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DIFFUSION IN THE SPACE OF COMPLEX HERMITIAN MATRICES — MICROSCOPIC PROPERTIES OF THE AVERAGED CHARACTERISTIC POLYNOMIAL AND THE AVERAGED INVERSE CHARACTERISTIC POLYNOMIAL*

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We show that the averaged characteristic polynomial and the averaged inverse characteristic polynomial, associated with the Hermitian matrices whose elements perform a random walk in the space of complex numbers, satisfy certain partial differential, diffusion-like equations. These equations are valid for matrices of arbitrary size and for any initial condition assigned to the process. The solutions have compact integral representation that allows for a simple study of their asymptotic behavior, uncovering the Airy and Pearcey functions.

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1. Introduction

As two of us have argued [1], a particularly interesting, hydrodynamic-like picture of the spectral evolution emerges if one exploits Dyson's idea [2] of introducing temporal dynamics into random matrix ensembles. Such dynamics appears in random matrix models in a broad context: in the physical

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applications, "time" can correspond to the length of a mesoscopic wire, area of the string/Wilson loop or the inverse temperature (see *e.g.* [3]): in mathematical studies, such dynamics appears in the study in non-intersecting Brownian paths and in relations to random skew planar partitions [4–6], to mention a few.

Here, we consider perhaps the simplest case of random matrix model, *i.e.* a Hermitian, $N \times N$ matrix whose entries perform a properly normalized, continuous random walk in the space of complex numbers. Let this process be initiated with a matrix filled with zeros. For this particular choice, a time-dependent, monic Hermite polynomial satisfies a complex diffusion equation with a diffusion constant equal to $-\frac{1}{2N}$. A key feature of this setting is that the polynomial is equal to the averaged characteristic polynomial (hereafter ACP) associated with the random matrix. The latter can be, moreover, transformed into a different function (by taking its logarithmic derivative) which, in turn, fulfills the viscid Burgers equation. In the large N (inviscid) limit, it admits solutions exhibiting shocks, whose positions coincide with the edges of the eigenvalue spectrum. In the finite N (viscid) case, one can perform an expansion around the shocks to obtain the well-known Airy asymptotic behavior of the orthogonal polynomial.

This analysis can be carried over to the case of diffusing Wishart matrices [7], for which the relevant monic orthogonal polynomials are given in terms of the Laguerre polynomials, corresponding to trivial initial condition for the evolution. We have shown [8] that for the evolution corresponding to non-trivial initial conditions, one can also rely directly on the ACP, despite it is not equal in this case to the orthogonal polynomial. By representing the determinant as a Berezin integral, we have derived the complex diffusion equation independently of the initial conditions, which allowed us to study a new, microscopic universal behavior at the spectral shock reaching the so-called wall at the origin (resulting in a Bessoid type function).

In this paper, we demonstrate that this strategy works also for the simplest case of Hermitian matrices, *i.e.* Gaussian Unitary Ensemble (hereafter GUE). In this way, we fill the certain logical gap between our early paper on GUE [1] and recent paper of Wishart ensemble [8]. In particular, we demonstrate that both the ACP and the averaged inverse characteristic polynomial (AICP) for GUE satisfy the same complex diffusion equation, except that for the former the diffusion constant is equal to $-\frac{1}{2N}$, whereas for the latter it is $\frac{1}{2N}$. As in the case of the Wishart ensemble, the new proof works regardless of the actual form of the initial condition imposed on the process and thus allows us to examine two different, generic scenarios. The solution of the diffusion equation leads to simple integral representations for both the ACP and the AICP, which makes it possible to study their asymptotic, large N, universal behaviors. In particular, we recover the known scaling property of the ACP and the AICP at the edge of the spectrum in terms of Airy functions. Moreover, for a process initiated with at least two distinct eigenvalues, a case we were not able to study within the orthogonal polynomials based approach, when two edges of the spectrum meet, the Pearcey functions emerge. Thus, the diffusion scenario provides a natural and simple way to re-derive the universal functions corresponding to the fold and cusp singularities in random matrix models. Certainly, some of the results presented in this paper are not new and have been derived by other methods by several authors. One can consider, therefore, this work as a pedagogical review of finite and infinite N effects in GUE, perhaps from a non-orthodox point of view. However, we believe that the scheme we are proposing here for GUE has much greater potential for broad class of random matrix models. Indeed, some recent results mentioned in the conclusions have already confirmed this rationale.

The paper is organized as follows. We start by introducing the stochastic evolution of the studied matrix. Using the representations of the determinant and its inverse as Gaussian integrals over, respectively, Grassmann or complex variables, we derive the diffusion equations for the ACP and the AICP. For the simplest scenario, in which the process is initialized with a null matrix, we crosscheck the equation for the AICP by exhibiting the equivalence of its solution with the Cauchy transform of the ACP. In such a way, we establish a connection to the well-known result of time-dilated Hermite polynomial and its Cauchy transform. In the following section, we derive the corresponding Burgers equation, which we solve in the large N limit with the method of complex characteristics and obtain the associated Green's function for two different generic examples. In the first one, the initial matrix is filled with zeros, whereas in the second, it has two distinct non-vanishing eigenvalues. We subsequently use the saddle point method to inspect how the ACP and AICP behave in the former scenario, at the points corresponding to the edges of the probability density function for the eigenvalues, asymptotically when N is increased. Whereas in the case of trivial initial conditions our result is equivalent to the well-known case of Airy asymptotics for Hermite polynomials, the case of non-trivial initial conditions offers a new perspective for the case of the so-called Pearcey kernel. Since the spectrum forms then two disjoint lumps of eigenvalues that eventually collide, more subtle saddle point analysis around the time and point of this collision is required. In particular, one has to consider rather refine coalescence of three saddle points. The studies of Pearcey asymptotics in the literature are either based on the introduction of the biorthogonal ensemble [22], or by application of the powerful Riemann–Hilbert approach [4]. We found it quite amusing, that a simple diffusion equation (or equivalent to it viscid Burgers equation) can lead in a straightforward way to both cases of universality. Moreover, all the subtleties of the double scaling limit in the vicinity of the critical points (here shock waves) are put in a nut-shell by the emergence of a spectral viscosity $1/2N$ in the complex Burgers equation. Finally, we make a link to the well-studied case of random matrix kernel for GUE, showing how the ACP and AICP can be used to reconstruct the kernel. The last section summarizes our results and puts them in a broader context.

