Remarks on transversally f-biharmonic maps

Yuan-Jen Chiang and Robert A. Wolak

Abstract. Transversally f-biharmonic maps are different from f-biharmonic maps, and they generalize transversally biharmonic maps [7]. We show that if the transversal f-tension field of a map ψ of foliated Riemannian manifolds is a transversal Jacobi field and ϕ is a transversally totally geodesic map, then the transversal f-tension field of the composition $\phi \circ \psi$ is a transversal Jacobi field. We also investigate the transversal stress f-bienergy of a map ψ of foliated Riemannian manifolds.

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1 Introduction

The theory of harmonic maps between Riemannian manifolds was first established by Eells and Sampson [11] in 1964. Chiang, Ratto, Sun and Wolak also studied harmonic and biharmonic maps in [2]–[8]. The f-harmonic maps between Riemannian manifolds which generalize harmonic maps, were first introduced by Lichnerowicz [19] in 1970. They were recently investigated by Course [9, 10], Huang and Tang [13], Li and Wang [20], etc. The f-harmonic maps relate the equations of the motions of continuous systems of spins with inhomogeneous neighbor Heisenberg interactions in mathematical physics.

Harmonic maps between manifolds with one manifold foliated by points were first explored by Eells and Verjovsky [12], and Kacimi and Gomez [16]. Transversally harmonic maps between manifolds with Riemannian foliations were first defined by Konderak and Wolak [17, 18] in 2003, and they were different from harmonic maps between Riemannian manifolds.

The f-biharmonic maps between Riemannian manifolds were first investigated by Ouakkas, Nasri and Djaa [22] in 2010, and they generalized biharmonic maps by Jiang [14, 15]. Transversally f-biharmonic maps between foliated Riemannian manifolds are different from f-biharmonic maps between Riemannian manifolds. Transversally f-biharmonic maps generalize transversally biharmonic maps by Chiang and Wolak [7] in 2008. This paper is the continuation of the previous paper of Chiang and Wolak

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[8], and it would be interesting to mathematicians who work on harmonic maps, biharmonic maps, the geometry of foliations, and mathematical physics.

In section two, we briefly review f-biharmonic maps and foliations. In section three, we define transversally f-biharmonic maps following the notions of transversally harmonic maps in [17, 18]. There are examples of transversally f-biharmonic maps which are not f-biharmonic maps, and vice versa. We prove in Theorem 3.3 that if the transversal f-tension field of a smooth map ψ of foliated Riemannian manifolds is a transversal Jacobi field and ϕ is transversally totally geodesic, then the transversal f-tension field of the composition $\phi \circ \psi$ is a transversal Jacobi field. As a corollary, if ψ is a transversally biharmonic map of foliated Riemannian manifolds and ϕ is transversally totally geodesic, then $\phi \circ \psi$ is a transversally biharmonic map (cf. [7]). In section four, we investigate the transversal stress f-bienergy tensor. If ψ is a transversally f-biharmonic of foliated Riemannian manifolds, then it usually does not satisfy the conservation law for the transversal stress f-bienergy tensor. However, we obtain in Theorem 4.2 that if the transversal f-tension field of a smooth map $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ between foliated Riemannian manifolds is a transversal Jacobi field, then ψ satisfies the conservation law for the transversal stress f-bienergy tensor. In particular, if ψ is a transversally biharmonic map between foliated Riemannian manifolds, then ψ satisfies the conservation law for the transversal stress bienergy tensor. We illustrate that the conservation law for transversal stress bienergy tensor is different from the conservation law for stress bienergy tensor. We also discuss applications concerning the vanishing of transversal stress f-bienergy tensor.

2 Preliminaries

2.1 f-biharmonic maps

Let $f : (M_1, g) \to (0, \infty)$ be a smooth function. The f-harmonic maps between Riemannian manifolds were first introduced in [19], and they were studied in [9, 10, 13, 20] recently. Let $\psi : (M_1, g) \to (M_2, h)$ be a smooth map from an m-dimensional Riemannian manifold $(M_1,\, g)$ into an n-dimensional Riemannian manifold $(M_2,\, h)$. A map $\psi : (M_1, g) \to (M_2, h)$ is *f-harmonic* iff ψ is a critical point of the f-energy $E_f(\psi) = \frac{1}{2}\int_{M_1} f|d\psi|^2 dv$, where dv is the volume form determined by the metric g of M_1. In terms of the Euler-Lagrange equation, ψ is *f-harmonic* iff the *f-tension field*

$$\tau_f(\psi) = f\tau(\psi) + d\psi(grad\, f) = 0, \tag{2.1}$$

where $\tau(\psi) = trace_g Dd\psi$ is the tension field of ψ.

The f-biharmonic maps between Riemannian manifolds were first studied in [22], and they generalized biharmonic maps by Jiang [14, 15]. A f-biharmonic map $\psi : (M_1, g) \to (M_2, h)$ between Riemannian manifolds is the critical point of the bi-energy functional

$$(E_2)_f(\psi) = \frac{1}{2}\int_{M_1} ||\tau_f(\psi)||^2 dv. \tag{2.2}$$

In terms of Euler-Lagrange equation, ψ is a f-biharmonic map iff the *f-bitension field* of ψ

$$(\tau_2)_f(\psi) = (-)(\triangle_2^f \tau_f(\psi) + fR'(d\psi, d\psi)\tau(\psi)) = 0, \tag{2.3}$$

where

$$\begin{aligned} \triangle_2^f \tau_f(\psi) &= D^\psi f D^\psi \tau_f(\psi) - f D^\psi{}_D \tau_f(\psi) \\ &= \sum_{i=1}^m (D^\psi{}_{e_i} f D\psi_{e_i} \tau_f(\psi) - f D^\psi_{D_{e_i} e_i}) \end{aligned}$$

for an orthonormal frame $\{e_i\}_{1\le i\le m}$ on M_1, and R' is the Riemannian curvature of M_2. There is a + or − sign convention in (2.3), and we take + sign for simplicity in the context. In particular, if $\tau_f(\psi) = 0$, then $(\tau_2)_f(\psi) = 0$.

