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Functor of extension in Hilbert cube and Hilbert space

Research Article

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Abstract: It is shown that if $\Omega = Q$ or $\Omega = \ell_2$, then there exists a functor of extension of maps between Z-sets in Ω to mappings of Ω into itself. This functor transforms homeomorphisms into homeomorphisms, thus giving a functorial setting to a well-known theorem of Anderson [Anderson R.D., On topological infinite deficiency, Michigan Math. J., 1967, 14, 365–383]. It also preserves convergence of sequences of mappings, both pointwise and uniform on compact sets, and supremum distances as well as uniform continuity, Lipschitz property, nonexpansiveness of maps in appropriate metrics.

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1. Introduction

Anderson in his celebrated paper [2] showed that if $\Omega = Q$ or $\Omega = \ell_2$, then every homeomorphism between two Z-sets in Ω can be extended to an autohomeomorphism of Ω (see also [1] or [8]). The theorem on extending homeomorphisms between Z-sets was generalized [3, 9] and settled in any manifold modelled on an infinite-dimensional Fréchet space [9] (which is, in fact, homeomorphic to a Hilbert space, see [16, 17]), and is one of the deepest results in infinite-dimensional topology. (For more information on Z-sets consult e.g. [8, Chapter V].) The aim of this paper is to strengthen Anderson's theorem in a functorial manner. To formulate our results, let us fix the notation.

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Notation and terminology

Below, Ω continues to denote the Hilbert cube $Q = [-1,1]^\omega$ or the Hilbert space ℓ_2, and for metrizable spaces X and Y we denote by $\mathcal{C}(X,Y)$ the set of all *maps* (that is, continuous functions) from X to Y. The closure operation is marked by an overline; in particular, $\overline{\operatorname{im}}(\varphi)$ denotes the closure of the image of a function φ. For a topological space X, $\mathcal{Z}(X)$ stands for the collection of all Z-sets in X; that is, $K \in \mathcal{Z}(X)$ if K is closed and $\mathcal{C}(Q, X \setminus K)$ is dense in $\mathcal{C}(Q,X)$ (in the uniform convergence topology). An embedding $u\colon (X,d) \to (Y,\varrho)$ between metric spaces is called *uniform* if both u and u^{-1} are uniformly continuous (with respect to the metrics d and ϱ). By a *compatible* metric on a metrizable space we mean any metric which induces the topology of the space. The collection of all compatible bounded metrics on a metrizable space X is denoted by $\operatorname{Metr}(X)$. For $d \in \operatorname{Metr}(Y)$ the supremum metric on $\mathcal{C}(X,Y)$ induced by d is denoted by $d_{\sup}$. The category of continuous functions between topological spaces is denoted by Top. Whenever $\mathcal{K}$ is a class of topological spaces, $\mathsf{Top}\restriction_{\mathcal{K}}$ denotes the category of (all) maps between members of $\mathcal{K}$ (thus $\mathcal{K}$ is the class of all objects in $\mathsf{Top}\restriction_{\mathcal{K}}$). The identity map on X is denoted by id_X.

Let $\mathfrak{Z} = \mathsf{Top}\restriction_{\mathcal{Z}(\Omega)}$ and $\mathfrak{C} = \mathsf{Top}\restriction_{\{\Omega\}}$. Notice that whenever $\mathcal{E}\colon \mathfrak{Z} \to \mathfrak{C}$ is a functor, then necessarily $\mathcal{E}(K) = \Omega$ and $\mathcal{E}(\varphi)$ is a map of Ω into itself for each $K \in \mathcal{Z}(\Omega)$ and every map φ between two Z-sets in Ω. Our main result is

Theorem 1.1.