2. Diffusion of Hermitian matrices

Let us introduce an $N \times N$ Hermitian matrix H by defining its complex entries according to

$$H_{ij} = \begin{cases} x_{ii}\,, & i = j\,, \\ \frac{1}{\sqrt{2}}(x_{ij} + iy_{ij})\,, & i \neq j\,, \end{cases} \tag{1}$$

where $x_{ij} = x_{ji}$ and $y_{ij} = -y_{ji}$, with x_{ij} and y_{ij} real. Furthermore, let x_{ij} and y_{ij} perform white-noise driven, independent random walks, such that

$$\langle \delta H_{ij} \rangle = 0\,, \qquad \left\langle (\delta H_{ij})^2 \right\rangle = \frac{1}{N}\delta\tau \tag{2}$$

for any i and j. Let $P(x_{ij},\tau)P(y_{ij},\tau)$ be the probability that the off-diagonal matrix entry H_{ij} will change from its initial state to $\frac{1}{\sqrt{2}}(x_{ij} + iy_{ij})$ after time τ. Analogically, $P(x_{ii},\tau)$ is the probability of the diagonal entry H_{ii} becoming equal to x_{ii} at τ. The evolution of these functions is governed by the following diffusion equations:

$$\begin{aligned} \frac{\partial}{\partial\tau}P(x_{ij},\tau) &= \frac{1}{2N}\frac{\partial^2}{\partial x_{ij}^2}P(x_{ij},\tau)\,, \\ \frac{\partial}{\partial\tau}P(y_{ij},\tau) &= \frac{1}{2N}\frac{\partial^2}{\partial y_{ij}^2}P(y_{ij},\tau)\,, \qquad i \neq j\,. \end{aligned} \tag{3}$$

Moreover, the joint probability density function

$$P(x,y,\tau) \equiv \prod_k P(x_{kk},\tau) \prod_{i<j} P(x_{ij},\tau)P(y_{ij},\tau) \tag{4}$$

satisfies the following equation [9]

$$\begin{aligned} \partial_\tau P(x,y,\tau) &= \mathcal{A}(x,y)P(x,y,\tau)\,, \\ \mathcal{A}(x,y) &= \frac{1}{2N}\sum_k \frac{\partial^2}{\partial x_{kk}^2} + \frac{1}{2N}\sum_{i<j}\left(\frac{\partial^2}{\partial x_{ij}^2} + \frac{\partial^2}{\partial y_{ij}^2}\right)\,. \end{aligned} \tag{5}$$

With the setting thus defined, let us proceed to the derivation of the partial differential equations governing the ACP and AICP.

2.1. Evolution of the averaged characteristic polynomial

Let $U_N(z,\tau)$ be the averaged characteristic polynomial associated with the diffusing matrix H: $U_N(z,\tau) \equiv \langle \det(z-H)\rangle$, where the angular brackets denote the averaging over the time-dependent probability density (4). In order to derive the partial differential equation governing the ACP, we write the determinant as a Gaussian integral over Grassmann variables $\eta_i, \bar{\eta}_i$

$$\det\, A = \int \prod_{i,j} d\eta_i d\bar{\eta}_j \exp\left(\bar{\eta}_i A_{ij} \eta_j\right) . \tag{6}$$

This allows us to express the averaged characteristic polynomial in the following way

$$U_N(z,t) = \int \mathcal{D}[\bar{\eta},\eta,x,y] P(x,y,\tau) \exp\left[\bar{\eta}_i \left(z\delta_{ij} - H_{ij}\right)\eta_j\right] , \tag{7}$$

where the joint integration measure is defined by

$$\mathcal{D}[\bar{\eta},\eta,x,y] \equiv \prod_{i,j} d\eta_i d\bar{\eta}_j \prod_{k} dx_{kk} \prod_{n<m} dx_{nm} dy_{nm} . \tag{8}$$

The Hermiticity condition ($H_{ij} = \bar{H}_{ji}$) allows us to write the argument of the exponent of (7) in a convenient form

$$\begin{aligned} T_f(\bar{\eta},\eta,x,y,z) \;\equiv\; & \sum_r \bar{\eta}_r \left(z - x_{rr}\right)\eta_r \\ & - \frac{1}{\sqrt{2}} \sum_{n<m} \left[x_{nm}\left(\bar{\eta}_n\eta_m - \eta_n\bar{\eta}_m\right) + i y_{nm}\left(\bar{\eta}_n\eta_m + \eta_n\bar{\eta}_m\right)\right] . \end{aligned}$$

Note that the time dependence of $\pi(z,\tau)$ resides entirely in $P(x,y,\tau)$. By differentiating Eq. (7) with respect to τ, and using Eq. (5), one ends up with an expression where the operator $\mathcal{A}(x,y)$ acts on the joint probability density function. Integrating by parts with respect to x_{ij} and y_{ij}, one obtains

$$\partial_\tau U_N(z,\tau) = \int \mathcal{D}[\bar{\eta},\eta,x,y] P(x,y,\tau) \mathcal{A}(x,y) \exp\left[T_f(\bar{\eta},\eta,x,y,z)\right] . \tag{9}$$

At this point, we differentiate with respect to the matrix elements (acting with $\mathcal{A}(x,y)$), exploit some simple properties of the Grassmann variables, and obtain

$$\partial_\tau U_N(z,\tau) = -\frac{1}{N} \int \mathcal{D}[\bar{\eta},\eta,x,y] P(x,y,\tau) \sum_{i<j} \bar{\eta}_i\eta_i\bar{\eta}_j\eta_j \exp\left[T_f(\bar{\eta},\eta,x,y,z)\right] . \tag{10}$$

It is easily verified that this expression, when multiplied by $-2N$, matches the double differentiation with respect to z of Eq. (7). We thus end up with

$$\partial_\tau U_N(z,t) = -\frac{1}{2N}\partial_{zz}U_N(z,\tau)\,. \tag{11}$$

This is the sought for diffusion equation for the ACP. Note that the same equation was already obtained in Ref. [1], albeit for a very specific initial condition, for which $U_N(z,t)$ is a scaled Hermite polynomial. The present derivation has the advantage of being independent of the choice of initial condition.

2.2. Evolution of the averaged inverse characteristic polynomial

We now turn to the averaged inverse characteristic polynomial

$$E_N(z,t) \equiv \left\langle \frac{1}{\det\left(z-H\right)}\right\rangle \tag{12}$$

to which we are going to apply a similar strategy. In this case, we use the fact that the inverse of a determinant has a well-known representation in terms of a Gaussian integral over complex variables ξ_i

$$\frac{1}{\det A} = \int\prod_{i,j} d\xi_i d\bar{\xi}_j \exp\left(-\bar{\xi}_i A_{ij}\xi_j\right). \tag{13}$$

As in the ACP case, we use this representation to express (12) as

$$E_N(z,\tau) = \int \mathcal{D}\left[\bar{\xi},\xi,x,y\right] P(x,y,\tau)\exp\left[\bar{\xi}_i\left(H_{ij}-z\delta_{ij}\right)\xi_j\right], \tag{14}$$

where, again, the proper notation for the joint integration measure was introduced. Performing the differentiation with respect to τ yields

$$\partial_\tau E_N(z,\tau) = \int \mathcal{D}\left[\bar{\xi},\xi,x,y\right] P(x,y,\tau)\mathcal{A}(x,y)\exp\left[T_b\left(\bar{\xi},\xi,x,y\right)\right], \tag{15}$$

with

$$\begin{aligned} T_b(\bar{\xi},\xi,x,y,z) \;\equiv\; & \sum_r \bar{\xi}_r\left(x_{rr}-z\right)\xi_r \\ & +\frac{1}{\sqrt{2}}\sum_{n<m}\left[x_{nm}\left(\bar{\xi}_n\xi_m+\xi_n\bar{\xi}_m\right)+iy_{nm}\left(\bar{\xi}_n\xi_m-\xi_n\bar{\xi}_m\right)\right], \end{aligned}$$

where we have used (5), the Hermiticity of H and we have performed integrations by parts. After differentiation with respect to the matrix elements, one obtains

$$\partial_\tau E_N(z,\tau) = \frac{1}{N}\int \mathcal{D}\left[\bar{\xi},\xi,x,y\right] P(x,y,\tau) \times\left(\sum_{i<j}\bar{\xi}_i\xi_i\bar{\xi}_j\xi_j+\frac{1}{2}\sum_k \bar{\xi}_k\xi_k\bar{\xi}_k\xi_k\right)\exp\left[T_b\left(\bar{\xi},\xi,x,y,z\right)\right], \quad (16)$$

which, multiplied by $2N$, matches the double differentiation of Eq. (14) with respect to z. The final result reads

$$\partial_\tau E_N(z,t) = \frac{1}{2N}\partial_{zz}E_N(z,t)\,, \quad (17)$$

the announced diffusion equation for the AICP. This equation is identical to that satisfied by the ACP except for the sign of the diffusion constant.

