

Enhancing Pseudo-Telepathy in the Magic Square Game

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Abstract

We study the possibility of reversing an action of a quantum channel. Our principal objective is to find a specific channel that reverses as accurately as possible an action of a given quantum channel. To achieve this goal we use semidefinite programming. We show the benefits of our method using the quantum pseudo-telepathy Magic Square game with noise. Our strategy is to move the pseudo-telepathy region to higher noise values. We show that it is possible to reverse the action of a noise channel using semidefinite programming.

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Introduction

Quantum game theory is an interdisciplinary field that combines game theory and quantum information. It lies at the crossroads of physics, quantum information processing, computer and natural sciences. Various quantizations of games were presented by different authors [1–5].

Quantum pseudo-telepathy games [6] form a subclass of quantum games. A game belongs to the pseudo-telepathy class providing that there are no winning strategies for classical players, but a winning strategy can be found if the players share a sufficient amount of entanglement. In these games quantum players can accomplish tasks that are unfeasible for their classical counterparts. It has been shown [7] that noise in a quantum channel can decrease the probability of winning the Magic Square game even below the classical threshold.

Noise is an unavoidable ingredient of a quantum system. Therefore its thorough investigation is a fundamental issue in quantum information processing. Quantum game theory has several potential applications (e.g. quantum auctions [8]) that may be hindered by noise effects. Our previous investigation of quantum noise effects in quantum games [7,9,10], and quantum algorithms performance [11] revealed several interesting issues that act as an incentive of the present work. The tools developed in this work can be used to analyse the behaviour of quantum channels in other settings.

Motivation

The motivation to study the Magic Square game and pseudo-telepathy games in general is that their physical implementation could provide convincing, even to a layperson, demonstration that the physical world is not local realistic. By *local* we mean that no action performed at some location X can have an effect on some remote location Y in a time shorter than that required by light to

travel from X to Y. *Realistic* means that a measurement can only reveal elements of reality that are already present in the system [6].

Given a pseudo-telepathy game, one can implement a quantum winning strategy for this game [6]. In an ideal case, the experiment should involve a significant number of rounds of the game. The experiment should be continued until either the players lose a single round or the players win such a great number of rounds, that it would be nearly impossible if they were using a classical strategy.

In the particular case of the magic square game the classical strategy allows the players to achieve the success rate no larger than $\frac{8}{9}$. In theory, the success rate of the quantum strategy is equal to one. But any physical implementations of a quantum protocol cannot be perfect because it is subject to noise.

In particular, the players, Alice and Bob, must be so far away from each other that the time between the question and their respective answers is shorter than the time required by light to travel between their locations. This set-up involves sending parts of an entangled quantum state to two remote locations. Sending qubits through a channel will inevitably add noise to the system. Our aim is to counteract this noise. In this paper we focus on the destructive aspects of the process of transmission of a qubit through a noisy separable quantum channel and introduce a scheme that allows the partial reversion of the channel action. This reversion gives rise to the players' success rate above the classical limit of $\frac{8}{9}$ for some parameters of noisy channels. Our scheme for reversing an action of a noisy channel may prove valuable in future experimental set-ups of such games.

Magic square game

The magic square is a 3×3 matrix filled with numbers 0 or 1 so that the sum of entries in each row is even and the sum of entries in each column is odd. Although such a matrix cannot exist (see Table 1) one can consider the following game.

Table 1. An illustrative filling of the magic square with numbers 0 and 1.

1	1	0
1	0	1
1	0	?

The question mark shows that it is not possible to put a number in the last field and satisfy both conditions of the game.
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The game setup is as follows. There are two players: Alice and Bob. Alice is given a row, Bob is given a column. Alice has to give the entries for a row and Bob has to give entries for a column so that the parity conditions are met. Winning condition is that the players' entries at the intersection must agree. Alice and Bob can prepare a strategy but they are not allowed to communicate during the game.

There exists a (classical) strategy that guarantees the winning probability of $\frac{8}{9}$. If the parties are allowed to share a quantum state they can achieve probability of success equal to one [6].

