Densities of the Raney Distributions

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Abstract. We prove that if $p \geq 1$ and $0 < r \leq p$ then the sequence $\binom{mp+r}{m}\frac{r}{mp+r}$ is positive definite. More precisely, it is the moment sequence of a probability measure $\mu(p,r)$ with compact support contained in $[0,+\infty)$. This family of measures encompasses the multiplicative free powers of the Marchenko-Pastur distribution as well as the Wigner's semicircle distribution centered at $x = 2$. We show that if $p > 1$ is a rational number and $0 < r \leq p$ then $\mu(p,r)$ is absolutely continuous and its density $W_{p,r}(x)$ can be expressed in terms of the generalized hypergeometric functions. In some cases, including the multiplicative free square and the multiplicative free square root of the Marchenko-Pastur measure, $W_{p,r}(x)$ turns out to be an elementary function.

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Introduction

For $p, r \in \mathbb{R}$ we define the *Raney numbers* (or *two-parameter Fuss-Catalan numbers*) by

$$A_m(p,r) := \frac{r}{m!} \prod_{i=1}^{m-1} (mp + r - i), \tag{1}$$

$A_0(p,r) := 1$. We can also write

$$A_m(p,r) = \binom{mp+r}{m} \frac{r}{mp+r}, \tag{2}$$

(unless $mp + r = 0$), where the generalized binomial is defined by

$$\binom{a}{m} := \frac{a(a-1)\dots(a-m+1)}{m!}.$$

Let $\mathcal{B}_p(z)$ denote the generating function of the sequence $\{A_m(p,1)\}_{m=0}^{\infty}$, the *Fuss numbers of order* p:

$$\mathcal{B}_p(z) := \sum_{m=0}^{\infty} A_m(p,1) z^m, \tag{3}$$

convergent in some neighborhood of 0. For example

$$\mathcal{B}_2(z) = \frac{2}{1 + \sqrt{1 - 4z}}. \tag{4}$$

Lambert showed that

$$\mathcal{B}_p(z)^r = \sum_{m=0}^{\infty} A_m(p,r) z^m, \tag{5}$$

see [9]. These generating functions also satisfy

$$\mathcal{B}_p(z) = 1 + z\mathcal{B}_p(z)^p, \tag{6}$$

which reflects the identity $A_m(p,p) = A_{m+1}(p,1)$, and

$$\mathcal{B}_p(z) = \mathcal{B}_{p-r}\big(z\mathcal{B}_p(z)^r\big). \tag{7}$$

Using the free probability theory (see [28, 18, 6]) it was shown in [16] that if $p \geq 1$ and $0 \leq r \leq p$ then the sequence $\{A_m(p,r)\}_{m=0}^{\infty}$ is positive definite, i.e. is the moment sequence of a probability measure $\mu(p,r)$ on $\mathbb{R}$. Moreover, $\mu(p,r)$ has compact support (and therefore is unique) contained in the positive half-line $[0,\infty)$ (for example $\mu(p,0) = \delta_0$). The measures $\mu(p,r)$ satisfy some interesting relations, for example

$$\mu(p_1, r) \boxtimes \mu(1 + p_2, 1) = \mu(p_1 + rp_2, r) \tag{8}$$

and

$$\mu(p,r) \triangleright \mu(p+s,s) = \mu(p+s,r+s), \tag{9}$$

see [16], where "$\boxtimes$" and "$\triangleright$" denotes the multiplicative free and the monotonic convolution (see [17]). A relation analogous to (9) is also satisfied by the three-parameter family of distributions studied by Arizmendi and Hasebe [4].

Among the measures $\mu(p,r)$ perhaps the most important is the *Marchenko-Pastur* (called also the *free Poisson*) *distribution*

$$\mu(2,1) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,dx \qquad \text{on } [0,4], \tag{10}$$

which plays an important role in the theory of random matrices, see [29, 10, 11, 2, 1, 5]. It was proved in [1] that the multiplicative free power $\mu(2,1)^{\boxtimes n} = \mu(n+1,1)$ is the limit of the distribution of squared singular values of the power G^n of a random matrix G, when the size of the matrix G goes to infinity. The moments of $\mu(2,1)$, $A_m(2,1) = \binom{2m+1}{m}/(2m+1)$, are called *Catalan numbers* and play an important role in combinatorics, see A000108 in OEIS [24].

In this paper we are going to prove positive definiteness of $\{A_m(p,r)\}_{m=0}^{\infty}$ using more classical methods. Namely, we show that if $p>1$, $0<r\le p$ and if p is a rational number then $\mu(p,r)$ is absolutely continuous and can be represented as Mellin convolution of modified beta measures. Next we provide a formula for the density $W_{p,r}(x)$ of $\mu(p,r)$ in terms of the Meijer G-function and of the generalized hypergeometric functions (cf. [30, 21], where p was assumed to be an integer). This allows us to draw graphs of these densities and, in some particular cases, to express $W_{p,r}(x)$ as an elementary function.

Let us mention that the measures $\mu(2,1)^{\boxtimes p} = \mu(1+p,1)$ were also studied by Banica, Belinschi, Capitaine and Collins [5] as a special case of the *free Bessel laws*. They showed in particular that for $p>0$ this measure is absolutely continuous and its support is $[0,(p+1)^{p+1}p^{-p}]$. Liu, Song and Wang [14] found a formula expressing the density of $\mu(2,1)^{\boxtimes n}$, n natural, as integral of a certain kernel over $[0,1]^n$. Recently Haagerup and Möller [12] studied a two-parameter family $\mu_{\alpha,\beta}$, $\alpha,\beta>0$, of probability measures. The measures $\mu_{\alpha,0}$ coincide with our $\mu(1+\alpha,1)$, but if $\beta>0$ then $\mu_{\alpha,\beta}$ has noncompact support, so it does not coincide with any of $\mu(p,r)$. The authors found a formula for the density function of $\mu_{\alpha,\beta}$, which in the case of $W_{1+p,1}$ reads as follows:

$$W_{1+p,1}\left(\frac{\sin^{p+1}((p+1)t)}{\sin t\,\sin^{p}(pt)}\right) = \frac{\sin^2 t\,\sin^{p-1}(pt)}{\pi \sin^{p}((p+1)t)}, \tag{11}$$

for $0<t<\pi/(p+1)$. It can be used for drawing the graph of $W_{1+p,1}(x)$ by computer.

1 Preliminaries

For probability measures μ_1, μ_2 on the positive half-line $[0,\infty)$ the *Mellin convolution* is defined by

$$(\mu_1 \circ \mu_2)(A) := \int_0^\infty \int_0^\infty \mathbf{1}_A(xy) d\mu_1(x) d\mu_2(y) \tag{12}$$

for every Borel set $A \subseteq [0,\infty)$. This is the distribution of product $X_1 \cdot X_2$ of two independent nonnegative random variables with $X_i \sim \mu_i$. In particular, $\mu \circ \delta_c$ $(c > 0)$ is the *dilation* of μ:

$$(\mu \circ \delta_c)(A) = \mathbf{D}_c\mu(A) := \mu\left(\frac{1}{c}A\right).$$

If μ has density $f(x)$ then $\mathbf{D}_c(\mu)$ has density $f(x/c)/c$.
If both the measures μ_1, μ_2 have all *moments*

$$s_m(\mu_i) := \int_0^\infty x^m \, d\mu_i(x)$$

finite then so has $\mu_1 \circ \mu_2$ and

$$s_m(\mu_1 \circ \mu_2) = s_m(\mu_1) \cdot s_m(\mu_2)$$

for all m.
If μ_1, μ_2 are absolutely continuous, with densities f_1, f_2 respectively, then so is $\mu_1 \circ \mu_2$ and its density is given by the Mellin convolution:

$$(f_1 \circ f_2)(x) := \int_0^\infty f_1(x/y) f_2(y) \frac{dy}{y}.$$

We will need the following *modified beta measures*:

Lemma 1.1. *Let $u, v, l > 0$. Then*

$$\left\{ \frac{\Gamma(u+n/l)\Gamma(u+v)}{\Gamma(u+v+n/l)\Gamma(u)} \right\}_{n=0}^\infty$$

is the moment sequence of the probability measure

$$\mathbf{b}(u+v,u,l) := \frac{l}{\mathrm{B}(u,v)} x^{lu-1} \left(1 - x^l\right)^{v-1} dx \tag{13}$$

on $[0,1]$, where B is the Euler beta function.