3. The integral representation

The main advantage of the equations derived above is that they have obvious solutions in terms of initial condition-dependent integrals. In this section, we explicitly state those representations and show additionally how, for the simplest initial condition, one is a Cauchy transform of the other. Let us also note here that these types of integrals were obtained [10] as representations of multiple orthogonal polynomials [11, 12] and, equivalently, as averaged characteristic polynomials of GUE matrices perturbed by a source [13]. We note that the presented integral representations can be also considered as a special case of the more general rations of characteristic polynomials, where supersymmetric methods have to be used [9].

3.1. The averaged characteristic polynomial

One can verify by a direct calculation that the expression

$$U_N(z,\tau) = \mathcal{C}\,\tau^{-1/2}\int_{-\infty}^{\infty}\exp\left(-N\frac{(q-iz)^2}{2\tau}\right)U_N(-iq,\tau=0)\,dq \quad (18)$$

satisfies the complex diffusion equation (11) governing the evolution of the averaged characteristic polynomial. The imaginary unit in the exponent and in the argument of the initial condition arises from the negative value of the diffusion constant in this equation. For finite N, the most general form of the initial condition is $U_N(z,\tau=0)=\prod_i(z-\lambda_i)$, where the λ_is are real due to the Hermiticity of the initial matrix $H(\tau=0)$. Exploiting the steepest descent method to match this with Eq. (18), one determines the constant term $\mathcal{C}$. The saddle point associated with $\tau\to 0$ is $u_0=iz$. Performing the Gaussian integration around the saddle point, we obtain $\mathcal{C}=\sqrt{\frac{N}{2\pi}}$ so that Eq. (18) reads

$$U_N(z,\tau)=\sqrt{\frac{N}{2\pi\tau}}\int\limits_{-\infty}^{\infty}\exp\left(-N\frac{(q-iz)^2}{2\tau}\right)U_N(-iq,\tau=0)\,dq\,. \tag{19}$$

3.2. The averaged inverse characteristic polynomial

The integral representation of the averaged inverse characteristic polynomial arising as a solution to the partial differential equation (17) is

$$E_N(z,\tau)=\mathcal{C}\int\limits_{\Gamma}\exp\left(-N\frac{(q-z)^2}{2\tau}\right)E_N(q,\tau=0)\,dq\,. \tag{20}$$

As in the case of the ACP, the initial condition has to be recovered. Here however, $E_N(z,\tau=0)$ has poles on the real axis and the contour Γ must avoid these poles. A first possibility is to choose Γ_+ parallel and slightly above the real axis. In this case, the saddle point analysis for $\tau\to 0$ is performed by moving Γ_+ upward so that it crosses the saddle point $q_0=z$. Obviously, this is possible only if $\mathrm{Im}z>0$. If instead $\mathrm{Im}z<0$, we need to choose Γ_- also parallel to the real axis but slightly below. Imposing an integration contour that would switch from the upper to the lower half plane (and *vice versa*) in between the poles, would results in a function no longer being the solution of the initial problem.

In the simple case of $U_N(z,\tau=0)=z^N$, we can cross check the above results using the well-known [14] Cauchy transform formula linking the ACP and the AICP. In particular, (19) coincides with the integral representation of the Hermite polynomial [15]

$$\pi_k(s,\tau)=(-i)^k\sqrt{\frac{N}{2\pi\tau}}\int\limits_{-\infty}^{\infty}q^k\exp\left(-\frac{N}{2\tau}(q-is)^2\right)dq \tag{21}$$

and the aforementioned Cauchy transform formula reads

$$E_N(z,\tau)=\frac{1}{c_{N-1}^2}\int\frac{ds}{z-s}\pi_{N-1}(s,\tau)\exp\left(-\frac{Ns^2}{2\tau}\right), \tag{22}$$

where the constant $c_k^2 = \sqrt{\frac{2\pi\tau}{N}}\left(\frac{\tau}{N}\right)^k k!$ is the normalization of the monic polynomials. Note that this is valid only for this particular, simplest initial condition. Analogical prescriptions for other cases are significantly more complicated [11].

To proceed, we plug (21) into (22). After transforming the q^{N-1} term into a differentiation of the exponent with respect to s, followed by integrating by parts, we get the result

$$E_N(z,\tau) = \sqrt{\frac{N}{2\pi\tau}} \int\limits_{-\infty+z}^{\infty+z} \frac{1}{u^N} \exp\left(-N\frac{(u-z)^2}{2\tau}\right) du\,. \tag{23}$$

The integration contour can be, in turn, deformed to the real axis and a half circle enclosing the pole at 0 from above (Im$z > 0$) or below (Im$z < 0$) — in complete agreement with equation (20).

4. Large N spectral dynamics

Let λ_is be the eigenvalues of the diffusing matrix $H(\tau)$. The connection between the spectral density $\rho(\lambda) \equiv \langle\sum_i \delta(\lambda-\lambda_i)\rangle$, in the limit of N going to infinity and the averaged characteristic polynomial, is established through the so-called Green's function, defined by

$$G(z,\tau) \equiv \frac{1}{N}\left\langle \mathrm{Tr}\,[z-H(\tau)]^{-1}\right\rangle\,. \tag{24}$$

Note that when N goes to infinity, the poles of this function merge, forming a cut on the complex plane. The link is made with the well-known Sokhotski–Plemelj formula $\rho(\lambda) = \frac{1}{\pi}\lim_{\epsilon\to 0_\pm} \mathrm{Im}G(\lambda \mp i\epsilon)$ and the relation

$$G(z,\tau) = \lim_{N\to\infty} \frac{1}{N}\partial_z \ln U_N(z,\tau)\,. \tag{25}$$

