ELECTRONIC RESEARCH ANNOUNCEMENTS
IN MATHEMATICAL SCIENCES
Volume 22, Pages 103–108 (December 30, 2015)
S 1935-9179 AIMS (2015)

doi:10.3934/era.2015.22.103

ON INTEGRAL ZARISKI DECOMPOSITIONS OF PSEUDOEFFECTIVE DIVISORS ON ALGEBRAIC SURFACES

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(Communicated by Igor Dolgachev)

Abstract. In this note we consider the problem of integrality of Zariski decompositions for pseudoeffective integral divisors on algebraic surfaces. We show that while sometimes integrality of Zariski decompositions forces all negative curves to be (-1)-curves, there are examples where this is not true.

1. Introduction

In this note we work over an arbitrary algebraically closed field K, unless otherwise specified. By a negative curve, we mean a reduced irreducible divisor C with $C^2 < 0$ on a smooth projective surface. By a $(-k)$-curve, we mean a negative curve C with $C^2 = -k < 0$.

There has been a recent resurgence of interest in the so-called local negativity for reduced curves on algebraic surfaces. One of the most important and intriguing conjectures around negativity questions is the Bounded Negativity Conjecture.

Conjecture 1.1 (BNC)**.** *Let X be a smooth projective surface over a field of characteristic 0. Then there exists an integer $b(X) \in \mathbb{Z}$ such that for all* reduced *curves $C \subset X$ one has $C^2 \geqslant b(X)$.*

Counterexamples are known in positive characteristics, but they are very special. In particular, none are known for rational surfaces, so even in positive characteristics the question of which surfaces have bounded negativity is of interest. Moreover, bounded negativity has connections to substantial open conjectures. For example, for a surface X obtained by blowing up $\mathbb{P}^2$ at any finite set of generic points, the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture (i.e., the SHGH Conjecture) [6] asserts that $h^1(X, \mathcal{O}_X(F)) = 0$ for every effective nef divisor F and in addition that all negative curves on X are (-1)-curves. The Bounded Negativity Conjecture (BNC) is another, even older, still open conjecture which asserts that smooth complex projective surfaces all have bounded negativity. It is also an open question

Received by the editors August 24, 2015 and, in revised form, November 27, 2015.
2010 *Mathematics Subject Classification.* 14C20, 14M25.
Key words and phrases. Zariski decomposition, K3 surfaces, blow ups, projective plane.
The second author would like to thank Roberto Laface for conversations on the topic of this paper. The second author is partially supported by National Science Centre Poland Grant 2014/15/N/ST1/02102 and SFB 45 "Periods, moduli spaces and arithmetic of algebraic varieties," and the third author is supported by National Science Centre Poland Grant 2014/15/B/ST1/02197. We also thank the referee for his careful reading of the paper.

in any characteristic whether all smooth projective rational surfaces have bounded negativity.

Recent work has established a connection of bounded negativity with a certain property of Zariski decompositions. Indeed, in [1] the second author with Th. Bauer and D. Schmitz studied the following question for algebraic surfaces. (Recall that on a surface X, a divisor D is pseudoeffective if $D \cdot B \geqslant 0$ for nef divisors B.)

Question. Let X be a smooth projective surface. Does there exist an integer $d(X) \geqslant 1$ such that for every pseudoeffective integral divisor D the denominators in the Zariski decomposition of D are bounded from above by $d(X)$?

Such a question is natural when one studies Zariski decompositions [8] of pseudoeffective divisors since we have the following geometric interpretation. Given a pseudoeffective integral divisor D on X with Zariski decomposition $D = P + N$, for every sufficiently divisible integer $m \geqslant 1$ we have the equality

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X, \mathcal{O}_X(mP)),$$

and hence $|mD| = |mP| + mN$. Sufficiently divisible is required in order to clear denominators in P and obtain Cartier divisors.

If such a bound $d(X)$ exists, then we say that X has *bounded Zariski denominators*. It is an intriguing question as to whether a given smooth surface satisfies this boundedness condition. Somewhat surprisingly, boundedness of Zariski denominators on a smooth projective surface X is equivalent to X having bounded negativity [1]. The equivalence of boundedness of Zariski denominators and bounded negativity provides a new perspective on these conjectures and also sheds some light on links between numerical information about divisors on a given surface X and the possible negative curves on X.

Let us say that a pseudoeffective integral divisor D has an *integral Zariski decomposition* $D = P + N$ if P and N are defined over the integers (i.e., all coefficients occurring in P, N are integers). An interesting criterion for surfaces to have bounded Zariski denominators was given in [1] as follows:

Proposition 1.2. *Let X be a smooth projective surface such that for every reduced and irreducible curve C one has $C^2 \geqslant -1$. Then all integral pseudoeffective divisors on X have integral Zariski decompositions.*

This raises the converse question:

Question 1.3. Let X be a smooth projective surface having the property that

every integral pseudoeffective divisor D has an integral Zariski decomposition. $(*)$

Is every negative curve then a (-1)-curve?

