IDENTIFICATION OF STRUCTURAL MODELS AS A PROBLEM OF GROUP REPRESENTATION THEORY

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1. Introduction.

There is a great diversity of theoretical frameworks in which the problem of structural identification has been formulated. In some cases this problem is solved as an inverse problem of differential equations, in other cases as a problem of optimization [cf. 1].

However, in such approaches, ill-conditioning and non-uniqueness in the solutions are inevitable difficulties. In order to overcome these obstacles, the present author applied a more geometrical method which is based on the group theory and the group representation theory [cf. 5,8].

In the first stage the group – theoretical analysis of some fundamental concepts of stochastic dynamics: stochastic processes and functional series of Volterra – Wiener type has been undertaken. The group representations (GR) theory has been introduced into stochastic dynamics. It has been shown that the symmetry is a typical phenomenon for the models of stochastic mechanics [cf. 2] in contrast to the geometrical symmetry of systems of bodies in the space.

The analysis of the symmetry (GR) of the moment functions of order *m* for stochastic processes is the basic, original concept of the work. The following groups: symmetric S_m, special affine SAff(m), general linear GL(n,$\Re$), GL(n,$\mathbb{C}$) and their subgroups as well as some gauge symmetries play the main role in the models considered.

In the second stage the informational entropy [cf. 3,4] has been introduced as a measure of the randomness in the identified models.

Some basic problems of identification of the systems can be effectively solved as problems in the GR theory. The theorems concerning the symmetries of moment functions and multispectra as well as the uniqueness of the system kernels have been proved.

The problem of ill-conditioning is formulated in terms of the information entropy. For the simple model of a discrete cantilever structure the observed regularities have been tested numerically.

2. Group properties of moment functions.

The present author has observed that symmetries of moment functions of stochastic

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processes play an important role in identification of systems. They provide the group-theoretical method of choice of the model structure and the model parameters. Let X(t), $t \in \Re$, be a scalar, real-valued stochastic process. The process is described by the moment functions of order m:

$$K(t_1, \ldots, t_m) = E(X(t_1) \ldots X(t_m)), \tag{2.1}$$

where E(-) is the expectation operator.
Let B be the group of all bijections, $B: \Re^m \to \Re^m$ [cf.6].
The symmetry group G_K of the moment functions is defined as follows:

$$G_K = \{B \mid K(B(t_1, \ldots, t_m)) = K(t_1, \ldots, t_m)\} \tag{2.2}$$

Every such a group includes the symmetric (permutation) group S_m but can be richer than it.
In $\Re^m$ also the group of special translations acts:

$$ST(m): (t_1, \ldots, t_m) \to (t_1+\tau, \ldots, t_m+\tau), \ \tau \in \Re. \tag{2.3}$$

The direct product of S_m and ST(m) is defined as the special affine group:

$$SAff(m) = S_m \times ST(m) \tag{2.4}$$

The reader can check
Proposition1.
The quotient group of the special affine group with respect to the subgroup of special translations is isomorphic to the symmetric group:

$$SAff(m) / ST(m) \cong S_m \tag{2.5}$$

The above proposition is an analogy of the so-called fundamental theorem of solid state physics for the space group and the point group.
Proposition 2.
The special affine group is the symmetry group of the moment functions of stationary stochastic processes.
Example 1.
For m = 3, the moment function of the stationary process follows from definition (2.1):

$$\forall t \in \Re, \ K(\tau_1, \tau_2) = E(X(t)X(t+\tau_1)X(t+\tau_2)) \tag{2.6}$$

The symmetry group of (2.6) is $SAff(3) = S_3 \times ST(3)$
The irreducible representation of this affine group is given by the following six matrices:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right\} \tag{2.7}$$

This matrix group is isomorphic to the point crystallographic group C_{3v}. □

For the vector stochastic processes, the moment functions of the order m are defined by:

$$\underbrace{K_{ij\,\dots\,z}}_{m-\text{times}}(t_1, \dots, t_m) = E(X_i(t_1) \dots X_z(t_m)), \tag{2.8}$$

where i,j, ..., z = 1, 2, … , n

In this case the symmetry group acts simultaneously on the indices and on the time arguments.

Example 2.

For m = 3, the moment functions of the vector stochastic process have the following symmetries:

$$\begin{aligned} K_{ijk}(\tau_1, \tau_2) &= K_{jik}(-\tau_1, \tau_2 - \tau_1) = \\ &= K_{ikj}(\tau_2, \tau_1) = K_{kji}(\tau_1 - \tau_2, \tau_2) = \\ &= K_{kij}(-\tau_2, \tau_1 - \tau_2) = K_{jki}(\tau_2 - \tau_1, -\tau_1), \end{aligned} \tag{2.9}$$

where

$$K_{ijk}(\tau_1, \tau_2) = E(X_i(t)\, X_j(t+\tau_1) X_k(t+\tau_2)), \quad \forall t \in \Re;\ i,j,k = 1,2, \dots, n. \tag{2.10}$$

This concludes the example.

