On the solution of spatial generalizations of Beltrami equations

Introduction

With the help of functional analytical methods complex analysis is a powerful tool in treating non-linear first-order partial differential equations in the plane. Some of the most important of these non-linear first-order partial differential equations are the Beltrami equations. This is due to the fact that the theory of Beltrami systems is related with many problems of geometry and analysis, like non-linear subsonic two-dimensional hydrodynamics, problems of conformal and quasiconformal mappings of two-dimensional Riemannian manifolds, or complex analytic dynamics. We illustrate this with a simple example. Let us consider the two-dimensional linear first-order partial differential equations:

$$a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial u}{\partial y}+b_{11}\frac{\partial v}{\partial x}+b_{12}\frac{\partial v}{\partial y}+a_1u+b_1v=f_1,$$

$$a_{21}\frac{\partial u}{\partial x}+a_{22}\frac{\partial u}{\partial y}+b_{21}\frac{\partial v}{\partial x}+b_{22}\frac{\partial v}{\partial y}+a_2u+b_2v=f_2.$$

This system of partial differential equations can be transformed into a generalized Cauchy-Riemann equation:

$$\frac{\partial w}{\partial \bar{\varsigma}}=a(\varsigma)w+b(\varsigma)\overline{w},$$

where the coordinate transform has to satisfy the Beltrami equation

$$\frac{\partial \varsigma}{\partial \bar{z}}=q(z)\frac{\partial \varsigma}{\partial z}.$$

Here $q(z)$ depends on the coefficients a_{ij} and b_{ij}. In general the Beltrami equation

$$\frac{\partial w}{\partial \bar{z}}=q(z)\frac{\partial w}{\partial z}$$

is the complex form of the first-order elliptic system

$$v_y=au_x+bu_y$$

$$-v_x=bu_x+du_y$$

where $w=u+iv,\ z=x+iy,\ ad-b^2=1,\ a>0$ and $b>0.$ Using the ansatz

$$w(z)=z+T_\Omega h$$

with $T_\Omega h(z)=\frac{1}{2\pi i}\int_\Omega\frac{h(\xi)}{\xi-z}d\xi_1d\xi_2$ we get the fixed-point equation:

$$h=q(z)(1+\Pi_\Omega h),$$

where $\Pi_\Omega h(z)=\frac{1}{2\pi i}\int_\Omega\frac{h(\xi)}{(\xi-z)^2}d\xi_1d\xi_2$ is the complex Π -operator, defined as the derivative of the T_Ω -operator. Obviously, the theory of Beltrami equations is strongly connected with the Π -operator. This singular integral operator is immediately recognized as two-dimesional Hilbert transform, known also under the name of integral operator with Beurling kernel, acting as an isometry from $L_2(C)$ onto $L_2(C)$. This means that the starting point for investigating spatial Beltrami equations is the generalization of the complex Π -operator. But before we deal with this problem we will make some necessary preliminaries.

Preliminaries

For all what follows we work in H, the skew-field of real quaternions. This means that if $z \in H$, then

$$z = \sum_{k=0}^{3} z_k \cdot i_k$$

where $i_0 \cdot i_k = i_k \cdot i_0 = i_k$, $i_k^2 = -i_0$, for $k \in \{1,2,3\}$, $i_1 \cdot i_2 = -i_2 \cdot i_1 = i_3$, $i_2 \cdot i_3 = -i_3 \cdot i_2 = i_1$, $i_3 \cdot i_1 = -i_1 \cdot i_3 = i_2$, $\{z_k\} \subset R.$ Quaternion i_0 is often called the real unit while all i_k are called imaginary units. Reals z_k are called the real coordinates of a quaternion z. The natural operations of addition and multiplication in H turn it into a non-communtantive, associative skew-field.

There is a main involution in $H,$ i.e. quaternionic conjugation, which plays an exceptionally important role. It is defined in such a way that for the basis units

$$\overline{i_0} = i_0, \ \overline{i_k} = -i_k, \qquad k \in \{1,2,3\}$$

and it is extended onto the whole H as an R-linear mapping: if $z \in H$ then

$$\overline{z} = \sum_{k=0}^{3} \overline{z_k i_k} = \sum_{k=0}^{3} z_k \overline{i_k} = z_0 - \sum_{k=1}^{3} z_k i_k \ .$$

If $\{a, b\} \in H$ then $\overline{ab} = \overline{b} \cdot \overline{a},$ and

$$\overline{a} \cdot a = a \cdot \overline{a} = \sum_{k=0}^{3} a_k^2 = |a|^2 \in R.$$

