QUATERNIONS IN APPLIED SCIENCES
A HISTORICAL PERSPECTIVE OF A MATHEMATICAL CONCEPT

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After more than hundred years of arguments in favor and against quaternions, of exciting odysseys with new insights as well as disillusions about their usefulness the mathematical world saw in the last 40 years a burst in the application of quaternions and its generalizations in almost all disciplines that are dealing with problems in more than two dimensions. Our aim is to sketch some ideas - necessarily in a very concise and far from being exhaustive manner - which contributed to the picture of the recent development. With the help of some historical reminiscences we firstly try to draw attention to quaternions as a special case of Clifford Algebras which play the role of a *unifying language* in the Babylon of several different mathematical languages. Secondly, we refer to the use of quaternions as a *tool for modelling problems* and at the same time for *simplifying the algebraic calculus* in almost all applied sciences. Finally, we intend to show that quaternions in combination with classical and modern analytic methods are a *powerful tool for solving concrete problems* thereby giving origin to the development of Quaternionic Analysis and, more general, of Clifford Analysis.

> *Could anything be simpler or more satisfactory? Don't you feel, as well as think, that we are on a right track, and shall be thanked hereafter. Never mind when.*
>
> W. R. Hamilton 1859

> *...there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into living communication with contemporary developments...*
>
> H. G. Grassmann 1861

1 Instead of an Introduction

The NOTICES OF THE AMS, VOL. 48 (2001), NUMBER 4, 411-412 announced the solution of a problem in fluid mechanics that had been open and actively investigated for a hundred years.

> *The Ruth Lyttle Satter Prize in Mathematics is awarded to* Sijue Wu *for her work on a long-standing problem in the water wave equation, in particular for the results in her papers (1) Well- posedness in Sobolev spaces of the full water wave problem in 2-D, Invent. Math. 130 (1997), 39-72; and (2) Well-posedness in Sobolev spaces of the full water wave problem in 3-D, J. Amer. Math. Soc. 12, no. 2 (1999), 445-495. By applying tools from harmonic analysis* (singular integrals and Clifford algebra), *she proves that the Taylor sign condition always holds and that there exists a unique solution to the water wave equations for a finite time interval when*

the initial wave profile is a Jordan surface.

The recognition of this outstanding contribution has been confirmed on the occasion of the Second International Congress of Chinese Mathematicians in Taipei in December, 2001. Sijue Wu was awarded the 2001 Morningside Silver Medal in Mathematics for her work on water wave problems.
An abstract from the Technical Report Server of the *Johnson Space Center* reveals in 1995 that

> *...To compensate for the required iteration methodology, all reference frame change definitions and calculations are performed with quaternions. Quaternion algebra significantly reduces the computational time required for the accurate determination of shadow terminator points.*
>
> [C. R. Ortiz Longo and St. L. Rickman, M*ethod for the Calculation of Spacecraft Umbra and Penumbra Shadow Terminator Points*]

In September 2002 the UNDERGRADUATE STUDENT MANUAL of the Department of Mechanical Engineering and Applied Mechanics at the University of Pennsylvania mentioned :

> 528. Advanced Kinematics. *(M) Prerequisites: Multivariate calculus, introductory abstract algebra, mathematical maturity.*
> *Differential geometry, Lie groups and rigid body kinematics; Lie algebra, screws, quaternions and dual number algebra; geometry of curves and ruled surfaces; trajectory generation and motion planning; applications will be to robotics and spatial mechanisms.*

In a typical abstract of a contribution on quaternions to a recent conference in mathematics and its applications (GAMM) the following remarks can be found:

> *... we consider applications of quaternions (a subcase of the more general Clifford Algebras) to different problems of orientation in engineering. It is shown that many problems from different engineering areas such as computer vision, robotics, navigation, photogrammetry, etc. have the same mathematical background and can be formulated as quaternion optimization problems. ... Thus, by using quaternions we have en elegant mathematical method for solutions of many complicated problems in different areas of engineering. The quaternion approach allows to clarify the essence of problems and to simplify numerical calculations.*

An important indicator for the dynamic in the field is the publication of books. Some examples from the last 6 years are

- Applications of Geometric Algebra in Computer Science and Engineering (Dorst, L., Doran, Ch., Lasenby, J. (eds.), Birkhäuser, 2002)[11]
- Geometric Algebra with Applications in Science and Engineering (Corrochano, E. B. and G. Sobczyk (eds.) Birkhäuser, 2001)[6]

- Geometric Computing with Clifford Algebras Theoretical Foundations and Applications in Computer Vision and Robotics (Sommer, G. (ed.), Springer, 2001)[41]
- Clifford Algebras and their Applications in Mathematical Physics, Vol. 1 Algebra and Physics (Abłamowicz, R., Fauser, B. (eds.)) Vol. 2 Clifford Analysis, (Ryan, J., Sprößig, W.(eds.) Birkhäuser, 2000)[1]
- Electrodynamics: A Modern Geometric Approach, (Baylis W. E., Birkhäuser, 1999)[3]
- Quaternions and Rotations Sequences, (Kuipers, J. B., Princeton Univ. Press, 1998)[34]
- Quaternionic and Clifford Calculus for Physicists and Engineers (Gürlebeck, K.,Sprößig, W., John Wiley &. Sons, 1997)[20]

To complete this list we still quote a notice on the history of the Mathematics Department at Union College (USA):

> *Gillespie not only stressed the importance of mathematics in engineering, he began a tradition of emphasizing the humanities for the engineers as well; this is a tradition that the College has wisely preserved to this day, to its enormous benefit. The Catalogue in 1882-83 listed, among many other things, a course in quaternions, which at the time was right at the frontier of mathematical research, and involved some of the biggest names in American mathematics (for example, Sylvester on one side and J. Willard Gibbs of Yale on the other).*

The aforementioned facts speak for themselves. As 120 years ago quaternions and their generalizations are present in recent mathematical research, in applications as well as in teaching. Surely, they are not an esoteric topic for pure mathematicians, but rather a mathematical tool for engineering. Nevertheless, is it not strange that a theory which was created for applications 160 years ago could stay (with only very few exceptions) almost dormant for about 120 years?