In the quantum version of this game [12,13] Alice and Bob are allowed to share an entangled quantum state. The winning strategy is following. Alice and Bob share an entangled state:

$$|\psi\rangle = \frac{1}{2}(|0011\rangle + |1100\rangle - |0110\rangle - |1001\rangle) \quad (1)$$

and apply local unitary operators forming operator $A_i \otimes B_j$, where

$$\begin{aligned} \bullet A_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix}, \\ \bullet A_2 &= \frac{1}{2} \begin{pmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{pmatrix}, \\ \bullet A_3 &= \frac{1}{2} \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \bullet B_1 &= \frac{1}{2} \begin{pmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & 1 & 1 \end{pmatrix}, \\ \bullet B_2 &= \frac{1}{2} \begin{pmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{pmatrix}, \\ \bullet B_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}. \end{aligned}$$

Indices i and j label rows and columns of the magic square. The state of this scheme before measurement is

$$\rho_f = (A_i \otimes B_j) |\psi\rangle \langle \psi| (A_i^\dagger \otimes B_j^\dagger). \quad (2)$$

The final step of the game consists of the measurement in the computational basis.

In [7], the situation where the initial state $|\psi\rangle$ is corrupted by the noise was investigated. Therefore, Eq. 2 is transformed into

$$\rho_f = (A_i \otimes B_j) \Phi_\alpha(|\psi\rangle \langle \psi|) (A_i^\dagger \otimes B_j^\dagger), \quad (3)$$

where Φ_α denotes one-parameter family of noisy quantum channels.

In such a case it is justified to inquire what is the mean probability of Alice and Bob's success given the amount of noise introduced by channel Φ_α . The mean probability $p(\alpha)$ of measuring the outcome yielding success in the state ρ_f is given by

$$p(\alpha) = \frac{1}{9} \sum_{i,j=1}^3 \sum_{\zeta \in \mathcal{S}_{ij}} \text{tr} \rho_f |\zeta\rangle \langle \zeta|, \quad (4)$$

where \mathcal{S}_{ij} is the set of right answers for the column and row ij (Table 2). The mean is taken over all pairs (i,j) .

A winning strategy exists for noiseless channels. In the case of noisy channel, the same strategy gives a higher probability of winning than in the classical case for low noise channels [7]. The objective of this work is to find local channels that partially reverse the action of the noise and therefore extends the pseudo-telepathy

Table 2. Sets \mathcal{S}_{ij} – plus sign (+) indicates that the given element belongs to the set, minus (–) sign indicates that the element does not belong to the set.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
\mathcal{S}_{11}	+	+	–	–	+	+	–	–	–	–	+	+	–	–	+	+
\mathcal{S}_{12}	+	+	–	–	–	–	+	+	+	+	–	–	–	–	+	+
\mathcal{S}_{13}	+	+	–	–	–	–	+	+	–	–	+	+	+	+	–	–
\mathcal{S}_{21}	+	–	+	–	+	–	+	–	–	+	–	+	–	+	–	+
\mathcal{S}_{22}	+	–	+	–	–	+	–	+	+	–	+	–	–	+	–	+
\mathcal{S}_{23}	+	–	+	–	–	+	–	+	–	+	–	+	+	–	+	–
\mathcal{S}_{31}	–	+	+	–	–	+	+	–	+	–	–	+	+	–	–	+
\mathcal{S}_{32}	–	+	+	–	+	–	–	+	–	+	+	–	+	–	–	+
\mathcal{S}_{33}	–	+	+	–	+	–	–	+	+	–	–	+	–	+	+	–

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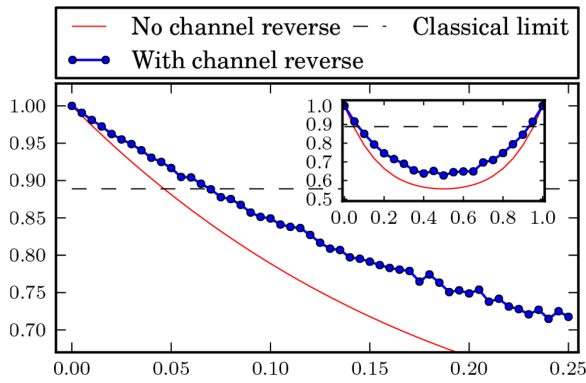