In particular, it means that any $z \in H \setminus \{0\}$ has an inverse z^{-1}:

$$z^{-1} = \frac{\overline{z}}{|z|^2}.$$

Also, for all what follows let $\Omega \subset H$ be a bounded domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$. Moreover, we will consider functions f defined on Ω with values in $H.$

Now, let $\psi := \{\psi^0, \psi^1, \psi^2, \psi^3\}$ be an orthonormalized (in the usual Euclidean sense) system of quaternions. Sometimes we call ψ a structural set or a structural 4-tupel. With our structural set ψ we can define the generalized Cauchy-Riemann operator by

$${}^{\psi}Df = \sum_{k=0}^{3} \psi^k \frac{\partial f}{\partial x_k}$$

and it’s conjugate operator by

$${}^{\overline{\psi}}Df = \sum_{k=0}^{3} \overline{\psi^k} \frac{\partial f}{\partial x_k}.$$

For this operator we have

$${}^{\psi}D\,{}^{\overline{\psi}}D = {}^{\overline{\psi}}D\,{}^{\psi}D = \Delta,$$

where Δ is the Laplacian. Of some mayor importance are the functions which lie in the kernel of ${}^{\psi}D$, or with other words satisfy the equation ${}^{\psi}Df = 0$. These functions are called ψ-hyperholomorphic functions. The Cauchy-Riemann operator has a right inverse of the form

$${}^{\psi}Tf(x) = -\frac{1}{2\pi^2} \int_{\Omega} \frac{\overline{(y-x)_{\psi}}}{|y-x|^4} f(y) d\Omega, \qquad x \in \Omega$$

where $(y-x)_{\psi} = \sum_{k=0}^{3} \psi^k (y_k - x_k).$

This operator acts continuously from $W_p^k(\Omega)$ into $W_p^{k+1}(\Omega)$, $1 < p < \infty, \quad k \in N \cup \{0\}$ (see [9]).
For more information about these topics and general quaternionic analysis we refer to [2].

Generalization of the complex Π-operator

As we stated above if we want to deal with spatial Beltrami equations then we have to generalize the complex Π-operator to the quaternionic case. On the first view this should be an easy task. In the complex plane we have defined the Π-operator as the derivative of the complex T-operator, so why we shall not go in the same way? But here a problem arises. Mehlikhzon proved in the 40's that a quaternionic derivative in the usual sense exist at most for linear functions [7]. So we have the big problem how we generalize the derivative in a way, that preserves most of it's useful properties. One way is to use the ${}^{\overline{\psi}}D$-operator. Mitelman and Shapiro have shown that this operator can be considered as the derivative-operator (hyperderivative called) in some sense [8].

Definition 1. Let ψ and φ be two structural sets. The operator ${}^{\psi,\varphi}\Pi$, defined by

$$ {}^{\psi,\varphi}\Pi f = {}^{\overline{\psi}}D^{\varphi}Tf $$

is called generalized Π-operator [11].

This operator was first investigated by Sprößig [10] in 1979 in the case of $\varphi = \psi = \{i_0, i_1, i_2, i_3\}$, the usual basis of the real quaternions. For this special case we refer to [3]. Our definition using the structural sets ψ and φ was first introduced by Shapiro and Vasilevski [9].
In general the generalized Π-operator has the representation formula:

$$ {}^{\psi,\varphi}\Pi f(z) = -\frac{1}{2\pi^2}\int_\Omega \left(\frac{-\sum_{k=0}^{3}\overline{\psi^k}\ \overline{\varphi^k}}{|\xi - z|^4} + \frac{4(\xi - z)_{\overline{\psi}}(\xi - z)_{\overline{\varphi}}}{|\xi - z|^6} \right) f(\xi)d\Omega - \frac{\pi^2}{2}\sum_{k=0}^{3}\overline{\psi^k}\ \overline{\varphi^k} f(z). $$

In the case of $\sum_{k=0}^{3}\overline{\psi^k}\ \overline{\varphi^k} = 0$ we have the representation formula:

$$ {}^{\psi,\varphi}\Pi f(z) = -\frac{1}{2\pi^2}\int_\Omega \frac{4(\xi - z)_{\overline{\psi}}(\xi - z)_{\overline{\varphi}}}{|\xi - z|^6} f(\xi)d\Omega. $$

This important special case was intensively studied by Shapiro and Vasilevski [9]. Up to know it is the only case where a relationship between the Π-operator and the Bergman-operator could be derived.