Note that $f_N(z,\tau) \equiv \frac{1}{N}\partial_z \ln U_N(z,\tau)$ is the famous Cole–Hopf transform [16]. One can easily verify that the diffusion equation derived for the ACP in the previous section corresponds to the following Burgers equation for $f_N(z,\tau)$

$$\partial_\tau f_N(z,\tau) + f_N(z,\tau)\partial_z f_N(z,\tau) = -\frac{1}{2N}\partial_z^2 f_N(z,\tau)\,, \tag{26}$$

in which the "spatial" variable z is complex and the role of "viscosity" is played by $-1/2N$, a negative number. In the large N limit, the viscosity vanishes, $f_N(z,\tau) \to G(z,\tau)$ and equation (26) becomes the inviscid Burgers equation

$$\partial_\tau G(z,\tau) + G(z,\tau)\partial_z G(z,\tau) = 0\,. \tag{27}$$

This equation may be solved by determining the characteristic lines, curves labeled ξ along which the solution is constant, that is $G(z,\tau) = G_0(\xi)$, where $G_0(z) \equiv G(z,\tau=0)$. In this particular case, these characteristic lines are given in the (z,τ) hyperplane by the equation

$$z = \xi + \tau G_0(\xi)\,, \tag{28}$$

which can be solved given the initial condition $G_0(\xi)$. The characteristic lines are tangent to the so-called caustics, whose location is given by the condition

$$0 = \left.\frac{dz}{d\xi}\right|_{\xi=\xi_\mathrm{c}} = 1 + \tau G_0'(\xi_\mathrm{c})\,. \tag{29}$$

Along the caustics, the mapping between z and ξ ceases to be one to one, and the characteristic method loses its validity.

We consider generic initial conditions of two types: $H(\tau=0) = 0$ and $H(\tau=0) = \mathrm{diag}(-a\ldots,a\ldots)$. The first one corresponds to $G_0 = \frac{1}{z}$, the second (for which we assume that N is even) amounts to $G_0 = \frac{1}{2(z-a)} + \frac{1}{2(z+a)}$. In the former scenario, for an infinitely large matrix, the spectrum forms a single connected interval throughout its evolution. In the latter, it initially occupies two separate domains of the real axis which, in turn, merge at some critical space-time point. Figure 1 pictures the time evolution of the two corresponding spectral densities. In both cases, the characteristic lines that are real at $\tau = 0$, remain on the plane $\mathrm{Im}z = 0$ throughout the time evolution — we depict them in Fig. 2. Those that are complex, on the other hand, are symmetric under complex conjugation and never cross each other until some time, when they hit the real line and end on the cut of the Green's function. The caustics live on the plane $\mathrm{Im}z = 0$ so long as they do not merge. Moreover, they move along the branching points of the resulting Green's function and mark, therefore, the edges of the spectra. Additionally, they constitute the positions of the shocks, curves in the (z,τ) space along which the characteristic lines have to be cut to ensure unambiguity of the solution for $G(z,\tau)$. Note finally, that if the complex characteristic lines were allowed to cross the cuts of the complex plane, they would form (in the second scenario) complex caustics evolving out of the merging point of the real ones. These are depicted by dashed lines in Fig. 2.

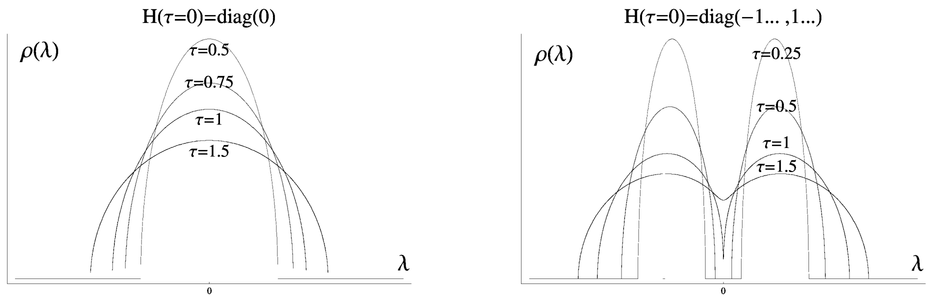

Fig. 1. The time evolution of the large N spectral density of the evolving matrices for two scenarios that differ in the imposed initial condition. The parameter a was set to one.

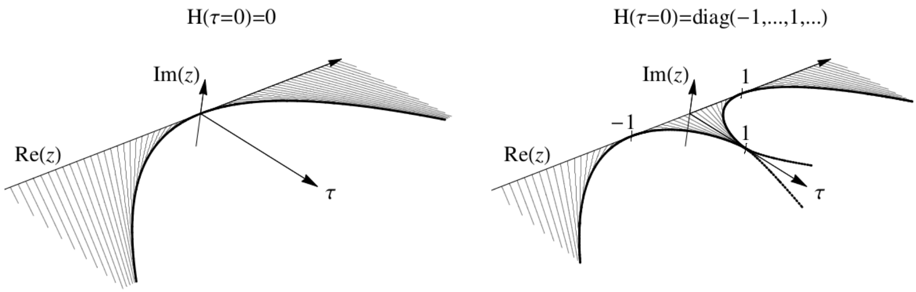

Fig. 2. The thin lines are characteristics that remain real throughout their temporal evolution. They finish at the bold lines which are caustics and shocks simultaneously. The dashed bold lines are the caustics that would be formed by the strictly complex characteristics (not depicted here) if they did not end on the branch cut.

5. Universal microscopic scaling

In this section, we inspect the ACP and the AICP in the vicinity of the points corresponding to the edges of the spectrum of $H(\tau)$, that is near the shocks. For the size of the matrix approaching infinity, one expects that the behaviors of the ACP and the AICP do not depend on the details of the stochastic process governing the evolution — a manifestation of the so-called microscopic universality.

As some of us have demonstrated [1], the asymptotic behavior of the ACP can be recovered by analyzing Eq. (26) through an expansion of f_N around the positions of the shocks. Following this approach, one could recover the Airy function describing the behavior of the ACP near the propagating edge. However, this method does not seem to be so effective when one examines a situation when two shocks collide. As we shall now see, it is more convenient in this case to return to the diffusion equation and realize that, irrespective of the initial condition, the integral representations of the ACP and AICP have the following generic structure

$$\int_\Gamma e^{Nf(p,z,\tau)}dp\,. \tag{30}$$

This is well-suited for a steepest descent analysis in the large N limit [18]. Moreover, the saddle point condition $\partial_p f(p,z,\tau)|_{p=p_i} = 0$ is equivalent to equation (28). For the ACP, that is, we have

$$f(p,z,\tau) = \frac{1}{N}\ln\left[U_0(-ip)\right] - \frac{1}{2\tau}(p-iz)^2\,, \tag{31}$$

where $U_0(-ip)\equiv U_N(-ip,\tau=0)$ and we identify ξ with $-ip_i$. For the AICP,

$$f(p,z,\tau) = \frac{1}{N}\ln\left[E_0(p)\right] - \frac{1}{2\tau}(p-z)^2\,, \tag{32}$$

(notice that $G_0(p) = -\frac{1}{N}\partial_p\ln\left[E_0(p)\right]$ and $E_0(p) \equiv E_N(p,\tau=0)$) with ξ identified as p_i. The fact that the labels of the characteristics and the saddle points are connected is clearly not coincidental. The viscid Burgers equation and the diffusion equation are equivalent through the Cole–Hopf transform. This induces an equivalence between the characteristics method used to solve the inviscid limit of the former and the saddle point method applied for the large N solution of the latter. Consider approaching a caustic in the (z,τ) (hyper-)plane. Through the equivalence just pointed out, the merging of characteristics implies the merging of two saddle points — this will be the scenario of the first example considered below. When the caustics merge, forming a cusp, three saddle points coalesce, which will be studied subsequently.