2.2 Foliations

Let $\mathcal{F}$ be a foliation on a Riemannian n-manifold (M, g). Then $\mathcal{F}$ is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_{i\in I}$ modeled on a q-manifold N_0 such that (1) $\{U_i\}_{i\in I}$ is an open covering of M, (2) $f_i : U_i \to N_0$ are submersions with connected fibres, (3) $g_{ij} : N_0 \to N_0$ are local diffeomorphisms of N_0 with $f_i = g_{ij} f_j$ on $U_i \cap U_j$. The connected components of the trace of any leaf of $\mathcal{F}$ on U_i consist of the fibres of f_i. The open subsets $N_i = f_i(U_i) \subset N_0$ form a q-manifold $N = \amalg N_i$, which can be considered as a transverse manifold of the foliation $\mathcal{F}$. The pseudogroup $\mathcal{H}_N$ of local diffeomorphisms of N generated by g_{ij} is called the holonomy pseudogroup of the foliated manifold $(M, \mathcal{F})$ defined by the cocycle $\mathcal{U}$. If the foliation $\mathcal{F}$ is Riemannian for the Riemannian metric g, then it induces a Riemannian metric $\bar{g}$ on N such that the submersions f_i are Riemannian submersions and the elements of the holonomy group are local isometries.

Let $\phi : U \to R^p \times R^q$, $\phi = (\phi^1, \phi^2) = (x_1, \cdots, x_p, y_1, \cdots, y_q)$ be an adapted chart on a foliated manifold $(M, \mathcal{F})$. Then on U the vector fields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_p}$ span the bundle $T\mathcal{F}$ tangent to the leaves of the foliation $\mathcal{F}$, the equivalence classes of $\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_q}$ denoted by $\frac{\bar\partial}{\partial y_1}, \cdots, \frac{\bar\partial}{\partial y_q}$ span the normal bundle $N(M, \mathcal{F}) = TM/T\mathcal{F}$ which is isomorphic to the subbundle $T\mathcal{F}^\perp$.

Suppose that $(M, \mathcal{F}, g)$ is a Riemannian foliation. The sheaf $\Gamma_b(T\mathcal{F}^\perp)$ of foliated sections of the vector bundle $T\mathcal{F}^\perp \to M$ may be described as follows: If U is an open subset of M, then $X \in \Gamma_b(U, T\mathcal{F}^\perp)$ if and only if for each local Riemannian submersion $\phi : U \to \bar{U}$ defining $\mathcal{F}$, the restriction of X to U is projectable via the map ϕ on a vector field $\bar{X}$ on $\bar{U}$.

Definition 2.1 [21]. A *basic partial connection* $(M, \mathcal{F}, g)$ is a sheaf operator $D : \Gamma_b(U, T\mathcal{F}^\perp) \times \Gamma_b(U, T\mathcal{F}^\perp) \to \Gamma_b(U, T\mathcal{F}^\perp)$ such that (1) $D_{fX+hY} Z = f D_X Y + h D_X Z$, (2) D_X is R-linear, (3) $D_X fY = X(f)Z + f D_X Y$ for any $X, Y, Z \in \Gamma_b(U, T\mathcal{F}^\perp)$ and any $f, h \in C_b^\infty(U)$, where U is any open subset of M.

Let ∇ be the Levi-Civita connection of g; then for any open subset U of M and $X, Y \in \Gamma_b(U, T\mathcal{F}^\perp)$ we define D as $D_X Y = (\nabla_X Y)^\perp$, where $(\nabla_X Y)^\perp$ is a local foliated section of $T\mathcal{F}^\perp$. It is easy to check that D is a basic partial connection on $(M, \mathcal{F}, g)$. Let $\phi : U \to \bar{U}$ be a Riemannian submersion defining the foliation $\mathcal{F}$ on an open set U. Let us assume that $X, Y \in \Gamma_b(U, T\mathcal{F}^\perp)$, and $\bar{X}, \bar{Y}$ are the push-forward vector fields via the map ϕ. Then there is a well-known property of Riemannian foliations from [23] that $d\phi(D_X Y) = \nabla^{\bar g}_{\bar X} \bar{Y}$, where $\nabla^{\bar g}$ is the Levi-Civita connection of the metric $\bar{g}$. Please see more details about foliations in [23, 24].

3 Transversally f-biharmonic maps

Let $(M_1, \mathcal{F}_1, g_1)$ and $(M_2, \mathcal{F}_2, g_2)$ be two foliated Riemannian manifolds, ∇^i be the Levi-Civita connections of the respective metrics, and D^i be the induced basic partial connections on the orthogonal complement bundles $T\mathcal{F}_i^{\perp} \to M_i$, $i = 1, 2$. Suppose that $\psi : M_1 \to M_2$ is a smooth foliated map, i.e., $d\psi(T\mathcal{F}_1) \subset T\mathcal{F}_2$. Then there are given natural bundle maps $I_i : T\mathcal{F}_i^{\perp} \to TM_i$, $P_i : TM_i \to T\mathcal{F}_i^{\perp}$ for $i = 1, 2$, where I_i is the inclusion of $T\mathcal{F}_i^{\perp}$ in TM_i and P_i is the orthogonal projection of TM_i onto $T\mathcal{F}_i^{\perp}$. If X is a local foliated section of $T\mathcal{F}_1^{\perp} \to M_1$, then $P_2 d\psi(X)$ is a foliated section of the bundle $\psi^{-1}T\mathcal{F}_2^{\perp}$. Thus $P_2 d\psi I_1$ is a foliated section of the bundle $(T\mathcal{F}_1^{\perp})^* \otimes \psi^{-1}T\mathcal{F}_2^{\perp}$. We define the transversally second fundamental form as the covariant derivative $D(P_2 d\psi I_1)$ which is a global section of the bundle $(T\mathcal{F}_1^{\perp})^* \otimes (T\mathcal{F}_1^{\perp})^* \otimes \psi^{-1}T\mathcal{F}_2^{\perp} \to M_1$, where D is the connection on the bundle $(T\mathcal{F}_1^{\perp})^* \otimes \psi^{-1}T\mathcal{F}_2^{\perp} \to M_1$ induced by D^1 and D^2.