The condition $(*)$ at first glance seems to be very restrictive, so it is plausible that Question 1.3 could have an affirmative answer. However, by our main result we see that the answer is negative.

Theorem A. *There exists a smooth complex projective surface X having the property that all integral pseudoeffective divisors have integral Zariski decompositions yet all negative curves on X have self-intersection -2.*

On the other hand, sometimes the answer is affirmative:

Theorem B. *Let X be a smooth projective surface such that every integral pseudo-effective divisor D has an integral Zariski decomposition (i.e., $d(X) = 1$) and such that $|\Delta(X)| = 1$, where $\Delta(X)$ is the determinant of the intersection form on the Néron-Severi lattice of X. Then all negative curves on X are (-1)-curves, that is, $b(X) = 1$.*

This follows from [1, Thm. 2.3], which gives the bound $b(X) \leqslant d(X)\, d(X)!\, |\Delta(X)|$. Thus, for example, if X is a blow up of $\mathbb{P}^2$ at a finite set of points, then $|\Delta(X)| = 1$, so if Zariski decompositions are integral on X, then $d(X)$ is also 1 and hence $b(X) = 1$. Because of the recent interest in blow ups of $\mathbb{P}^2$ at finite sets of points (see, for example, [3, 2, 4]), a direct proof in the special case of blow ups of $\mathbb{P}^2$ may be useful. We provide such a proof below in Theorem C.

2. Results

Before we present the main result of this note, let us recall the definition of Zariski decompositions.

Definition 2.1 (Fujita-Zariski decomposition [5, 8])**.** Let X be a smooth projective surface and D a pseudoeffective integral divisor on X. Then D can be written uniquely as a sum

$$D = P + N$$

of $\mathbb{Q}$-divisors such that

(i) P is nef,
(ii) N is effective with negative definite intersection matrix if $N \neq 0$, and
(iii) $P \cdot C = 0$ for every component C of N.

Now we are ready to produce the surface whose existence is asserted in Theorem A.

Theorem 2.2. *There exists a smooth complex $K3$ surface X of Picard number 2 having intersection form*

$$\begin{pmatrix} -2 & 4 \\ 4 & -2 \end{pmatrix}$$

such that all integral pseudoeffective divisors on X have integral Zariski decompositions.

Proof. The existence of such a surface X is a consequence of [7]. Indeed, by [7, Theorem 1.1], one can find a smooth hypersurface X of degree 4 in $\mathbb{P}^3_{\mathbb{C}}$ containing a smooth curve C_1 of degree 2 (hence C_1 is contained in a hyperplane section H) and such that $\mathrm{Pic}(X) = \mathbb{Z}H + \mathbb{Z}C_1$. By adjunction we have $C_1^2 = -2$. Since $H = C_1 + C_2$, where C_2 is also a plane conic, we have $C_1 \cdot C_2 = 4$ and $\mathrm{Pic}(X) = \mathbb{Z}C_1 + \mathbb{Z}C_2$. If C_2 were reducible, it would consist of two lines, say $C_2 = L_1 + L_2$, so $L_1 = mC_1 + nC_2$ and hence $1 = L_1 \cdot H = (mC_1 + nC_2) \cdot C_1$. But the latter is even, and therefore this is impossible. Thus C_2 is also smooth and irreducible with $C_2^2 = -2$. So, up to numerical equivalence, every prime divisor D (and hence every effective divisor) is of the form $mC_1 + nC_2$ with $m, n \geqslant 0$ (since any divisor of the form $mC_1 + nC_2$ with $m < 0$ or $n < 0$ will meet either C_2 or C_1, respectively, negatively and thus cannot be the class of a prime divisor other than C_1 or C_2). In particular, a divisor D is effective if and only if it is pseudoeffective, and no negative curve can meet both C_1 and C_2 nonnegatively (i.e., C_1 and C_2 are the only negative curves). Also, every nef divisor is in the cone dual to the effective cone; i.e., every nef divisor is a rational nonnegative linear combination of C_1+2C_2 and $2C_1+C_2$. It is not hard to check that therefore an integral divisor D is nef if and only if it is a nonnegative integer linear combination of C_1+C_2, $2C_1+C_2$ and C_1+2C_2.