We now take up the problem of symmetries of the multispectra defined by the Fourier transform:

$$S_{ij\dots z}(i\omega_1,\dots,i\omega_{m-1}) = \int_{-\infty}^{\infty}\dots\int K_{ij\dots z}(\tau_1,\dots,\tau_{m-1}) \exp\left(-i\sum_{k=1}^{m-1}\omega_k\tau_k\right) d\tau_1 \dots d\tau_{m-1}, \tag{2.11}$$

i,j,...z = 1,2,...n.

We should not to mix up the index „i” with the imaginary unit.

Let $\mathbf{D}(\pi)$ be the irreducible representation of the moment functions in (2.11), $\pi \in S_m$.

Proposition 3.

The irreducible ((m-1) dimensional) representation of the symmetry group of the multispectra consists of the matrices:

$$(\mathbf{D}^T(\pi))^{-1},\ \forall\ \pi \in S_m \tag{2.12}$$

Example 3.

In the case m=2, $S_2 \cong \mathbf{D}(\pi) = \{1, -1\}$

and

$$S_{ij}(-i\omega) = S_{ji}(i\omega) \tag{2.13}$$

Moreover

$$S_{ij}(-i\omega) = \bar{S}_{ij}(i\omega), \tag{2.14}$$

where the bar denotes the complex conjugation.

Therefore the spectral matrix is Hermitian which follows, as we have shown, from the invariance of the correlation function in (2.11) with respect to the group SAff(2) = $S_2 \times ST(2)$.

3. Identification of the Volterra kernels in the vector case.

Let the following functional series be the constitutive equation of a nonlinear, n DOF system with the vector stochastic inputs $X_j(t)$:

$$Y_i(t) = \int_{-\infty}^{\infty} h_{ij}(\tau_1) X_j(t-\tau_1) d\tau_1 + \iint_{-\infty}^{\infty} h_{ijk}(\tau_1,\tau_2) X_j(t-\tau_1) X_k(t-\tau_2) d\tau_1 d\tau_2 + \ldots \quad (3.1)$$

where i,j = 1,2,…,n

and the summation convention has been assumed.

Functions $\underbrace{h_{ij\ldots z}}_{m\text{ - times}}(\tau_1,\ldots,\tau_{m-1})$ are the deterministic Volterra kernels of order m-1.

We shall identify the kernels by a generalization of the Wiener approach developed for scalar stochastic processes (cf. [7]).

Let us consider, without restricting generality, the case m = 3 (i.e. only the kernel of second order is nonzero).

We assume the inputs as the Gaussian white noises with the correlation matrix of the form $K_{rs}(\tau) = \delta_{rs}\delta(\tau)$, where δ_{rs} is the Kronecker delta and $\delta(\tau)$ is the Dirac delta.

After some calculations one obtains:

$$E(Y_i(t)X_s(t-\tau_1)X_r(t-\tau_2)) = h_{isr}(\tau_1,\tau_2) + h_{irs}(\tau_2,\tau_1) \quad (3.2)$$

It is seen that in order to get the unique solution one has to assume in the kernel the symmetry of the type described by eq. (2.9).

4. Model and parametric identification in the frequency domain.

Let $\mathbf{H}(i\omega)$ be the Fourier transform of the first – order kernel $h_{ij}(\tau)$ (see eq. (3.1)):

$$\mathbf{H}(i\varpi) = \int_{-\infty}^{\infty} \mathbf{h}(\tau) \exp(-i\omega\tau) d\tau \quad (4.1)$$

It can be shown that in structural dynamics models, $\mathbf{H}(i\omega)$ is an element of the general linear group GL(n, $\mathbb{C}$) (for every $\omega \in \Re$).

Let the spectral matrix $\mathbf{S}_x(i\omega)$ at the input be an element of the set of the Hermitian matrices (see eqs. (2.13), (2.14)) of dimension $n \times n$.

We shall consider the action of the group GL(n $\mathbb{C}$) on the set of the Hermitian matrices, for every $\omega \in \Re$.

