Archiv der Mathematik

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On behaviour of holomorphically contractible systems under non-monotonic sequences of sets

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Abstract. The new results concerning the continuity of holomorphically contractible systems treated as set functions with respect to non-monotonic sequences of sets are given. In particular, continuity properties of Kobayashi and Carathéodory pseudodistances, as well as Lempert and Green functions with respect to sequences of domains converging in the Hausdorff metric are delivered.

Mathematics Subject Classification. Primary 32F45; Secondary 32H02.

Keywords. Invariant pseudodistances, Kobayashi pseudodistance, Carathéodory pseudodistance, Pluricomplex Green function, Hausdorff distance.

1. Introduction. It is known that both Carathéodory and Kobayashi pseudodistances depend continuously on increasing and decreasing sequences of domains (in the latter case, adding some regularity assumptions on the limiting domain; cf. [3] and references therein). The pseudodistances mentioned above are particular examples of a wider class of holomorphically contractible systems, i.e. systems of functions

$$\boldsymbol{d}_D \colon D \times D \to [0, +\infty),$$

D running through all domains in all $\mathbb{C}^n$'s, such that $\boldsymbol{d}_{\mathbb{D}}$ is forced to be $\boldsymbol{p}$, the hyperbolic distance on $\mathbb{D}$, the unit disc on the plane and all holomorphic mappings are contractions with respect to the system $(\boldsymbol{d}_D)$ (cf. Definition 2.1). The question about the behaviour of holomorphically contractible systems under not necessarily monotonic sequences of sets seems to be natural and important. In the present note, inspired by [1], we shall give a very general result stating the continuity of holomorphically contractible systems under the sequences of domains convergent with respect to the Hausdorff distance (for

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two nonempty bounded sets A, B it is defined as

$$\mathfrak{H}(A, B) := \inf\{\delta > 0 \colon A \subset B^{(\delta)} \text{ and } B \subset A^{(\delta)}\},$$

where for a set S and a positive number ε, the set $S^{(\varepsilon)} := \bigcup_{s \in S} \mathbb{B}(s, \varepsilon)$ is the ε-envelope of S; $\mathbb{B}(x, r)$ denotes the open Euclidean ball of center x and radius r). Namely, our main result reads as follows:

Theorem 1.1. *Let $(\boldsymbol{d}_D)$ be a holomorphically contractible system, and let $D \subset \mathbb{C}^m$ be a bounded domain. Assume that there exist two sequences $(I_n)_{n \in \mathbb{N}}$, $(E_n)_{n \in \mathbb{N}}$ of domains such that*

$$E_{n+1} \subset\subset E_n, n \in \mathbb{N}, \bigcap_{n \in \mathbb{N}} E_n = \overline{D}, I_n \subset\subset I_{n+1}, n \in \mathbb{N}, \bigcup_{n \in \mathbb{N}} I_n = D$$

and such that for each $z, w \in D$ there is

$$\lim_{n \to \infty} \boldsymbol{d}_{E_n}(z, w) = \lim_{n \to \infty} \boldsymbol{d}_{I_n}(z, w) = \boldsymbol{d}_D(z, w).$$

Let $(D_n)_{n \in \mathbb{N}}$ be a sequence of bounded domains in $\mathbb{C}^m$ such that

$$\lim_{n \to \infty} \mathfrak{H}(\overline{D}_n, \overline{D}) = 0$$

and with the property that for each compact $K \subset D$ there exists an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0, K \subset D_n$. Then for any $z, w \in D$

$$\lim_{n \to \infty} \boldsymbol{d}_{D_n}(z, w) = \boldsymbol{d}_D(z, w).$$

Definition 1.2. In what follows, we shall say that the sequence of bounded domains $(D_n)_{n \in \mathbb{N}}$ *has the property* $\mathfrak{C}$ *with respect to* D if it satisfies the assumptions of Theorem 1.1, i.e. if

$$\lim_{n \to \infty} \mathfrak{H}(\overline{D}_n, \overline{D}) = 0$$

and if for each compact $K \subset D$ there exists an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0, K \subset D_n$.