The trace of the transversally second fundamental form is called the *transversal tension field* of ψ, and it is denoted by $\tau_b(\psi)$. If $X_{1x}, \cdots, X_{q_1x}$ is an orthonormal basis of the space $T_x\mathcal{F}_1^{\perp}$, then

$$\tau_b(\psi)_x = trace_{T\mathcal{F}_1^{\perp}} D(P_2 d\psi I_1) = \sum_{\alpha=1}^{q_1} D(P_2 d\psi I_1)(X_\alpha, X_\alpha)$$

is a section of the bundle $\psi^{-1}T\mathcal{F}_2^{\perp} \to M_1$. Let $f : (M_1, \mathcal{F}_1) \to (0, \infty)$ be a smooth basic function. We define the *transversal f-tension field* of ψ by

$$(\tau_f)_b(\psi) = f\tau_b(\psi) + (P_2 d\psi I_1)(grad\, f), \tag{3.1}$$

which is a section of the bundle $\psi^{-1}T\mathcal{F}_1^{\perp} \to M_1$. In particular, if $f = 1$, then $(\tau_f)_b(\psi) = \tau_b(\psi)$.

Let $\psi : (M_1, \mathcal{F}_1, g_1) \to (M_2, \mathcal{F}_2, g_2)$ be a smooth foliated map between two foliated Riemannian manifolds, $U_i \subset M_i$ be open subsets, and $\phi_i : (U_i, g_i) \to (\bar{U}_i, \bar{g}_i)$ be Riemannian submersions on U_i which define locally the Riemannian foliations $\mathcal{F}_i$ for $i = 1, 2$. Let $f : (M_1, \mathcal{F}_1) \to (0, \infty)$ be a smooth basic function, and $\bar{f}$ be the corresponding holonomy invariant function on the transverse manifold N_1 such that $f = \bar{f} \circ \phi_1$. Suppose that $\psi(U_1) \subset U_2$. Let $X_1, \cdots, X_{q_1}$ and $Y_1, \cdots, Y_{q_2}$ be two local bases of foliated sections of $T\mathcal{F}_1{}^{\perp}$ and $T\mathcal{F}_2{}^{\perp}$ over U_1 and U_2, respectively. Then $X_1, \cdots, X_{q_1}$ are projectable via ϕ_1 on the frame sections $\bar{X}_1, \cdots, \bar{X}_{q_1}$, and $Y_1, \cdots, Y_{q_2}$ are projectable via the map ϕ_2 on the frame sections $\bar{Y}_1, \cdots, \bar{Y}_{q_2}$. Then there exists the unique map $\bar{\psi} : \bar{U}_1 \to \bar{U}_2$ such that the diagram

$$\begin{array}{ccc} U_1 & \xrightarrow{\;f\;} & U_2 \\ {\scriptstyle\phi_1}\downarrow & & {\scriptstyle\phi_2}\downarrow \\ \bar{U}_1 & \xrightarrow{\;\bar{f}\;} & \bar{U}_2 \end{array}$$

Diagram 1.

commutes.

Let X, Y, and ξ be the foliated sections of $T\mathcal{F}_2^{\perp}$, and $D' = D^2$ be the basic partial connection on $T\mathcal{F}_2^{\perp}$. Then the Riemannian curvature $R'(X, Y)\xi = D'_X D'_Y \xi -$

$D'_Y D'_X \xi - D'_{[X,Y]}\xi$ is a section of the bundle $T\mathcal{F}_2^{\perp} \to M_2$. The transversal f-bi-tension field of $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is defined by

$$((\tau_2)_f)_b(\psi) = \triangle_2^f (\tau_f)_b(\psi) + fR'((\tau_f)_b(\psi), d\psi)d\psi, \tag{3.2}$$

where

$$\triangle_2^f (\tau_f)_b(\psi) = D^{\psi} f D^{\psi} (\tau_f)_b(\psi) - f D^{\psi}{}_D (\tau_f)_b(\psi).$$

Following the similar notion as in [17], there is a close relationship between the transversal f-bitension field of ψ and the $\bar{f}$-bitension fields of the induced maps $\bar{\psi}$, obtained by using the local submersions defining the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. Then by Diagram 1 we have

$$d\phi_2(\tau_{2f})_b(\psi)_x = (\tau_2)_{\bar{f}}(\bar{\psi})_{\phi_1(x)}, \tag{3.3}$$

which holds for each of the foliations defining local submersions $\phi_i : U_i \to \bar{U}_i$ $(i = 1, 2)$ such that $\psi(U_1) \subset U_2$. Here,

$$(\tau_2)_{\bar{f}}(\bar{\psi})_{\phi_1(x)} = \triangle_2 \tau_{\bar{f}}(\bar{\psi}) + \bar{f}R'(\tau_{\bar{f}}(\bar{\psi}), d\bar{\psi})d\bar{\psi}, \tag{3.4}$$

where

$$\triangle_2 \tau_{\bar{f}}(\bar{\psi}) \quad = \quad \nabla^{\bar{\psi}} \bar{f} \nabla^{\bar{\psi}} \tau_{\bar{f}}(\bar{\psi}) - \bar{f} \nabla^{\bar{\psi}}_{\nabla} \tau_{\bar{f}}(\bar{\psi}), \tag{3.5}$$

and $\bar{R}'$ is the Riemannian curvature in each $\bar{U}_1$. Notice that the definition of a transversally f-biharmonic map between foliated Riemannian manifolds does not depend on the choices of local Riemannian submersions defining the Riemannian foliations.

Theorem 3.1. *Let $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ be a smooth foliated map between foliated Riemannian manifolds. Then ψ is transversally f-biharmonic if and only if the induced map $\bar{\psi} : \bar{U}_1 \to \bar{U}_2$ is $\bar{f}$-biharmonic locally.*

Proof. The assertion follows from Diagram 1 and (3.3).

Theorem 3.2. [8]. *Let $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ be a transversally f-biharmonic map from a compact foliated Riemannian manifold $(M_1, \mathcal{F}_1)$ into a foliated Riemannian manifold $(M_2, \mathcal{F}_2)$ with non-positive transverse Riemannian curvature. If the induced map $\bar{\psi} : \bar{U}_1 \to \bar{U}_2$ satisfies*

$$\bar{f}(\nabla_{\bar{X}_i} \nabla_{\bar{X}_i} \tau_{\bar{f}}(\bar{\psi})) - \nabla_{\bar{X}_i} \bar{f} \nabla_{\bar{X}_i} \tau_{\bar{f}}(\bar{\psi}) \geq 0, \tag{3.6}$$

for a local frame $\{\bar{X}_1, \cdots, \bar{X}_{q_1}\}$ in $\bar{U}_1$, then ψ is a transversally f-harmonic map.