Proof. Using the substitution $t = x^l$ we obtain:

$$\begin{aligned} \frac{\Gamma(u+n/l)\Gamma(u+v)}{\Gamma(u+v+n/l)\Gamma(u)} &= \frac{\mathrm{B}(u+n/l,v)}{\mathrm{B}(u,v)} = \frac{1}{\mathrm{B}(u,v)} \int_0^1 t^{u+n/l-1}(1-t)^{v-1} dt \\ &= \frac{l}{\mathrm{B}(u,v)} \int_0^1 x^{lu+n-1} \left(1 - x^l\right)^{v-1} dx. \end{aligned}$$

□

Note that if X is a positive random variable whose distribution has density $f(x)$ and if $l > 0$ then the distribution of $X^{1/l}$ has density $lx^{l-1}f(x^l)$. In particular, if the distribution of a random variable X is $\mathbf{b}(u+v,u,1)$ then the distribution of $X^{1/l}$ is $\mathbf{b}(u+v,u,l)$. For $u,l>0$ we also define

$$\mathbf{b}(u,u,l) := \delta_1. \tag{14}$$

2 Applying Mellin convolution

From now on we assume that $p > 1$ is a rational number, say $p = k/l$, with $1 \le l < k$, and that $0 < r \le p$. We will show that then $A_m(p,r)$ is the moment sequence of a probability measure $\mu(p,r)$, which can be represented as Mellin convolution of modified beta measures. In particular, $\mu(p,r)$ is absolutely continuous and we will denote its density by $W_{p,r}$. The case when p is an integer was studied in [21, 30].

First we need to express the numbers $A_m(p,r)$ in a special form.

Lemma 2.1. *If $p = k/l$, where k,l are integers, $1 \le l < k$ and $0 < r \le p$ then*

$$A_m(p,r) = \frac{r}{l\sqrt{2\pi k(p-1)}}\left(\frac{p}{p-1}\right)^r \frac{\prod_{j=1}^k \Gamma(\beta_j + m/l)}{\prod_{j=1}^k \Gamma(\alpha_j + m/l)} c(p)^m, \tag{15}$$

where $c(p) = p^p(p-1)^{1-p}$,

$$\alpha_j = \begin{cases} \dfrac{j}{l} & \text{if } 1 \le j \le l,\\[2mm] \dfrac{r+j-l}{k-l} & \text{if } l+1 \le j \le k, \end{cases} \tag{16}$$

$$\beta_j = \frac{r+j-1}{k}, \qquad 1 \le j \le k. \tag{17}$$

Proof. First we write:

$$\binom{mp+r}{m}\frac{r}{mp+r} = \frac{r\Gamma(mp+r)}{\Gamma(m+1)\Gamma(mp-m+r+1)}. \tag{18}$$

Now we apply the Gauss's multiplication formula:

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right)$$

to get:

$$\begin{aligned}\Gamma(mp+r) &= \Gamma\left(k\left(\frac{m}{l}+\frac{r}{k}\right)\right)\\ &= (2\pi)^{(1-k)/2}k^{mk/l+r-1/2}\prod_{j=1}^{k}\Gamma\left(\frac{m}{l}+\frac{r+j-1}{k}\right),\end{aligned}$$

$$\Gamma(m+1) = \Gamma\left(l\frac{m+1}{l}\right) = (2\pi)^{(1-l)/2} l^{m+1/2}\prod_{j=1}^{l}\Gamma\left(\frac{m}{l}+\frac{j}{l}\right)$$

and

$$\Gamma(mp - m + r + 1) = \Gamma\left((k-l)\left(\frac{m}{l} + \frac{r+1}{k-l}\right)\right)$$
$$= (2\pi)^{(1-k+l)/2}(k-l)^{m(k-l)/l+r+1/2} \prod_{j=l+1}^{k} \Gamma\left(\frac{m}{l} + \frac{r+j-l}{k-l}\right).$$

It remains to use them in (18). □

In order to apply Lemma 1.1 we need to modify enumeration of α's.

Lemma 2.2. *For* $1 \leq i \leq l+1$ *denote*

$$j_i := \left\lfloor \frac{(i-1)k}{l} \right\rfloor + 1,$$

where $\lfloor\cdot\rfloor$ *is the floor function, so that*

$$1 = j_1 < j_2 < \ldots < j_l < k < k+1 = j_{l+1}.$$

For $1 \leq j \leq k$ *define*

$$\widetilde{\alpha}_j = \begin{cases} \dfrac{i}{l} & \text{if } j = j_i,\ 1 \leq i \leq l, \\ \dfrac{r+j-i}{k-l} & \text{if } j_i < j < j_{i+1}. \end{cases} \tag{19}$$

Then the sequence $\{\widetilde{\alpha}_j\}_{j=1}^k$ *is a rearrangement of* $\{\alpha_j\}_{j=1}^k$.
Moreover, if $0 < r \leq p = k/l$ *then we have* $\beta_j \leq \widetilde{\alpha}_j$ *for all* $j \leq k$.

Proof. It is easy to verify the first statement.
Assume that $j = j_i$ for some $i \leq l$. We have to show that

$$\frac{r + j_i - 1}{k} \leq \frac{i}{l},$$

which is equivalent to

$$lr + l\left\lfloor \frac{k(i-1)}{l} \right\rfloor \leq ki.$$

The latter is a consequence of the fact that $\lfloor x\rfloor \leq x$ and the assumption that $r \leq p = k/l$.
Now assume that $j_i < j < j_{i+1}$. We ought to show that

$$\frac{r+j-1}{k} \leq \frac{r+j-i}{k-l},$$

which is equivalent to

$$lr + lj + k - l - ki \geq 0.$$

Using the inequality $\lfloor x \rfloor + 1 > x$ we obtain

$$\begin{aligned} lj + k - l - ki &\geq l(j_i + 1) + k - l - ki \\ &= lj_i + k - ki > k(i-1) + k - ki = 0, \end{aligned}$$

which completes the proof, as $r > 0$. $\square$

Now we are ready to prove the main theorem of this section.

Theorem 2.3. *Suppose that $p = k/l$, where k, l are integers, $1 \leq l < k$, and that r is a real number such that $0 < r \leq p$. Then there exists a unique probability measure $\mu(p,r)$ such that (1) is its moment sequence. Moreover $\mu(p,r)$ can be represented as the following Mellin convolution:*

$$\mu(p,r) = \mathbf{b}(\widetilde{\alpha}_1, \beta_1, l) \circ \ldots \circ \mathbf{b}(\widetilde{\alpha}_k, \beta_k, l) \circ \delta_{c(p)},$$

where

$$c(p) := \frac{p^p}{(p-1)^{p-1}}.$$

Consequently, $\mu(p,r)$ is absolutely continuous and its support is $[0, c(p)]$.