2 Clifford Analysis and Geometric Algebras - Quaternions on the Right Track

Hamilton believed that he was "on the right track". But it is a historical fact that in the very beginning the true nature of quaternions was misunderstood, also by Hamilton, and the consequences were "Missed opportunities" (F. Dyson,[12]) which caused a delay of about 40 years in what concerns their application in Physics. In pure mathematics the progress was never completely stopped, but mainly going on in Number Theory and Abstract Algebra with controversial academic discussions about *the right* understanding of quaternions. In engineering the vector, tensor and matrix calculus dominate everywhere, at least since the beginning of the 20th century.

From the historical point of view it is very difficult to do justice to the pioneers of that time, but it seems to us that two events marked significantly some changes. One was the newly started research on hypercomplex function theory by Iftimie [32](in the tradition of works done by his romanian co-patriots Moisil and Toedorescu in 1931 [38]) and by Delanghe [8](following the line of Fueter's school in Zurich between 1930 and 1950). But whereas the motivation of Fueter [13]came from number theoretic problems, Delanghe and his soon growing Gent school on *Clifford Analysis* concentrated on problems arising from harmonic analysis, coming much closer to significant applications in Physics. The other event was initiated by the physicist D. Hestenes who had worked for NASA and published *"Space-time algebra"*[26] and *"Multivector calculus"* [27] in 1966 resp. 1968. His work was mainly concerned with physical problems expressed in the language of Geometry. Following Clifford's original idea he coined the term *"Geometric Algebra"* for his approach to the use of Clifford algebras in applications. *Clifford Analysis* and *Geometric Algebras* are nowadays synonyms for the two main ways of dealing with Clifford algebras: the first with emphasis in the analytic theory as generalization of Complex Analysis in a wide sense, including modelling and numerical methods, the second one with emphasis in geometric-algebraic models of very diversified physical or technical problems.

Finally, it was less than 20 years ago, in 1985, that the first international conference on "Clifford Algebras and Their Applications in Mathematical Physics" took place in Kent, Canterbury, U.K., joining mathematicians, physicist and engineers. Clifford Analysis and Geometric Algebras met each other on that occasion. D. Hestenes [28]noticed in his address to the conference:

> *This first international workshop on Clifford Algebras testifies to an increasing awareness of the importance of Clifford Algebras in mathematics and physics. Nevertheless, Clifford Algebra is still regarded as a narrow mathematical speciality , and few mathematicians or physicists are likely to characterize Clifford Algebra as merely the algebra of a quadratic form, while the physicists are likely to regard it as a specialized matrix algebra.*
> *...*
> *The fact that Clifford Algebra keeps popping up in different places throughout mathematics and physics shows that it has a universal significance transcending narrow specialities.*

Meanwhile a lot of disciplines in science and engineering got increasingly interested and contribute intensively to the spectra of applications of quaternions and their generalizations. We will mention several of them but, of course, are not able to draw a complete picture. The books we mentioned in the preceding section can serve as a guide for getting some overview about recent developments.

To get closer to the general perspective and the role of quaternions as a mathematical concept in theory and applications we will review some basic algebraic relations and few analytic methods in more detail. We begin to recall some questions that arose in the very beginning of the creation of quaternions which needed a long time to be understood and re-interpreted.

3 Hamilton - his Discovery and his Obsession

Though the early history of quaternions is more or less well known, several basic facts are best explained in their historical context, particularly their *origin as a tool*, not as a theoretical concept. The discovery of quaternions by the Royal Astronomer of Ireland, William Rowan Hamilton [22], on the 16th of October 1843 was motivated by the hope to create a type of *hypercomplex* numbers related to the three dimensional space of our visual intuition like complex numbers are related to the plane. At that time, one of the most striking facts about the complex numbers was their geometric visualization (Wessel, Argand, Gauss) and the discovery that the simple algebraic operation of multiplication could be interpreted in terms of rotations in the plan. It became an intensely studied question whether one could discover other number systems which would model rotations in three dimensional space. Hamilton succeeded to go an important (now seemingly trivial) step forward to the solution. His formal definition of complex numbers as ordered pairs of real numbers (1835) suggested to him the idea to attack the problem as an algebraic one for ordered triplets (α, β, γ) of real numbers combined to $z = \alpha + \beta i + \gamma j$ with $i^2 = j^2 = -1$. But it took him almost ten years to understand that without a fourth dimension and without dropping the commutativity of multiplication no such system exists. Indeed, in a letter to J. Graves [23](17th of October 1843) he wrote:

> *...I made therefore* $ij = k, ji = -k$, *reserving to myself to inquire whether* k *was 0 or not...*

and also

> *...and there dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triplets...*

After having arrived to the basis representation of his quaternions as expressions of the form

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \;\; q_m \in \mathbb{R}, \; m = \overline{0,3} \text{ with } \; \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \quad (1)$$

Hamilton showed very carefully that besides commutativity all other properties that characterize quaternions as a number system (in particular the associativity of multiplication) are fulfilled. Since the inverse of a nonzero quaternion q is given by

$$q^{-1} = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{-1}(q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}) = |q|^{-2}\bar{q} \quad (2)$$

Hamilton [25] constructed with the set $\mathbb{H}$ of all quaternions a real finite-dimensional *associative division algebra* unknown so far, which later was recognized by Frobenius (1877) as the only one besides $\mathbb{R}$ and $\mathbb{C}$. In fact, his discovery heralded the golden age of algebra when instead of algebraic equations the study of algebraic structures became the main concern. But it seems that Hamilton's concern was not this essential contribution to modern algebra.

In fact, Hamilton was already at that time a recognized physicist working in geometric optics on extremal principles and successfully extending these ideas to dynamics in 1834/35. With the introduction of the principle of least action, the Hamilton function and his canonical equations of motion, he had already reached the Hall of Fame, having been knighted in 1835 and elected for President of the Royal Irish Academy in 1837. This explains why his prevailing attitude towards quaternions was their applicability, their geometric (physical) nature, not their algebraic properties. From the very beginning of his trials he was obsessed by the quixotic idea that quaternions would play a key role in physics, being on a par with the creation of the infinitesimal calculus. Consequently, for the last twenty years of his life, Hamilton concentrated all his power on the study of quaternions.

