ANALYTICAL SOLUTIONS
OF HEAT CONDUCTION PROBLEMS

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CERTIFICATE

This is to certify that the thesis titled "Analytical Solutions of Heat Conduction Problems" submitted by J. Rahul, Roll No. 10503032 in partial fulfillment for the degree of Bachelor of Technology in Mechanical Engineering at National Institute of Technology, Rourkela is an authentic work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other University/Institute for the award of any Degree or Diploma.

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ABSTRACT

The following thesis deals with the analytical methods which are in vogue for solving problems to the area of heat conduction. There have been discussed two methods, an old method known as the HEAT BALANCE INTEGRAL METHOD, and a relatively newer method christened as the DIFFERENTIAL TRANSFORMATION METHOD. The latter is dealt with first, as it is easier of the two. Dealing involves the basic idea of the method used, followed by the general theorems adopted. Two problems follow, illustrating the ease of use of this method, along with a comparison with the solutions of the problem using the numerical methods.

The former method, on the other hand, is more of an assumptive method, where one has to guess a temperature profile for proceeding. This is, nonetheless, a very accurate method, albeit a long one. Similar comparisons have been made for this method, like the ones made for the DT method.

The reader may use either method with ease, as it was for the simplification of the problem that these methods were developed.

CONTENTS

1). INTRODUCTION

The heat conduction problem is one of the most frequently encountered problems by scientists. The wide varieties of problems that are covered under conduction also make it one of the most researched and thought about problems in the field of engineering and technology. This variety of problems can thus be solved in a variety of methods, all of which can be broadly categorized into the following:-

1) Numerical Methods
2) Analytical Methods

1) Numerical Methods: These methods are used when we require exact solutions, involving several parameters. They may be a little complicated in their usage, but the advantage they offer is that the results obtained are definitely precise. These methods may also involve the usage of special functions such as Bessel functions.
2) Analytical Methods: These methods are relatively easier and less complicated, and are precise to a certain extent. They are used for smaller studies or researches, and they do not involve the use of special functions.

There are several analytical methods available for solving a heat conduction problem. One of these is the DIFFERENTIAL TRANSFORM METHOD or the DT method, which has been discussed below.

2). THE DIFFERENTIAL TRANSFORM METHOD

The DT method is a relatively newer, exact series method of solution. Unlike many popular methods, however, it is an exact method and yet it does not require the use of Bessel or other special functions.

The 2-dimesional differential transform of a function f(x, y) is defined as

$$F(k, h) = [\partial^{k+h} f(x, y)/\partial x^k \partial y^h]/[k!\, h!]_{(0, 0)}$$

Where F (k, h) is the DT of the original function f(x, y).

The inverse DT is defined as

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} F(k, h)\,(x^k y^h)$$

Using the two equations we can thus find the expression for f(x, y).

The following theorems are stated without proof.

1) If $f(x, y) = \partial u(x, y)/\partial x$, then $F(k, h) = (k+1) * U(k+1, h)$

2) If $f(x, y) = \partial u(x, y)/\partial y$, then $F(k, h) = (h+1) * U(k, h+1)$

3) If $f(x, y) = \partial^{r+s} u(x,y)/\partial x^r \partial y^s$, then $F(k, h) = (k+1)\ldots.(k+r)(h+1)\ldots(h+s)U(k+r, h+s)$

4) If $f(x, y) = u(x, y)v(x, y)$, then $F(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} U(r, h-s)V(k-r, s)$

5) If $f(x, y) = x^m y^n$, then $F(k, h) = \delta(k-m)\delta(h-n)$

6) If $f(x, y) = \left[\frac{\partial u(x,y)}{\partial x}\right]\left[\frac{\partial v(x,y)}{\partial x}\right]$, then

$$F(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (r+1)(k-r+1)U(r+1, h-s)V(k-r+1, s)$$

The same analogy will be applied when the two functions are differentiated partially w.r.t. y.

7) If $f(x, y) = \left[\frac{\partial u(x,y)}{\partial x}\right]\left[\frac{\partial v(x,y)}{\partial y}\right]$, then

$$F(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (k - r + 1)(h - s + 1)U(k - r + 1,\ s)V(r,\ h - s + 1)$$

These theorems will be of great help in solving heat conduction problems.

2.1) APPLICATION OF THE DIFFERENTIAL TRANSFORM METHOD IN TAPERED FINS

We will now apply the differential transform method for heat conduction in a triangular profile fin to see its utility.