It is easy to see that the density function is positive on $(0, c(p))$. The representation of densities in the form of Mellin convolution of modified beta measures was used in different context in [8], see its Appendix A.

Example. For the Marchenko-Pastur measure we get the following decomposition:

$$\mu(2,1) = \mathbf{b}(1, 1/2, 1) \circ \mathbf{b}(2,1,1) \circ \delta_4, \tag{20}$$

where $\mathbf{b}(1,1/2,1)$ has density $1/(\pi\sqrt{x - x^2})$ on $[0,1]$, the arcsine distribution with the moment sequence $\binom{2m}{m}4^{-m}$, and $\mathbf{b}(2,1,1)$ is the Lebesgue measure on $[0,1]$ with the moment sequence $1/(m+1)$.

Proof. In view of Lemma 2.1 and Lemma 2.2 we can write

$$A_m(p,r) = D \prod_{j=1}^{k} \frac{\Gamma(\beta_j + m/l)\Gamma(\widetilde{\alpha}_j)}{\Gamma(\widetilde{\alpha}_j + m/l)\Gamma(\beta_j)} \cdot c(p)^m$$

for some constant D. Taking $m = 0$ we see that $D = 1$. $\square$

Note that a part of the theorem illustrates a result of Kargin [13], who proved that if μ is a compactly supported probability measure on $[0, \infty)$, with expectation 1 and variance V, and if L_n denotes the supremum of the support of the multiplicative free convolution power $\mu^{\boxtimes n}$, then

$$\lim_{n\to\infty} \frac{L_n}{n} = eV, \tag{21}$$

where $e = 2.71\ldots$ is the Euler's number. The Marchenko-Pastur measure $\mu(2,1)$ has expectation and variance equal to 1 and $\mu(2,1)^{\boxtimes n} = \mu(n+1, 1)$, so in this case $L_n = (n+1)^{n+1}/n^n$ (this was also proved in [29] and [11]) and (21) holds.

The density function for $\mu(p,r)$ will be denoted by $W_{p,r}(x)$. Since $A_m(p,p) = A_{m+1}(p,1)$, we have

$$W_{p,p}(x) = x \cdot W_{p,1}(x), \tag{22}$$

for example

$$W_{2,2}(x) = \frac{1}{2\pi}\sqrt{x(4-x)} \qquad \text{on } [0,4], \tag{23}$$

which is the semicircle Wigner distribution with radius 2, centered at $x = 2$.

Now we can reprove the main result of [16].

Theorem 2.4. *Suppose that p, r are real numbers satisfying $p \geq 1$, $0 \leq r \leq p$. Then there exists a unique probability measure $\mu(p,r)$, with compact support contained in $[0, c(p)]$, such that $\{A_m(p,r)\}_{m=0}^{\infty}$ is its moment sequence.*

Proof. It follows from the fact that the class of positive definite sequence is closed under pointwise limits. □

Remark. In view of Theorem 2.1 in [5], for every $p > 1$ the measure $\mu(p,1)$ is absolutely continuous and its support is equal $[0, c(p)]$, see also [14, 12].

3 Applying Meijer G-function

The aim of this section is to describe the density function $W_{p,r}(x)$ of $\mu(p,r)$ in terms of the Meijer G-function (see [19] for example) and consequently, as a linear combination of generalized hypergeometric functions. We will see that in some particular cases $W_{p,r}$ can be represented as an elementary function.

For $p > 1$, $r > 0$ define an analytic function

$$\phi_{p,r}(\sigma) = \frac{r\Gamma\big((\sigma-1)p+r\big)}{\Gamma(\sigma)\Gamma\big((\sigma-1)(p-1)+r+1\big)},$$

which is well defined whenever $(\sigma-1)p+r$ is not a nonpositive integer. Note that $\phi_{p,1}(\sigma+1) = \phi_{p,p}(\sigma)$ and if m is a natural number then

$$\phi_{p,r}(m+1) = \binom{mp+r}{m}\frac{r}{mp+r}.$$

Then we define $W_{p,r}$ as the inverse Mellin transform:

$$W_{p,r}(x) = \frac{1}{2\pi \mathrm{i}}\int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} x^{-\sigma}\phi_{p,r}(\sigma)\,d\sigma,$$

$x > 0$, if exists, see [25] for details. It turn out that if $p > 1$ is a rational number then $W_{p,r}$ can be expressed in terms of the Meijer G-function and its Mellin transform is $\phi_{p,r}$. For the theory of the Meijer G-functions we refer to [15, 23, 19].

Theorem 3.1. *Suppose that $p = k/l$, where k, l are integers, $1 \le l < k$ and $r > 0$. Then $W_{p,r}(x)$ is well defined and*

$$W_{p,r}(x) = \frac{rp^r}{x(p-1)^{r+1/2}\sqrt{2k\pi}}\, G^{k,0}_{k,k}\left(\frac{x^l}{c(p)^l}\,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_k \\ \beta_1, \ldots, \beta_k\end{matrix}\right), \tag{24}$$

$x \in (0, c(p))$, where $c(p) = p^p(p-1)^{1-p}$ and the parameters α_j, β_j are given by (16) and (17). Moreover, $\phi_{p,r}$ is the Mellin transform of $W_{p,r}$, namely

$$\phi_{p,r}(\sigma) = \int_0^{c(p)} x^{\sigma-1} W_{p,r}(x)\, dx, \tag{25}$$

for $\Re\sigma > 1 - r/p$.

If $0 < r \le p$ then $W_{p,r}(x) > 0$ for $0 < x < c(p)$ and therefore $W_{p,r}$ is the density function of the probability distribution $\mu(p, r)$.

Proof. Putting $m = \sigma - 1$ in (15) we get

$$\phi_{p,r}(\sigma) = \frac{r(p-1)^{p-r-3/2}}{lp^{p-r}\sqrt{2k\pi}} \frac{\prod_{j=1}^k \Gamma(\beta_j - 1/l + \sigma/l)}{\prod_{j=1}^k \Gamma(\alpha_j - 1/l + \sigma/l)} c(p)^\sigma. \tag{26}$$

Writing the right hand side as $\Phi(\sigma/l - 1/l)c(p)^\sigma$, using the substitution $\sigma = lu + 1$ and the definition of the Meijer G-function (see [19] for example), we obtain

$$\begin{aligned}
W_{p,r}(x) &= \frac{1}{2\pi \mathrm{i}} \int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} \Phi(\sigma/l - 1/l) c(p)^\sigma x^{-\sigma} d\sigma \\
&= \frac{lc(p)}{2\pi x \mathrm{i}} \int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} \Phi(u) \left(x^l/c(p)^l\right)^{-u} du \\
&= \frac{rp^r}{x(p-1)^{r+1/2}\sqrt{2k\pi}} G^{k,0}_{k,k}\left(z\,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_k \\ \beta_1, \ldots, \beta_k\end{matrix}\right),
\end{aligned}$$

where $z = x^l/c(p)^l$. Recall that for the Meijer function of type $G^{k,0}_{k,k}$ there is no restriction on the parameters and the integral converges for $0 < x < c(p)$ (see 16.17.1 in [19]).