Note: It should be mentioned that, at first sight, the quaternion formalism might seem awkward to a physicist or engineer, since the square of the unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are negative; besides, invoking a fourth dimension beyond our ability of visualization for treating 3-dimensional problems also looks rather strange. Indeed, many of the obstacles in gaining a firm reputation for easy applicability of quaternions were (and still are) caused by these "strange" properties. Hamilton's own struggle seems to show that he was aware of it. But the man who succeeded to explain the secrets of the new number system with relations to dimension three and four was a man of a new generation, born after 1843. His name was William Kingdom Clifford (1845-1879).

4 Quaternions and Vector Calculus

The most natural example of an elementary tool that is widely used in geometry, physics and any technical science, without showing directly its relationship to quaternions, is vector calculus. Nevertheless, and almost forgotten, vector algebra as well as vector analysis have their origin in the theory of quaternions, i.e. historically quaternions were the first.

It is very curious, may be in some sense ironic and even of philosophical significance, that the appearance of quaternions also marked the beginning of modern vector analysis, which later on prevailed over the use of quaternions. The fact that Hamilton started with triplets and ended up with quaternions implied immediately his concern about the special role of $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathrm{Im}\mathbb{H}$ which he called *vector* in the sense of the similar term *radius vector* that had been used for many years before. Hamilton wrote:

> *Regarded from a geometrical point of view, this algebraically imaginary part of a quaternion has thus so natural and simple a signification or representation in space, that the difficulty is transferred to the algebraically real part; and we tempted to ask what this last can denote in geometry, or what in space might have suggested it.*

For the algebraically real part he introduced the word scalar part or simply the scalar of q and prefixed both components of q by V resp. S, i.e. $q = Sq + Vq$. The use of S and V applied to the product of two quaternions being "vectors" $\alpha = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\alpha' = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$ then leads to

$$S\alpha\alpha' = -(xx'+yy'+zz') \text{ resp. } Vaa' = \mathbf{i}(yz'-zy')+\mathbf{j}(zx'-xz')+\mathbf{k}(xy-'yx'). \quad (3)$$

But formula (3) shows that the quaternion product is the sum of the modern vector (cross) product and the negative of the modern scalar (dot) product of α and α'. The present formulation of vector algebra comes from this quaternion product of triplets/vectors in the sense of Hamilton, extracted by Gibbs 1881-84 and first published in 1901.

With the introduction of the Nabla-operator ∇ in the form

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{4}$$

in 1846/1847, Hamilton also invented the other essential technical ingredient for the vector calculus: the vector differential operator which is used to describe the gradient of a scalar function as well as the divergence and the curl of a vector valued function.

Long time after Hamilton the vector calculus found in Gibbs and Heaviside their most prominent promoters. Remarkable and surely not fair, that both strongly tried to deny that inheritance as it is described in Crowe's book on the History of Vector Analysis [7]. Perhaps because of the objective of this book (expressed by its title) the role of another, probably the most important, mathematician in this story, W. K. Clifford, is only described in there as a "transition figure". In fact, Clifford's insights in the relationship between Hamilton's quaternions and Grassmann's "algebra of extensions" are the key to a fair appreciation of Hamilton's work as well as the milestone for the progress in this field of algebra.

But before we discuss the role of Clifford we have to mention the use of quaternions for describing rotations (one of the most important procedures in all technical applications). Hamilton soon found out how the desired description of rotations in $\mathbb{R}^3$ can be related to a vector (nowadays also called "pure quaternion") $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. He first wrote a rotation $\mathbf{a} \mapsto \mathbf{a_1}$ in the form $\mathbf{a_1} = u\mathbf{a}$ where $u \in \mathbb{H}$, $|u| = 1$, is a unit quaternion. But in such a rotation, the imaginary part $\mathbf{u}$ of u (which describes the axis of the rotation) had to be perpendicular to the vector $\mathbf{a}$. Altmann extensively discussed this insufficiency in Hamilton's early approach to rotations in ([2]). Of course, this was the form that Hamilton expected from the complex multiplication and it is also the form of the matrix representation of rotations in $\mathbb{R}^3$. Indeed, using the 3×3-rotation matrix U corresponding to a rotation of φ about the axis given by $\mathbf{u}^T$ there holds $\mathbf{a}_1 = U\mathbf{a}$.

But the concept of a matrix was also motivated by quaternions: matrix algebra was introduced by Cayley only in 1858. Cayley, seriously concerned with Hamilton's creation of quaternions, published in 1845 the right formula for rotations represented by quaternions (he assigns the priority to Hamilton):

Every rotation (properly orthogonal mapping) of $\mathbf{a} \in \mathbb{R}^3$ has the form

$$a \mapsto u\mathbf{a}u^{-1}, \text{ where } u = \cos\frac{\varphi}{2} + \frac{\mathbf{u}}{\varphi}\sin\frac{\varphi}{2} \tag{5}$$

runs through all nonzero quaternions and $\varphi = |u|$.

Hamilton's particular problem was to explain the appearance of the half of the rotation angle in this general rotation formula. According to (2) this means that a rotation of $\mathbf{a} \in \mathbb{R}^3$ is realized by multiplication with a unit quaternion from the left and its inverse from the right (almost like rotations in the complex plane, except that there commutativity allows the multiplication with $\exp(i\varphi)$ from one side only). Notice that two unit quaternions u and $-u$ represent the same rotation and that the composition of two rotations in $\mathbb{R}^3$ in general is not commutative (this, of course, has its true counterpart in the non-commutativity of the quaternionic multiplication). In modern group theoretic terms (cf. the next section) this means that unit quaternions are elements of the double covering group $\mathbf{Spin}(3)$ of the rotation group $SO(3)$ in $\mathbb{R}^3$. Finally we should notice that later on, in 1855 Cayley remarked that every properly orthogonal mapping of $\mathbb{R}^4 \cong \mathbb{H}$ has the form

$$q \mapsto aqb$$

where a, b independently of each other run through all unit quaternions.

