1.0380	1.0833	1.1429	1.2143	
Rao[42]	1.097	1.0381	1.0838	1.1443	1.2174	

Table 6.1 presents the non-linear frequency ratios for the fundamental mode for SSSS, CCCC, SSCC, boundary conditions for a square plate for amplitude ratio $\left(\frac{w}{h}\right)$ values ranging

from 0.2 to 1.0. The results have been compared with different papers. It can be seen that the values of the references are on the lower side. This is because of the different techniques chosen by them for the solution of the non-linear equations. Their formulations have been based on appropriate linearization of the non-linear strain-displacement relations. They have also neglected the in-plane deformation terms. In the present investigation, in-plane deformation terms have been considered and no approximating procedure is used. Hence the present result may be deemed as more accurate.

6.2 FREE FLEXURAL VIBRATION OF LAMINATED COMPOSITE PLATE

In order to verify the accuracy of the results obtained in the free flexural vibration of laminated composite plate, a comparison has been done with published ones. Table 6.2 shows the comparison of linear natural frequencies with Liew [28].

Table 6.2

Non-dimensional **linear frequency parameter** $\overline{\omega} = (\omega b^2/\pi^2)\sqrt{(\rho h/D_0)}$ for three ply $(0^{\cdot}/90^{\cdot}/0^{\cdot})$ simply supported SSSS square laminated composite plate of different thickness ratio (b/h).(**Material 1**)

MODES SEQUENCE NUMBER								
h/b		1	2	3	4	5	6	
0.001	Present	6.6288	9.4567	16.2977	25.2305	26.5811	26.8823	SSSS
	Liew[28]	6.6252	9.4470	16.2051	25.1146	26.4982	26.6572	
0.20	Present	3.5646	5.7920	7.3218	8.6121	9.2617	10.9987	
	Liew[28]	3.5939	5.7691	7.3972	8.6876	9.1451	11.2080	
0.001	Present	14.6951	17.6501	24.6492	36.0814	39.3690	41.0343	CCCC
	Liew[28]	14.6655	17.6138	24.5114	35.5318	39.1572	40.7685	
0.02	Present	4.3291	6.6619	7.5813	9.1514	9.8627	11.2049	
	Liew[28]	4.4468	6.6419	7.6996	9.1852	9.7378	11.3991	

6.3 LARGE AMPLITUDE FREE VIBRATIONS OF LAMINATED COMPOSITE PLATE

Material 1: Laminated composite plate

$$E_1 = 40E_2, G_{12} = G_{31} = 0.6E_2, G_{23} = 0.5E_2, E_2 = 1, \rho = 1, \nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

Material 2: Laminated composite plate

$$E_1 = 25E_2, G_{12} = G_{31} = 0.2E_2, G_{23} = 0.5E_2, E_2 = 1, \rho = 1, \nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

A simply supported cross-ply $(0^{\cdot}/90^{\cdot}/90^{\cdot}/0^{\cdot})$ with in-plane motions at the boundary restricted with a/h=10 &1000 and the **Material 1** and **Material 2** are considered. The present results for the ratio of non-linear to linear frequency against the amplitude ratio (w_0/h) are compared with Ganapati et al. [16] and Kant et al.[18] in **Table 6.3.**

Table 6.3 non-linear frequency ratio $(\boldsymbol{\omega_{NL}/\omega_L})$ of a simple supported Cross-ply $(\mathbf{0^{\cdot}/90^{\cdot}/90^{\cdot}/0^{\cdot}})$ laminate at different aspect ratio (a/h) and amplitude ratio (a/h)

a/h	W_0/h	Material	Kant et al.[18]	Ganapati et al. [16]	present
1000	0.2	1	1.02843	1.04125	1.04355
	0.4		1.14575	1.15093	1.15769
	0.6		1.29166	1.31825	1.32836
	0.8		1.48372	1.51495	1.51832
	1.0		1.70091	1.73828	1.73831
1000	0.2	2	1.02808	1.04108	1.04201
	0.4		1.13436	1.15029	1.15288
	0.6		1.28324	1.31653	1.31793
	0.8		1.47890	1.51394	1.51937
	1.0		1.68399	1.73650	1.73411
10	0.2	1	1.04843	1.06453	1.06772
	0.4		1.21575	1.22915	1.23089
	0.6		1.42617	1.44215	1.44144
	0.8		1.63372	1.66125	1.66225
	1.0		1.82091	1.85671	1.86631
10	0.2	2	1.04082	1.06016	1.06303
	0.4		1.20057	1.21973	1.22413
	0.6		1.40401	1.43125	1.44174
	0.8		1.60105	1.65078	1.66406
	1.0		1.81449	1.85126	1.85629

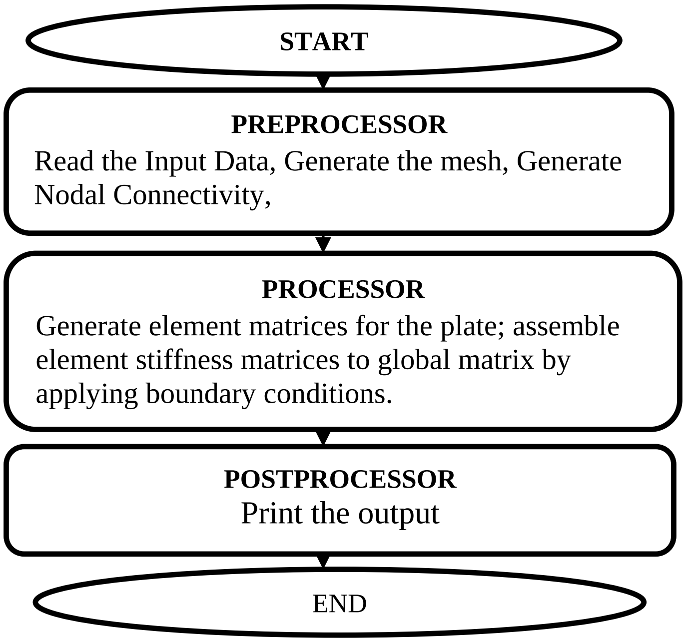

Fig.5.2: **Basic Elements of the Computer Programmes**

5.3.2 Processor

This module of the programmes performs the following tasks:

1. Generation of the element matrices.
2. Assembly of the element matrices into global matrices.
3. Imposition of the boundary conditions.
4. Determination of eigenvalues and eigenvectors for the free vibration analysis using simultaneous vector iteration technique.