On the other hand, substituting $x = c(p)t^{1/l}$ we can write

$$\begin{aligned}
&\int_0^{c(p)} x^{\sigma-1} W_{p,r}(x)\, dx \\
&\quad = \frac{rp^r}{(p-1)^{r+1/2}\sqrt{2k\pi}} \int_0^{c(p)} x^{\sigma-2} G^{k,0}_{k,k}\left(\frac{x^l}{c(p)^l}\,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_k \\ \beta_1, \ldots, \beta_k\end{matrix}\right) dx. \\
&\quad = \frac{rp^r c(p)^{\sigma-1}}{l(p-1)^{r+1/2}\sqrt{2k\pi}} \int_0^1 t^{(\sigma-1)/l-1} G^{k,0}_{k,k}\left(t\,\middle|\, \begin{matrix}\alpha_1, \ldots, \alpha_k \\ \beta_1, \ldots, \beta_k\end{matrix}\right) dt.
\end{aligned}$$

Since $\sum_{j=1}^{k}(\beta_j-\alpha_j)=-3/2<0$, so the assumptions of (2.24.2.1) in [23], the third case, are satisfied and therefore the last integral is convergent provided

$$-\frac{r}{k}=-\min\beta_j<\Re\frac{\sigma-1}{l},$$

(equivalently: $\Re\sigma>1-r/p$) and the whole expression is equal to the right hand side of (26).

For the last statement we note that in view of Theorem 2.3, of the uniqueness part of the Riesz representation theorem for linear functionals on $\mathcal{C}[0,c(p)]$ and of the Weierstrass approximation theorem, for $0<r\le p$ the density function of $\mu(p,r)$ must coincide with $W_{p,r}$. ☐

Now applying Slater's formula we can express $W_{p,r}$ as a linear combination of hypergeometric functions.

Theorem 3.2. *For $p=k/l$, with $1\le l<k$, $r>0$, and $x\in(0,c(p))$ we have*

$$W_{p,r}(x)=\gamma(k,l,r)\sum_{h=1}^{k}c(h,k,l,r)\,{}_kF_{k-1}\left(\left.\begin{matrix}\mathbf{a}(h,k,l,r)\\ \mathbf{b}(h,k,l,r)\end{matrix}\right|z\right)z^{(r+h-1)/k-1/l}, \tag{27}$$

where $z=x^l/c(p)^l$,

$$\gamma(k,l,r)=\frac{r(p-1)^{p-r-3/2}}{p^{p-r}\sqrt{2k\pi}}, \tag{28}$$

$$c(h,k,l,r)=\frac{\prod_{j=1}^{h-1}\Gamma\left(\frac{j-h}{k}\right)\prod_{j=h+1}^{k}\Gamma\left(\frac{j-h}{k}\right)}{\prod_{j=1}^{l}\Gamma\left(\frac{j}{l}-\frac{r+h-1}{k}\right)\prod_{j=l+1}^{k}\Gamma\left(\frac{r+j-l}{k-l}-\frac{r+h-1}{k}\right)}, \tag{29}$$

and the parameter vectors of the hypergeometric functions are

$$\mathbf{a}(h,k,l,r)=\left(\left\{\frac{r+h-1}{k}-\frac{j-l}{l}\right\}_{j=1}^{l},\left\{\frac{r+h-1}{k}-\frac{r+j-k}{k-l}\right\}_{j=l+1}^{k}\right), \tag{30}$$

$$\mathbf{b}(h,k,l,r)=\left(\left\{\frac{k+h-j}{k}\right\}_{j=1}^{h-1},\left\{\frac{k+h-j}{k}\right\}_{j=h+1}^{k}\right). \tag{31}$$

Proof. Putting $z=x^l/c(p)^l$, and hence $x=c(p)z^{1/l}$, we can rewrite (24) as

$$W_{p,r}(x)=\frac{r(p-1)^{p-r-3/2}}{z^{1/l}p^{p-r}\sqrt{2k\pi}}\,G^{k,0}_{k,k}\left(z\left|\begin{matrix}\alpha_1,\ \dots,\ \alpha_k\\ \beta_1,\ \dots,\ \beta_k\end{matrix}\right.\right), \tag{32}$$

$x\in(0,c(p))$. Observe that for $1\le i<j\le k$ the difference $\beta_j-\beta_i=(j-i)/k$ is never an integer. Therefore we can apply formula (8.2.2.3) in [23] (see also (16.17.2) in [19] or formula (7) on page 145 in [15]), so that

$$c(h,k,l,r)=\frac{\prod_{j\neq h}\Gamma(\beta_j-\beta_h)}{\prod_{j=1}^{k}\Gamma(\alpha_j-\beta_h)},$$

which gives (29). For the parameter vectors we have

$$\mathbf{a}(h,k,l,r)_j = 1+\beta_h-\alpha_j$$

and

$$\mathbf{b}(h,k,l,r)_j = 1+\beta_h-\beta_j, \qquad j\neq h,$$

which leads to (30) and (31). Finally, the summand with index h is in addition multiplied by $z^{\beta_h-1/l}$. □

Theorem 3.1 and Theorem 3.2 are sufficient for drawing graphs of the functions $W_{p,r}$ with help of computer programs. In some cases however it is possible to express $W_{p,r}$ as an elementary function. The most tractable case is $p=2$. We know already that

$$W_{2,1}(x)=\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, \qquad W_{2,2}(x)=\frac{1}{2\pi}\sqrt{x(4-x)}.$$

Now we can give a simple formula for $W_{2,r}$.

Corollary 3.3. *For $p=2$, $r>0$, the function $W_{2,r}$ is*

$$W_{2,r}(x)=\frac{\sin\left(r\cdot\arccos\sqrt{x/4}\right)}{\pi x^{1-r/2}}, \tag{33}$$

$x\in(0,4)$. If $0<r\le 2$ then $W_{2,r}$ is the density function of the measure $\mu(2,r)$. In particular for $r=1/2$ and $r=3/2$ we have

$$W_{2,1/2}(x)=\frac{\sqrt{2-\sqrt{x}}}{2\pi x^{3/4}}, \tag{34}$$

$$W_{2,3/2}(x)=\frac{(\sqrt{x}+1)\sqrt{2-\sqrt{x}}}{2\pi x^{1/4}}. \tag{35}$$

Note that if $r>2$ then $W_{2,r}(x)<0$ for some values of $x\in(0,4)$.

Proof. We take $k=2$, $l=1$ so that $c(2)=4$, $z=x/4$ and $\gamma(2,1,r)=r2^r/(8\sqrt{\pi})$. Using the Euler's reflection formula and the identity $\Gamma(1+r/2)=\Gamma(r/2)r/2$ we get

$$c(1,2,1,r)=\frac{\Gamma(1/2)}{\Gamma(1-r/2)\Gamma(1+r/2)}=\frac{2\sin(\pi r/2)}{r\sqrt{\pi}},$$

$$c(2,2,1,r)=\frac{\Gamma(-1/2)}{\Gamma\big((1-r)/2\big)\Gamma\big((1+r)/2\big)}=\frac{-2\cos(\pi r/2)}{\sqrt{\pi}}.$$

We also need formulas for two hypergeometric functions, namely

$$ {}_2F_1\left(\frac{r}{2}, \frac{-r}{2}; \frac{1}{2} \middle| z\right) = \cos(r \arcsin\sqrt{z}), $$

$$ {}_2F_1\left(\frac{1+r}{2}, \frac{1-r}{2}; \frac{3}{2} \middle| z\right) = \frac{\sin(r \arcsin\sqrt{z})}{r\sqrt{z}}, $$

see 15.4.12 and 15.4.16 in [19]. Now we can write

$$ W_{2,r}(x) = \frac{\sin(\pi r/2)\cos\left(r \arcsin\sqrt{x/4}\right) - \cos(\pi r/2)\sin\left(r \arcsin\sqrt{x/4}\right)}{\pi x^{1-r/2}} $$

$$ = \frac{\sin\left(\pi r/2 - r \arcsin\sqrt{x/4}\right)}{\pi x^{1-r/2}} = \frac{\sin\left(r \arccos\sqrt{x/4}\right)}{\pi x^{1-r/2}}. $$