Corollary 3.3. [7]. *If $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a transversally biharmonic map from a compact foliated Riemannian manifold $(M_1, \mathcal{F}_1)$ into a foliated Riemannian manifold $(M_2, \mathcal{F}_2)$ with non-positive transverse Riemannian curvature, then ψ is transversally harmonic* (taking $f = 1$ and $\bar{f} = 1$, (3.6) is automatically satisfied).

Example 1. Let (B_1, g_1), (B_2, g_2), (F_1, h_1) and (F_2, h_2) be Riemannian manifolds, and the foliations on $B_1 \times F_1$ and $B_2 \times F_2$ be given by the projections on the first component $\pi_1 : B_1 \times F_1 \to B_1$ and $\pi_2 : B_2 \times F_2 \to B_2$. The projections π_1 and π_2 are Riemannian submersions and the foliations are also Riemannian. Let $\psi : B_1 \times F_1 \to B_2 \times F_2$ be a smooth map which preserves the leaves of the foliations. Then ψ must be of the form $\psi(x, y) = (\psi_1(x), \psi_2(x, y))$, $x \in B_1$, $y \in F_1$, where $\psi_1 : B_1 \to B_2$ and $\psi_2 : B_1 \times F_1 \to F_2$ are smooth. Let $f : B_1 \times F_1 \to (0, \infty)$ be a smooth basic function which induces $\bar{f} : B_1 \to (0, \infty)$ such that $f = \bar{f} \circ \pi_1$. For the product Riemannian metrics on $B_1 \times F_1$ and $B_2 \times F_2$, the bi-tension field of f can be expressed as

$$(\tau_2)_f(\psi) = ((\tau_2)_{\bar{f}}(\psi_1), (\tau_2)_f(\psi_2|_{B_1}) + (\tau_2)_f(\psi_2|_{F_1})), \tag{3.7}$$

where $(\tau_2)_{\bar{f}}(\psi_1)$ is the $\bar{f}$-bitension field at x of $\psi_1 : B_1 \to B_2$, $(\tau_2)_f(\psi_2|_{B_1})$ is the f-bitension field at x of the map $x \to \psi_2(x, y)$ with y fixed, and $(\tau_2)_f(\psi_2|_{F_1})$ is the f-bitension field at y of the map $y \to \psi_2(x, y)$ with x fixed. On one hand, by (3.7) the biharmonicity of $\psi = (\psi_1, \psi_2)$ is equivalent to ψ_1 being $\bar{f}$-biharmonic and $(\tau_2)_f(\psi_2|_{B_1}) + (\tau_2)_f(\psi_2|_{F_1}) = 0$, i.e., the vertical and horizontal contributions to the f-bitension field annihilate each other. On the other hand, if ψ_1 is $\bar{f}$-biharmonic and $\psi_2|_{B_1}$, $\psi_2|_{F_1}$ are f-biharmonic for $x \in B_1, y \in F_1$, then ψ is f-biharmonic. Hence, it follows that there are maps ψ which are transversally f-biharmonic, but not f-biharmonic. □

Example 2. Following the setting of example 1, let $\psi : B_1 \times F_1 \to B_2 \times F_2$ be a smooth map preserving the leaves of the foliations such that $\psi(x, y) = (\psi_1(x), \psi_2(x, y))$, where $B_1 = B_2 = F_1 = F_2 = \mathbf{R}$. By [17], choosing $\alpha_1(x) = 0$ and $\alpha_2(x) = x$ as two warping functions in $\mathbf{R}$ and letting $f = e^{4x}$ and $\psi_1(x) = e^{-4x} + x$, $\psi_2(x, y) = y$, we have

$$\begin{aligned} \tau_f(\psi) &= f\tau(\psi) + d\psi(grad\, f) \\ &= f[\tau(\psi_1) + \tau(\psi_2|_{B_1}) + \tau(\psi_2|_{F_1}) \\ &- ||d\psi_2||^2(grad_{g_2}\alpha_2) \circ \psi_1] + d\psi(grad\, f) = 12, \end{aligned} \tag{3.8}$$

and then $(\tau_2)_f\psi = 0$. It implies that ψ is f-biharmonic non f-harmonic. However, $(\tau_2)_{\bar{f}}(\psi_1) \neq 0$ which implies that ψ is not transversally f-biharmonic. It follows that the f-biharmonicity of the map ψ does not imply the transversal f-biharmonicity of the map either. □

It is known from [11] that if $\psi : (M_1, g) \to (M_2, h)$ is a harmonic map of Riemannian manifolds and $\phi : (M_2, h) \to (M_3, k)$ is a totally geodesic map of Riemannian manifolds, then $\phi \circ \psi : (M_1, g) \to (M_3, k)$ is harmonic. However, if $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a transversally f-biharmonic map and $\phi : (M_2, \mathcal{F}_2) \to (M_3, \mathcal{F}_3)$ is transversally totally geodesic (i.e., the induced map $\phi : \bar{U}_1 \to \bar{U}_2$ is totally geodesic locally), then $\phi \circ \psi : (M_1, \mathcal{F}_1) \to (M_3, \mathcal{F}_3)$ is not necessarily a transversally f-biharmonic map. We obtain the following theorem instead. If $\tau_{\bar{f}}(\bar{\psi})$ is a Jacobi field for the induced map $\bar{\psi} : N_1 = \cup\bar{U}_1 \to N_2 = \cup\bar{U}_2$ between transverse manifolds, it is called a *transversal Jacobi field* of $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$.

Theorem 3.4. *If $\tau_{\bar{f}}(\bar{\psi})$ is a transversal Jacobi field of a smooth foliated map $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ of foliated Riemannian manifolds and $\phi : (M_2, \mathcal{F}_2) \to (M_3, \mathcal{F}_3)$*

is a transversally totally geodesic map of foliated Riemannian manifolds, then $\tau_{\bar f}(\bar\phi\circ\bar\psi)$ *is a transversal Jacobi field of* $\phi\circ\psi$.