*There exists a **functor** $\mathcal{E}\colon \mathfrak{Z} \to \mathfrak{C}$ such that for any $\varphi \in \mathcal{C}(K,L)$ with $K, L \in \mathcal{Z}(\Omega)$,*

($\mathcal{E}0$) *$\mathcal{E}(\varphi)$ extends φ;*

($\mathcal{E}1$) *$\mathcal{E}(\varphi)$ is an injection (resp. a surjection or an embedding) iff φ is so;*

($\mathcal{E}2$) *the image of $\mathcal{E}(\varphi)$ is closed in Ω iff the image of φ is closed in L, and similarly with "dense" in place of "closed";*

($\mathcal{E}3$) *for an arbitrary sequence $\varphi_1, \varphi_2, \ldots \in \mathcal{C}(K,L)$, the maps $\mathcal{E}(\varphi_n)$ converge to $\mathcal{E}(\varphi)$ pointwise (resp. uniformly on compact sets) iff the maps φ_n converge so to φ.*

Condition ($\mathcal{E}0$) of the above result asserts that $\mathcal{E}$ (being a functor) extends homeomorphisms to autohomeomorphisms of Ω. The functor $\mathcal{E}$ has also additional properties listed in the following proposition.

Proposition 1.2.

Under the notation of Theorem 1.1,

($\mathcal{E}4$) *$\overline{\operatorname{im}}(\mathcal{E}(\varphi))$ is homeomorphic to Ω; and $\operatorname{im}(\varphi)$ is completely metrizable iff so is $\operatorname{im}(\mathcal{E}(\varphi))$, iff $\operatorname{im}(\mathcal{E}(\varphi))$ is homeomorphic to Ω;*

($\mathcal{E}5$) *$\overline{\operatorname{im}}(\mathcal{E}(\varphi))$ either is a Z-set in Ω or coincides with Ω;*

($\mathcal{E}6$) *the image of $\mathcal{E}(\varphi)$ and its closure in Ω intersect L along $\operatorname{im}(\varphi)$ and $\overline{\operatorname{im}}(\varphi)$, respectively.*

Our method also enables extending metrics in a way that the extensor for metrics harmonize with the functor $\mathcal{E}$ discussed above, as shown by

Theorem 1.3.

Under the notation of Theorem 1.1, to each (complete) metric $d \in \operatorname{Metr}(L)$ one may assign a [complete] metric $\mathcal{E}(d) \in \operatorname{Metr}(\Omega)$ such that:

($\mathcal{E}7$) *$\mathcal{E}(d)$ extends d;*

($\mathcal{E}8$) *for any $K \in \mathcal{Z}(\Omega)$, the map $(\mathcal{C}(K,L), d_{\sup}) \ni \xi \mapsto \mathcal{E}(\xi) \in (\mathcal{C}(\Omega,\Omega), \mathcal{E}(d)_{\sup})$ is isometric;*

($\mathcal{E}9$) *if $K \in \mathcal{Z}(\Omega)$ and $\varrho \in \operatorname{Metr}(K)$, then a map $\varphi \in \mathcal{C}(K,L)$ is uniformly continuous (resp. a uniform embedding, (bi-)Lipschitz, nonexpansive, isometric) with respect to ϱ and d iff $\mathcal{E}(\varphi)$ is so with respect to $\mathcal{E}(\varrho)$ and $\mathcal{E}(d)$.*

Our proofs are surprisingly easy and use simple ideas. However, the main tools of this paper are Anderson's result (mentioned at the beginning of the section) and the well-known theorem of Keller [11] (for $\Omega = Q$) and the result of Bessaga and Pełczyński [7] on spaces of measurable functions (for $\Omega = \ell_2$). What is more, the main proof is nonconstructive and — in contrast to homeomorphism extension theorems — the extensor will not be continuous in the limitation topologies (see [16] for the definition) in case $\Omega = \ell_2$.

In the next section we shall prove the above results in case of $\Omega = Q$, while Section 3 deals with the Hilbert space settings.

2. Hilbert cube

Throughout this section we assume that $\Omega = Q$. In this case, the main tool to build the functor $\mathcal{E}$ will be the well-known theorem of Keller [11] (the proof may also be found in [8, Chapter III, § 3]).