5.3.3 Postprocessor

In this part of the programme, all the input data are echoed to check for their accuracy. The function **print-disp()** is used to print the output data in terms of **non-linear frequency ratio, eigenvalue** etc. The results are stored in a series of separate output files for each category of problems analyzed and those values are used to prepare tables and graphs etc.

5.4 different functions and variables used

5.4.1 input variables

length : length of the composite plate

breadth : breath of the composite plate

thichness : thickness of plate

$E1$: young's modulus in the longitudinal direction

$E2$: young's modulus in the transverse direction

$G12$: shear modulus on plane 1 along direction 2

$G13$: shear modulus on plane 1 along direction 3

$G23$: shear modulus on plane 2 along direction 3

ν_{12} : poisson's ration on plane 1 along direction 2

ν_{13} : poisson's ration on plane 1 along direction 3

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theta : vector of ply orientation, angle is taken $+ve$ in counter clockwise direction.

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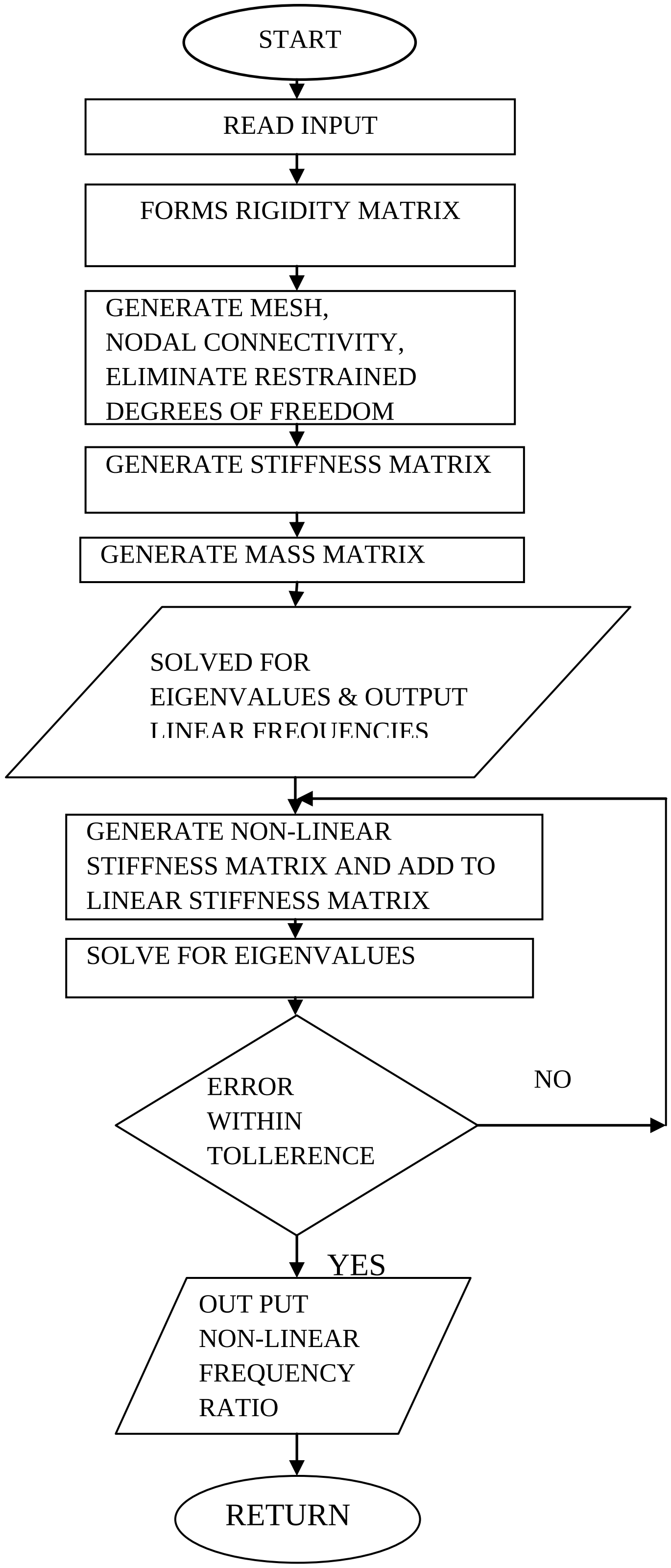

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6.2 FREE FLEXURAL VIBRATION OF LAMINATED COMPOSITE PLATE

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Non-dimensional **linear frequency parameter** $\bar{\omega} = (\omega b^2/\pi^2)\sqrt{(\rho h/D_0)}$ for three ply $(0^{\cdot}/90^{\cdot}/0^{\cdot})$ simply supported SSSS square laminated composite plate of different thickness ratio (b/h).(**Material 1**)

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	Liew[28]	4.4468	6.6419	7.6996	9.1852	9.7378	11.3991	

6.3 LARGE AMPLITUDE FREE VIBRATIONS OF LAMINATED COMPOSITE PLATE

Material 1: Laminated composite plate

$$E_1 = 40E_2, G_{12} = G_{31} = 0.6E_2, G_{23} = 0.5E_2, E_2 = 1, \rho = 1, \nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