5 Clifford's Geometric Algebra

Now we will see how a deeper theoretical background on the algebra of quaternions allows to benefit more from their properties. Thanks to Clifford [5] this approach started soon after Hamilton, and seems to be still in progress. The variety of Clifford algebra representations is very big and a lot of different approaches exist. Specially in theoretical physics, the use of Clifford algebras is meanwhile abundant and the number of different possibilities of writing Maxwell's equations by using Clifford algebras is very high. The same concerns the equations of motion of the spinning electron in quantum mechanics, first obtained by Dirac when trying to linearize the Klein-Gordon equation [10].

Clifford has been considered as one of the four mathematicians that very soon understood the ideas of Grassmann, after their publication in the confusing work "Die lineale Ausdehnungslehre - ein neuer Zweig der Mathematik" [18]. He succeeded to combine in his "Geometric Algebra" the ideas of both, Hamilton and Grassmann. Indeed, in 1844 the exterior algebra $\bigwedge \mathbb{R}^3$ of the linear space $\mathbb{R}^3$ was constructed by Grassmann with the basis

$$\begin{array}{lll}
1 & & \\
e_1, & e_2, & e_3, \\
e_1 \wedge e_2, & e_1 \wedge e_3, & e_2 \wedge e_3 \\
e_1 \wedge e_2 \wedge e_3 & &
\end{array}$$

satisfying the multiplication rules

$$\begin{array}{ll}
(i) & e_i \wedge e_j = -e_j \wedge e_i \;\text{ for }\; i \neq j, \\
(ii) & e_i \wedge e_i = 0.
\end{array} \tag{6}$$

The Grassmann exterior algebra has no inner product and does not require a metric. Therefore Clifford, motivated by Hamilton's ideas, introduced in 1878 a new product where metric relations for products of vectors (realized by areas, volumes etc.) also have place. He kept the first multiplication rule and replaced the second by

$$
\begin{aligned}
e_i e_j &= -e_i e_j, \\
e_i e_i &= -1.
\end{aligned}
\tag{7}
$$

In the concrete case of $\mathbb{R}^2$ this led him to the basis

1		the scalar
$e_1,$	$e_2,$	vectors
$e_1e_2,$		bivector or area element

of the real associative 4-dimensional Clifford algebra $\mathcal{C}\ell_{0,2}$. The dimension grading introduces a multi-vector structure and the multiplication rules show that $\mathcal{C}\ell_{0,2} \cong \mathbb{H}$, i.e. Clifford obtained an algebra which is isomorphic to Hamilton's quaternions with the basis $\{1, e_1 = \mathbf{i}, e_2 = \mathbf{j}, e_1e_2 = \mathbf{k}\}$. But from the geometric point of view this approach was not satisfactory, because the nature of the basis elements $\{1, e_1 = \mathbf{i}, e_2 = \mathbf{j}, e_1e_2 = \mathbf{k}\}$ seems different (2 vectors and 1 bivector) and the square of an element of the underlying vector space $\mathbb{R}^2$ remains negative. Clifford found only later the way out to an algebra which fulfills all demands. His work was published posthumously in 1882. Therefore he changed the multiplication rules to

$$
\begin{aligned}
e_i e_j &= -e_i e_j, \\
e_i e_i &= +1.
\end{aligned}
\tag{8}
$$

In the concrete case of $\mathbb{R}^3$ this led him to the basis

1			the scalar
$e_1,$	$e_2,$	$e_3,$	vectors
$e_1e_2,$	$e_1e_3,$	e_2e_3	bivectors
$e_1e_2e_3$			trivector or volume element.

Here the quaternions can be recognized as isomorphic to the *even subalgebra* $\mathcal{C}\ell_{3,0}^{+}$ constituted by *scalars and bivectors* (visualized by oriented areas) of the real associative 8-dimensional Clifford algebra $\mathcal{C}\ell_{3,0}$ according to the following correspondences:

$$
\mathbf{i} \to -e_2e_3, \quad \mathbf{j} \to -e_3e_1, \quad \mathbf{k} \to -e_1e_2
$$

Indeed, for example $\mathbf{i}^2 = (-e_2e_3)^2 = -e_2e_3e_3e_2 = -e_2^2e_3^2 = -1$ and $\mathbf{ij} = (-e_2e_3)(-e_3e_1) = -e_2e_1 = \mathbf{k}$ as well as $\mathbf{ijk} = -e_2e_3e_3e_1e_1e_2 = -1$.

In this case all basis elements $\mathbf{i}, \mathbf{j}, \mathbf{k}$ have an interpretation as basic bivectors. Therefore the set of pure quaternions $\mathrm{Im}\mathbb{H}$ as the set of real linear combinations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ geometrically represents the set of bivectors and *not* the set of vectors in the sense of radius vectors like Hamilton had supposed. This explains also, why Hamilton met problems when trying to explain the appearance of the half of the rotation angle in the general formula of rotation (see [2]).

The embedding of $\mathbb{H}$ in the even subalgebra $\mathcal{C}\ell_{3,0}^{+}$ of the Clifford algebra $\mathcal{C}\ell_{3,0}$ is also of relevance to the relation between the quaternionic product of pure quaternions and the dot- and cross-products of vector analysis. Indeed, it follows from

(3) that

$$\begin{aligned}(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}) &= -(a_1b_1+a_2b_2+a_3b_3)+(a_2b_3-a_3b_2)\mathbf{i}\\ &\quad +(a_3b_1-a_1b_3)\mathbf{j}++(a_1b_2-a_2b_2)\mathbf{k} \qquad (9)\\ &= -(\mathbf{a}\cdot\mathbf{b})+(\mathbf{a}\times\mathbf{b})\end{aligned}$$

but the Clifford product between two arbitrary vectors $\mathbf{a},\mathbf{b}\in\mathcal{C}\ell_{3,0}\cong\mathbb{H}$ is obtained as

$$\begin{aligned}&\mathbf{ab}=(a_1e_1+a_2e_2+a_3e_3)(b_1e_1+b_2e_2+b_3e_3)=\\ &(a_1b_1+a_2b_2+a_3b_3)+(a_2b_3-a_3b_2)e_2e_3+(a_3b_1-a_1b_3)e_3e_1+(a_1b_2-a_2b_2)e_1e_2\\ &\quad=(\mathbf{a}\cdot\mathbf{b})+(\mathbf{a}\wedge\mathbf{b})\\ &\quad=(\mathbf{a}\cdot\mathbf{b})+(\mathbf{a}\times\mathbf{b})e_1e_2e_3\end{aligned}$$