For the special cases we use the identity $\sin\left(\frac{1}{2}\arccos(t)\right) = \sqrt{(1-t)/2}$, which is valid for $0 \le t \le 1$. ◻

Remark. Note that

$$ \frac{W_{2,1}\left(\sqrt{x}\right)}{2\sqrt{x}} = \frac{1}{4} W_{2,1/2}\left(\frac{x}{4}\right) = \frac{\sqrt{4-\sqrt{x}}}{4\pi x^{3/4}}. \tag{36} $$

It means that if X, Y are random variables such that $X \sim \mu(2,1)$ and $Y \sim \mu(2,1/2)$ then $X^2 \sim 4Y$. This can be also derived from the relation $A_m(2,1/2)4^m = A_{2m}(2,1) = \binom{4n+1}{2n}/(4n+1)$, A048990 in OEIS [24]. Hence A048990 is the moment sequence of the density function (36), $x \in (0,16)$.

4 Some particular cases

In this part we will see that for $k = 3$ some densities still can be represented as elementary functions. We need two families of formulas (cf. 15.4.17 in [19]).

Lemma 4.1. *For $c \neq 0, -1, -2, \ldots$ we have*

$$ {}_2F_1\left(\frac{c}{2}, \frac{c-1}{2}; c \middle| z\right) = 2^{c-1}\left(1+\sqrt{1-z}\right)^{1-c}, \tag{37} $$

$$ {}_2F_1\left(\frac{c+1}{2}, \frac{c-2}{2}; c \middle| z\right) = \frac{2^{c-1}}{c}\left(1+\sqrt{1-z}\right)^{1-c}\left(c-1+\sqrt{1-z}\right). \tag{38} $$

Proof. We know that ${}_2F_1(a, b; c|\, z)$ is the unique function f which is analytic at $z = 0$, with $f(0) = 1$, and satisfies the *hypergeometric equation*:

$$ z(1-z)f''(z) + \left[c - (a+b+1)z\right]f'(z) - abf(z) = 0 $$

(see [3]). Now one can check that this equation is satisfied by the right hand sides of (37) and (38) for given parameters a, b, c. ◻

Now consider $p = 3/2$.

THEOREM 4.2. *Assume that $p = 3/2$. Then for $r = 1/2, 1, 3/2$ we have*

$$W_{3/2,1/2}(x) = \frac{\left(1+\sqrt{1-4x^2/27}\right)^{2/3} - \left(1-\sqrt{1-4x^2/27}\right)^{2/3}}{2^{5/3}3^{-1/2}\pi x^{2/3}}, \tag{39}$$

$$W_{3/2,1}(x) = 3^{1/2}\frac{\left(1+\sqrt{1-4x^2/27}\right)^{1/3} - \left(1-\sqrt{1-4x^2/27}\right)^{1/3}}{2^{4/3}\pi x^{1/3}} \tag{40}$$
$$+3^{1/2}x^{1/3}\frac{\left(1+\sqrt{1-4x^2/27}\right)^{2/3} - \left(1-\sqrt{1-4x^2/27}\right)^{2/3}}{2^{5/3}\pi}$$

and, finally, $W_{3/2,3/2}(x) = x \cdot W_{3/2,1}(x)$, with $x \in (0, 3\sqrt{3}/2)$.

Proof. For arbitrary r we have

$$W_{3/2,r}(x) = \frac{2^{1-2r/3}\sin\left(2\pi r/3\right)}{3^{3/2-r}\pi}\,{}_3F_2\left(\frac{3+2r}{6}, \frac{r}{3}, \frac{-2r}{3}; \frac{2}{3}, \frac{1}{3}\middle|\, z\right)z^{r/3-1/2}$$
$$-\frac{2^{(4-2r)/3}r\sin\left((1-2r)\pi/3\right)}{3^{3/2-r}\pi}\,{}_3F_2\left(\frac{5+2r}{6}, \frac{1+r}{3}, \frac{1-2r}{3}; \frac{4}{3}, \frac{2}{3}\middle|\, z\right)z^{(r+1)/3-1/2}$$
$$-\frac{r(1+2r)\sin\left((1+2r)\pi/3\right)}{2^{(1+2r)/3}3^{3/2-r}\pi}\,{}_3F_2\left(\frac{7+2r}{6}, \frac{2+r}{3}, \frac{2-2r}{3}; \frac{5}{3}, \frac{4}{3}\middle|\, z\right)z^{(r+2)/3-1/2},$$

where $z = 4x^2/27$. If $r = 1/2$ or $r = 1$ then one term vanishes and in the two others the hypergeometric functions reduce to ${}_2F_1$.

For $r = 1/2$ we apply (37) to obtain:

$$W_{3/2,1/2}(x) = \frac{z^{-1/3}}{2^{1/3}3^{1/2}\pi}\,{}_2F_1\left(\frac{1}{6}, \frac{-1}{3}; \frac{1}{3}\middle|\, z\right) - \frac{z^{1/3}}{2^{5/3}3^{1/2}\pi}\,{}_2F_1\left(\frac{5}{6}, \frac{1}{3}; \frac{5}{3}\middle|\, z\right)$$

$$= \frac{z^{-1/3}}{2^{1/3}3^{1/2}\pi}\,2^{-2/3}\left(1+\sqrt{1-z}\right)^{2/3} - \frac{z^{1/3}}{2^{5/3}3^{1/2}\pi}\,2^{2/3}\left(1+\sqrt{1-z}\right)^{-2/3}$$

$$= \frac{z^{-1/3}}{2\cdot 3^{1/2}\pi}\left(1+\sqrt{1-z}\right)^{2/3} - \frac{z^{1/3}}{2\cdot 3^{1/2}\pi}\left(\frac{1-\sqrt{1-z}}{z}\right)^{2/3}$$

$$= \frac{z^{-1/3}}{2\cdot 3^{1/2}\pi}\left(1+\sqrt{1-z}\right)^{2/3} - \frac{z^{-1/3}}{2\cdot 3^{1/2}\pi}\left(1-\sqrt{1-z}\right)^{2/3}$$

and this yields (39).