Material 2: Laminated composite plate

$$E_1 = 25E_2, G_{12} = G_{31} = 0.2E_2, G_{23} = 0.5E_2, E_2 = 1, \rho = 1, \nu_{12} = \nu_{13} = \nu_{23} = 0.25$$

A simply supported cross-ply $(0^{\cdot}/90^{\cdot}/90^{\cdot}/0^{\cdot})$ with in-plane motions at the boundary restricted with a/h=10 &1000 and the **Material 1** and **Material 2** are considered. The present results for the ratio of non-linear to linear frequency against the amplitude ratio (w_0/h) are compared with Ganapati et al. [16] and Kant et al.[18] in **Table 6.3.**

Table 6.3 non-linear frequency ratio (ω_{NL}/ω_L) of a simple supported Cross-ply $(0^{\cdot}/90^{\cdot}/90^{\cdot}/0^{\cdot})$ laminate at different aspect ratio (a/h) and amplitude ratio (a/h)

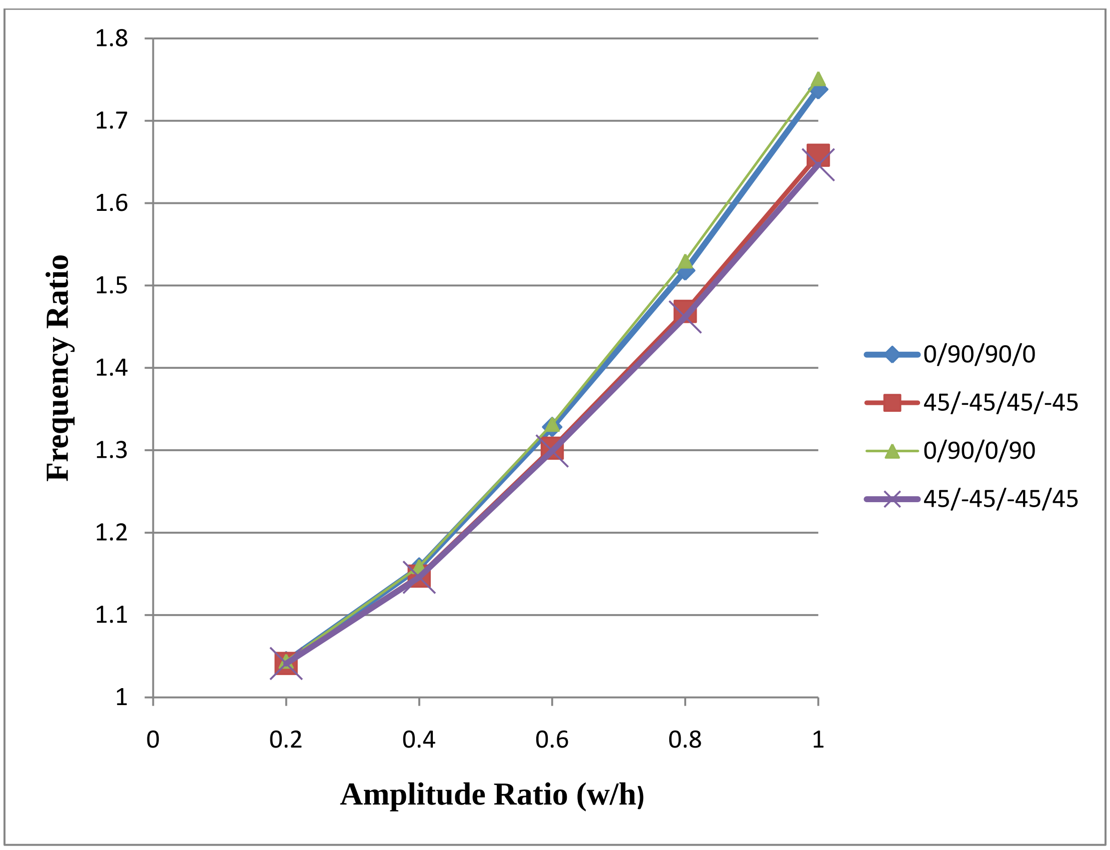
a/h	W_0/h	Material	Kant et al.[18]	Ganapati et al. [16]	present
1000	0.2	1	1.02843	1.04125	1.04355
	0.4		1.14575	1.15093	1.15769
	0.6		1.29166	1.31825	1.32836
	0.8		1.48372	1.51495	1.51832
	1.0		1.70091	1.73828	1.73831
1000	0.2	2	1.02808	1.04108	1.04201
	0.4		1.13436	1.15029	1.15288
	0.6		1.28324	1.31653	1.31793
	0.8		1.47890	1.51394	1.51937
	1.0		1.68399	1.73650	1.73411
10	0.2	1	1.04843	1.06453	1.06772
	0.4		1.21575	1.22915	1.23089
	0.6		1.42617	1.44215	1.44144
	0.8		1.63372	1.66125	1.66225
	1.0		1.82091	1.85671	1.86631
10	0.2	2	1.04082	1.06016	1.06303
	0.4		1.20057	1.21973	1.22413
	0.6		1.40401	1.43125	1.44174
	0.8		1.60105	1.65078	1.66406
	1.0		1.81449	1.85126	1.85629

The **table 6.3** shows that the results of higher order shear deformation theory (HSDT) of kant et al.[18] is somehow lower than that of first order shear deformation theory, by Ganapati et al. [16] and the present one. This may be due to the better representation of in-plane deformation in higher order shear deformation theory by kant et al.[18] which is important in non-linear analysis and also the real parabolic distribution of shear stress which is important for moderately thick laminated plates.