Proof. The map $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ induces $\bar\psi : \bar U_1 \to \bar U_2$ locally. Let $\nabla, \nabla', \bar\nabla, \bar\nabla', \bar\nabla'', \hat\nabla, \hat\nabla', \hat\nabla''$ be the connections on $T\bar U_1, T\bar U_2$, $\bar\psi^{-1}T\bar U_2$, $\bar\phi^{-1}T\bar U_3$, $(\bar\phi\circ\bar\psi)^{-1}T\bar U_3$, $T^*\bar U_1\otimes\psi^{-1}T\bar U_2$, $T^*\bar U_2\otimes\bar\phi^{-1}T\bar U_3$, $T^*\bar U_1\otimes(\bar\phi\circ\bar\psi)^{-1}T\bar U_3$, respectively. We first have

$$\bar\nabla''_{\bar X} d(\bar\phi\circ\bar\psi)(\bar Y) = (\hat\nabla'_{d\bar\psi(\bar X)}d\bar\phi)d\bar\psi(\bar Y) + d\bar\phi\circ\bar\nabla_{\bar X}d\psi(\bar Y) \tag{3.9}$$

for $\bar X$, $\bar Y \in \Gamma(T\bar U_1)$. We also have

$$R^{\bar U_3}(d\bar\phi(\bar X'), d\bar\phi(\bar Y'))d\bar\phi(Z') = R^{\bar\phi^{-1}T\bar U_3}(\bar X', \bar Y')d\bar\phi(\bar Z') \tag{3.10}$$

for $\bar X'$, $\bar Y'$, $\bar Z' \in \Gamma(T\bar U_2)$.

By [11] we get

$$\tau(\bar\phi\circ\bar\psi) = d\bar\phi(\tau(\bar\psi)) + tr_{\bar g}\nabla d\bar\phi(d\bar\psi, d\bar\psi) = d\bar\phi(\tau(\bar\psi)),$$

because ϕ is transversally totally geodesic (i.e., $\bar\phi$ is totally geodesic). Then we have

$$\tau_f(\bar\psi\circ\bar\phi) = d\bar\phi(\tau_f(\bar\psi)) + fTr_{\bar g}\nabla d\bar\phi(d\bar\psi, d\bar\psi) = d\bar\phi(\tau_f(\bar\psi)),$$

since $\bar\phi$ is totally geodesic. Recall that $\{\bar X_i\}_{i=1}^{q_1}$ is a local orthonormal frame at a point in $\bar U_1$, and let $\bar\nabla^*\bar\nabla = \bar\nabla_{\bar X_k}\bar\nabla_{\bar X_k} - \bar\nabla_{\nabla_{\bar X_k}\bar X_k}$ and $\bar\nabla''^*\bar\nabla'' = \bar\nabla''_{\bar X_k}\bar\nabla''_{\bar X_k} - \bar\nabla''_{\nabla_{\bar X_k}\bar X_k}$. Thus we obtain

$$\begin{aligned}\bar\nabla''^*\bar\nabla''\tau_{\bar f}(\bar\phi\circ\bar\psi) &= \bar\nabla''^*\bar\nabla''(d\bar\phi\circ\tau_{\bar f}(\bar\psi))\\ &= \bar\nabla''_{\bar X_k}\bar\nabla''_{\bar X_k}(d\bar\phi\circ\tau_{\bar f}(\bar\psi)) - \bar\nabla''_{\nabla_{\bar X_k}\bar X_k}(d\bar\phi\circ\tau_{\bar f}(\bar\psi)).\end{aligned} \tag{3.11}$$

We derive from (3.9) that

$$\bar\nabla''_{\bar X_k}(d\bar\phi\circ\tau_{\bar f}(\bar\psi)) = (\hat\nabla'_{\bar\nabla_{\bar X_j}d\bar\psi(\bar X_k)}d\bar\phi)(\tau_{\bar f}(\bar\psi)) + d\bar\phi\circ\bar\nabla_{\bar X_k}(\tau_{\bar f}(\bar\psi)) = d\bar\phi\circ\bar\nabla_{\bar X_k}\tau_{\bar f}(\bar\psi),$$

since $\bar\phi$ is totally geodesic. Therefore, we arrive at

$$\bar\nabla''_{\bar X_k}\bar\nabla''_{\bar X_k}(d\bar\phi\circ\tau_{\bar f}(\bar\psi)) = \bar\nabla''_{\bar X_k}(d\bar\phi\circ\bar\nabla_{\bar X_k}\tau_{\bar f}(\bar\psi)) = d\bar\phi\circ\bar\nabla_{\bar X_k}\bar\nabla_{\bar X_k}\tau_{\bar f}(\bar\psi), \tag{3.12}$$

and

$$\bar\nabla''_{\nabla_{\bar X_k}\bar X_k}(d\bar\phi\circ\tau(\bar\psi)) = d\bar\phi\circ\bar\nabla_{\nabla_{\bar X_k}\bar X_k}\tau_{\bar f}(\nabla\bar\psi). \tag{3.13}$$

Substituting (3.12), (3.13) into (3.11), we deduce

$$\bar\nabla''^*\bar\nabla''\tau_{\bar f}(\bar\phi\circ\bar\psi) = d\bar\phi\circ\bar\nabla^*\bar\nabla\tau_{\bar f}(\bar\psi). \tag{3.14}$$

On the other hand, it follows from (3.10) that

$$\begin{aligned}& R^{\bar U_3}(d(\bar\phi\circ\bar\psi)(\bar X_i), \tau_{\bar f}(\bar\phi\circ\bar\psi))d(\bar\phi\circ\bar\psi)(\bar X_i)\\ =\ & R^{\bar\phi^{-1}T\bar U_3}(d\bar\psi(\bar X_i), \tau_{\bar f}(\bar\psi))d\bar\phi(d\bar\psi(\bar X_i))\\ =\ & d\bar\phi\circ R^{\bar U_2}(d\bar\psi(\bar X_i), \tau_{\bar f}(\bar\psi))d\bar\psi(\bar X_i).\end{aligned} \tag{3.15}$$