Let x be the position co-ordinate along the fin axis, θ be the dimensionless temperature above ambient. The governing differential equation is given by:

$$x d^2\theta(x)/dx^2 + d\theta(x)/dx - m^2\theta(x) = 0 \qquad \text{--- (1)}$$

Where $\theta(x)$ the temperature is measured above ambient and normalized by the base temperature and m^2 is the fin parameter given by

$$m = \sqrt{(h1 + h2)AR/k} \qquad \text{--- (2)}$$

AR is the plate aspect ratio (length/depth), h1 and h2 are respectively the convective heat transfer co-efficients of the top and the bottom surfaces, and k is the thermal conductivity.

Boundary Conditions: There is no heat flux at the tip

$$\lim_{x\to 0} \frac{x d\theta(x)}{dx} = 0 \qquad \text{--- (3)}$$

And a dimensionless temperature of unity at the base (x=1) i.e.

$$\theta(1) = 1 \qquad \text{--- (4)}$$

We now take the DT of each term in (1).

For $xd^2\theta(x)/dx^2$, $\sum_{l=0}^{k}\delta(l-1)(k-l+1)(k-l+2)T(k-l+2)$ --- (5)

$$\frac{d\theta(x)}{dx} = (k+1)T(k+1) \quad \text{--- (6)}$$

$$-\ m^2\theta(x) = -m^2T(k) \quad \text{--- (7)}$$

Thus (1) is transformed into

$$\sum_{l=0}^{k}\delta(l-1)(k-l+1)(k-l+2)T(k-l+2) + (k+1)T(k+1) - m^2\,T(k) = 0 \qquad (8)$$

Transforming (2) and (3), we get

$\sum_{l=0}^{k}\delta(l-1)(k+1)T(k+1) = 0$ And $\sum_{l=0}^{k}T(k) = 1$ (9)

Expanding (9) for various values of k, we can obtain all the T (k) values as zero which is a trivial solution. So substituting various values of k in (8), we can obtain the T (k) values.

Then finally we can express the temperature function as

$$\theta(x) = T(0) + xT(1) + x^2\,T(2) + x^3\,T(3)$$

For up to n=3.

This was a simple problem chosen to instruct as to apply the DT method. We can easily put the requisite values of the various parameters to find that the method is highly precise and exact.

2.2) THE DIFFUSION PROBLEM

Now we will apply the differential transform method to solve another class of problems, which is known as the diffusion problem. The diffusion process (non-linear) with initial condition is given by the following differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \partial/\partial x[\frac{D(u)\partial u}{\partial x}]$$

With the initial condition that

$$u(x, 0) = f(x)$$

The process is defined by the diffusion term of the form

$$D(u) = u^n \qquad n>0$$

We will now solve a problem with the diffusion term as u i.e. for n=1.

Therefore the governing equation is reduced to

$$\frac{\partial u}{\partial t} = u\partial^2 u/\partial x^2 + (\frac{\partial u}{\partial x})^2$$

Also given is the initial condition

$$u(x, 0) = x^2/c \qquad x>0, c>0$$

Transforming both sides of the governing equation, we get

$$(h+1)U(k,h+1) = \sum_{r=0}^{k} \quad \sum_{s=0}^{h}(k-r+1)(k-r+2)U(r,h-s)U(k-r+2,s) + \sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)\,U(k-r+1,s)$$

By applying the initial condition, we get

$$u\,(i,0) = 0 \qquad \text{for all I except } i=2$$

$$u\,(2,0) = 1/c$$

Substituting into the transformed equation, we find out the values of the various co-efficients.

$u\,(2,1) = 6/c^2$

$u\,(2,2) = 36/c^3$

$u\,(2,3) = 216/c^4$

Finally substituting all the co-efficients, and by using the definition of the inverse of the DT , we obtain the series solution as

$$u\,(x,t) = x^2(1/c + 6t/c^2 + 36t^2/c^3 + 216t^3/c^4 + \ldots\ldots.)$$

A closed form solution of the equation may also be given by

$$u\,(x,t) = x^2/c\text{-}6t$$

3) **THE HEAT BALANCE INTEGRAL METHOD**

The next method we will encounter in this thesis is the heat balance integral method or HBIM in an abbreviated form. It is a semi-analytical method, and is an earlier invention than the previously discussed DT method. It is analogous to the classical integral technique used for fluid flow problems.

The HBIM is also used to find approximate solutions to transient diffusion problems, solving one dimensional linear and non-linear problems involving temperature dependent thermal properties and phase change problems such as freezing.

We will use this method to solve some classical problems.

3.1) METHOD ADOPTED FOR SOLVING THE PROBLEM USING THE HBIM

To illustrate the subtle nuances of using this method, we will take a simple problem of a slab with no internal heat source. Other specifications include that the slab is semi-infinite. The slab is assumed to have constant thermal conductivity k, specific heat c, density ρ and thermal diffusivity a= k / (ρ*c).