6.3.1 Effect of Amplitude to Thickness Ratios on non-linear to linear frequency ratios

Fig. 6.1 depicts the effect of Amplitude Ratio (w/h) on the non-linear Frequency Ratio $\left({}^{\omega_{NL}}/_{\omega_L}\right)$ for different laminates $\left((0^{\cdot}/90^{\cdot}/90^{\cdot}/0^{\cdot}), (45^{\cdot}/-45^{\cdot}/45^{\cdot}/-45^{\cdot}), (0^{\cdot}/90^{\cdot}/0^{\cdot}/90^{\cdot}), (45^{\cdot}/-45^{\cdot}/-45^{\cdot}/45^{\cdot})\right)$ for a square plate with SSSS. From the graphs it can be seen that there is an increase in frequency ratio with an increase in amplitude ratio. The increase in frequency ratio is more in Cross-Ply laminates as compared to Angle-Ply laminates.

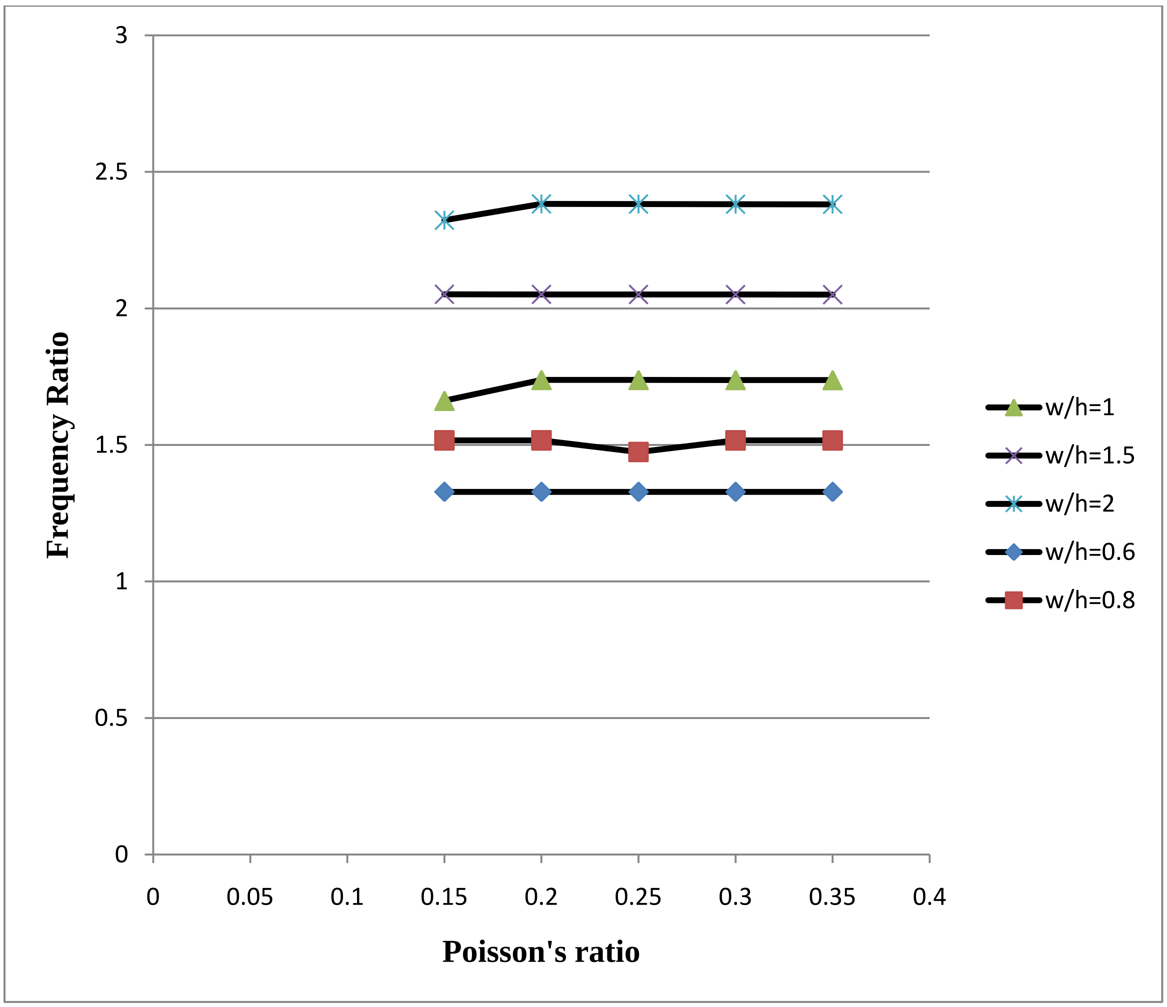
Fig. 6.1 Variation of non-linear frequency ratio $\left({}^{\omega_{NL}}/_{\omega_L}\right)$ with amplitude ratio of a square laminates (SSSS). (Material 1)



6.3.2 Effect of Poisson's Ratio on non-linear to linear frequency ratios

Fig. 6.2 depicts the effect of Amplitude Ratio (w/h) on the non-linear Frequency Ratio $\left({}^{\omega_{NL}}/_{\omega_L}\right)$ for different laminates $(0^{\cdot}/90^{\cdot}/90^{\cdot}/0^{\cdot})$ for a square plate with SSSS. From the graphs it can be seen that there is an increase in frequency ratio with an increase in Poisson' ratio, but increment is very much less.