Say D is pseudoeffective and integral; i.e., $D = mC_1 + nC_2$ for $m, n \geqslant 0$. By symmetry, it is enough to assume that $m \geqslant n$. If $2n \geqslant m$, then D is nef and the Zariski decomposition $D = P + N$ of D is integral since $P = D$ and $N = 0$. Now assume $m > 2n$. Take $P = n(2C_1 + C_2)$ and $N = (m-2n)C_1$. Then P is nef, N clearly has negative definite intersection matrix, and $P \cdot N = 0$, so $D = P + N$ is again an integral Zariski decomposition of D. □

Now we provide a proof of Theorem B in the special case mentioned above.

Theorem C. *Let $\pi : X \to \mathbb{P}^2$ be the blow up (over an algebraically closed ground field K of arbitrary characteristic) of a finite set of points $p_1, \ldots, p_s$ (possibly infinitely near). Suppose that every integral pseudoeffective divisor D has an integral Zariski decomposition. Then all negative curves on X have self-intersection -1, i.e., are (-1)-curves.*

Proof. Denote $\pi^{-1}(p_i)$ by E_i and the total transform of a line by H. We will consider two cases: in the first case we assume that none of the points p_i is infinitely near any other (so the points p_i are distinct points of $\mathbb{P}^2$), and in the second case we assume that some point is infinitely near another. To accommodate this second case, we define X_1 to be the blow up of $X_0 = \mathbb{P}^2$ at any point $p_1 \in X_0$; X_2 the blow up of X_1 at any point $p_2 \in X_1$, so p_2 can be infinitely near to p_1; etc. Continuing in this way we eventually have that $X = X_s$ is the blow up of X_{s-1} at any point $p_s \in X_{s-1}$. In order to avoid confusion, we indicate the exceptional curve for the blow up of $p_i \in X_{i-1}$ by $E_{i,i} \subset X_i$ and its total transform on X_j for $j > i$ by $E_{i,j}$. For simplicity, we denote $E_{i,s}$ by E_i. Thus $E_{i,i}$ is always irreducible, and $E_{i,j}$ is irreducible if and only if no point p_ℓ for $i < \ell \leqslant j$ is infinitely near to p_i.

We begin with case 1. Suppose to the contrary that X has a $(-k)$-curve C with $k > 1$. First note that $C \cdot H > 0$. (If $C \cdot H = 0$, then, since C is a prime divisor and none of the points are infinitely near, we would have $C = E_i$ for some i and hence $C^2 = -1$.) Now, since the classes of $H, E_1, \ldots, E_s$ give a basis for the divisor class group of X, we can (up to linear equivalence) write $C = dH - \sum_{j=1}^{r} b_{i_j} E_{i_j}$, where $b_{i_j} > 0$ and the sum is over the $r \leqslant s$ exceptional curves E_i with $C \cdot E_i > 0$. In particular, $C^2 = d^2 - \sum_{j=1}^{r} b_{i_j}^2 = -k$.

We claim that there is a big integral divisor D such that $D = P + aC$ with $a \in \mathbb{Q} \setminus \mathbb{Z}$. To see this, note that if $d' \gg 0$ and $a_i > 0$ are integers, then

$$A = d'H - \sum_{i=1}^{s} a_i E_i$$

will be an integral ample divisor. (We pause to justify this. We may assume that $d' > \sum_i a_i$. Note that H and $H - E_i$ are nef with $H^2 > 0$, hence $A^2 > 0$. Now assume G is some effective, nonzero divisor. Since H is nef, we have $G \cdot H \geqslant 0$. If $G \cdot H = 0$, then, since the E_i are effective prime divisors of negative self-intersection, we have $G = \sum_i g_i E_i$ for integers $g_i \geqslant 0$ with some $g_j > 0$. Thus $G \cdot (-E_i) \geqslant 0$ for all i and $G \cdot (-E_j) > 0$, hence $G \cdot A \geqslant G \cdot (-E_j) > 0$. So assume $G \cdot H > 0$. Since $B = \sum_i a_i(H - E_i)$ is nef, we have $G \cdot A = G \cdot ((d' - \sum_i a_i)H + B) \geqslant G \cdot (d' - \sum_i a_i)H > 0$. Thus $A \cdot G > 0$ for all effective divisors G, hence A is ample.)

For D, we take $D = A + eC$ for $e \in \mathbb{Z}$, $e > 0$, where we choose a number e such that $D \cdot C < 0$ and so that we have

$$0 > D \cdot C = (A + eC) \cdot C = A \cdot C - ek = d'd - \sum_{j=1}^{r} a_{i_j} b_{i_j} - ke.$$

Finding the Zariski decomposition of D boils down to computing a. Observe that

$$a = \frac{dd' - \sum_{j=1}^{r} a_{i_j} b_{i_j} - ke}{-k} = e + \frac{\sum_{j=1}^{r} a_{i_j} b_{i_j} - dd'}{k}.$$

We just need to show that k does not divide $\sum_j a_{i_j} b_{i_j} - dd'$.