For $r = 1$ we use (38):

$$W_{3/2,1}(x) = \frac{z^{-1/6}}{2^{2/3}\pi}\,{}_2F_1\left(\frac{5}{6}, \frac{-2}{3}; \frac{2}{3}\middle|\, z\right) + \frac{z^{1/6}}{2^{1/3}\pi}\,{}_2F_1\left(\frac{7}{6}, \frac{-1}{3}; \frac{4}{3}\middle|\, z\right)$$

$$= \frac{z^{-1/6}}{4\pi}\left(1+\sqrt{1-z}\right)^{1/3}\left(3\sqrt{1-z}-1\right) + \frac{z^{1/6}}{4\pi}\left(1+\sqrt{1-z}\right)^{-1/3}\left(3\sqrt{1-z}+1\right)$$

$$= \frac{z^{-1/6}}{4\pi}\left(1+\sqrt{1-z}\right)^{1/3}\left(3\sqrt{1-z}-1\right) + \frac{z^{-1/6}}{4\pi}\left(1-\sqrt{1-z}\right)^{1/3}\left(3\sqrt{1-z}+1\right).$$

Now we have

$$\left(1+\sqrt{1-z}\right)^{1/3}\left(3\sqrt{1-z}-1\right) = -\left(1+\sqrt{1-z}\right)^{1/3}\left(3-3\sqrt{1-z}-2\right)$$

$$= -3z^{1/3}\left(1-\sqrt{1-z}\right)^{2/3} + 2\left(1+\sqrt{1-z}\right)^{1/3}$$

and similarly

$$\left(1-\sqrt{1-z}\right)^{1/3}\left(3\sqrt{1-z}+1\right) = 3z^{1/3}\left(1+\sqrt{1-z}\right)^{2/3} - 2\left(1-\sqrt{1-z}\right)^{1/3}.$$

Therefore

$$W_{3/2,1}(x) = \frac{z^{-1/6}}{2\pi}\left(\left(1+\sqrt{1-z}\right)^{1/3} - \left(1-\sqrt{1-z}\right)^{1/3}\right)$$

$$+\frac{3z^{1/6}}{4\pi}\left(\left(1+\sqrt{1-z}\right)^{2/3} - \left(1-\sqrt{1-z}\right)^{2/3}\right),$$

which entails (40). The last statement is a consequence of (22). □

The dilation $\mathbf{D}_2\mu(3/2,1/2)$, with the density $W_{3/2,1/2}(x/2)/2$, is known as the *Bures distribution*, see (4.4) in [26]. The integer sequence

$$4^m A_m(3/2,1/2) = \binom{3m/2+1/2}{n}\frac{4^m}{3m+1},$$

moments of the density function $W_{3/2,1/2}(x/4)/4$ on the interval $(0,6\sqrt{3})$, appears as A078531 in [24] and counts the number of symmetric noncrossing connected graphs on $2n+1$ equidistant nodes on a circle. The axis of symmetry is a diameter of a circle passing through a given node, see [7].

The measure $\mu(3/2,1)$ is equal to $\mu(2,1)^{\boxtimes 1/2}$, the multiplicative free square root of the Marchenko-Pastur distribution and the integer sequence

$$4^m A_m(3/2,1) = \binom{3m/2+1}{n}\frac{4^m}{3m/2+1},$$

moments of the density function $W_{3/2,1}(x/4)/4$ on $(0,6\sqrt{3})$, appears in [24] as A214377.

For the sake of completeness we also include the densities for the sequences $A_m(3,1)$ (A001764 in [24]) and $A_m(3,2)$ (A006013), which have already appeared in [20, 21].

Theorem 4.3. *Assume that $p = 3$. Then for $r = 1, 2, 3$ we have*

$$W_{3,1}(x) = \frac{3\left(1+\sqrt{1-4x/27}\right)^{2/3} - 2^{2/3}x^{1/3}}{2^{4/3}3^{1/2}\pi x^{2/3}\left(1+\sqrt{1-4x/27}\right)^{1/3}}, \tag{41}$$

$$W_{3,2}(x) = \frac{9\left(1+\sqrt{1-4x/27}\right)^{4/3} - 2^{4/3}x^{2/3}}{2^{5/3}3^{3/2}\pi x^{1/3}\left(1+\sqrt{1-4x/27}\right)^{2/3}} \tag{42}$$

and, finally, $W_{3,3}(x) = x \cdot W_{3,1}(x)$, with $x \in (0, 27/4)$.

Proof. For arbitrary r we have

$$\begin{aligned} W_{3,r}(x) = {} & \frac{2^{(6-2r)/3}\sin\left(\pi r/3\right)}{3^{3-r}\pi}\,{}_3F_2\left(\frac{r}{3}, \frac{3-r}{6}, \frac{-r}{6}; \frac{2}{3}, \frac{1}{3}\middle| z\right) z^{(r-3)/3} \\ & - \frac{2^{(4-2r)/3} r \sin\left((1+r)\pi/3\right)}{3^{3-r}\pi}\,{}_3F_2\left(\frac{1+r}{3}, \frac{5-r}{6}, \frac{2-r}{6}; \frac{4}{3}, \frac{2}{3}\middle| z\right) z^{(r-2)/3} \\ & + \frac{r(r-1)\sin\left((1-r)\pi/3\right)}{2^{(1+2r)/3}3^{3-r}\pi}\,{}_3F_2\left(\frac{2+r}{3}, \frac{7-r}{6}, \frac{4-r}{6}; \frac{5}{3}, \frac{4}{3}\middle| z\right) z^{(r-1)/3}, \end{aligned}$$

where $z = 4x/27$. For $r = 1$ and $r = 2$ we have similar reduction as in the previous proof. Here we will be using only (37).

Taking $r = 1$ we get

$$\begin{aligned} W_{3,1}(x) &= \frac{2^{1/3}z^{-2/3}}{3^{3/2}\pi}\,{}_2F_1\left(\frac{1}{3}, \frac{-1}{6}; \frac{2}{3}\middle| z\right) - \frac{z^{-1/3}}{2^{1/3}3^{3/2}\pi}\,{}_2F_1\left(\frac{2}{3}, \frac{1}{6}; \frac{4}{3}\middle| z\right) \\ &= \frac{z^{-2/3}}{3^{3/2}\pi}\left(1+\sqrt{1-z}\right)^{1/3} - \frac{z^{-1/3}}{3^{3/2}\pi}\left(1+\sqrt{1-z}\right)^{-1/3} \\ &= \frac{\left(1+\sqrt{1-z}\right)^{2/3} - z^{1/3}}{3^{3/2}\pi z^{2/3}\left(1+\sqrt{1-z}\right)^{1/3}}, \end{aligned}$$

which implies (41).

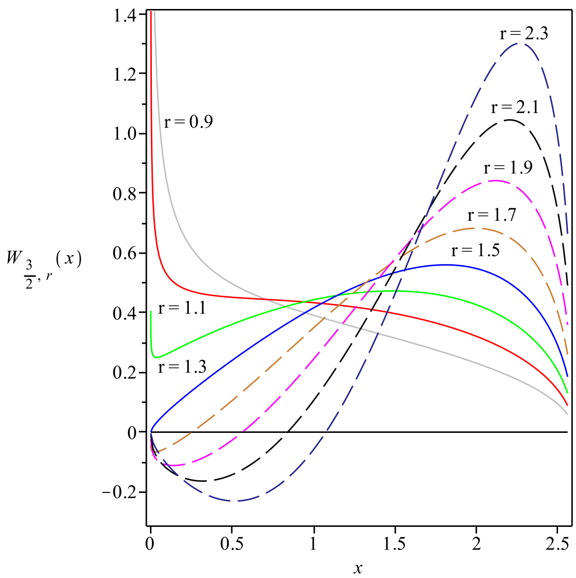
Now we take $r = 2$:

$$\begin{aligned} W_{3,2}(x) &= \frac{z^{-1/3}}{2^{1/3}3^{1/2}\pi}\,{}_2F_1\left(\frac{1}{6}, \frac{-1}{3}; \frac{1}{3}\middle| z\right) - \frac{z^{1/3}}{2^{5/3}3^{1/2}\pi}\,{}_2F_1\left(\frac{5}{6}, \frac{1}{3}; \frac{5}{3}\middle| z\right) \\ &= \frac{z^{-1/3}}{2\cdot 3^{1/2}\pi}\left(1+\sqrt{1-z}\right)^{2/3} - \frac{z^{1/3}}{2\cdot 3^{1/2}\pi}\left(1+\sqrt{1-z}\right)^{-2/3} \\ &= \frac{\left(1+\sqrt{1-z}\right)^{4/3} - z^{2/3}}{2\cdot 3^{1/2}\pi z^{1/3}\left(1+\sqrt{1-z}\right)^{2/3}}, \end{aligned}$$

and this gives us (42). Finally we apply (22). □

Recall that the measure $\mu(3,1)$ is equal to $\mu(2,1)^{\boxtimes 2}$, the multiplicative free square of the Marchenko-Pastur distribution.