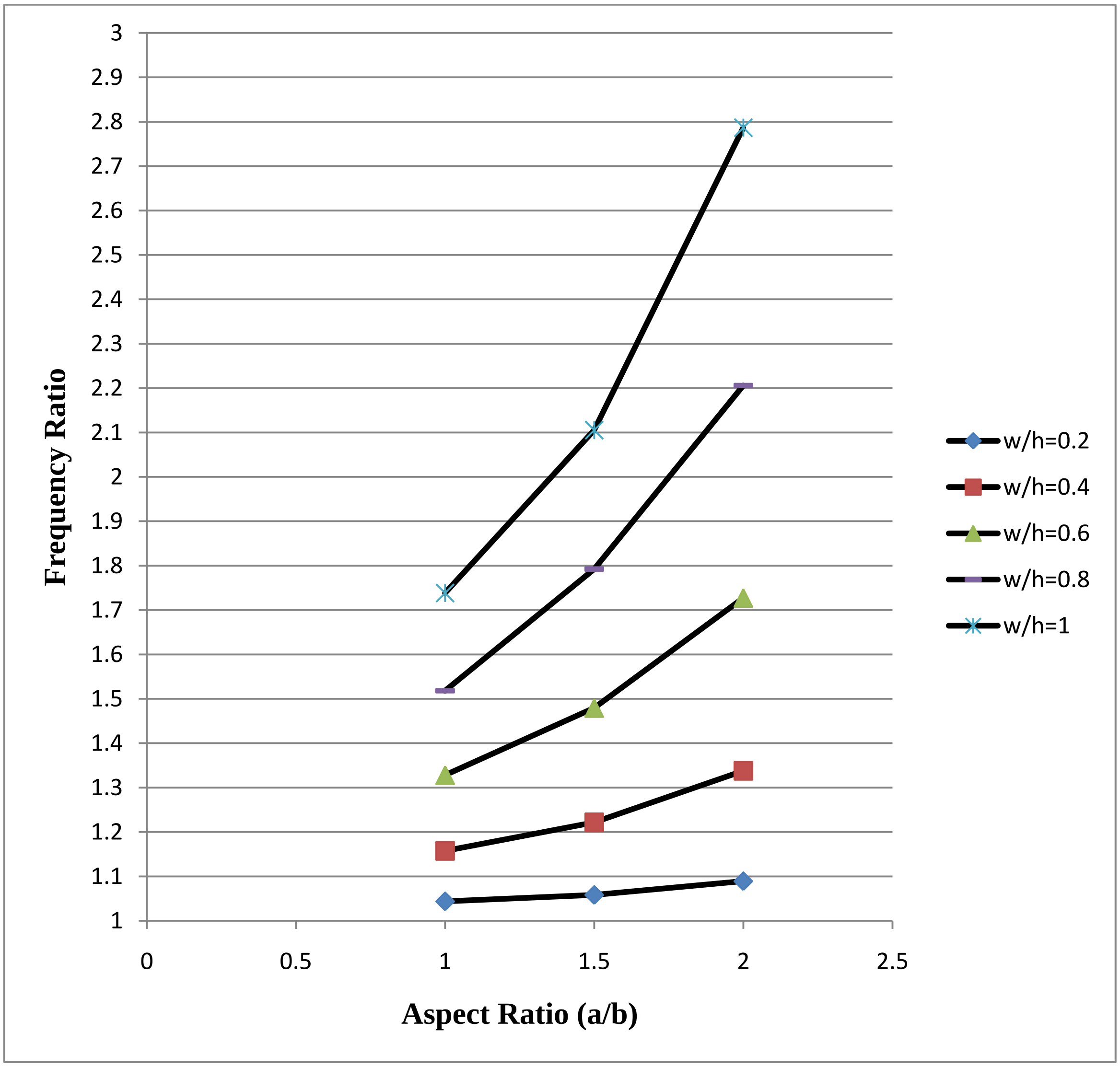
Fig. 6.2 Variation of non-linear Frequency ratio $\left({}^{\omega_{NL}}/_{\omega_L}\right)$ with Poisson's ratio Ratio (a/b) **of a square plate (SSSS) for the fundamental mode**



6.3.3 Effect of Aspect Ratios on non-linear to linear frequency ratios

Fig. 6.3 shows the variation of the non-linear frequency ratio$\left({}^{\omega_{NL}}/_{\omega_L}\right)$for simply supported laminate $(0^{\circ}/90^{\circ}/90^{\circ}/0^{\circ})$ plates of different aspect ratios (a/b)**.** It may be observed that there is an increase of the frequency ratio with the increase of the aspect ratio for the plates at all the amplitude levels.