In particular, we get the results in this spirit for Carathéodory and Kobayashi pseudodistances as well as for Green and Lempert functions (cf. Corollaries 2.3, 2.4, and 2.6). We believe they are interesting in their own right.

In [1] all the results are settled in the context of complex Banach spaces, yet under a strong assumption about the convexity of the approximating domains together with the limiting one. Our results are free from this restrictive assumption.

In Section 2 we give the formal definition of a holomorphically contractible system and both list and prove the corollaries from Theorem 1.1 while the proof of the main result itself comes in Section 3.

In what follows, $\mathcal{O}(D, G)$ stands for the family of all holomorphic mappings between open sets D, G and $\mathcal{PSH}(D)$ abbreviates the family of all plurisubharmonic functions on an open set D.

2. Holomorphically contractible systems. Let us start with the precise definition of holomorphically contractible system.

Definition 2.1. (*Cf.* [2, Section 4.1]) A family $(\boldsymbol{d}_D)$ of functions

$$\boldsymbol{d}_D \colon D \times D \to [0, +\infty),$$

where D runs over all domains in $\mathbb{C}^n$ with arbitrary n, is called a *holomorphically contractible system* if the following two conditions are satisfied:

1. $\boldsymbol{d}_{\mathbb{D}} = \boldsymbol{p}$
2. for any two domains $D \subset \mathbb{C}^n, G \subset \mathbb{C}^m$ and any mapping $f \in \mathcal{O}(D, G)$ there is

$$\boldsymbol{d}_G(f(z), f(w)) \leq \boldsymbol{d}_D(z, w), \quad z, w \in D.$$

Remark 2.2. If in the above definition we replace $\boldsymbol{p}$ by $\boldsymbol{m}$, the Möbius distance on $\mathbb{D}$, then we speak of a $\boldsymbol{m}$-*contractible system*. This distinction is however somewhat artificial, since having $(\boldsymbol{d}_D)$, a holomorphically contractible system, we may define $\boldsymbol{d}_D^* := \tanh \boldsymbol{d}_D$ and then the operator sending $(\boldsymbol{d}_D)$ to $(\boldsymbol{d}_D^*)$ is a bijection between the class of contractible systems and the class of $\boldsymbol{m}$-contractible systems (see [2, Section 4.1]).

The most important examples of holomorphically contractible systems are the following:

1. *Carathéodory pseudodistance*:

   $$\boldsymbol{c}_D(z, w) := \sup\{\boldsymbol{p}(0, f(w)) \colon f \in \mathcal{O}(D, \mathbb{D}), f(z) = 0\}, \quad z, w \in D.$$

2. *Lempert function*:

   $\boldsymbol{l}_D(z, w)$
   $:= \inf\{\boldsymbol{p}(0, \lambda) \colon \lambda \in \mathbb{D} \colon \text{there exists a } \varphi \in \mathcal{O}(\mathbb{D}, D) \colon \varphi(0) = z, \varphi(\lambda) = w\},$
   $z, w \in D.$

3. *Kobayashi pseudodistance*:

   $\boldsymbol{k}_D(z, w)$
   $$:= \inf\left\{\sum_{j=1}^{N} \boldsymbol{l}_D(z_{j-1}, z_j) \colon N \in \mathbb{N}, z_1, \ldots, z_N \in D, z_0 = z, z_N = w\right\},$$
   $z, w \in D.$

4. *Green function*:

   $$\boldsymbol{g}_D(z, w) := \sup\{u(z) \colon u \colon D \to [0, 1) \colon \log u \in \mathcal{PSH}(D), \text{ there exist } M, r > 0$$
   $$: u(z) \leq M\|\zeta - w\|, \zeta \in \mathbb{B}(w, r) \subset D\}$$

   forms an example of $\boldsymbol{m}$-contractible system.