Figure 1. Phase flip channel. Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter α for the phase flip channel. The inset shows the probability of winning for $\alpha \in [0; 1]$. doi:10.1371/journal.pone.0064694.g001

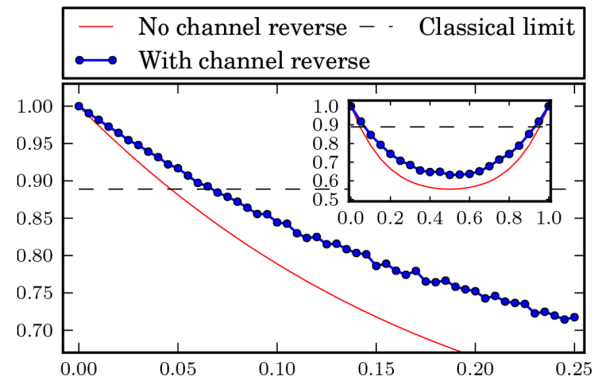


Figure 3. Bit phase flip channel. Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter α for the bit phase flip channel. The inset shows the probability of winning for $\alpha \in [0; 1]$. doi:10.1371/journal.pone.0064694.g003

to channels with higher noise. In order to achieve this, Eq. 3 is transformed into

$$\rho_f = (A_i \otimes B_j) \Psi_\alpha(\Phi_\alpha(|\psi\rangle\langle\psi|))(A_i^\dagger \otimes B_j^\dagger), \quad (5)$$

where Ψ_α denotes local channel with respect to Alice and Bob's subsystems that allows to raise their probability of winning $p(\alpha)$. In order to achieve that a series of semi-definite optimization programs has to be numerically solved.

Quantum channels

In the most general case, the evolution of a quantum system can be described using the notion of a *quantum channel* [14–16]. A quantum channel is a mapping acting on density operators $\rho \in D(\mathcal{H})$, i.e., operators where $\rho \geq 0$ and $\text{tr}(\rho) = 1$ on a Hilbert space \mathcal{H}_1 and transforming them into operators on a another Hilbert space \mathcal{H}_2 . Thus

$$\Phi : L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2), \quad (6)$$

where $L(\mathcal{H}_i)$ denotes the set of linear operators on \mathcal{H}_i . To form a

proper quantum channel, the mapping Φ must satisfy the following restrictions:

1. Φ must be *trace-preserving*, that is $\text{tr}(\Phi(\rho)) = \text{tr}(\rho)$,
2. Φ must be *completely positive*, that is $\Phi \otimes \mathbb{I}_{L(\mathcal{H}_3)}$ is a positive mapping, thus

$$(\Phi \otimes \mathbb{I}_{L(\mathcal{H}_3)})(\rho) \in D(\mathcal{H}_2 \otimes \mathcal{H}_3), \quad (7)$$

for every choice of $\rho \in D(\mathcal{H}_1 \otimes \mathcal{H}_3)$ and every choice of finite-dimensional Hilbert space \mathcal{H}_3 , where $\mathbb{I}_{L(\mathcal{H}_3)}$ is an identity channel on the space $L(\mathcal{H}_3)$.

The notion of a *product* quantum channel is introduced as follows [17]. For any choice of quantum channels that satisfy

$$\Phi_1 : L(\mathcal{H}_1^1) \rightarrow L(\mathcal{H}_2^1), \dots, \Phi_N : L(\mathcal{H}_1^N) \rightarrow L(\mathcal{H}_2^N), \quad (8)$$

we define a linear mapping

$$\Phi_1 \otimes \dots \otimes \Phi_N : L(\mathcal{H}_1^1 \otimes \dots \otimes \mathcal{H}_1^N) \rightarrow L(\mathcal{H}_2^1 \otimes \dots \otimes \mathcal{H}_2^N), \quad (9)$$