Obviously, we have the following continuous mapping property

$$ {}^{\psi,\varphi}\Pi : W_p^k(\Omega) \to W_p^k(\Omega), \quad k \geq 0, p > 1. $$

By the help of the theory of Calderon and Zygmund we can estimate the norm of the Π-operator in the form

$$ \left\| {}^{\psi,\varphi}\Pi \right\|_{[L_2(\Omega), L_2(\Omega)]} \leq \frac{8\sqrt{c_4}}{\sqrt[4]{2\pi^2}} $$

where c_4 is a constant [3], [5].

In the next section we see how we can solve non-linear first-order systems of partial differential equations using our spatial generalization of the complex Π-operator.

Case of non-linear first-order partial differential equations

Let $w:\Omega \to H.$ We will consider the non-linear first-order system of partial differential equations

$$ {}^{\varphi}Dw = F\left(z, w, {}^{\bar{\varphi}}Dw\right). \tag{1} $$

Furthermore, let $w = w(z)$ be a solution of (1), then we can define $\Phi := w - {}^{\varphi}TF\left(z, w, {}^{\bar{\varphi}}Dw\right).$

Obviously, Φ is φ-hyperholomorphic and we have the equation

$$ w = \Phi + {}^{\varphi}TF\left(z, w, {}^{\bar{\varphi}}Dw\right). \tag{2} $$

From this equation we obtain the result, that for each solution w there exist a φ-hyperholomorphic function $\Phi,$ such that (2) is fulfilled. Also, it holds the other way around: If Φ is (φ-hyperholomorphic) arbitrarily choosen and (2) is solvable with this Φ, then w is a solution of (1).

This means we have the possibility to transform boundary value problems for (1) to boundary value problems of φ-hyperholomorphic functions. Now, let us introduce the operator

$$ W = \Phi + {}^{\varphi}TF\left(z, w, {}^{\bar{\varphi}}Dw\right) $$

in $W_2^1(\Omega)$. We remark that Φ depends on w. Solutions of (2) are fixed-points of this operator. Using the condition, that $W_2^1(\Omega)$ has to be mapped into itself, we get:

$$ {}^{\varphi}DW = 0 + F\left(z, w, {}^{\bar{\varphi}}Dw\right), $$

$$ {}^{\bar{\varphi}}DW = {}^{\bar{\varphi}}D\Phi + {}^{\varphi,\varphi}\Pi F\left(z, w, {}^{\bar{\varphi}}Dw\right), $$

or with other words, we have

$$ W = \Phi(w,h) + {}^{\varphi}TF(\cdot, w, h) $$

$$ H = {}^{\bar{\varphi}}D\,\Phi(w,h) + {}^{\varphi,\varphi}\Pi F(\cdot, w, h) $$

where $h = {}^{\bar{\varphi}}Dw,\; H = {}^{\bar{\varphi}}DW$, and our solutions of (2) are also fixed-points of the operator (W, H).

We remark that in the case of a boundary condition $Bw = g$, we get the boundary condition

$$ B\Phi(w,h) = g - B\,{}^{\varphi}TF(\cdot, w, h) = Bw - B\,{}^{\varphi}TF(\cdot, w, h). $$

Now we deal with an important special case of these non-linear first-order system's of partial differential equations.

Generalization of Beltrami equations

As we have seen before complex Beltrami equations are a starting point for the application of the theory of generalized analytic functions in the sense of Bers or Vekua. The same holds for the hypercomplex case [6]. Spatial generalizations of Beltrami equations are also useful for the theory of quasiconformal mappings.

Now, let $q:\Omega \to H$ be a measurable function, $w \in W_p^1(\Omega)$, $1 < p < \infty$, ψ and φ are structural sets. Then we call the equation

$$ {}^{\psi}Dw = q\,{}^{\bar{\varphi}}Dw \tag{3} $$

generalized Beltrami equation. In the same way as in the last section we make the ansatz

$$ w = \Phi + {}^{\psi}Th $$

where Φ is a ψ-hyperholomorphic function, and transform (3) into the integral equation:

$$ h = q\left({}^{\bar{\varphi}}D\Phi + {}^{\varphi,\psi}\Pi h\right). $$

Investigating the norm of the operator $q\,{}^{\varphi,\psi}\Pi$ we have that in the case of

$$ \|q\| \le \frac{1}{\left\|{}^{\varphi,\psi}\Pi\right\|} $$

this operator is contractive. That means we can get a solution of our integral equation using Banach’s fixed-point theorem (the other conditions of Banach’s fixed-point theorem can easily be verifed). Applying our norm estimate for the generalized Π -operator we get the sufficient condition

$$\|q\| \leq \frac{\sqrt[4]{2\pi^2}}{8\sqrt{c_4}}\,.$$

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