The final issue to resolve before engaging the calculations is the question of what precisely we mean by the "vicinity" of the edges. If the width of the studied interval remains constant or shrinks too slowly, as the number of the eigenvalues grows to infinity, we will deal with an infinite number of eigenvalues and most of them will not "feel" that they are "close" to edge. On the other hand, if the interval shrinks too fast, in the end there will not be any eigenvalues left inside the interval. This is a heuristic explanation of why the studied vicinity of the edge should have a span proportional to the average spacing of the eigenvalues near the shock. This quantity can be derived by inspecting the large N limit of the spectral density that can be obtained from the Green's function. To proceed, one expands G around $\xi_{\rm c}$

$$G_0(\xi) = G_0(\xi_{\rm c}) + (\xi-\xi_{\rm c})G_0'(\xi_{\rm c}) + \tfrac{1}{2}(\xi-\xi_{\rm c})^2G_0''(\xi_{\rm c}) + \tfrac{1}{6}(\xi-\xi_{\rm c})^3G_0'''(\xi_{\rm c}) + \ldots \tag{33}$$

From equation (28), we have $G_0 = (z-\xi)/\tau$ and $G_0'(\xi_{\rm c}) = -1/\tau$, so that

$$z - z_{\rm c} = \frac{\tau}{k!}(\xi - \xi_{\rm c})^k G_0^{(k)}(\xi_{\rm c}) + \dots\,, \tag{34}$$

where k in $G_0^{(k)}(\xi_{\rm c})$ indicates the power of the first after (G_0'), non-vanishing derivative of G_0 taken in $\xi_{\rm c}$, for a given critical point. This leads to

$$G(z,\tau) \simeq G_0(\xi_{\rm c}) + G_0'(\xi_{\rm c})\left[\frac{k!(z-z_{\rm c})}{\tau G_0^{(k)}(\xi_{\rm c})}\right]^{1/k}\,. \tag{35}$$

Now, let N_Δ be the average number of eigenvalues located in an interval of width Δ near $\xi_{\rm c}$. We have

$$N_\Delta \sim N\int\limits_{z_{\rm c}}^{z_{\rm c}+\Delta}(z-z_{\rm c})^{1/k}dz \sim N\Delta^{1+1/k}\,, \tag{36}$$

which for fixed N_Δ, implies that Δ, or equivalently the average eigenvalue spacing, should scale with N as $N^{-k/(1+k)}$.

As we shall perform the rest of the calculations using the saddle point method, let us finally show how the proper scalings arise in that framework. This will be done, after [19], through the condition for the merging of saddle points p_i. In this context, the deviation s from $z_{\rm c}$ has to scale with the size of the matrix in such a way that, when N grows to infinity, the value of the integrand is not concentrated at separate p_is but in a single point $p_{\rm c}$ (corresponding to the merging of the saddle points). This is equivalent to requiring that for such an s, the distance between the saddle points p_i is of the order of the width of the Gaussian functions arising from expanding $f(p)$ around the respective p_is in $\exp[Nf(p)]$. In particular, the condition

$$|p_i - p_n| \sim \left[Nf''(p_j)\right]^{-1/2}\,, \qquad i \neq n\,, \tag{37}$$

(with p_j being any of the saddle points merging) gives the relevant order of magnitude of s, that is N^α. In other words, by plugging the formulas for the saddle points into (37) and substituting z with $z_{\rm c}+s$ one obtains $s \sim N^\alpha$, with α depending on the number of saddle points merging. We, therefore, set $z = z_{\rm c} + N^\alpha\eta$ and η is of the order of one. This subsequently sets the scale for the distance probed by the deviation from $p_{\rm c}$ and so, in the same manner, the condition $|p_i - p_{\rm c}| \sim N^\beta$ defines the substitution $p = p_{\rm c} + N^\beta t$.

The connection between the saddle points and characteristic lines allows us to relate β and α through k. First, note that by definition, α and k are related through $\alpha(1+k) = -k$. Near the critical point, we see, on the other hand, that $|p_i - p_c| \sim |\xi - \xi_c|$ (through the equivalence of the saddle point condition and the prescription for the characteristics). Using Eq. (34), we thus obtain $\beta = \alpha/k = -(1+\alpha)$, which can be used as a consistency check.

In the example of Subsection 3.2, the merging of the saddle points happens in a particular critical time τ_c and there exists a time scale of the order of N^γ for which, asymptotically p_i are not distinguishable. This exponent is calculated by expanding the condition for the merging of saddle points around the critical value $\tau = \tau_c + N^\gamma \kappa$.

5.1. Soft, Airy scaling

Let us start by considering the simplest initial condition, namely $U_N(z,\tau = 0) = z^N$, for which $H(\tau = 0)$ is filled with zeros

$$U_N(z,\tau) = (-i)^N \sqrt{\frac{N}{2\pi\tau}} \int_{-\infty}^{\infty} q^N \exp\left(-\frac{N}{2\tau}(q-iz)^2\right) dq\,. \tag{38}$$

In this setting, the two edges of the spectrum plainly move in the opposite directions along the real line (see Fig. 1). We conduct a large N steepest descent analysis. As a first step, we set $f = \ln q - \frac{1}{2\tau}(q-iz)^2$ and obtain the saddle point equation as

$$\tau = q(q-iz)\,.$$

This relation has the role analogous to the equation governing the characteristics, as was introduced in the analysis of the Burgers equation and described above. The positions of the saddle points are given by $q_\pm = \frac{1}{2}\left(iz \pm \sqrt{4\tau - z^2}\right)$. Their merging, at $q_c = i\sqrt{\tau}$, marks the locations of the spectral edges. For simplicity, we focus on just one of them, in particular the right edge, for which $z_c = 2\sqrt{\tau}$. It is easily verified that this points to the crossing of the characteristic lines.

The new contour, going through q_c is depicted (for $\tau = 1$) in the left plot of Fig. 3. The contour deformation is constrained by two conditions: (a) the real part of the function f reaches a maximum along the contour and (b) the imaginary part must obey the condition $\mathrm{Im} f = \mathrm{const}$. By imposing the latter, we guarantee the steepest descent of $\mathrm{Re} f$ upon integrating along the contour. These requirements fix uniquely the path marked in bold in the left plot of Fig. 3.