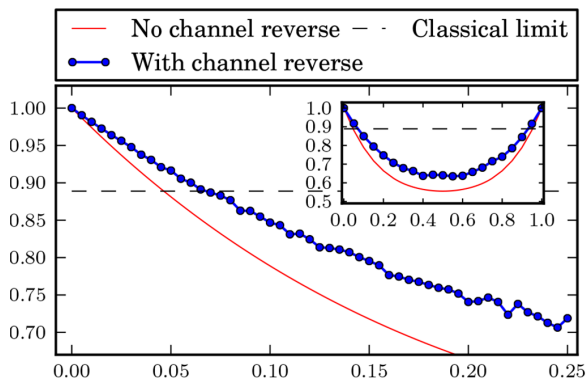


Figure 2. Bit flip channel. Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter α for the bit flip channel. The inset shows the probability of winning for $\alpha \in [0; 1]$. doi:10.1371/journal.pone.0064694.g002

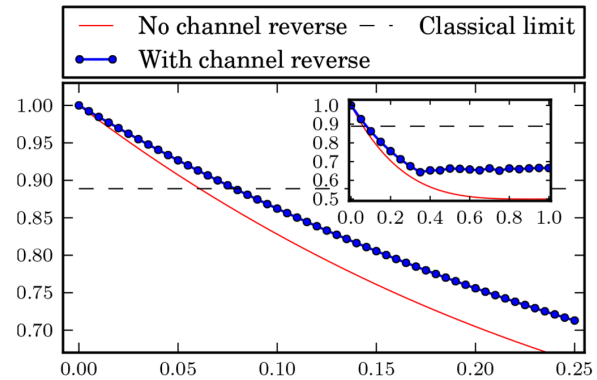


Figure 4. Depolarising channel. Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter α for the depolarising channel. The inset shows the probability of winning for $\alpha \in [0; 1]$. doi:10.1371/journal.pone.0064694.g004

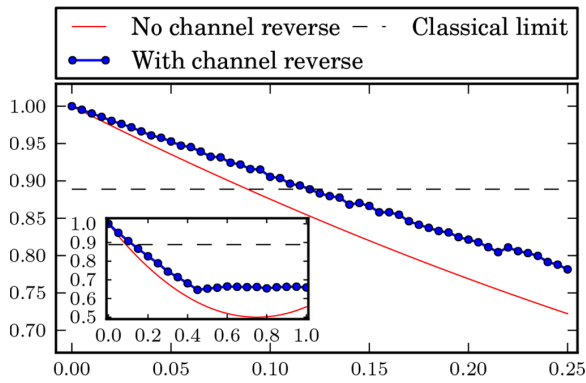


Figure 5. Amplitude damping channel. Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter α for the amplitude damping channel. The inset shows the probability of winning for $\alpha \in [0; 1]$. doi:10.1371/journal.pone.0064694.g005

to be the unique mapping that satisfies the equation

$$(\Phi_1 \otimes \dots \otimes \Phi_N)(A_1 \otimes \dots \otimes A_N) = \Phi_1(A_1) \otimes \dots \otimes \Phi_N(A_N), \quad (10)$$

for all operators $A_1 \in L(\mathcal{H}_1^1), \dots, A_N \in L(\mathcal{H}_1^N)$.

Many different representations of quantum channels can be chosen, depending on the application. Among these are the Jamiolkowski representation, the Kraus representation and the Stinespring representation. These three representations will be used throughout this paper.

The Jamiolkowski representation of a quantum channel Φ is given by

$$\mathcal{J}(\Phi) = \sum_{a,b} \Phi(E_{a,b}) \otimes E_{a,b}, \quad (11)$$

where $E_{a,b} \in L(\mathcal{H}_1)$ are operators with all entries equal to zero, except the entry a,b equal to one. From this definition, it is straightforward to observe that $\mathcal{J}(\Phi) \in L(\mathcal{H}_2 \otimes \mathcal{H}_1)$. By the Choi's [14] theorem a channel is completely positive if and only if $\mathcal{J}(\Phi) \geq 0$. It is trace-preserving if and only if