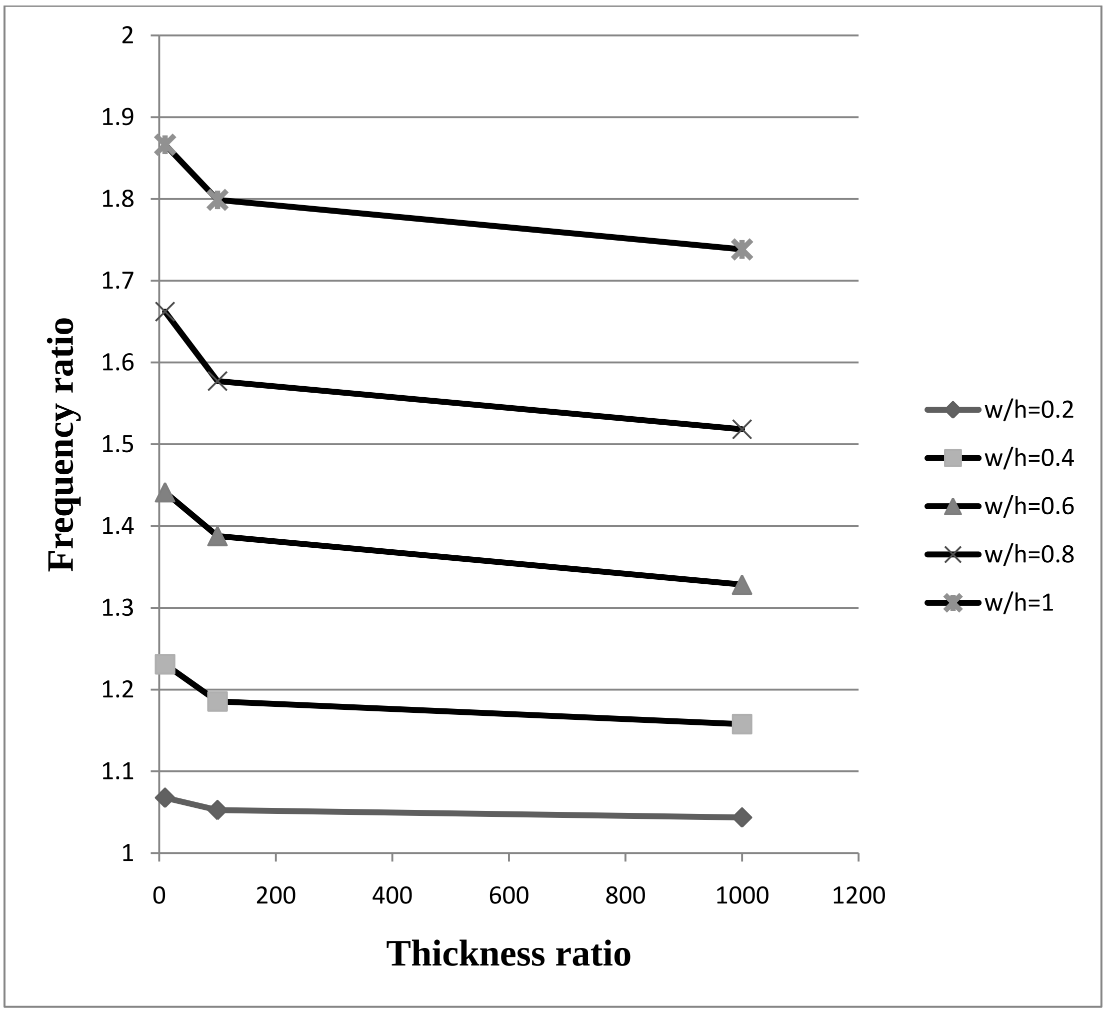
Fig. 6.3 Variation of non-linear Frequency ratio $\left({}^{\omega_{NL}}/_{\omega_L}\right)$ with Aspect Ratio (a/b) **of a square plate (SSSS) for the fundamental mode**



6.3.4 Effect of Breadth to Thickness Ratios on non-linear to linear frequency ratios

Fig. 6.4 shows the variation of the non-linear frequency ratio$\left({}^{\omega_{NL}}/_{\omega_L}\right)$for simply supported laminate $(0^{\cdot}/90^{\cdot}/90^{\cdot}/0^{\cdot})$ plates of different thickness ratios (b/h). It may be observed that there is an decrease of the frequency ratio with the increase of the thickness ratio for the plates at all the amplitude levels.

Fig. 6.4 Variation of non-linear Frequency ratio $\left({}^{\omega_{NL}}/_{\omega_L}\right)$ with Thickness Ratio (b/h) **of a square plate (SSSS) for the fundamental mode**



CONCLUSIONS

The following conclusions may be made from the investigation of the large amplitude free vibration analysis of composite plate by finite element method:

1. For the materials having higher poisson's ratio, the non-linear to linear frequency ratios have been observed to be higher.
2. At lower values of amplitude to thickness ratios the Poisson's ratio has no significant effect on the non-linear to linear frequency ratio of vibration. But as the amplitude to thickness ratios increase there is remarkable increase in the non-linear to linear frequency ratios.
3. For the composite plate having higher breadth to thickness ratio, the non-linear to linear frequency ratio has been observed to be lower.
4. The non-linear to linear frequency ratio varies directly with respect to the plate aspect ratios. The increase in the aspect ratio shifts the non-linear to linear frequency ratio towards the higher side.
5. Non-linear to linear frequency ratios are higher in cross-ply laminates as compared to angle-ply laminates of same geometric and material properties.

FUTURE SCOPE OF RESEARCH

- Material nonlinearity may be taken into account in the formulation for further extension of the laminated composite plate configurations.
- The present formulation can be extended to include the large amplitude free vibration of stiffened composite plates.
- The plates studied here are of uniform thickness. The elements can be modified to incorporate the composite plates of varying thickness.

REFERENCES

[1] Abir , Humayun R.H. “On free vibration response and mode shapes of arbitrarily laminated rectangular plates” Composite Structures 65 (2004) 13–27.

[2] Ahmed, S., Irons, B. M., and Zienkiewicz, O. C. “Analysis of thick and thin structures by curved finite elements” Comput. Methods Appl. Mech. Eng. 501970:121-145.

[3] Allahverdizadeh, A., Naei, M. H., and Nikkhah, B. M. “Vibration amplitude and thermal effects on the nonlinear behavior of thin circular functionally graded plates” International Journal of Mechanical Sciences 50 (2008) 445–454.

[4] Amabili, M. “Nonlinear vibrations of rectangular plates with different boundary conditions theory and experiments” Computers and Structures 82 (2004) 2587–2605.

[5] Amabili, M. “Theory and experiments for large-amplitude vibrations of rectangular plates with geometric imperfections” Journal of Sound and Vibration 291 (2006) 539–565.

[6] Barik, M. and Mukhopadhyay, M. “Finite element free flexural vibration analysis of arbitrary plates” Finite element in Analysis and Design 29(1998)137-151.

[7] Barik, M. and Mukhopadhyay, M. “A new stiffened plate element for the analysis of arbitrary plates” Thin-Walled Structures 40 (2002) 625–639.

[8] Bhavikatti, S. S. “finite element analysis”.

[9] Bhimaraddi, A. and Chandrasekhara, K.” Nonlinear vibrations of heated antisymmetric angle-ply laminated plates” Int. J. Solids Structures Vol. 30, No. 9, pp. 1255-1268, 1993.

[10] Bikri, K. El., Benamar, R., and Bennouna, M. “Geometrically non-linear free vibrations of clamped simply supported rectangular plates. Part I: the effects of large vibration amplitudes on the fundamental mode shape” Computers and Structures 81 (2003) 2029–2043.