The scaling exponent α is equal to $-\frac{2}{3}$ and we have $\eta = (z - 2\sqrt{\tau})N^{2/3}$. Moreover, $\beta = -\frac{1}{3}$ and the change of variables in the integral is given by $t = (q - i\sqrt{\tau})N^{1/3}$. These scaling parameters were obtained from equation (37) which compares the spacing between the saddle points $q_\pm$ to the width of the

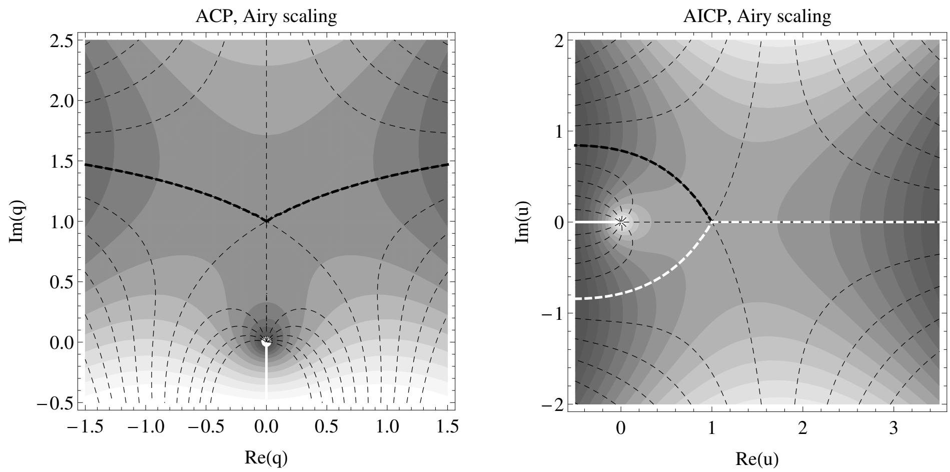

Fig. 3. The gray scale gradient in the graphs above portrays the value of $\mathrm{Re}f(p)$ (growing with the brightness), whereas the dashed lines depict the curves of constant $\mathrm{Im}f(p)$. The left figure is plotted for the ACP, with $p \equiv q$, the right one for the AICP, with $p \equiv u$. The initial condition is $H(\tau = 0) = 0$ and time τ is fixed to 1 for both. Dashed bold curves indicate contours of integration suitable for the saddle point analysis, for the AICP, we identify the black and white line with the contours Γ_+ and Γ_- respectively.

Gaussian approximation. The same result can be read out from the spectral density expanded around the edge of the bulk $\rho \sim \sqrt{|z - z_{\mathrm{c}}|}$. Expanding the logarithm and taking the large matrix size limit yields

$$U_N\left(z = 2\sqrt{\tau} + \eta N^{-2/3}, \tau\right) \approx \tau^{N/2}\frac{N^{1/6}}{\sqrt{2\pi}} \exp\left(\frac{N}{2} + \frac{\eta N^{1/3}}{\sqrt{\tau}}\right) \mathrm{Ai}\left(\frac{\eta}{\sqrt{\tau}}\right), \tag{39}$$

where

$$\mathrm{Ai}(x) = \int\limits_{\Gamma_0} dt \exp\left(\frac{it^3}{3} + itx\right) \tag{40}$$

is the well-known Airy function. The contour Γ_0 is formed by the rays $-\infty \times e^{5i\pi/6}$ and $\infty \times e^{i\pi/6}$ emerging as N goes to infinity. Along these rays, integral (40) is convergent as can be seen by substituting $t \to te^{i\pi/6}$.

In the case of the AICP, the initial condition used above takes the form of $E_N(z,\tau=0)=z^{-N}$ and (20) reads

$$E_N(z,\tau)=\sqrt{\frac{N}{2\pi\tau}}\int\limits_{\Gamma_\pm} u^{-N}\exp\left(-N\frac{(u-z)^2}{2\tau}\right)du\,, \tag{41}$$

where the contours avoid the pole at zero from above Γ_+ (for $\mathrm{Im}z>0$) or below Γ_- (for $\mathrm{Im}z<0$), as explained in Subsection 3.2. This time, the saddle point merging for the right spectral edge occurs for $u_{\rm c}=\sqrt{\tau}$. It also marks the position of the cusps of the new integration contours (depicted for $\tau=1$ in the second plot of Fig. 3). The transformation of variables is given by $\eta=(z-2\sqrt{\tau})N^{2/3}$ and $it=(u-\sqrt{\tau})N^{1/3}$. Notice that the complex plane of t is rotated by $\pi/2$ with respect to the one of u. By expanding the logarithm and taking the large matrix size limit, we obtain

$$\begin{aligned}E_N\left(z=2\sqrt{\tau}+\eta N^{-2/3},\tau\right) \;\approx\; & i\tau^{-N/2}\frac{N^{1/6}}{\sqrt{2\pi}}\exp\left(-\frac{N}{2}-\frac{\eta N^{1/3}}{\sqrt{\tau}}\right)\\ & \times\mathrm{Ai}\left(e^{i\phi_\pm}\frac{\eta}{\sqrt{\tau}}\right)\,,\end{aligned} \tag{42}$$

the asymptotic behavior in terms of the Airy function, yet with its argument rotated by $\phi_+=-2\pi/3$, for $\mathrm{Im}z>0$, and by $\phi_-=2\pi/3$, for $\mathrm{Im}z<0$ in accordance with previous results for static matrices [20].

5.2. Pearcey scaling

To observe a collision of the edges of the spectrum, in the large N limit, one has to consider a slightly different initial condition. Let $U_N(z,\tau=0)=(z^2-a^2)^{N/2}$, with N even. This corresponds to the initial matrix eigenvalues set to $\pm a$ with equal degeneracy $N/2$. In this case, the ACP takes the form

$$U_N(z,\tau)=i^N\sqrt{\frac{N}{2\pi\tau}}\int\limits_{-\infty}^{+\infty}dq\exp\left[-\frac{N}{2\tau}(q-iz)^2+\frac{N}{2}\log\left(a^2+q^2\right)\right]. \tag{43}$$

We determine the saddle point equation

$$\frac{q}{q^2+a^2}-\frac{q-iz}{\tau}=0 \tag{44}$$

and calculate its three solutions $q_{1,2,3}$. In this scenario, the two parts of the spectra join at $z_{\rm c}=0$ at $\tau_{\rm c}=a^2$ and this is reflected in the saddle points merging for $q_c = 0$. The contour does not have to be deformed in this case as seen in the left plot of Fig. 4. Subsequently, the leading term of equation (37) relating the distance between the solutions $q_{1,2,3}$ to the Gaussian width of the saddle point approximation is used to extract the scaling $z \sim N^{-3/4}$. Further analysis of the distance between the critical saddle point q_c and the

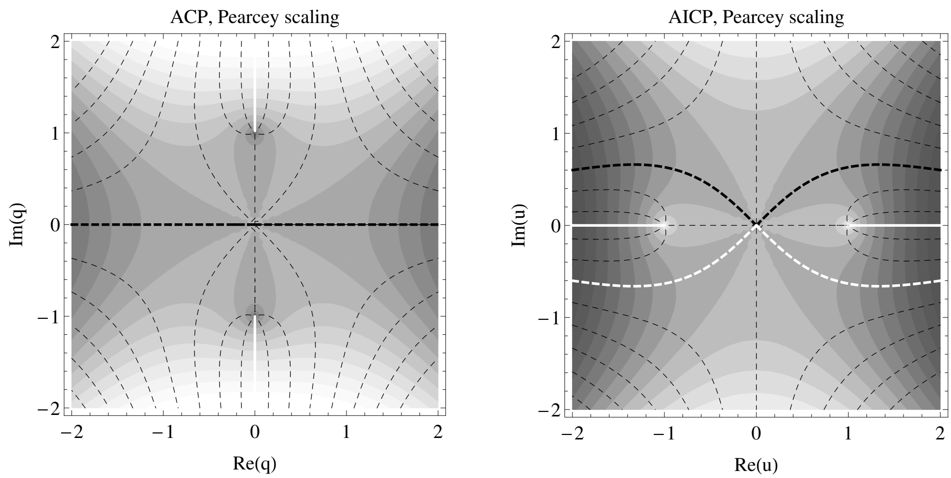

Fig. 4. In the two above graphs, the gray scale gradient portrays the value of $\mathrm{Re} f(p)$ (growing with the brightness), whereas the dashed lines depict the curves of constant $\mathrm{Im} f(p)$. The left figure is plotted for the ACP, with $p \equiv q$, the right one for the AICP, with $p \equiv u$. The initial condition is $H(\tau = 0) = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ and time τ is fixed to 1 for both. Dashed bold curves indicate contours of integration suitable for the saddle point analysis, for the AICP, we identify the black and white line with the contours Γ_+ and Γ_- respectively.