Note that $(\boldsymbol{c}_D)$ and $(\boldsymbol{k}_D)$ are extremal holomorphically contractible systems of pseudodistances, i.e. if $(\boldsymbol{d}_D)$ is any holomorphically contractible system of pseudodistances, it verifies the inequalities

$$\boldsymbol{c}_D \leq \boldsymbol{d}_D \leq \boldsymbol{k}_D$$

for all domains D. Similarly, if $(\boldsymbol{d}_D)$ is any holomorphically contractible system of functions, then

$$\boldsymbol{c}_D \leq \boldsymbol{d}_D \leq \boldsymbol{l}_D$$

for all domains D (see [2, Section 4.1]).

Having Theorem 1.1 we may settle the continuity results for particular objects.

Corollary 2.3. *Let $D \subset \mathbb{C}^m$ be a bounded taut domain with boundary of class $\mathcal{C}^{1,1}$. Let $(D_n)_{n \in \mathbb{N}}$ be a sequence with the property $\mathfrak{C}$ with respect to D. Then for any $z, w \in D$*

$$\lim_{n \to \infty} \boldsymbol{k}_{D_n}(z, w) = \boldsymbol{k}_D(z, w)$$

as well as

$$\lim_{n \to \infty} \boldsymbol{l}_{D_n}(z, w) = \boldsymbol{l}_D(z, w)$$

Proof. Indeed, in virtue of the regularity assumption, one can take

$$E_n = D^{\left(\frac{1}{N_0 + n}\right)} := \left\{ z \in \mathbb{C}^m : \operatorname{dist}(z, D) < \frac{1}{N_0 + n} \right\}$$

and

$$I_n = D^{\left(-\frac{1}{N_0 + n}\right)} := \left\{ z \in D : \operatorname{dist}(z, \partial D) > \frac{1}{N_0 + n} \right\}$$

with $N_0 \in \mathbb{N}$ large enough and make use of the continuity of the Kobayashi pseudodistance and the Lempert function with respect to monotonic sequences of domains (see [3] and references therein). $\square$

Corollary 2.4. *Let $D \subset \mathbb{C}^m$ be a bounded strictly pseudoconvex domain. Let $(D_n)_{n \in \mathbb{N}}$ be a sequence with the property $\mathfrak{C}$ with respect to D. Then for any $z, w \in D$*

$$\lim_{n \to \infty} \boldsymbol{c}_{D_n}(z, w) = \boldsymbol{c}_D(z, w).$$

Proof. The proof goes along the same lines as the proof of Corollary 2.3. $\square$

In the case of Green functions, things become a little bit more complicated. Let us see the details.

Definition 2.5. Let $D \subset \mathbb{C}^m$ be a bounded domain.

1. D is *hyperconvex* if there exists a continuous and negative plurisubharmonic exhaustive function on D.
2. D is *strictly hyperconvex* if there exist a bounded domain Ω and a continuous function $\rho \in \mathcal{PSH}(\Omega)$ with values in $(-\infty, 1)$ such that $D = \{z \in \Omega : \rho(z) < 0\}$, ρ is exhaustive for Ω, and the sublevel sets $\{z \in \Omega : \rho(z) < \alpha\}$ are connected for $\alpha \in [0, 1]$.

One can observe that strictly hyperconvex domain is a hyperconvex domain with negative continuous exhaustive function that can be plurisubharmonically and continuously extended to some open neighbourhood of the closure of the domain. The examples of such domains are bounded strictly pseudoconvex domains.

Corollary 2.6. *Let $D \subset \mathbb{C}^m$ be a strictly hyperconvex domain. Let ρ be as in Definition* 2.5. *Assume that D^k is a hyperconvex domain given by $\{z \in \Omega \colon \rho(z) < \frac{1}{k}\}, k \in \mathbb{N}$. Let $(D_n)_{n \in \mathbb{N}}$ be a sequence with the property $\mathfrak{C}$ with respect to D. Then for any $z, w \in D$*

$$\lim_{n \to \infty} \boldsymbol{g}_{D_n}(z, w) = \boldsymbol{g}_D(z, w).$$

Proof. By [4] we know that the Green function is continuous with respect to increasing sequensces of domains. Therefore, $(I_n)_{n \in \mathbb{N}}$ may be chosen as some exhausting sequence of smoothly bounded strictly pseudoconvex relatively compact open subsets of D. Also, using results of [5], it is clear that the good candidate for the "exterior" sequence is $(E_n)_{n \in \mathbb{N}} := (D^n)_{n \in \mathbb{N}}$. $\square$