Therefore this authomorphism gives us the governing equation in the frequency domain. Moreover it can be regarded as a gauge symmetry in this domain.

$$\mathbf{S}_y(i\omega) = \mathbf{H}(i\omega)\,\mathbf{S}_x(i\omega)\overline{\mathbf{H}}^T(i\omega) \quad (4.2)$$

The invariant of such an action is the rank of the matrices:

$$\forall\omega \in \Re,\ \mathrm{rank}(\mathbf{S}_y(i\omega)) = \mathrm{rank}(\mathbf{S}_x(i\omega)) \quad (4.3)$$

This relation can be useful in description of experiments on real structures.

By using the modal decomposition of the dynamical system it is possible to distinguish an Abelian subgroup in the group GL(n, C). The elements of this subgroup have the following form:

$$\mathbf{H}_0\,(i\omega) = (-\omega^2\mathbf{I} + \mathbf{2}\omega_0\,\mathbf{c}\,i\omega + \omega_0^2)^{-1} \tag{4.4}$$

where

$$\omega_0 = \mathrm{diag}(\omega_{01}, \omega_{02},..., \omega_{0n}) \tag{4.5}$$

is the diagonal matrix of the eigenvalues,

$$\mathbf{c} = \mathrm{diag}(c_{11}, c_{22},..., c_{nn}) \tag{4.6}$$

is the diagonal matrix of the modal damping coefficients.

This formulation gives us the possibility of parametric identification of the system, under environmental excitation of the white noise type with unknown intensities.

Example 4.

The cantilever offshore structure as a two-degree-of-freedom model has been considered.

The spectral densities of the response in some interval are given in Fig. 1. From eqs. (4.2) and (4.4) an algebraic nonlinear equation for the modal parameters follows. Application of the assumption of the input invariance in the frequency domain leads to the unique solution:

$$\omega_0 = \mathrm{diag}(9{,}0;\ 10{,}35)\ \frac{\mathrm{rad}}{\mathrm{s}};$$

$$\mathbf{c} = \mathrm{diag}(0{,}12;\ 0{,}10)\ [-]$$

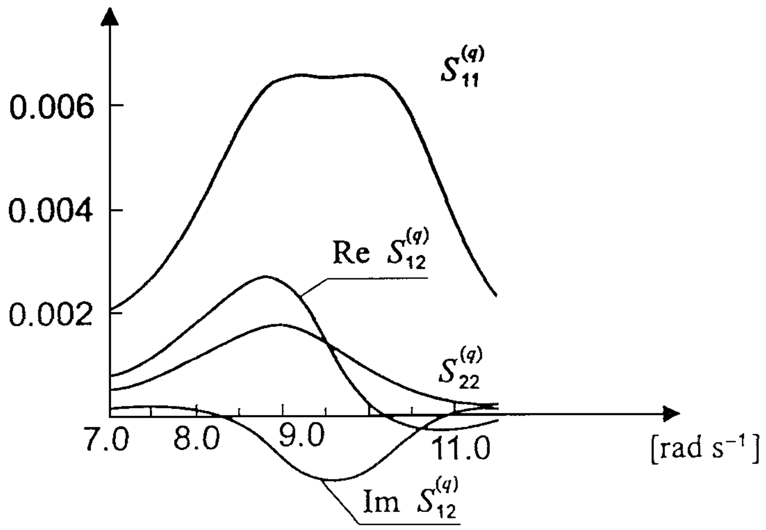

Fig. 1. Spectral densities at the output of the system: $\mathbf{q} = (y_1, y_2)$

5. Information entropy and symmetry.

We can regard the information entropy as a measure of randomness in the structural models (cf [3], [4]).

However, there is a special group-theoretical argument making clear that the very concept of the entropy has some connection with the symmetry.

We recall that the determinant of a matrix can be defined as:

$$\det \mathbf{K} = \sum_{\pi} (\operatorname{sgn} \pi) k_{\pi(1)1} k_{\pi(2)2} \ldots k_{\pi(n)n}, \tag{5.1}$$

where the sum is taken over all the elements π of the symmetric group S_n.
In the range of the second – order theory the entropy H(n) is defined by the determinant of the covariance matrix $\mathbf{K}_{n\times n}$:

$$H(n) = \frac{1}{2} \ln\left((2\pi e)^n \det \mathbf{K}_{n\times n}\right) \tag{5.2}$$

n denotes here the product of the number of mesh nodes and the dimension of the vector random field (cf. [3]).
Eq. (5.2) describes the entropy of a continuous distribution with the normal probability density function. An unexpected result is that in some cases, the greater randomness at the input (measured by the entropy) does not lead to the greater randomness at the output (cf. [4]). Moreover, the ill-conditioning can be caused by the choice of differentiable covariance functions, as the study of the entropy reveals.

6. Concluding remarks.

The symmetries of the moment functions play an important role in identification of systems. The group-theoretic approach underlines the unity of the mathematical description of the symmetries and gives some results of the practical value.

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