points $q_{1,2,3}$ shows that the main contribution to the integral for $z \sim N^{-3/4}$ comes from q of the order of $N^{-1/4}$ and τ of the order of $N^{-1/2}$. Based on this, we set $q = tN^{-1/4}$, $\tau = a^2 + \kappa N^{-1/2}$, $z = \eta N^{-3/4}$. In the limit of $N \to \infty$, we expand the logarithm arising in the exponent through the initial condition and find

$$U_N\left(z = \eta N^{-3/4}, \tau \approx a^2 + \kappa N^{-1/2}\right) \approx \frac{N^{1/4}}{\sqrt{2\pi}} (ia)^N \mathrm{P}\left(\frac{\kappa}{2a^2}, \frac{\eta}{a}\right) , \tag{45}$$

where we define the Pearcey integral by

$$\mathrm{P}(x, y) = \int\limits_{-\infty}^{\infty} dt \ \exp\left(-\frac{t^4}{4} + xt^2 + ity\right) . \tag{46}$$

This conclusion to the analysis of the microscopic behavior of the ACP is not a surprise — the Pearcey kernel was first derived [21] in the context of GUE matrices perturbed by a source [22], *i.e.* in a setting analogical to that considered here, in which the sources constitute the initial condition and their critical adjustment plays the role of the critical time.

In the case of the AICP, the initial condition reads $E_N(z,\tau=0) = (z^2-a^2)^{-N/2}$ and the solution to the complex diffusion equation is

$$E_N(z,\tau) = \sqrt{\frac{N}{2\pi\tau}} \int\limits_{\Gamma_\pm} du \left(u^2-a^2\right)^{-N/2} \exp\left(-\frac{N}{2\tau}(u-z)^2\right) , \tag{47}$$

where $\Gamma_\pm$ denotes contours circling the poles $u_i = \pm a$ from above (Γ_+, $\mathrm{Im}z > 0$) or from below (Γ_-, $\mathrm{Im}z < 0$). The right plot in Fig. 4 depicts the types of curves the contours are deformed to. We parametrize the saddle point expansion by $\tau = a^2 + \kappa N^{-1/2}$, $z = \eta N^{-3/4}$ and $u = e^{i\pi/4} t N^{-1/4}$. We then obtain

$$\begin{aligned} E_N\left(z=\eta N^{-3/4}, \tau \approx a^2+\kappa N^{-1/2}\right) &\approx \frac{N^{1/4}}{\sqrt{2\pi}}(ia)^{-N} \\ \times \int\limits_{\Gamma_\pm} dt\ \exp\left(-t^4/4 - \frac{i\kappa}{2a^2}t^2 + it\frac{e^{-i\pi/4}\eta}{a}\right) , \end{aligned} \tag{48}$$

that is, a Pearcey type integral along contours Γ_+, Γ_- depending on the choice of sign of the imaginary part of z. The former is defined by rays with phase $\pi/2$ and 0 whereas the latter starts at $-\infty$ and after reaching zero forms a ray along a phase $-\pi/2$.

6. Constructing the kernel

The ACP and the AICP are the building blocks of the matrix kernel, which, in turn, contains, for arbitrary N, all the information about the matrix model. Obtaining the kernel was the aim of many previous works [10–12]. By making use of results of this paper, we can easily write down its form for the case of the studied diffusing matrix. We have

$$K_N(x,y,\tau) = \sum_{i=0}^{N-1} \Theta_i(x,\tau)\Pi_i(y,\tau) , \tag{49}$$

where $\Theta_i(x,\tau)$ and $\Pi_i(y,\tau)$ are defined as follows. First, let

$$\Theta_{\vec{m}}(x,\tau) \equiv E^{+}_{|\vec{m}|}(x,\tau) - E^{-}_{|\vec{m}|}(x,\tau) = \sqrt{\frac{N}{2\pi\tau}} \oint\limits_{\Gamma_0} du \times \exp\left(-N\frac{(u-x)^2}{2\tau}\right) E_0(u)\,, \tag{50}$$

$$\Pi_{\vec{m}}(x,\tau) \equiv U_{|\vec{m}|}(x,\tau) = \sqrt{\frac{N}{2\pi\tau}} \int\limits_{-\infty}^{\infty} dq \exp\left(-N\frac{(q-ix)^2}{2\tau}\right) U_0(-iq)\,. \tag{51}$$

Here, $E_0(x) = \prod_{i=1}^{d}(x-a_i)^{-m_i}$, $U_0(x) = \prod_{i=1}^{d}(x-a_i)^{m_i}$ are the initial conditions and $\vec{m}$ is a corresponding arbitrary eigenvalue multiplicity vector. Moreover, the contour Γ_0 encircles all the sources a_i clockwise and $E^{+}(z,\tau)$, $E^{-}(z,\tau)$ denote the different solutions of AICP diffusion equation valid for $\mathrm{Im}z > 0$ and $\mathrm{Im}z < 0$ respectively. Finally, the functions labeled by the index i in (49) arise through an ordering of the multiplicities according to their increasing norm $|\vec{m}| \equiv \sum_{j=1}^{d} m_j$ (see (A.6))

$$\Theta_i \equiv \Theta_{\vec{n}^{(i+1)}}\,, \qquad \Pi_i \equiv \Pi_{\vec{n}^{(i)}}\,, \qquad i = 0,\ldots,N-1\,. \tag{52}$$

The details concerning the construction of (49) are delivered in Appendix. It is shown in this appendix that the kernel for the Pearcey process that derives from the present construction is identical to that obtained by Brezin and Hikami [21]

$$K_{\mathrm{BH}}(x,y,\tau) = -\frac{N}{2\pi\tau} \oint\limits_{\Gamma_0} du \int\limits_{-\infty}^{\infty} dq \frac{\left(-q^2-a^2\right)^{N/2}}{\left(u^2-a^2\right)^{N/2}} \frac{1}{u+iq} \times \exp\left(-N\frac{(q-iy)^2}{2\tau} - N\frac{(u-x)^2}{2\tau}\right)\,. \tag{53}$$

7. Conclusions

In this work, we have studied the behavior of averaged characteristic polynomials and averaged inverse characteristic polynomials associated with Hermitian matrices filled with entries performing Brownian motion in the space of complex numbers. A key new step of our analysis was a derivation of partial differential equations governing the matrix-valued evolution independently of the initial conditions. These turned out to be complex heat equations with diffusion coefficients inversely proportional to the size of the matrices, and thus provided us with integral representations of the polynomials. By using the saddle point method, we were able to examine their so-called critical microscopic behavior, which is known to be universal. In particular, the asymptotics are driven by the Airy functions, at the edges of the spectrum, and by the Pearcey functions, when those edges meet. The first case holds for any frozen moment of time, however, one can easily modify the free diffusion into an Ornstein–Uhlenbeck problem to obtain the Airy behavior in the stationary limit at large time.