3. Proof of Theorem 1.1.

Proof of Theorem 1.1. There exists an $m_1 \in \mathbb{N}$ such that for $m \geq m_1$ we have

$$I_1 \subset\subset D_m \subset\subset E_1.$$

We may choose the smallest possible such m_1. In what follows, we shall construct two sequences of sets, $(L_n)_{n \in \mathbb{N}}, (U_n)_{n \in \mathbb{N}}$, such that $L_n \subset L_{n+1}, n \in \mathbb{N}, \bigcup_{n=1}^{\infty} L_n = D, U_{n+1} \subset\subset U_n, n \in \mathbb{N}, \bigcap_{n=1}^{\infty} U_n = \overline{D}$ and

$$L_n \subset D_{m_1+n-1} \subset U_n, n \in \mathbb{N}.$$

Then for n large enough, $z, w \in L_n$ and

$$\boldsymbol{d}_{U_n}(z, w) \leq \boldsymbol{d}_{D_{m_1+n-1}}(z, w) \leq \boldsymbol{d}_{L_n}(z, w).$$

Finally, letting $n \to \infty$ and using the assumptions concerning continuity of the system $(\boldsymbol{d}_D)$ with respect to monotonic sequences of domains $(I_n)_{n \in \mathbb{N}}, (E_n)_{n \in \mathbb{N}}$, we reach the conclusion of Theorem 1.1. Let us pass to the construction. Let $L_1 := I_1, U_1 := E_1$. We proceed as follows:

Choose the smallest $m_2 \in \mathbb{N}$ such that for any $m \geq m_2$ we have

$$I_2 \subset\subset D_m \subset\subset E_2.$$

There are two cases to be considered:
Case 1. $m_2 \in \{m_1, m_1 + 1\}$. Then

$$I_2 \subset\subset D_{m_2} \subset\subset E_2$$

and we put $L_2 := I_2, U_2 := E_2$.
Case 2. $m_2 = m_1 + s$ with some $s \geq 2$. Then

$$I_1 \subset\subset D_l \subset\subset E_1, l = m_1, \dots, m_1 + s,$$

and so

$$I_1 \subset\subset \bigcup_{l=m_1+1}^{m_1+s-1} D_l \subset\subset E_1.$$

We define $L_2 = \cdots = L_s := I_1, L_{s+1} := I_2$. Further, as U_2 we choose a domain relatively compact in E_1, containing in its interior $\overline{E_2 \cup \bigcup_{l=m_1+1}^{m_1+s-1} D_l}$. Inductively, for $k = 2, \ldots, s$, a domain U_k is chosen as a domain relatively compact in U_{k-1}, containing in its interior $\overline{E_2 \cup \bigcup_{l=m_1+k-1}^{m_1+s-1} D_l}$. Finally, we put $U_{s+1} := E_2$.

Suppose we have constructed domains $L_1 \subset \cdots \subset L_r$ and $U_1 \subset\subset \cdots \subset\subset U_r$ such that
$$L_j \subset D_{m_1+j-1} \subset U_j, j = 1, \ldots, r$$
and $L_r = I_M, U_r = E_M, m_1 + r - 1 = m_M$ with some $M \in \mathbb{N}$. We choose the smallest $m_{M+1} \in \mathbb{N}$ with
$$I_{M+1} \subset\subset D_m \subset\subset E_{M+1}, m \geq m_{M+1}.$$
Similarly as before, there are two cases to be considered:

Case 1. $m_{M+1} \in \{m_M, m_M + 1\}$. Then we put $L_{r+1} := I_{M+1}, U_{r+1} := E_{M+1}$.
Case 2. $m_{M+1} = m_M + s$ with some $s \geq 2$. Then we mimic the previously presented construction with necessary modifications.

$\square$

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