[11] Chandrashekhara, K. and Tenneti, R. “Non-linear static and dynamic analysis of heated laminated plates: a finite element approach” Composites Science and Technology 51 (1994) 85-94.

[12] Chandrasekhar et al. “Nonlinear vibration analysis of composite laminated and sandwich plates with random material properties” International Journal of Mechanical Sciences, 8 march 2010.

[13] Das, D., Sahoo, P., and Saha, K. “Large-amplitude dynamic analysis of simply supported skew plates by a variational method” Journal of Sound and Vibration 313 (2008) 246–267.

[14] Ferreira, A.J.M. “MATLAB codes for finite element analysis”.

[15] Gajbir, S. and Venkateswara rao, G. “Nonlinear oscillations of laminated plates using an accurate four-node rectangular shear flexible material finite element” Sddhan & Vol. 25, Part 4, August 2000, pp. 367-380.

[16] Ganapathi, M., and Varadan, T. K., and Sarma, B. S. “Nonlinear flexural vibrations of laminated orthotropic plates” Computers and structures Vol. 39. No. 6. pp. 685-688, 1991.

[17] Gang Wang Y., Ling Shi, J., and Zhi Wang, X. “Large amplitude vibration of heated corrugated circular plates with shallow sinusoidal corrugations” Applied Mathematical Modelling 33 (2009) 3523– 3532.

[18] Kant T. and Kommineni, J. R. “Large Amplitude Free Vibration analysis of cross-ply Composite and sandwich laminates with a refined theory and C^0 finite elements” Comput. Struct. 50(1):123-134.

[19] Harras, B. and Benamar, R. “Geometrically non-linear free vibration of fully clamped symmetrically laminated rectangular composite plates” Journal of Sound and vibration (2002) 251(4), 579-619.

[20] Haterbouch, M. and Benamar, R. “ Geometrically nonlinear free vibrations of simply supported isotropic thin circular plates” Journal of Sound and Vibration 280 (2005) 903–924.

[21] Hinton, E. and Campbell, J. S. “Local and global smoothing of discontinuous finite element functions using a least squares method” Int. Z Numer. Methods Eng. 8: (1974)461-480.

[22] Houmat, A. “Large amplitude free vibration of shear deformable laminated composite annular sector plates by a sector p-element “International Journal of Non-Linear Mechanics 43 (2008) 834 – 843.

[23] Jayakumar , K., Yadav , D., and Nageswara Rao, B. “Nonlinear free vibration analysis of simply supported piezo-laminated plates with random actuation electric potential difference and material properties” Communications in Nonlinear Science and Numerical Simulation 14 (2009) 1646–1663.

[24] Krishnamoorthy, C.S. “Finite Element Analysis-Theory and Programming, Tata McGraw Hill Publishing Company Ltd., New Delhi, 1987.

[25] Lal, A., Singh, B.N., and Kumar, R. “Nonlinear free vibration of laminated composite plates on elastic foundation with random system properties” International Journal of Mechanical Sciences 50 (2008) 1203– 1212 .

[26] Leissa, A. W. " Nonlinear analysis of plate and shell vibrations " 1984 Proceedings of the Second International Conference on Recent Advances in Structural Dynamics, Southampton.

[27] Li, J. J. and Cheng, C. J. "Differential quadrature method for nonlinear vibration of orthotropic plates with finite deformation and transverse shear effect" Journal of Sound and Vibration 281 (2005) 295–309.

[28] Liew, K.M. "Solving the vibation of thick symmetric laminates by reissner/mindlin plate theory and the p-ritz method" Journal of Sound and Vibration (1996) 198(3), 343-360.

[29] Lin Huang, Xiao. and Shen Shen, H. "Nonlinear vibration and dynamic response of functionally graded plates in thermal environments" International Journal of Solids and Structures 41 (2004) 2403–2427.

[30] Malekzadeh, P. "A differential quadrature nonlinear free vibration analysis of laminated composite skew thin plates" Thin-Walled Structures 45 (2007) 237–250.

[31] Malekzadeh, P. "Differential quadrature large amplitude free vibration analysis of laminated skew plates based on FSDT" Composite Structures 83 (2008) 189–200.

[32] Malekzadeh, P. and Vosoughi, A. R. "DQM large amplitude vibration of composite beams on nonlinear elastic foundations with restrained edges" Communications in Nonlinear Science and Numerical Simulation 14 (2009) 906–915.

[33] Mei, C. "Finite element displacement method for large amplitude free flexural vibrations of beams and plates" Computers & Structures, Vol. 3, pp. 163-174. Pergamon Press 1973. Printed in Great Britain.

[34] Middleton, D.H. "Composite Materials in Aircraft Structures. Harlow: Longman Scienti"c& Technical.

[35] Mindlin R. D., Influence of Rotary Inertia and Shear on Flexural Motions of Isoparametric Elastic Plates, Jouranl of Applied Mechanics, Vol 18, 1951.

[36] Moussaoui , F., Benamar, R., and White, R.G. "The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic shells. part ii: a new approach for free transverse constrained vibration of cylindrical shells" Journal of Sound and Vibration (2002) 255(5), 931-963.

[37] Narita , Y. "Frequencies and mode shapes of cantilevered laminated composite plates" Journal of Sound and Vibration (1992) 154(l), 161-172.

[38] Panda, S.K. and Singh, B.N. "Nonlinear free vibration of spherical shell panel using higher order shear deformation theory – A finite element approach" International Journal of Pressure Vessels and Piping 86 (2009) 373–383.

[39] Pandit, M. K., Haldar, S., and Mukhopadhyay, M. “Free Vibration Analysis of Laminated Composite Rectangular Plate Using Finite Element Method” Journal of Reinforced Plastics and Composites 2007; 26; 69.

[40] Patel, B.P., Ganapathi, M., Makhecha, D.P., and Shah, P. “ Large amplitude free flexural vibration of rings using finite element approach” International Journal of Non-Linear Mechanics 37 (2003) 911-921.