It is worth mentioning that the Pearcey function type behavior of the ACP arises also in the model of multiplicatively diffusing unitary matrices [23]. Moreover, the associated critical point marks the universality class of the Durhuus–Olesen type transition of Wilson loops in the Yang–Mills theory, in the limit of infinite number of colors [24–26]. Interestingly, our approach allowed us to recover the microscopic, critical behavior of the ACP for a diffusing chiral matrix. The resulting Bessoid function, an axially symmetric version of the Pearcey, is conjectured to describe the partition function of Euclidean QCD at the moment of spontaneous chiral symmetry breaking [27]. Finally, we note that an analogous, yet more intricate Burgers-like picture, arises also in the case of diffusing non-Hermitian matrices [28, 29], leading to novel, duality-type relations [30].

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Appendix

The kernel structure

To obtain (49), the formula for the kernel, we first present the connection between the diffusive model considered in this paper and the matrix model with a source introduced in [22]. First, let us notice that at time τ, the ensemble of the diffusing matrices H is equivalent to the ensemble of matrices defined by

$$X_\tau = H_0 + X\sqrt{\tau}\,, \tag{A.1}$$

where H_0 is a fixed matrix and X is random and given by the complex ($\beta = 2$) GUE measure

$$P(X)dX \sim \exp\left(-\frac{N}{2}\,\mathrm{Tr}\,X^2\right)dX\,. \tag{A.2}$$

Since (A.1) is a linear transformation, we have $X = \frac{X_\tau - H_0}{\sqrt{\tau}}$ and $dX \sim dX_\tau$. We, therefore, recover a matrix model with a source in which τ is just a parameter

$$P(X_\tau)dX_\tau \sim \exp\left(-\frac{N}{2\tau}\,\mathrm{Tr}(X_\tau - H_0)^2\right)dX_\tau\,. \tag{A.3}$$

The matrix H_0, corresponding in our formalism to the initial condition at $\tau = 0$, is the source matrix. From now on, we follow closely the works on random matrices with a source [10–12]. The matrix H_0 can be written in a diagonal form as

$$H_0 = \mathrm{diag}\left(\underbrace{a_1\ a_1\ \ldots}_{n_1};\underbrace{a_2\ a_2\ \ldots}_{n_2};\ldots;\underbrace{a_d\ \ldots}_{n_d}\right)\,,$$

with d eigenvalues a_i of multiplicities n_i. Out of the degeneracies, we form a vector $\vec{n} = (n_1, \ldots, n_d)$ which has a norm $|\vec{n}| \equiv \sum_{i=1}^{d} n_i = N$ dictated by the matrix size. We subsequently introduce, after [12], the multiple orthogonal polynomials of type I and II. The functions of type I are defined on the real line through

$$\begin{aligned}\varTheta_{\vec{m}}(x,\tau) \ &\equiv\ E^{+}_{|\vec{m}|}(x,\tau) - E^{-}_{|\vec{m}|}(x,\tau) = \sqrt{\frac{N}{2\pi\tau}}\oint\limits_{\varGamma_0} du \\ &\times \exp\left(-N\frac{(u-x)^2}{2\tau}\right)E_{0,\vec{m}}(u)\,,\end{aligned} \tag{A.4}$$

with an arbitrary multiplicity vector $\vec{m}$, an initial condition $E_{0,\vec{m}}(x) = \prod_{i=1}^{d}(x-a_i)^{-m_i}$ and the contour $\varGamma_0$ encircling all a_is clockwise. This contour arose since the AICP was defined in (20) by two different contours $\varGamma_-$ and $\varGamma_+$ for $\mathrm{Im}z < 0$ and $\mathrm{Im}z > 0$ respectively.

Analogously, polynomials of type II are defined through the averaged characteristic polynomial by

$$\varPi_{\vec{m}}(x,\tau) \equiv U_{|\vec{m}|}(x,\tau) = \sqrt{\frac{N}{2\pi\tau}}\int\limits_{-\infty}^{\infty} dq\ \exp\left(-N\frac{(q-ix)^2}{2\tau}\right)U_{0,\vec{m}}(-iq)\,, \tag{A.5}$$

with an initial condition $U_{0,\vec{m}}(x) = \prod_{i=1}^{d}(x-a_i)^{m_i}$. We stress the dependency of the polynomials on the multiplicity vector $\vec{m}$ of arbitrary norm $|\vec{m}| \neq N$. As a last step, we introduce an ordering of the vector $\vec{n}$

$$\begin{aligned}
\vec{n}^{(0)} &= (0,0,\ldots,0)\,,\\
\vec{n}^{(1)} &= (1,0,\ldots,0)\,,\\
&\vdots\\
\vec{n}^{(n_1)} &= (n_1,0,\ldots,0)\,,\\
\vec{n}^{(n_1+1)} &= (n_1,1,\ldots,0)\,,\\
&\vdots\\
\vec{n}^{(N)} &= (n_1,n_2,\ldots,n_d)\,,
\end{aligned} \tag{A.6}$$

which forms a "nested" sequence increasing in norm. This sequence is exploited to compose N pairs of type I and type II polynomials

$$\Theta_i \equiv \Theta_{\vec{n}^{(i+1)}}\,, \qquad \Pi_i \equiv \Pi_{\vec{n}^{(i)}}\,, \qquad i = 0,\ldots,N-1\,. \tag{A.7}$$

This stack of functions forms a kernel valid for an arbitrary source H_0

$$K_N(x,y) = \sum_{i=0}^{N-1} \Theta_i(x)\Pi_i(y)\,. \tag{A.8}$$

As an example, we consider the case of $a_1 = a, a_2 = -a$ and multiplicities $n_1 = n_2 = N/2$. We plug in the integral representations (A.4) and (A.5)

$$K_N(x,y) = \frac{N}{2\pi\tau}\oint_{\Gamma_0} du \int_{-\infty}^{\infty} dq \exp\left(-N\frac{(q-iy)^2}{2\tau} - N\frac{(u-x)^2}{2\tau}\right) I(q,u)\,, \tag{A.9}$$

where the sum over the initial conditions is denoted by $I(q,u)$. In our example, it is equal to

$$\begin{aligned}
I(q,u) &= \sum_{j=0}^{\frac{N}{2}-1} \frac{(-iq-a)^j}{(u-a)^{j+1}} + \frac{(-iq-a)^{N/2}}{(u-a)^{N/2}} \sum_{j=0}^{\frac{N}{2}-1} \frac{(-iq+a)^j}{(u+a)^{j+1}}\\
&= \frac{1}{u+iq}\left(1 - \frac{\left(-q^2-a^2\right)^{N/2}}{\left(u^2-a^2\right)^{N/2}}\right)\,.
\end{aligned}$$

By noticing that, under the integral, the first term vanishes, we arrive at the formula (53) given by Brezin and Hikami [21].

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