Figure 1: Raney distributions $W_{3/2,r}(x)$ with values of the parameter r labeling each curve. For $r > p$ solutions drawn with dashed lines are not positive.



5 Graphical representation of selected cases

The explicit form of $W_{p,r}(x)$ given in Theorem 3.2 permits a graphical visualization for any rational $p > 0$ and arbitrary $r > 0$. We shall represent some selected cases in Figures 1–9. These graphs which are partly negative are drawn as dashed curves. In Fig. 1 the graphs of the functions $W_{3/2,r}(x)$ for values of r ranging from 0.9 to 2.3 are given. For $r \leq 3/2$ these functions are positive, otherwise they develop a negative part. In Fig. 2 we represent $W_{5/2,r}(x)$ for r ranging from 2 to 3.4. In Fig. 3 we display the densities $W_{p,p}(x)$ for $p = 6/5, 5/4, 4/3$ and $3/2$. All these densities are unimodal and vanish at the extremities of their supports. They can be therefore considered as generalizations of the Wigner's semicircle distribution $W_{2,2}(x)$, see equation (23). In Fig. 4 we depict the functions $W_{4/3,r}(x)$, for values r ranging from 0.8 to 2.4. Here for $r \geq 1.4$ negative contributions clearly appear. In Fig. 5 and 6 we present six densities expressible through elementary functions, namely $W_{3/2,r}(x)$ for $r = 1/2, 1, 3/2$, see Theorem 4.2 and $W_{3,r}(x)$ for $r = 1, 2, 3$, see Theorem 4.3. In Fig. 7 the set of densities $W_{p,1}(x)$ for five fractional values of p is presented. The appearance of maximum near $x = 1$ corresponds to the fact that $\mu(p, 1)$ weakly converges to δ_1 as $p \to 1^+$. In Fig. 8 the fine details of densities $W_{p,1}(x)$ for $p = 5/2, 7/3, 9/4, 11/5$, on a narrower range $2 \leq x \leq 4.5$ are presented. In Fig. 9 we display the densities $W_{p,1}(x)$ for $p = 2, 5/2, 3, 7/2, 4$, near the upper edge of their respective supports, for $3 \leq x \leq 9.5$.

Figure 2: As in Fig. 1 for Raney distributions $W_{5/2,r}(x)$.

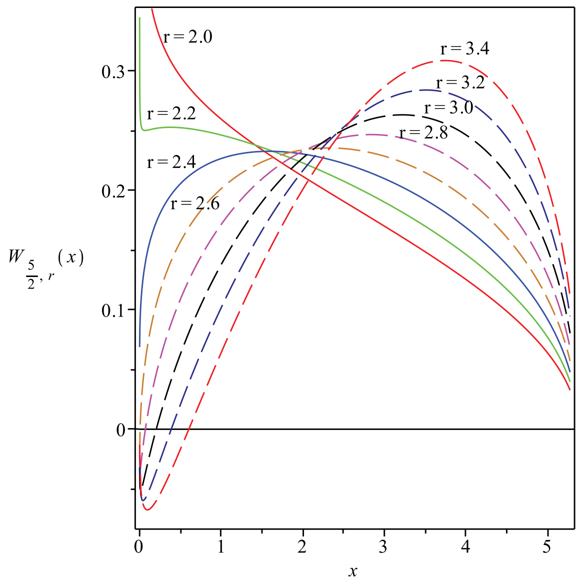


Figure 3: Diagonal Raney distributions $W_{p,p}(x)$ with values of the parameter p labeling each curve.

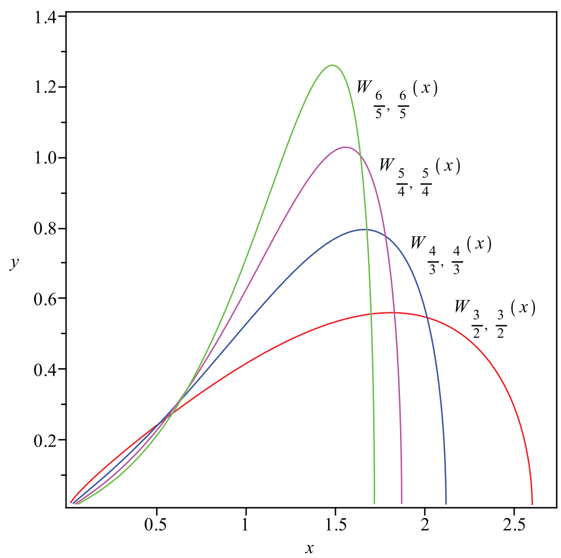


Figure 4: The functions $W_{4/3,r}(x)$ for r ranging from 0.8 to 2.4.

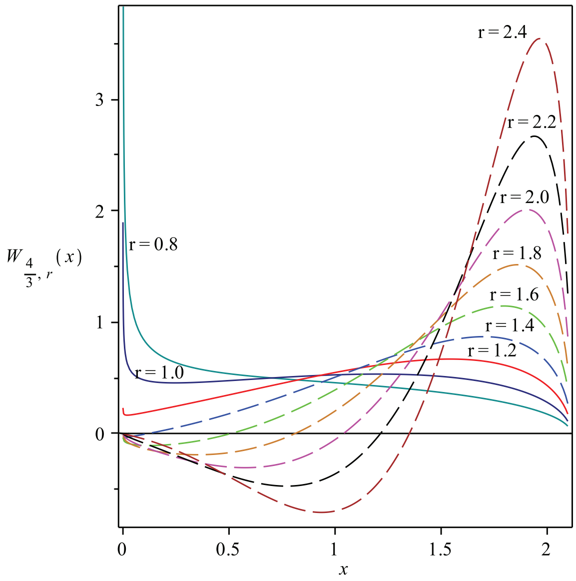


Figure 5: Raney distributions $W_{3/2,r}(x)$ with values of the parameter r labeling each curve. The case $W_{3/2,1}(x)$ represents $MP^{\boxtimes 1/2}$, the multiplicative free square root of the Marchenko-Pastur distribution.

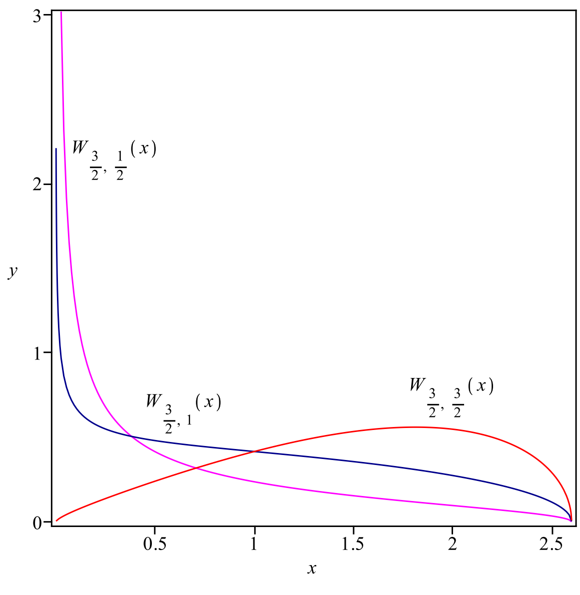


Figure 6: Raney distributions $W_{3,r}(x)$ with values of the parameter r labeling each curve. The case $W_{3,1}(x)$ represents $MP^{\boxtimes 2}$, the multiplicative free square of the Marchenko-Pastur distribution.