By (3.14) and (3.15), we obtain

$$\begin{aligned}\bar{\nabla}''^{*}\bar{\nabla}''\tau_{\bar{f}}(\bar{\phi}\circ\bar{\psi}) \quad &+ \quad R^{\bar{U}_3}(d(\bar{\phi}\circ\bar{\psi})(\bar{X}_i),\tau_{\bar{f}}(\bar{\phi}\circ\bar{\psi}))d(\bar{\phi}\circ\bar{\psi})(\bar{X}_i)\\ &= \quad d\bar{\phi}\circ[\bar{\nabla}^{*}\bar{\nabla}\tau_{\bar{f}}(\psi)+R^{\bar{U}_2}(d\bar{\psi}(\bar{X}_i),\tau_{\bar{f}}(\bar{\psi}))d\bar{\psi}(\bar{X}_i)].\end{aligned} \tag{3.16}$$

Consequently, if $\tau_{\bar{f}}(\bar{\psi})$ is a transversal Jacobi field of ψ, then $\tau_{\bar{f}}(\bar{\phi}\circ\bar{\psi})$ is a transversal Jacobi field of $\phi\circ\psi$. □

Corollary 3.5. *If $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a transversally biharmonic map between foliated Riemannian manifolds and $\phi : (M_2, \mathcal{F}_2) \to (M_3, \mathcal{F}_3)$ is transversally totally geodesic, then $\phi\circ\psi : (M_1, \mathcal{F}_1) \to (M_3, \mathcal{F}_3)$ is a transversally biharmonic map.*

Proof. Taking $f = 1$ ($\bar{f} = 1$), the transversal biharmonic map $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ induces the biharmonic map $\bar{\psi} : \bar{U}_1 \to \bar{U}_2$ locally. Applying analogous arguments to the proof of Theorem 3.4, (3.16) yields

$$\begin{aligned}\bar{\nabla}''^{*}\bar{\nabla}''\tau(\bar{\phi}\circ\bar{\psi}) \quad &+ \quad R^{\bar{U}_3}(d(\bar{\phi}\circ\bar{\psi})(\bar{X}_i),\tau(\bar{\phi}\circ\bar{\psi}))d(\bar{\phi}\circ\bar{\psi})(\bar{X}_i)\\ &= \quad d\bar{\phi}\circ[\bar{\nabla}^{*}\bar{\nabla}\tau(\bar{\psi})+R^{\bar{U}_2}(d\bar{\psi}(\bar{X}_i),\tau(\bar{\psi}))d\bar{\psi}(\bar{X}_i)],\end{aligned}$$

i.e., $\tau_2(\bar{\phi}\circ\bar{\psi}) = d\bar{\phi}\circ(\tau_2(\bar{\psi}))$, where $\tau_2(\bar{\psi})$ is the bi-tension field of $\bar{\psi}$ (i.e., $\tau\bar{\psi}$ is a Jacobi field). Hence, the result follows. □

4 Transversal stress f-bienergy tensors

Let $f : (M_1, \mathcal{F}_1) \to (0,\infty)$ be a smooth basic function, and $\bar{f}$ be the corresponding holonomy invariant function on the transverse manifold $N_1 = \cup\bar{U}_1$ such that $f = \bar{f}\circ\phi_1$. Consider a smooth map $\psi : (M_1, \mathcal{F}_1) \to (M_2,\mathcal{F}_2)$ between foliated Riemannian manifolds, which induces $\bar{\psi} : (N_1 = \cup\bar{U}_1, \bar{g}_1) \to (N_2 = \cup\bar{U}_2, \bar{g}_2)$ (i.e., $\bar{\psi} : \bar{U}_1 \to \bar{U}_2$ locally). Following [1], the transversal stress energy tensor was defined by $S(\bar{\psi}) = e(\bar{\psi})\bar{g}_1 - \psi^*\bar{g}_2$, where $e(\bar{\psi}) = \frac{|d\bar{\psi}|^2}{2}$ for $\bar{\psi} : (\bar{U}_1, \bar{g}_1) \to (\bar{U}_2, \bar{g}_2)$, and we have $divS(\bar{\psi}) = -<\tau(\bar{\psi}), d\bar{\psi}>$. Hence, if ψ is transversally harmonic, then ψ satisfies the conservation law for the transversal stress energy (i.e., $div\, S(\bar{\psi}) = 0$). However, if we use the idea of [22], the transversal stress f-energy tensor of the smooth map ψ was similarly defined by $S^{\bar{f}}(\bar{\psi}) = \bar{f}e(\bar{\psi})\bar{g}_1 - \bar{f}\bar{\psi}^*\bar{g}_2$, and we have

$$div\, S^{\bar{f}}(\bar{\psi}) = -<\tau_{\bar{f}}(\bar{\psi}), d\bar{\psi}> + e(\bar{\psi})d\bar{f}.$$

Therefore, a transversal f-harmonic map usually does not satisfy the conservation law for transversal stress f-energy in this case.

The stress bienergy tensors and the conservation laws of biharmonic maps between Riemannian manifolds were first studied by Jiang [15] in 1987. Following his notion, we define the transversal stress f-bienergy tensor of a smooth foliated map as follows.

Definition 4.1. Let $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ be a smooth foliated map between foliated Riemannian manifolds which induces $\bar{\psi} : (N_1 = \cup\bar{U}_1, \bar{g}_1) \to (N_2 = \cup\bar{U}_2, \bar{g}_2)$ between transverse manifolds. The transversal stress f-bienergy tensor of ψ is defined by

$$\begin{aligned} S_2^{\bar f}(\bar X,\bar Y) &= \frac{1}{2}|\tau_{\bar f}(\psi)|^2 <\bar X,\,\bar Y> + < d\bar\psi,\,\bar\nabla(\tau_{\bar f}(\bar\psi)><\bar X,\,\bar Y> \\ &- < d\bar\psi(\bar X),\,\bar\nabla_{\bar Y}\tau_{\bar f}(\bar\psi)> - < d\bar\psi(\bar Y),\,\bar\nabla_{\bar X}\tau_{\bar f}(\bar\psi)>, \end{aligned} \tag{4.1}$$

for $\bar X,\,\bar Y\in\Gamma(T\bar U_1)$ in each $\bar U_1$.