Let subscripts denote partial differentiation w.r.t the particular variable. So, $T_{xx}(x, t) = \delta^2 T / \delta x^2$.

Let q(x, t) be the heat flow defined in the +ve x-direction per unit transverse area per unit time. Then from Fourier's equation,

$$q(x, t) = -kT_x(x, t) \qquad , x>=0, t>=0 \quad (1)$$

Since there is no internal heat source, equation (1) and the conservation of energy yield

$$kT_{xx}(x, t) = \rho c T_t(x, t) \qquad (2)$$

Initial and boundary conditions are as follows:

$$T(x, 0) = T_{initial} = T_i = \text{constant} \qquad , x>0 \qquad (3)$$

$$T(0, t) = T_{surface} = T_s = \text{constant} \qquad , t>0 \qquad (4)$$

$$T(\infty, t) = T_{initial} = T_i = \text{constant} \qquad , t \text{ finite} \qquad (5)$$

Equations (2) to (5) constitute a classical problem with the solution given by the equation

$$(T-T_i) / (T_s-T_i) = [1-\text{erf}(\text{sqrt}(x^2/4at))] \qquad (6)$$

where,

$$Q_0(t) = q_{exact}(0, t) = -kT_x(0, t) = k(T_s-T_i) / \text{sqrt}(\pi at) \qquad (7)$$

3.2) THE APPROXIMATION PROCEDURE FOR THE FOREGOING PROBLEM

Assume there exists a function U such that

$$U(x, t) \sim T(x, t) \qquad (8)$$

The actual temperature distribution T(x, t) will satisfy the partial differential equation (2) and also the following integral equation (note that the summation sign is used instead of the integral sign with limits mentioned, and will be used as that only, unless otherwise specified).

$$\Sigma_0^t\, q(0, t')\, dt' = \Sigma^t{}_0 -kT_x(0, t')\, dt' = \Sigma_0^\infty\, \rho c\, [T(x, t) - T(0, t)]\, dx \qquad (9)$$

This is simply a "balance" of the heat energy input on the left against its measurable effect on the right; this is the "heat balance integral".

Obviously, since U is an approximated function, it is not possible for it to satisfy (2) but it is required for it to satisfy (9)

$$\Sigma_0^t -k\, U_x\, (0, t')\, dt' = \Sigma_0^\infty\, \rho c\, [U(x, t) - T_i]\, dx \qquad (10)$$

Now, the next assumption is that the significant measurable effects of the boundary disturbance (3) and (4) do not penetrate beyond some finite distance x = p (t). This assumption can be mathematically stated as:

$$U\, (p(t), t) = T_i \qquad t>0 \qquad (11)$$

$$U(x, t) = T_i \qquad x>p\, (t) \qquad (12)$$

Using (12) in (10) yields the following integral

$$\Sigma_0^t -k\, U_x\, (0, t')\, dt' = \Sigma_0^{p(t)}\, \rho c\, [U(x, t) - T_i]\, dx \qquad (13)$$

Using (13) instead of (2), and modifying the boundary condition (3), we get

$$p(0) = 0 \qquad (14)$$

and the conditions (4) and (5) on T(x, t) are now replaced by the following conditions on U(x, t)

$$U\, (0, t) = T_s \qquad t>0 \qquad (15)$$

And $$U(p(t), t) = T_i \qquad t>0 \qquad (16)$$

Hence this procedure is reduced to finding a function U(x, t) which satisfies (15) and (16), which is then subsequently used in (13) to find the penetration depth p(t), subject to the boundary condition (14).

It then becomes highly convenient to restrict the search to polynomial functions of the form

$$U_n(x, t) = A_n(t) + B_n(t)x + C_n(t)x^2 + \ldots \qquad (17)$$

For the first approximation, we take the polynomial of first order, and along with conditions (15) and (16), we get an equation in U and T. This newest equation is then used in the equation obtained as (13), which upon differentiation and re-arrangement yields a simple differential equation, which upon integration yields the value of the penetration p (t).

3.3) ANALYSIS OF THE INVERSE STEFAN PROBLEM

We will now discuss the use of the HBIM in solving the inverse Stefan Problem, involving one region. The problem of the one-dimensional inward solidification or freezing is considered here. It is assumed that the entire domain is initially the liquid at the phase change temperature. The domain is separated into the solid and liquid phase by the phase change interface. The liquid phase being at a constant phase change temperature *T*ph throughout, the temperature is unknown only in the solid phase, so the problem is a one-region problem. The location of the moving interface is a monotonic function of time. As for the inverse Stefan problem, the variation of the interface with time is specified and controlled, i.e., it is a known function of time.