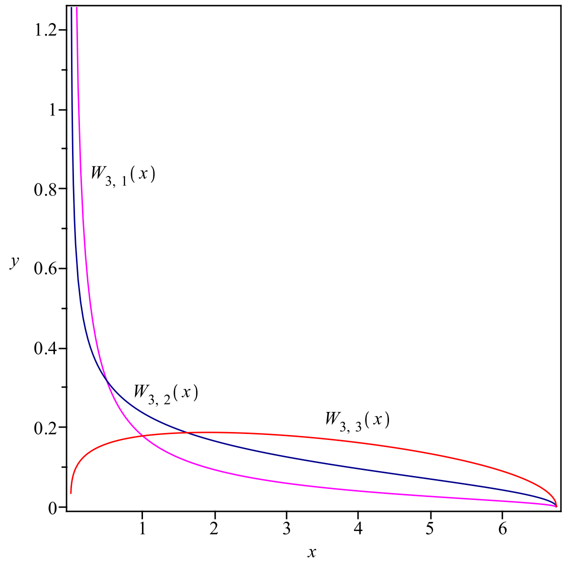


Figure 7: Raney distributions $W_{p,1}(x)$ with values of the parameter p labeling each curve. The case $W_{3/2,1}(x)$ represents the multiplicative free square root of the Marchenko–Pastur distribution, $MP^{\boxtimes 1/2}$.

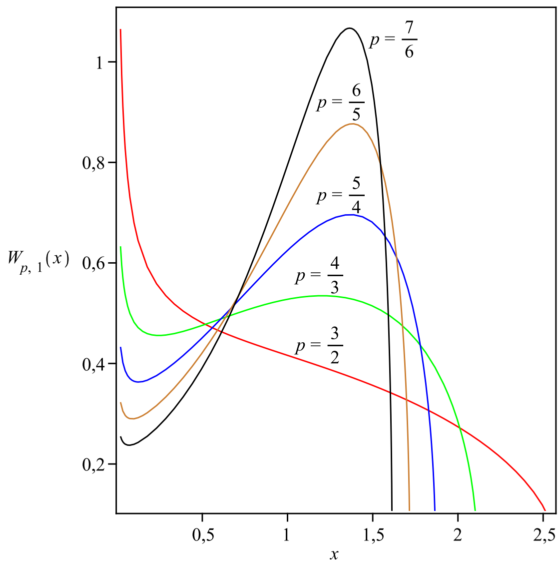


Figure 8: Tails of the Raney distributions $W_{p,1}(x)$ with values of the parameter p labeling each curve.

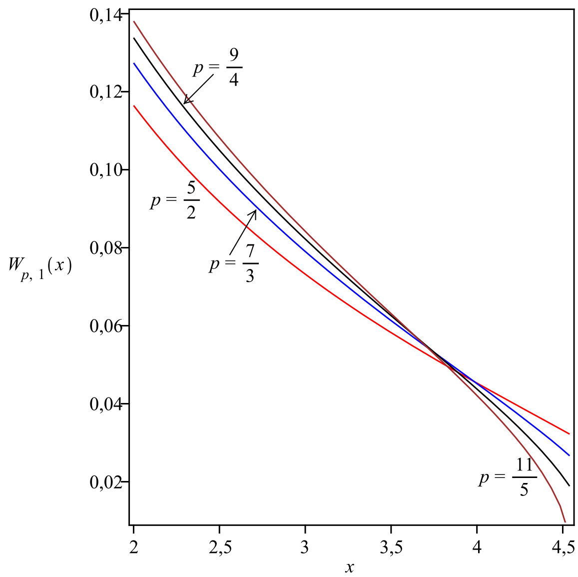


Figure 9: As in Fig. 8 for larger values of the parameter p.

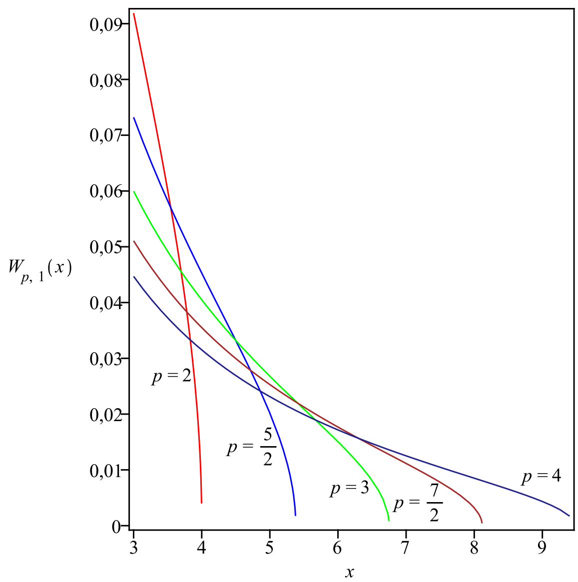
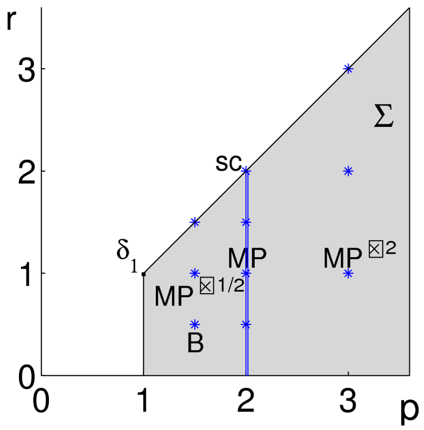


Figure 10: Parameter plane (p, r) describing the Raney numbers. The shaded set Σ corresponds to nonnegative probability measures $\mu(p, r)$. The vertical line $p = 2$ and the stars represent values of parameters for which $W_{p,r}(x)$ is an elementary function. Here MP denotes the Marchenko–Pastur distribution, $\mathrm{MP}^{\boxtimes s}$ its s-th free mutiplicative power, B-the Bures distribution while SC denotes the semicircle law. For $p > 1$ the points (p, p) on the upper edge of Σ represent the generalizations of the Wigner semicircle law, see Fig. 3.



The Fig. 10 summarizes our results in the $p > 0, r > 0$ quadrant of the (p, r) plane, describing the Raney numbers (c.f. Fig. 5.1 in [16] and Fig. 7 in [21]). The shaded region Σ indicates the probability measures $\mu(p, r)$ (i.e. where $W_{p,r}(x)$ is a nonegative function). The vertical line $p = 2$ and the stars indicate the pairs (p, r) for which $W_{p,r}(x)$ is an elementary function, see Corollary 3.3, Theorem 4.2 and Theorem 4.3. The points $(3/2, 1)$ and $(3, 1)$ correspond to the multiplicative free powers $\mathrm{MP}^{\boxtimes 1/2}$ and $\mathrm{MP}^{\boxtimes 2}$ of the Marchenko-Pastur distribution MP. Symbol B at $(3/2, 1/2)$ indicates the Bures distribution and SC at $(2, 2)$ denotes the semicircle law centered at $x = 2$, with radius 2.

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