Observe that if $\psi:(M_1,\,\mathcal F_1)\to(M_2,\,\mathcal F_2)$ is a transversal f-biharmonic map between foliated Riemannian manifolds, then ψ does not necessarily satisfy the conservation law for the transversal stress f-bienergy tensor. Instead, we obtain the following theorem.

Theorem 4.1. (1) *Let $\psi:(M_1,\,\mathcal F_1)\to(M_2,\,\mathcal F_2)$ be a smooth map between foliated Riemannian manifolds which induces $\bar\psi:(N_1=\cup\bar U_1,\bar g_1)\to(N_2=\cup\bar U_2,\bar g_2)$ between transverse manifolds. Then we have*

$$(4.2)\quad div\, S_2^{\bar f}(\bar Y) = (-) < J_{\tau_{\bar f}(\bar\psi)}(\bar Y),\ d\psi(\bar Y)> \quad for\ \bar Y\in\Gamma(T\bar U_1)\ in\ each\ \bar U_1,$$

where $J_{\tau_{\bar f}(\bar\psi)}$ is the Jacobi field of $\tau_{\bar f}(\bar\psi)$. (2) *If $\tau_{\bar f}(\bar\psi)$ is a transversal Jacobi field for a map $\psi:(M_1,\,\mathcal F_1)\to(M_2,\,\mathcal F_2)$ between foliated Riemannian manifolds, then it satisfies the conservation law (i.e., $div\,S_2^{\bar f}=0$) for the transversal stress f-bienergy tensor.*

Proof. The map $\psi:(M_1,\mathcal F_1)\to(M_2,\mathcal F_2)$ of foliated Riemannian manifolds induces $\bar\psi:(\bar U_1,\bar g_1)\to(\bar U_2,\bar g_2)$ locally. Set $S_2^{\bar f}=H_1+H_2$, where H_1 and H_2 are (0, 2)-tensors defined by

$$\begin{aligned} H_1(\bar X,\bar Y) &= \frac{1}{2}|\tau_{\bar f}(\bar\psi)|^2 <\bar X,\bar Y> + < d\bar\psi,\bar\nabla\tau_{\bar f}(\bar\psi)><\bar X,\bar Y>, \\ H_2(\bar X,\bar Y) &= -< d\bar\psi(\bar X),\,\bar\nabla_{\bar Y}\tau_{\bar f}(\bar\psi)> - < d\bar\psi,\,\bar\nabla_{\bar X}\tau_{\bar f}(\bar\psi)>. \end{aligned}$$

Let $\{\bar X_i\}$ be the geodesic frame at a point $a\in\bar U_1$, and write $\bar Y=\bar Y^i\bar X_i$ at the point a. We first calculate

$$\begin{aligned} div\,H_1(\bar Y) &= \sum_i(\bar\nabla_{\bar X_i}H_1)(\bar X_i,\bar Y) = \sum_i(\bar X_i(H_1(\bar X_i,\bar Y) - H_1(\bar X_i,\,\bar\nabla_{\bar X_i}\bar Y)) \\ &= \sum_i(\bar X_i(\frac{1}{2}|\tau_{\bar f}(\bar\psi)|^2\bar Y^i + \sum_k < d\psi(\bar X_k),\,\bar\nabla_{\bar X_k}\tau_{\bar f}(\bar\psi)>\bar Y^i) \\ &- \frac{1}{2}|\tau_{\bar f}(\bar\psi)|^2\bar Y^i\bar X_i - \sum_k < d\bar\psi(\bar X_k),\,\bar\nabla_{\bar X_k}\tau_{\bar f}(\bar\psi)>\bar Y^i\bar X_i)) \\ &= <\bar\nabla_{\bar Y}\tau_{\bar f}(\bar\psi),\tau_{\bar f}(\bar\psi)> + \sum_i < d\bar\psi(\bar Y,\bar X_i),\,\bar\nabla_{\bar X_i}\tau_{\bar f}(\bar\psi)> \\ &+ \sum_i < d\bar\psi(\bar X_i),\,\bar\nabla_{\bar Y}\bar\nabla_{\bar X_i}\tau_{\bar f}(\bar\psi)> \\ &= <\bar\nabla_{\bar Y}\tau_{\bar f}(\bar\psi),\tau_{\bar f}(\bar\psi)> + trace<\bar\nabla d\bar\psi(\bar Y,\cdot),\,\bar\nabla_{\cdot}\tau_{\bar f}(\bar\psi)> \\ &+ trace< d\bar\psi(\cdot),\bar\nabla^2\tau_{\bar f}(\bar\psi)(\bar Y,\cdot)>. \end{aligned} \tag{4.3}$$

We then calculate

$$
\begin{aligned}
div\, H_2(\bar{Y}) &= \sum_i (\bar{\nabla}_{\bar{X}_i} H_2)(\bar{X}_i, Y) = \sum_i (\bar{X}_i(H_2(\bar{X}_i, \bar{Y}) - H_2(\bar{X}_i, \bar{\nabla}_{\bar{X}_i}\bar{Y})) \\
&= -<\bar{\nabla}_{\bar{Y}}\tau_{\bar{f}}(\bar{\psi}),\, \tau_{\bar{f}}(\bar{\psi})> - \sum_i <\bar{\nabla} d\bar{\psi}(\bar{Y}, \bar{X}_i),\, \bar{\nabla}_{\bar{X}_i}\tau_{\bar{f}}(\bar{\psi})> \\
&- \sum_i <d\bar{\psi}(\bar{X}_i),\, \bar{\nabla}_{\bar{X}_i}\bar{\nabla}_{\bar{Y}}\tau_{\bar{f}}(\bar{\psi}) - \bar{\nabla}_{\nabla_{\bar{X}_i}\bar{Y}}\tau_{\bar{f}}(\bar{\psi})> + <d\bar{\psi}(\bar{Y}), \triangle\tau_{\bar{f}}(\bar{\psi})> \\
&= -<\bar{\nabla}_Y\tau_{\bar{f}}(\bar{\psi}), \tau_{\bar{f}}(\bar{\psi})> - trace <\bar{\nabla} d\bar{\psi}(\bar{Y}, \cdot),\, \bar{\nabla}_{\cdot}\tau_{\bar{f}}(\bar{\psi})> \\
&- trace <d\bar{\psi}(\cdot), \bar{\nabla}^2\tau_{\bar{f}}(\bar{\psi})(\cdot, \bar{Y})> + <d\bar{\psi}(\bar{Y}), \triangle\tau_{\bar{f}}(\bar{\psi})>\,. \qquad (4.4)
\end{aligned}
$$