[41] Raju, S., Venkateswara Rao, G., and Kanaka Raju, K. “ Effect of longitudinal or inplane deformation and inertia on the large amplitude flexural vibrations of slender beams and thin plates” Journal of Sound and Vibration (1976) 49(3), 415-422.

[42] Rao, S. R., Sheikh, A. H., and Mukhopadhyay, M. “Large-amplitude finite element flexural vibration of plates/ stiflened plates” Department of Naval Architecture Indian Institute of Technology Kharagpur 721302, India, accepted 14 January 1993.

[43] Ray, A.K., Banerjee, B., and Bhattacharjee, B. “large amplitude free vibrations of skew plates including transverse shear deformation and rotatory inertia a new approach” journal of Sound and Vibration (1995) 180(4), 669-681.

[44] Ribeiro, P. “Asymmetric solutions in large amplitude free periodic vibrations of plates” Journal of Sound and Vibration 322 (2009) 8-14.

[45] Ribeiro, P. “Forced periodic vibrations of laminated composite plates by a p-version, first order shear deformation finite element” Composites Science and Technology 66 (2006) 1844-1856.

[46] Roylance, D. “Laminated Composite Plates” Department of Materials Science and Engineering ,Massachusetts Institute of Technology Cambridge, MA 02139 February 10, 2000.

[47] Sarma, M. S., Venkateshwar Rao, A., Pillai, S. R. R., and Nageswara Rao, B. “Large Amplitude Vibrations of Laminated Hybrid Composite Plates.” Journal of Sound and Vibration, 159(3), (1992): p. 540-545.

[48] Sathyamoorthy, M. and Chia, C.Y. “Non-linear vibration of anisotropic rectangular plates including shear and rotatory inertia” Fibre Science and Technology 13 (1980) 337-361.

[49] Sheikh , A.H. and Mukhopadhyay, M. “Linear and nonlinear transient vibration analysis of stiffened plate structures” Finite Elements in Analysis and Design 38 (2002) 477–502.

[50] Sheng Chen, C. and Ping Fung, Chin. “Large-amplitude vibration of an initially stressed plate on elastic foundations” Computers and Structures 82 (2004) 689–701.

[51] Shu, C., Wu, W.X., Ding, H., and Wang, C.M. "Free vibration analysis of plates using least-square-based finite difference method" Comput. Methods Appl. Mech. Engrg. 196 (2007) 1330–1343.

[52] Singh, G. and Venkateswara Rao, G. "Nonlinear oscillations of laminated plates using an accurate four-node rectangular shear flexible material finite element" Sadhana, Vol. 25, Part 4, August 2000, pp. 367-380.

[53] Singha, Maloy K. and Ganapathi, M. "Large amplitude free flexural vibrations of laminated composite skew plates" International Journal of Non-Linear Mechanics 39 (2004) 1709-1720.

[54] Springer, G.S. "Mechanics of composite structures".

[55] Srinivas, S. "A refined analysis of composite laminates" journal of sound and vibration (1973), 30(4) 495-507.

[56] Singha, M. K. and Daripa R. "Nonlinear vibration of symmetrically laminated composite skew plates by finite element method" International Journal of Non-Linear Mechanics 42 (2007) 1144-1152.

[57] Sundararajan, N., Prakash, T., and Ganapathi, M. "Nonlinear free flexural vibrations of functionally graded rectangular and skew plates under thermal environments" Finite Elements in Analysis and Design 42 (2005) 152-168.

[58] Taazount, M., Zinai, A., and Bouazzouni, A., "Large free vibration of thin plates: Hierarchic finite Element Method and asymptotic linearization" European Journal of Mechanics of Solids 28 (2009) 155–165.

[59] Viswanathan, K. K. and Kim, K. S. "Free vibration of antisymmetric angle-ply-laminated plates including transverse shear deformation: Spline method International Journal of Mechanical Sciences 50 (2008) 1476–1485.

[60] Wang,W., Mottershed, J. E., and mares, C. "Vibration modeshape recognition using image processing "Journal of Sound and Vibration 326 (2009) 909–938.

[61] White, R. G. "A comparison of some statistical properties of the responses of aluminium and CFRP plates to acoustic excitation" 1978 Composites 9, 251-258.

[62] Wua, W. X., Shua, C., and Wang , C.M. "Mesh-free least-squares-based finite difference method for large amplitude free vibration analysis of arbitrarily shaped thin plates" Journal of Sound and Vibration 317 (2008) 955–974.

[63] Yang a, J., Kitipornchai, S., and Liew, K.M. "Large amplitude vibration of thermo-electro-mechanically stressed FGM laminated plates" Comput. Methods Appl. Mech. Engg. 192 (2003) 3861–3885.

[64] Yang, P. C., Norris, C. M., and Stavsky 1966 Elastic wave propagation in heterogeneous plates. Int. J. Solid Struct. 2:665-684.

[65] Yongsheng, R. and Shuangshuang, S. “Large Amplitude Flexural Vibration of the Orthotropic Composite Plate Embedded with Shape Memory Alloy Fibers” Chinese Journal of Aeronautics 20(2007) 415-424.

[66] Zhou, D., et.al “3-D vibration analysis of skew thick plates using Chebyshev–Ritz method” International Journal of Mechanical Sciences 48 (2006) 1481–1493.

[67] Zhou, L. and Zheng, W.X. “Vibration of skew plates by the MLS-Ritz method” International Journal of Mechanical Sciences 50 (2008) 1133– 1141.

[68] Zhang, Y.X. and Yang, C.H. “Recent developments in finite element analysis for laminated composite plates” Composite Structures 88 (2009) 147–157.

[69] Zienkiewicz, O. C.“The finite element method” 3rd edn 1977 (London: McGraw-Hill).