The last formula shows that

1. The Clifford product of two vectors belongs to $\mathcal{C}\ell_{3,0}$ and is a sum of the usual dot-product (scalar product) of vector analysis, as its scalar part, and the corresponding cross-product (vector-product) times the so called pseudo-scalar $e_1e_2e_3$, as the bivector part.

2. The Clifford product of two vectors from $\mathcal{C}\ell_{3,0}$ is the sum of an inner (scalar-valued) and an outer product (bivector-valued). Since the inner product commutes and the outer product anti-commutes both can be derived from the Clifford product in the form

$$(\mathbf{a}\cdot\mathbf{b})=\frac{1}{2}(\mathbf{ab}+\mathbf{ba})\text{resp.}(\mathbf{a}\wedge\mathbf{b})=\frac{1}{2}(\mathbf{ab}-\mathbf{ba})$$

Grassmann and Clifford belong to first group of mathematicians who overcame the barriers of the 3-dimensional space of our intuition. Clifford introduces his new algebras not only in $\mathbb{R}^3$ but in $\mathbb{R}^n$. In the modern treatment of Clifford algebras it is usual to permit that some of the elements of the orthonormal basis $e_1, e_2, \ldots e_n$ of $\mathbb{R}^n=\mathbb{R}^{p+q}$ have positive squares whereas other have negative squares, corresponding to the consideration of Euclidean or Pseudo-Euclidean spaces (like the Minkowski space $\mathbb{R}^{1+3}$). The multiplication rules

$$e_ie_j=-e_ie_j,\ \text{ where }\ e_ie_j=e_i\wedge e_j \qquad (10)$$

$$e_ie_i=1,\quad i=1,2,\ldots p \qquad (11)$$

$$e_je_j=-1,\quad j=1,2,\ldots q,\ \ p+q=n,$$

again produce a real 2^n -dimensional basis of the corresponding Clifford algebra $\mathcal{C}\ell_{p,q}$ given by

$$\begin{array}{llll}1 & & & \\ e_1, & e_2, & \cdots & e_n,\\ e_1e_2, & e_1e_3, & \cdots & e_{n-1}e_n,\\ e_1e_2e_3, & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ e_1e_2e_3\cdots e_n. & & & \end{array}$$

Note: It has been shown that Clifford himself studied already two different types of algebras which allow a geometric interpretation that goes behind the usual vector concept and makes Grassmann's extension ideas (in form of the outer product and the concept of bivectors) manifest. In the beginning of this section we also mentioned that in applications several variants of Clifford algebras are in use. The most accepted for practical purpose in the 3-dimensional Euclidean space $\mathcal{E}_3$ seems to be a variant that was proposed almost 40 years ago by D. Hestenes [28]. He called it simply the *"Geometric Algebra"* and based it on $\mathcal{G}_3 := \mathcal{C}\ell_{3,0}$ with the distinguished element $i = e_1 e_2 e_3 = e_1 \wedge e_2 \wedge e_3$ which is called *pseudoscalar*.

Given a vector $\mathbf{x}$ in $\mathcal{G}_3$, he defined

$$\mathcal{E}_3 := \{\mathbf{x} : \mathbf{x} \wedge i = 0\}.$$

The even subalgebra of scalars and bivectors $\mathcal{G}_3^+$ is called *spinor algebra* and through the isomorphism given by the correspondences

$$e_2 e_3 \to \mathbf{i}, \quad e_3 e_1 \to \mathbf{j}, \quad e_1 e_2 \to \mathbf{k}$$

(notice that in this case $\mathbf{ijk} \cong e_2 e_3 e_3 e_1 e_1 e_2 = 1$) the Quaternion algebra is interpreted as a Spinor algebra. This is motivated by the spinor representation theory of the rotation groups $SO(n)$ which first systematically has been developed in 1913 by Elie Cartan.

It is worth noticing that Freeman J. Dyson [12], who delivered in 1972 the famous annual J.W.Gibbs Lectures (Bull. AMS, Vol. 78 (1972), 635-652) discussed , among other themes, under the title "Missed Opportunities", also the lost time when quaternionists and anti-quaternionists were fighting against each other. He said:

> *Gibbs had not really succeded in unifying the notions of quaternion and vector. On the contrary, by putting the two notions side by side he had made explicit the lack of any real compatibility between them. His lecture* On multiple algebra*[17] ought to have suggested to any attentive mathematician the question,*
> *"How can it happen that the properties of three-dimensional space are represented equally well by two quite different and incompatible algebraic structures?"*
> *If this question had once been clearly asked, the answer would almost certainly have been forthcoming. And the answer would have led inevitably to a complete theory of the single valued and double-valued representations of the three-dimensional rotation group.The vectors are the simplest nontrivial single-valued representation, and the quaternions are the simplest double-valued representation. Also, the quaternions are the prototype of what later were called spinor representations. The development of spinor representations, which was actually begun by Elie Cartan in 1913 and completed during the 1930's with substantial help from the physicists Pauli and Dirac, might have been accelerated by approximately 40 years. It is impossible to say what effects such an accelerated development would have had on other branches of pure mathematics, but the effects could hardly have failed to be substantial.*

W. Pauli (1927) and P.A.M. Dirac rediscovered quaternions by their matrix representations in $Mat(2, \mathbb{C}) \cong \mathcal{C}\ell_{3,0}$ resp. $Mat(4, \mathbb{C}) \cong \mathbb{C} \otimes \mathcal{C}\ell_{1,3}$. Later on the mathematicians R. Brauer and H. Weyl (1935) as well as C. Chevalley (1954) contributed essentially to this important field of research in quantum theory and specialists in Physics and Clifford algebras are still very active in this field [1].