We deduce the following equation by adding(4.3) and (4.4)

$$
\begin{aligned}
divS_2^{\bar{f}}(\bar{Y}) &= (-)\Big(<d\bar{\psi}(\bar{Y}),\, \triangle\tau_{\bar{f}}(\bar{\psi}) + \sum_i <d\bar{\psi}(\bar{X}_i), R'(\bar{Y}, \bar{X}_i)\tau_{\bar{f}}(\psi)> \Big) \\
&= (-) <J_{\tau_{\bar{f}}(\bar{\psi})}(\bar{Y}),\, d\bar{\psi}(\bar{Y})>, \qquad (4.5)
\end{aligned}
$$

where $J_{\tau_{\bar{f}}(\bar{\psi})}$ is the Jacobi field of $\tau_{\bar{f}}(\bar{\psi})$ and there is $-$ or $+$ sign convention in the formula. Consequently, we can conclude both results. □

Corollary 4.2. *If $\psi : (M_1,\, \mathcal{F}_1) \to (M_2,\, \mathcal{F}_2)$ is transversally biharmonic between foliated Riemannian manifolds, then it satisfies the conservation law for the transversal stress bienergy tensor.*

Proof. Taking $f = 1$ ($\bar{f} = 1$), the transversally biharmonic map $\psi : (M_1,\, \mathcal{F}_1) \to (M_2,\, \mathcal{F}_2)$ induces the biharmonic map $\bar{\psi} : \bar{U}_1 \to \bar{U}_2$ locally. Then (4.5) yields

$$
\begin{aligned}
div\, S_2(\bar{Y}) &= (-) <d\bar{\psi},\, \triangle\tau(\bar{\psi}) + \sum_i (d\bar{\psi}(e_i), R'(\bar{Y}, e_i)\tau(\psi)> \\
&= (-) <J_{\tau(\bar{\psi})}(\bar{Y}),\, d\bar{\psi}(\bar{Y})> = (-) <\tau_2(\bar{\psi}),\, d\bar{\psi}(\bar{Y})>,
\end{aligned}
$$

where $\tau_2(\bar{\psi})$ is the bi-tension field of $\bar{\psi}$ (i.e., $\tau(\bar{\psi})$ is a Jacobi field). This completes the proof. □

The conservation law for the transversal stress bienergy tensor is different from the conservation law for stress bienergy tensor. By [15], if $\psi : M_1 \to M_2$ is a biharmonic map of Riemannian manifolds, then it satisfies the conservation law for the stress bienergy tensor. On one hand, by Example 1 and Corollary 4.3, there are transversally biharmonic maps which satisfy the conservation law for transversal stress bienergy tensor, but not for stress bienergy tensor. On the other hand, by Example 2 and [15], there are biharmonic maps which satisfy the conservation law for stress bienergy tensor, but not for transversal stress bienergy tensor either.

We next discuss applications of the vanishing of transversal f-bienergy tensor.

Proposition 4.3. *If $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ of foliated Riemannian manifolds induces $\bar{\psi} : (N_1^{q_1} = \cup\bar{U}_1, \bar{g}_1) \to (N_2^{q_2} = \cup\bar{U}_2, \bar{g}_2)$ with $S_2^{\bar{f}} = 0\,(q_1 \neq 2)$, then*

$$
\frac{1}{q_1-2} \quad |\tau_f(\bar{\psi})|^2(\bar{X}, \bar{Y}) + <\bar{\nabla}_{\bar{X}}\tau_{\bar{f}}(\bar{\psi}), d\bar{\psi}(\bar{Y})> + <\bar{\nabla}_{\bar{Y}}\tau_{\bar{f}}(\bar{\psi}), d\bar{\psi}(\bar{X})> = 0 \qquad (4.6)
$$

for $\bar{X}$, $\bar{Y} \in \Gamma(T(\bar{U}_1))$.

Proof. Suppose that $S_2^{\bar{f}} = 0$, it implies trace $S_2^{\bar{f}} = 0$. Therefore,

$$< \bar{\nabla}\tau_{\bar{f}}(\bar{\psi}), d\bar{\psi} >= -\frac{q_1}{2(q_1-2)}|\tau_f(\bar{\psi})|^2 (q_1 \neq 2). \tag{4.7}$$

Substituting it into the definition of $S_2^{\bar{f}}$, we derive

$$\begin{aligned} 0 &= S_2^{\bar{f}}(\bar{X},\bar{Y}) = -\frac{1}{q_2-2}|\tau_{\bar{f}}(\bar{\psi})|^2(\bar{X},\bar{Y}) \\ &- < \bar{\nabla}_{\bar{X}}\tau_{\bar{f}}(\bar{\psi} >, d\bar{\psi}(\bar{Y}) > - < \bar{\nabla}_{\bar{Y}}\tau_{\bar{f}}(\bar{\psi}), d\bar{\psi}(\bar{X}) > . \end{aligned} \tag{4.8}$$

Corollary 4.4. *If $\psi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ induces $\bar{\psi} : N_1^{q_1} = \cup \bar{U}_1 \to N_2^{q_2} = \cup \bar{U}_2$ with $S_2^{\bar{f}} = 0$ ($q_1 > 2$) and rank $\bar{\psi} \leq q_1 - 1$, then ψ is transversally f-harmonic.*

Proof. Since $rank\, \bar{\psi}(a) \leq q_1 - 1$, for a point $a \in \bar{U}_1$ there exists a unit vector $\bar{X}_a \in Ker\, d\bar{\psi}_a$. Letting $\bar{X} = \bar{Y} = \bar{X}_a$, (4.6) yields $\tau_{\bar{f}}(\bar{\psi}) = 0$, i.e., ψ is transversally f-harmonic.

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