Suppose that k divides $\sum_j a_{i_j} b_{i_j} - dd'$. Then we replace A by $A + H$ so d' becomes $d' + 1$. If k does not divide $\sum_j a_{i_j} b_{i_j} - dd' - d$, then we are done. If k divides this new number, then it means that $k \mid d$. In this case we replace A instead by $A + H - E_{i_1}$. Since $H - E_{i_1}$ is nef, $A + H - E_{i_1}$ is ample, and we get $d' + 1$ in place of d' and $a_{i_1} + 1$ in place of a_{i_1}. If k does not divide the number $\sum_j a_{i_j} b_{i_j} + b_{i_1} - d(d'+1)$, then we are done. If k divides this number, then it means that $k \mid b_{i_1}$. We proceed along the same lines for all j; we are done unless $k \mid b_{i_j}$ for all j. But this is impossible because then k would divide d and each b_{i_j}, so k^2 would divide $k = -(d^2 - \sum_j b_{i_j}^2)$, where k is an integer bigger than 1.

Now consider the case in which one of the points $p_1, \dots, p_s$ is infinitely near another. Let p_j be the first such point, and let p_i be the point p_j is infinitely near to. Thus $p_1, \dots, p_{j-1}$ are points of $\mathbb{P}^2$ and p_j is on the exceptional locus $E_{i,j-1}$ of p_i for some $i < j$. After reindexing, we may assume that $i = 1$ and $j = 2$. (The only constraint on reindexing is that if p_v is infinitely near to p_u, then $v > u$.) On X_2, the curve $E_{1,2}$ has two irreducible components, $E_{2,2}$ and $E = E_{1,2} - E_{2,2}$, so $E_{1,2} = E + E_{2,2}$. Thus we have $E^2 = -2$.

Here we take $D = 3H - E_{1,2} - 2E_{2,2}$ so up to linear equivalence we can write $2D = 6H - 2E_{1,2} - 4E_{2,2} = 6(H - E_{1,2}) + 4E = (6(H - E_{1,2}) + 3E) + E$ and $D = 2(H - E_{1,2} - E_{2,2}) + H + E_{1,2}$. Since all three terms of the latter are the classes of effective divisors, D is pseudoeffective. Since $H - E_{1,2}$ is nef and $(6(H - E_{1,2}) + 3E) \cdot E = 0$, we see that $D = P + N$ for $P = (6(H - E_{1,2}) + 3E)/2$ and $N = E/2$ is a non-integral Zariski decomposition on X_2.

However, π factors as $\pi : X \xrightarrow{\pi_2} X_2 \to \mathbb{P}^2$, where π_2 is the sequential blow up of the points $p_3, \dots, p_s$. Take the pull-back

$$2\pi_2^*(D) = \pi_2^*(2D) = \pi_2^*(2P + 2N) = \pi_2^*(2P) + \pi_2^*(2N)$$

and note that this is a Zariski decomposition. Indeed, observe that the pull-back of a nef divisor is nef, so $\pi_2^*(2P)$ is nef and $\pi_2^*(2P) \cdot \pi_2^*(2N) = 4P \cdot N = 0$. It only remains to show that the intersection matrix of $\pi_2^*(2N)$ is negative definite and that $\pi_2^*(2N)/2$ is not integral. However, up to linear equivalence, $N = E = E_{1,2} - E_{2,2}$, and the total transforms of $E_{1,2}$ and $E_{2,2}$ under π_2 are in the span (in the divisor class group) of $E_1, \dots, E_s$. (In the same way that the components of the total transform of $E_{1,1}$ under $X_2 \to X_1$ are $E_{2,2}$ and $E = E_{1,2} - E_{2,2}$, every component of $E_i = E_{i,s}$ on $X = X_s$ is a linear integer combination of $E_i = E_{i,s}, \dots, E_s = E_{s,s}$.)

Since $\pi_2^*(2N)$ is in the span of $E_i, \dots, E_s$ and this span is negative definite, the intersection matrix for the components of $\pi_2^*(2N)$ is also negative definite. Since $((\pi_2^*(2N))/2)^2 = N^2 = (E/2)^2 = -1/2$, we see that $\pi_2^*(2N))/2$ is not integral.

Thus $\pi_2^*(D) = \pi_2^*(2P)/2 + \pi_2^*(2N)/2$ gives a non-integral Zariski decomposition on X. □

We end by posing the following problem.

Problem 2.3. Classify all algebraic surfaces with $d(X) = 1$.

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