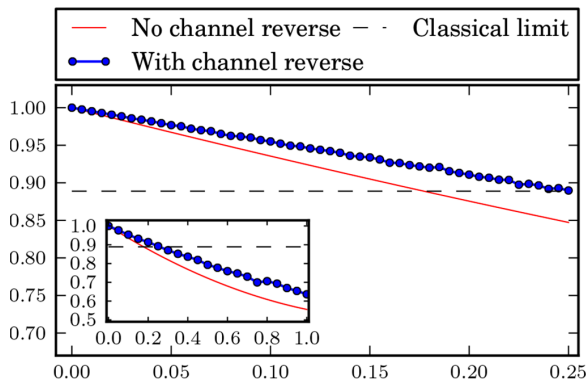


Figure 6. Phase damping channel. Probability of winning the pseudo-telepathy game with and without the use of our approach as a function of the noise parameter α for the phase damping channel. The inset shows the probability of winning for $\alpha \in [0; 1]$. doi:10.1371/journal.pone.0064694.g006

$$\text{tr}_{\mathcal{H}_2}(\mathcal{J}(\Phi)) = \mathbb{I}_{\mathcal{H}_1}. \quad (12)$$

Finally, the action of a quantum channel in the Jmiolkowski representation is given by

$$\Phi(\rho) = \text{tr}_{\mathcal{H}_1}(\mathcal{J}(\Phi)(\mathbb{I}_{\mathcal{H}_2} \otimes \rho^T)). \quad (13)$$

The Kraus representation of a quantum channel is given by a set of operators $E_k \in L(\mathcal{H}_1, \mathcal{H}_2)$. The action of quantum channel Φ is given by:

$$\Phi(\rho) = \sum_k E_k \rho E_k^\dagger. \quad (14)$$

This form ensures that the quantum channel is completely positive. For it to be also trace-preserving we need to impose the following constraint on the Kraus operators

$$\sum_k E_k^\dagger E_k = \mathbb{I}_{\mathcal{H}_1}. \quad (15)$$

Finally, given a mapping $\Phi : L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2)$ let us take another Hilbert space \mathcal{H}_3 such that $\dim(\mathcal{H}_3) = \text{rank}(J(\Phi))$ and a linear isometry $A \in U(\mathcal{H}_1, \mathcal{H}_2 \otimes \mathcal{H}_3)$. The action of a quantum channel is given by

$$\Phi(\rho) = \text{tr}_{\mathcal{H}_3}(A \rho A^\dagger). \quad (16)$$

This representation is called the Stinespring representation of Φ . For further discussion of quantum channels see e.g. [15] or [17].

Quantum noise

In the literature, several one-parameter families of qubit noisy channels are discussed [15]. For all the families of channels listed below the parameter $\alpha \in [0, 1]$ represents the amount of noise introduced by the channel. The symbols $\sigma_x, \sigma_y, \sigma_z$ denote Pauli operators. The Kraus operators for typical noisy channels are for

- • depolarising channel:

$$\left\{ \sqrt{1 - \frac{3\alpha}{4}} \mathbb{I}, \sqrt{\frac{\alpha}{4}} \sigma_x, \sqrt{\frac{\alpha}{4}} \sigma_y, \sqrt{\frac{\alpha}{4}} \sigma_z \right\},$$

- • amplitude damping: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\alpha} \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{\alpha} \\ 0 & 0 \end{pmatrix} \right\}$,

- • phase damping: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\alpha} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix} \right\}$,

- • phase flip: $\left\{ \sqrt{1-\alpha} \mathbb{I}, \sqrt{\alpha} \sigma_z \right\}$,

- • bit flip: $\left\{ \sqrt{1-\alpha} \mathbb{I}, \sqrt{\alpha} \sigma_x \right\}$,

```

input : A random Jamiołkowski matrix,  $Y$ , the matrices  $W$ ,  $\sigma$ ,  $\tau$  and the number of runs  $n$ 
output: Optimized values of the parts of the product channel

calculate  $M$ ;
for  $i \leftarrow 1$  to  $n$  do
     $Z = \text{SolveSDP}(\text{Eqn. (24)}, Y, M)$ ;
     $Y = \text{SolveSDP}(\text{Eqn. (25)}, Z, M)$ ;
end
return  $W(Y \otimes Z)W$ 
    
```

Figure 7. SDP optimization algorithm. SDP optimization of a product channel with a random Jamiołkowski matrix [20] as input. doi:10.1371/journal.pone.0064694.g007

- bit-phase flip: $\{\sqrt{1-\alpha}\mathbb{I}, \sqrt{\alpha}\sigma_y\}$.

In order to apply noise operators to multiple qubits we form new set of operators acting on a larger Hilbert space.