6 Trends in Geometric Algebra

Since about 40 years, the field of Geometric Algebra has been increasing and broadening itself in such a way that it is no longer possible to make a simple classification or analysis of its trends, even of the main trends regarding the concrete objects of research and application. It seems obvious that the natural and profound relationship to practical computer sciences, in the sense of computational geometry, turned out to be the main impulse for the development of new applications of Geometric Algebras. But also the re-interpretation of classical theories, like for instance the work of Hestenes on "Point groups and space groups" [29] which implies a better understanding of molecular modelling and crystallography still plays an important role. In this sense Hestenes' remarks at the first international conference on "Clifford Algebras and Their Applications in Mathematical Physics" should be remembered. He noticed that several different systems which provide similar geometric concepts, need to have a common language. Without trying to reduce or to question the advances and the usefulness of disciplines like

Vector Analysis	Tensor Analysis
Matrix Algebra	Clifford Algebra
Differential Forms	Coordinate Geometry
Synthetic Geometry	Grassmann Algebra
Spinor Calculus	Multivector Algebra,

all together constitute a set of highly redundant theories which sometimes need enormous technical skills. It seems to be rather evident that the coordinate-free tools provided by Geometric Algebra could serve in some sense as a unifying language for several different mathematical languages that are used in different fields of application. Further success will be the guarantee for progress in this direction.

What concerns modern trends we should mention the following. It is well known that the use of quaternions for instance in the fields of
- aircraft orientation,
- spacecraft stabilization,

and other areas of relevance for defense were a well hidden secret during the cold war on both sides of the iron curtain.

Meanwhile one can say that mathematical models based on quaternions have been very much appreciated in all high technologies with need of calculations in real time.

Software development for video games, for which quaternions allow efficient computations with minimal storage for smooth rotations of solids is a field of enjoyment which seems to increase without any limitations.

Another examples are the advances in
- robotics,
- computer vision,
- virtual reality,
- calculations in crystallography,
- electrical engineering,
- quantum information processing by nuclear magnetic resonance,
- neural computing etc.

where meanwhile the use of quaternions or derived algebraic structures has become indispensable.

7 New Horizons - Clifford Analysis

Not more than 20 years have passed since the book [4]of Brackx, Delanghe and Sommen "Clifford Analysis" from 1982 coined the name of a new discipline. In Kluwer's "Encyclopedia of Mathematics" from 1997 one reads

> *Clifford analysis studies functions with values in a Clifford algebra, and, as such, is a direct generalization to higher dimensions of the classical theory of functions of one complex variable .... It has its roots in quaternionic analysis, which was developed from the 1920s onwards as an, albeit modest, counterpart of the theory of two complex variables. The latter was to evolve into the vast and rich theory of several complex variables. The former gained renewed interest in the 1950s and led to hypercomplex function theory (cf. also Hypercomplex functions), renamed Clifford analysis in the 1980s, when it grew into an autonomous discipline.*

Indeed, almost at the same time when Pauli and Dirac studied Schrödinger's and Klein-Gordon's equations with the help of quaternion algebras the quaternions appeared also in the focus of a new type of research. It is not so well known that the inspiration for this came from a number theoretic problem.

7.1 Fueter's approach

A great part of number theoretic research on the quaternion algebra as associative division algebra had already been realized by Hurwitz [31]. The same time, in 1920, an important problem in the main stream of number theory, influenced by Hilbert's twelfth problem, had just been solved by the Japanese mathematician Tagaki. It was related to the class formula for Abelian number fields over an imaginary quadratic field and the so-called *problem of complex multiplication.* The Swiss mathematician Rudolf Fueter (1880-1950), as a former student of Hilbert [30], was interested in the generalization of this problem to the quaternionic case. Since the main tools in this field of analytic number theory are results from complex function theory he tried (since 1928) to develop a *hypercomplex function theory* by considering quaternion valued functions $f = f(z)$ of a quaternion variable $z \in \mathbb{H}$ [13, 14, 16]. (In the last decade of his life he also considered the general case of $\mathcal{C}\ell_{0,n}$-valued functions (cf. [15]).

Due to the fact that a quaternion $z \in \mathbb{H}$ can be represented by a pair of two complex variables $z_1 = x_0 + x_1 i$ and $z_2 = x_2 + x_3 i$ in the form

$$z = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = z_1 + z_2\mathbf{j}.$$

his work was also a contribution to the very intensive discussion in the 1920th about the "right" way for generalizing holomorphic function to higher dimensions. Of course, the algebraic properties of the quaternion algebra lead to another generalization than the consideration of a complex valued functions of two or more complex variables. In this sense, the challenge for Fueter to develop quaternionic analysis (or hypercomplex function theory as he called it) was twofold.

The fact that the quaternion form a division algebra promised that it simply could be sufficient to follow a generalization of Cauchy's approach to holomorphic complex functions by demanding the existence of a quaternionic differential quotient as the limit of the form

$$\lim_{\Delta z \to 0} \Big(f(z + \Delta z) - f(z)\Big)\Big(\Delta z\Big)^{-1}. \tag{12}$$

or

$$\lim_{\Delta z \to 0} \Big(\Delta z\Big)^{-1}\Big(f(z + \Delta z) - f(z)\Big). \tag{13}$$

But in fact, limits of the expressions (12) or (13) independent from the direction of convergence, exist only for *right* resp. *left linear functions* of a quaternionic variable. This has rigorously been proven by several author's independently from Fueter's research and until many years later, even until the 1990's! But Fueter was aware of a paper of Scheffers [40] where such difficulties had been announced (not well proved), and used another approach for defining generalized holomorphic functions (which he called regular).