Since the constant thermo-physical properties are assumed, the problem can be formulated by the following governing equation of the solid and the boundary conditions at the phase change interface:

$$(1 / r^i)\ (\delta^2 / \delta r^2)\ (r^i T_s) = (1 / \alpha_s)\ (\delta T_s / \delta t) \qquad , s(t) < r < r_o\ , t>0 \qquad (1)$$

$$T_s = T_{ph} \qquad , r = s(t), \quad t>0 \qquad (2)$$

$$\lambda_s\ \delta T_s / \delta r = \rho L\ (ds / dt) \qquad , r = s(t), \quad t>0 \qquad (3)$$

$$s(0) = r_0 \qquad (4)$$

where r_0 is the position of the fixed boundary.

Now define the following non-dimensional parameters:

$$u_s = (T_s - T_{ph}) / (T_0 - T_{ph}),\ \ R = r / r_0\ ,\ \tau_s = \alpha_s t / r_0^2\ ,\ \ S = s / s_0\ ,\ \ Ste = c_p\ (T_o - T_{ph}) / L \qquad (5)$$

where T_0 is an external reference temperature. A new variable is defined as

$$\theta_s = R^i\ u_s \qquad (6)$$

Then equations (1) to (4) can be modified as:

$$\delta^2\theta_s / \delta R^2 = \delta\theta_s / \delta\tau \qquad (7)$$

$$\theta_s = 0, \qquad R = S(\tau), \qquad \tau > 0 \tag{8}$$

$$\delta\theta_s / \delta R = (S^i / Ste)(dS / d\tau), \qquad R = S(\tau), \qquad \tau > 0 \tag{9}$$

$$S(0) = 1 \tag{10}$$

When the heat-balance integral method is applied to the inverse Stefan problem, it is not necessary to determine the functional relation between the position of the moving boundary and the time, because it has been specified *a priori*. Therefore, a polynomial temperature profile which satisfies all the moving boundary conditions and consists of a number of adjustable coefficients can be directly inserted into the heat-balance integral equation. A selected polynomial temperature profile, wherein some of the coefficients are determined by the moving boundary conditions, is then substituted into the heat-balance integral equation. The resulting equation is an ordinary differential equation for the undetermined coefficient. Solving this ordinary differential equation, the undetermined coefficient and the temperature distribution satisfied the moving boundary conditions and the heat-balance integral equation can be obtained finally.

The heat balance integral equation can be obtained by the integration of the governing equation (7) from R = 1 to R = S (τ).

$$(\delta\theta_s / \delta R)_{R = S} - (\delta\theta_s / \delta R)_{R = 1} = d\Theta / d\tau \tag{11}$$

Now, we assume θ_s to be a quadratic polynomial of the form:

$$\theta_s = a + b\,(R - S) / (1 - S) + c\,(R - S)^2 / (1 - S)^2 \tag{12}$$

where the co-efficients are the functions of S (τ) in general. The co-efficients a and b can be obtained by using (8) and (9). Then for further conditions we can differentiate (8) w.r.t τ.

$$(\delta\theta_s / \delta R)(dS / d\tau) + (\delta\theta_s / \delta\tau) = 0 \tag{13}$$

Using conditions (7) and (9) and substituting in (13), we get another condition for the phase change interface, which is

$$(\delta^2\theta_s / \delta R^2) = - (S^i / Ste)\ (dS / d\tau)^2 \qquad (14)$$

Thus we have calculated the values of all the three co-efficients used for θ_s.

We have from Leibnitz's condition:

$$\Theta_s = \Sigma_1{}^S\ \theta_s\, dR \qquad (15)$$

When the variation of the phase change interface with time is given, Θ_s can be easily solved.

4) CONCLUSION

Over the course of the last few pages we have observed two key things about the methods that we have encountered. The first thing is the enormous transforming power of both the methods, and in particular the DT method. Both were able to change the class of problems from a complex equation to a simpler, more easily solvable linear equation. The second thing was the wide ambit of problems for which these methods serve their purpose. The DT method was first put to use in solving problems relating to electrical circuits, whereas the HBIM was an offshoot of the classical integral method (as discussed in the literature) used to solve fluid flow problems. In the examples chosen, we have seen that they are effective enough with a high degree of accuracy. In fact, this is a third, and perhaps the most intriguing, aspect of these methods. Normally, such methods sacrifice accuracy for simplicity, which is a dogma shattered by these methods. Not only are they infinitely simpler than normal methods, they also ensure that accuracy is given a prime preference.

There is every possibility that these methods, along with many newer methods that may be developed in the future, can be used in solving even tougher problems with simplicity and accuracy.

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