Given a set of n one-qubit Kraus operators $\{e_k\}_{k=1}^n$ it is possible to construct new set of n^N operators $\{E_k\}_{k=1}^{n^N}$ that act on a Hilbert space of dimension 2^N by taking Cartesian product of one-qubit Kraus operators in the following way

$$\{E_k\}_{k=1}^{n^N} = \{e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N}\}_{i_1, i_2, \dots, i_N=1}^n. \tag{17}$$

Application of the above to Kraus operators listed before gives one-parameter families of local noisy channels. This form will be used in further investigations.

Results and Discussion

We propose the following scheme for reversing an action of a channel using semidefinite programming (SDP). In our case, the most useful formulation of a semidefinite program is as follows (after Watrous [17]).

A semidefinite program is a triple (Φ, A, B) where $\Phi : L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2)$ is a Hermiticity-preserving map and $A \in L(\mathcal{H}_1)$ and $B \in L(\mathcal{H}_2)$ are Hermitian operators for some choice of Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 . Two optimization problems are associated with the triple (Φ, A, B) , the primal and dual problems. We will focus our attention on the primal problem, which has the form:

$$\text{maximize : } \text{tr}(AX),$$

$$\text{subject to : } \Phi(X) = B,$$

$$X \geq 0.$$

In the case of the pseudo-telepathy game, it seems appropriate to look for a channel in a product form. This is due to the fact, that Alice and Bob are separated and each of them must apply a local channel. To model this situation, let us consider the Jamiołkowski representations of Alice's and Bob's channels, denoted Y and Z respectively. The resulting channel is given by

$$T = W(Y \otimes Z)W, \tag{18}$$

where W is an operator defined as follows

$$W = \mathbb{I}_{\mathcal{H}_2^A} \otimes U \otimes \mathbb{I}_{\mathcal{H}_1^B}, \tag{19}$$

where $U \in L(\mathcal{H}_1^A \otimes \mathcal{H}_2^B, \mathcal{H}_2^B \otimes \mathcal{H}_1^A)$ is the swap operation of subsystems \mathcal{H}_1^A and \mathcal{H}_2^B , defined as

$$U = \sum_{ij} |f_j e_i\rangle \langle e_i f_j|, \tag{20}$$

for e_i, f_j being elements of orthonormal bases of \mathcal{H}_1^A and \mathcal{H}_2^B respectively.

Next, let us denote by Ψ_N the noise channel and we put $\sigma = |\psi\rangle\langle\psi|$. For simplicity of further calculations, let us write $\tau = \Psi_N(\sigma)$ and $T = \mathcal{J}(\Psi_N)$. Consider the following maximization criterion problem

$$\text{maximize : } \text{tr}(\text{tr}_2(W(Y \otimes Z)W(\mathbb{I} \otimes \tau^T))\sigma), \tag{21}$$

which means we aim to find a channel that reverses the action of the noise channel as accurately as possible. Unfortunately, a maximization criterion in this form does not yield an SDP problem. To formulate this problem as an SDP, we first conduct some simple calculations that allow us to rewrite the maximization condition (21) as

$$\text{maximize : } \text{tr}((Y \otimes Z)M), \tag{22}$$

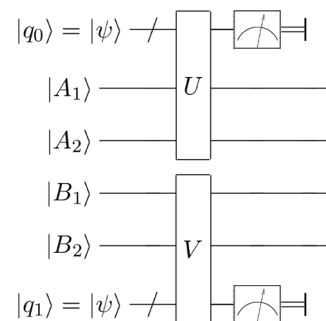


Figure 8. Quantum circuit. A quantum circuit showing the implementation of our scheme. A_i and B_j denote Alice's and Bob's qubits. q_0 and q_1 are the ancillary qubits they need to add. doi:10.1371/journal.pone.0064694.g008

$$\text{subject to : } M = W(\sigma \otimes \tau)W. \tag{23}$$

Considering the value of Y to be fixed and using the equation $\text{tr}((\mathbb{I} \otimes A)B) = \text{tr}(A \text{tr}_1(B))$, allows us to write the following SDP