Following Riemann's approach in the complex case, he defined quaternion valued regular functions $f(z) = f_0(z) + f_1(z)\mathbf{i} + f_2(z)\mathbf{j} + f_3(z)\mathbf{k}$ as belonging to the kernel of the quaternionic Cauchy-Riemann operator

$$\mathcal{D} := \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}\mathbf{i} + \frac{\partial}{\partial x_2}\mathbf{j} + \frac{\partial}{\partial x_3}\mathbf{k}. \tag{14}$$

This means that f is called a *regular from the right* function if $f\mathcal{D} = 0$ resp. a *regular from the left* function if $\mathcal{D}f = 0$.

Nowadays in Clifford Analysis regular functions are called *monogenic* (sometimes also hypercomplex holomorphic, Clifford holomorphic etc.).

The far reaching analogy between complex function theory and quaternionic function theory can be illustrated by some of the basic properties of monogenic functions.

Indeed, using a quaternion valued surface-element of the form

$$dZ := dx_1dx_2dx_3 - dx_0dx_2dx_3\mathbf{i} + dx_0dx_1dx_3\mathbf{j} - dx_0dx_1dx_2\mathbf{k}$$

and $\omega = f(z)dZg(z)$ with $f, g \in \mathcal{C}^1(\Omega)$ he noticed that it is possible to derive the quaternionic form of Stokes' formula over a 4-dimensional positively oriented domain Ω as

$$\int_{\partial\Omega} f(z)d\sigma g(z) = \int_{\Omega} (f\mathcal{D}g + f\mathcal{D}g)dV \tag{15}$$

where dV stands for the volume element $dV := dx_0 dx_1 dx_2 dx_3$.

This hypercomplex form of Stokes' formula in $\mathbb{R}^4$ immediately suggests the

Generalized Cauchy theorem
Let f be a function monogenic from the right (i.e. $f\mathcal{D} = 0$) and g be a function monogenic from the left (i.e. $\mathcal{D}g = 0$) in the 4-dimensional positively oriented domain with boundary Ω than

$$\int_{\partial\Omega} f(z)d\sigma g(z) = 0. \tag{16}$$

Other important properties of monogenic functions follow like in the complex plane from the fact that the Laplace operator can be factorized, namely as $\mathcal{D}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{D} = \Delta_4$, where

$$\overline{\mathcal{D}} := \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}\mathbf{i} - \frac{\partial}{\partial x_2}\mathbf{j} - \frac{\partial}{\partial x_3}\mathbf{k}$$

denotes the conjugated Cauchy-Riemann operator.

This shows that monogenic functions and their components are *harmonic functions*. Furthermore, the real analyticity of harmonic functions together with special functions methods (Legendre polynomials or Gegenbauer polynomials) can be used for defining *generalized power series*. A generalization of the concept of the *areolar derivative in the sense of Pompeiu,*[39] which relies on measure theoretic relations between higher-dimensional volume and surface-integrals, enables to show that $\frac{1}{2}\overline{\mathcal{D}}f$ can be considered as the generalized hypercomplex derivative of the monogenic function f (see [19]). This answered the question about the existence of a generalized Cauchy-approach as mentioned in the beginning of this section. In fact, it is also a hint to the fact that hypercomplex analysis can be considered as function theory in co-dimension 1.
Examples:
Using the shorthand notations $e_1 := \mathbf{i}, e_2 := \mathbf{j}, e_3 := \mathbf{k}$ there holds

1. ([35]) $f_k(z) = z_k := x_k - x_0 e_k = -\frac{1}{2}[ze_k + e_k z]$, $(k = 1, \ldots, n)$ are right- and left-regular (*totally regular variables*). Identifying $e_k \cong i$ they are 3 copies of the complex variable z multiplied by $-e_k$, i.e. $z_k \in \mathbb{C}_k := -e_k\mathbb{C}$.

2. The identity $f(z) = z$ is not monogenic since $\mathcal{D}f = f\mathcal{D} = -2$. Powers of z, i.e. $f(z) = z^n$ as well as simple products like $z_j \cdot z_k$, $j \neq k$ are also not monogenic.

3. [36] The symmetrized products $\frac{1}{2}[z_j \cdot z_k + z_k \cdot z_j] = x_j x_k - x_0 x_k e_j - x_0 x_j e_k$ are right- and left-monogenic. The same is true for m-fold symmetric products which are homogeneous of degree m and can serve as a complete basis in questions of interpolation or approximation.

4. If ω_3 stands for the area of the unit sphere S^3 in $\mathbb{R}^4$ then

$$E(z,\zeta) = \frac{1}{\omega_3} \frac{\overline{z-\zeta}}{|z-\zeta|^4}$$

generalizes the Cauchy kernel

$$C(z,\zeta) = \frac{1}{2\pi i} \frac{1}{z-\zeta}$$

in the plane, more precisely $E(z,\zeta)$ is the right (resp. left) fundamental solution (in the distributional sense) of the operator $\mathcal{D}$ in the unit ball $B(1,\zeta) = \{z : |z-\zeta| \leq 1\}$. With other words, applying (15) with f or g chosen equal to $E(z,\zeta)$ a generalized Cauchy integral formula of the form

$$f(z) = \int_{\partial\Omega} f(z) d\sigma E(z,\zeta) \ \text{ resp. } \ g(z) = \int_{\partial\Omega} E(z,\zeta) d\sigma g(z)$$

is valid in Ω for monogenic from the right (resp. left) functions $f, g \in \mathcal{C}\overline{\Omega}$.

5. Monogenic functions $f = f(z) = \sum_0^3 f_k(x) e_k$ with $f_0 = 0$ and $\frac{\partial f_k}{\partial x_0} = 0, k = 1,2,3$ are monogenic from the left as well as from the right and describe an irrotational fluid flow without sources nor sinks. This follows from the fact that, in this case, the corresponding generalized Cauchy-Riemann systems are equal, i.e. $\mathcal{D}f = f\mathcal{D} = 0$ and at the same time equivalent to the vector system:

$$\begin{aligned} \operatorname{div} \overrightarrow{\mathbf{f}} &= 0 \\ \operatorname{curl} \overrightarrow{\mathbf{f}} &= 0. \end{aligned} \tag{17}$$

where we identified in am obvious manner $\operatorname{Im} f = \mathbf{f} \cong \overrightarrow{\mathbf{f}}^T$.