$$\begin{aligned} &\text{maximize : } \text{tr}(Z \text{tr}_{\mathcal{H}_2^B, \mathcal{H}_1^B}(M(Y \otimes \mathbb{I}_{\mathcal{H}_2^B \otimes \mathcal{H}_1^B}))), \\ &\text{subject to : } \text{tr}_{\mathcal{H}_2^B}(Z) = \mathbb{I}_{\mathcal{H}_1^B}, \end{aligned} \tag{24}$$

$$Z \geq 0.$$

Fixing the value of Z and following a similar calculation give the following SDP problem

$$\begin{aligned} &\text{maximize : } \text{tr}(Y \text{tr}_{\mathcal{H}_2^A, \mathcal{H}_1^A}(\mathbb{I}_{\mathcal{H}_2^A \otimes \mathcal{H}_1^A} \otimes Z)M), \\ &\text{subject to : } \text{tr}_{\mathcal{H}_2^A}(Y) = \mathbb{I}_{\mathcal{H}_1^A}, \end{aligned} \tag{25}$$

$$Y \geq 0.$$

Now, we use the following algorithm to find an optimal channel. The algorithm in each iteration optimizes only a single part of the product channel. This algorithm was implemented using the SDPLR library [18,19].

Analysis

The numerical results are gathered in form of plots at the end of the paper. Figs. 1, 2, 3, 4, 5, 6 show the results of the optimization scheme shown in Fig. 7. The application of the SDP allowed us to achieve greater winning probability for all types of noisy channels.

In the case of the flip channels the obtained results are depicted in Figs. 1, 2 and 3. These plots show that it is possible to reverse the action of the noise channel for all values of the noise parameter α . Hence, we are able to observe quantum pseudo-telepathy for higher noise channels. Furthermore, the use of our optimization method results in a plot of probability of winning as a function of the noise parameter α which has a shape similar to the case when we do not try to reverse the action of a channel.

Next, we move to the depolarising channel. The results obtained in this case are shown in Fig. 4. Likewise, in this case our method has also allowed us to achieve pseudo-telepathy for higher values of the noise parameter α . The details are depicted in the inset in Fig. 4. Additionally, for values of the noise parameter

$\alpha \geq 0.45$ the probability of winning the game stabilizes around 0.65, opposed to the case with no channel action inverse, where it decreases to 0.5. Hence, we are able to retrieve some information in the case of high noise, local depolarising channels acting on many qubits.

Finally, we switch to the damping channels. Numerical results for this case are depicted in Figs. 5 and 6. Moreover, in this case we are able to reverse the action of a noise channel and broaden the pseudo-telepathy region. In the case of high values of the noise parameter α , results for the amplitude damping channel resemble those obtained for depolarising channel, as the probability of winning stabilizes around 0.65 for $\alpha \geq 0.45$ instead of decreasing to approximately 0.5.

Conclusions

The principal result of this paper is a methodology of partial denoising with the usage of local quantum channels. The presented tool can be used in the cases in which

- the parameters of the noise are accessible,
- the noisy channel is separable and acts independently on each qubit,
- the entangled quantum state the parties use is known in advance,
- the parties have access to quantum computers but
- are not allowed to communicate.

We have proposed a method to reverse an action of a quantum channel using semidefinite programming. The method allows us to find a product channel which partially reverses a given channel. We use the following scheme to achieve this goal. First, we fix all parts of the product, except for one, which is being optimized. After the SDP optimization, we move on to optimize the next part of the product channel, using the value obtained in the earlier step. We repeat this for all parts of the product channel. We run the process a great number of times to obtain a converging solution.

Obtained channel may be implemented on a real physical system using the Stinespring representation. An example of the quantum circuit implementing this scheme is shown in Fig. 8. Alice and Bob each add ancillary qubits to their original ones. Then they apply a unitary operator on their respective systems. Finally, they perform a measurement on the ancillary qubits, leaving their starting qubits in a less noisy state.

As an example of usage of this optimization scheme we present the quantum pseudo-telepathy magic square game. We obtained results showing an improvement in the players' success rate in the game. Specifically, we were able to broaden the range of the noise parameter α for which the quantum effect occurs.

Author Contributions

Conceived and designed the experiments: PG. Performed the experiments: LP. Analyzed the data: LP ZP. Contributed reagents/materials/analysis tools: PG ZP. Wrote the paper: LP PG ZP JS.

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