7.2 The Moisil-Teodorescu approach

Independently from Fueter, but also motivated by the idea to create a spatial holomorphic function theory two Romanian mathematicians, G. C. Moisil and N. Teodorescu, published in 1931 a paper [38] where they used a traceless matrix differential operator of the form

$$\mathbf{D} = \begin{pmatrix} 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix}. \tag{18}$$

Applying a matrix form of Stokes' theorem in $\mathbb{R}^3$ they studied the system

$$\mathbf{D}\overrightarrow{\mathbf{f}} = 0 \ \text{ where } \ \overrightarrow{\mathbf{f}}^T = (f_0, f_1, f_2, f_3) \tag{19}$$

which can also be written in the form

$$\begin{aligned} \operatorname{div} \overrightarrow{\mathbf{f}} &= 0 \\ \operatorname{grad} f_0 + \operatorname{curl} \overrightarrow{\mathbf{f}} &= 0. \end{aligned} \tag{20}$$

Comparing this with system (17) and its relation to monogenic functions one can see that Moisil e Teodorescu studied a less restrictive case (f_0 need not to vanish) of the equation $\mathcal{D}f = 0, \;\; f : \mathbb{H} \to \mathbb{H}$.

Besides its historical value, this contribution it also has one feature which became important later on as a general tool in Clifford Analysis: the fact that the considered differential operator is *not* the generalized Cauchy-Riemann operator $\mathcal{D}$, but the Dirac operator

$$D = \frac{\partial f}{\partial x_1}\mathbf{i} + \frac{\partial f}{\partial x_2}\mathbf{j} + \frac{\partial f}{\partial x_3}\mathbf{k}. \tag{21}$$

It follows immediately that the Laplace operator in $\mathbb{R}^3$ can now be factorized in the simple form $D\overline{D} = \overline{D}D = -D^2\Delta_3$ which implies several advantages in the applications to physical problems, where, from the viewpoint of spatial symmetry, the choice of a distinguished variable like x_0 is not motivated. Obviously, the use of the hypercomplex derivative in form of $\mathcal{D}f$ also becomes senseless. The Dirac operator is the central differential operator in fluid dynamics and in the theory of heat conduction, but it also plays an important role in the description of the electromagnetic field: Maxwell's equations rely on the Dirac operator. But the application from which the Dirac operator derives its name is quantum mechanics, as it was mentioned before. In non-relativistic quantum mechanics usually the Dirac operator on a three dimensional space is used as defined in (21). In relativistic mechanics however one uses the Dirac operator on the Minkowski space $\mathbb{R}^{1+3}$. This operator is no longer a linearization of the Laplacian $\Delta = \nabla^2$, but of the d'Alembertian , the operator of the wave equation.

From the mathematical point of view the use of the Dirac operator seems to be particularly adequate for refinements in harmonic analysis and all problems were only one differential operator is needed.

7.3 Trends in applied Clifford Analysis

About 75 years ago, in the beginning of Quaternionic analysis, the intention dominated to create general methods for solving problems in other fields, for instance in number theory and partial differential equations applied to special problems in Mathematical Physics. Problems in dimensions higher than two have been the main motivation for developing function theoretic tools in algebras more general than the algebra of complex numbers. Soon it became obvious that the combination with the algebra of quaternions or, more general, with Clifford algebras permits to develop a whole theory, but only at the end of the 1960th started a systematical research in this field.

In 1982, the book of Brackx, Delanghe and Sommen [4] marked the first period of advances in theoretical research whereas the book of Gürlebeck and Sprössig [20] already systematically described applications for solving linear and nonlinear boundary value problems of the most important partial differential equations of mathematical physics (Laplace and Helmholtz equations, equations of linear elasticity, Maxwell equations, Navier-Stokes equations and others). The authors studied questions of existence, uniqueness, regularity, and general representation of their solutions in a unified form. Furthermore they introduced and developed new boundary collocation methods as well as represented a discrete model of the quaternionic function theory for constructing finite difference methods.

In 1992 Delanghe, Sommen and Souček [9] studied applications in quantum physics, particularly in form of a function theory of the Dirac operator.

M. Mitrea [37] published in 1994 Lecture Notes dealing with Clifford wavelets, thereby showing that this modern branch of applied mathematics can also take profit from tools naturally adapted to higher dimensions.

Developing further ideas of [33] the application of integral representations for spatial models of mathematical physics (e.g. general Helmholtz equation, electrodynamical models and massive spinor fields, including the MIT bag model in the theory of quarks) was considered in 1996 in the book of Kravchenko and Shapiro [33].

An almost complete picture of the state of the art in theory and applications of Clifford analysis up to 1997 is contained in [21] which we mentioned already in the first section.

Without being able to overview all recent trends in the vast research on applications of Clifford analysis it seems to us that those aforementioned "traditional" fields, mainly related to mathematical physics, are continuing and undergoing a refinement. New results about Clifford-Hermite wavelets in Euclidean Spaces or Cauchy Transforms on Rectifiable Surfaces which will be presented at this IKM are two examples of such a development. Several other announced contributions join this trend. Unfortunately it is not possible to mention all of them explicitly.

Of more theoretical nature are contributions in Clifford Analysis on Projective Hyperbolic Space, but they are also aiming to future applications. It is our opinion that they are the expression of an undergoing reinforcement in the fundamentals of Clifford Analysis, which besides all progress are still not exhaustively constructed. This concerns also topics in classical complex function theory which until now have not yet been the target of Clifford Analysis research. As an example we refer to the fact that a derivative concept is one of the main classical function theoretic concepts for the qualitative and quantitative characterization of functions. Generalizations in approximation theory, the characterization and solution of mapping problems on higher dimensional manifolds or the qualitative investigation on scales of function spaces are some of the fields where the hypercomplex derivative mentioned in section 7.1 has proved to be useful.

After 20 years of remarkable dynamics and success in several areas of theoretic and applied research, Clifford Analysis will continue to approve its importance as a combination of algebra, geometry and analysis, particularly adapted to higher dimensions.

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