Theorem 2.1.

Every compact, metrizable, infinite-dimensional convex subset of a locally convex topological vector space is homeomorphic to Q.

We shall build the functor $\mathcal{E}$ using two additional functors, denoted by $\mathcal{I}$ and $\mathcal{M}$. Below we will write $A \oplus B$ for the topological disjoint union (or the direct sum) of topological spaces A and B. For simplicity, we shall write $A \cong B$ if A and B are homeomorphic, and we follow the convention that $A \subset A \oplus B$.

In order to define $\mathcal{I}$, let us fix a homeomorphic copy Ω' of Q with a metric $d' \in \mathrm{Metr}(\Omega')$ such that $\mathrm{diam}(\Omega', d') = 1$. For any $K, L \in \mathcal{Z}(Q)$, $d \in \mathrm{Metr}(L)$ and $\varphi \in \mathcal{C}(K, L)$ let $\mathcal{I}(K) = K \oplus \Omega'$, and let $\mathcal{I}(d) \in \mathrm{Metr}(\mathcal{I}(L))$ and $\mathcal{I}(\varphi) \in \mathcal{C}(\mathcal{I}(K), \mathcal{I}(L))$ be defined as follows: $\mathcal{I}(d)$ coincides with d on $K \times K$, with d' on $\Omega' \times \Omega'$ and $\mathcal{I}(d)(x, y) = \max(\mathrm{diam}(K, d), 1)$ if one of x and y belongs to K and the other to Ω'; $\mathcal{I}(\varphi)(x) = \varphi(x)$ for $x \in K$ and $\mathcal{I}(\varphi)(x) = x$ for $x \in \Omega'$. Notice that

$$\mathcal{I}(K) \text{ is a compact metrizable space having infinitely many points} \tag{1}$$

(this fact shall be used later), K is a closed subset of $\mathcal{I}(K)$, and $\mathcal{I}(\varphi)$ and $\mathcal{I}(d)$ extend φ and d, respectively. Now we turn to the definition of the functor $\mathcal{M}$.

Fix a compact metrizable space K. Let $\mathcal{M}(K)$ be the set of all probabilistic Borel measures on K equipped with the standard weak topology (inherited, thanks to the Riesz characterization theorem, from the weak-* topology of the dual Banach space of $\mathcal{C}(K, \mathbb{R})$), i.e. the topology with the basis consisting of finite intersections of sets of the form

$$B(\mu; f, \varepsilon) = \left\{ \lambda \in \mathcal{M}(K) : \left| \int_K f \, \mathrm{d}\mu - \int_K f \, \mathrm{d}\lambda \right| < \varepsilon \right\},$$

where $\mu \in \mathcal{M}(K)$, $f \in \mathcal{C}(K, \mathbb{R})$ and $\varepsilon > 0$. The space $\mathcal{M}(K)$ is compact, convex and metrizable. What is more, $\mathcal{M}(K)$ is infinite-dimensional provided K is an infinite set, and hence, by Theorem 2.1,

$$\mathrm{card}(K) \geqslant \aleph_0 \quad \Longrightarrow \quad \mathcal{M}(K) \cong Q. \tag{2}$$

For $a \in K$, let $\epsilon_a \in \mathcal{M}(K)$ denote the Dirac measure at a; that is, ϵ_a is the probabilistic measure on K such that $\epsilon_a(\{a\}) = 1$. It is clear that the map $\gamma_K : K \ni a \mapsto \epsilon_a \in \mathrm{im}(\gamma_K) \subset \mathcal{M}(K)$ is a homeomorphism. What is more,

$$\mathrm{card}(K) > 1 \quad \Longrightarrow \quad \mathrm{im}(\gamma_K) \in \mathcal{Z}(\mathcal{M}(K)). \tag{3}$$

Although (3) is elementary and simple, we will see in the sequel that it is a crucial property.

Further, for a metric $d \in \mathrm{Metr}(K)$ let $\mathcal{M}(d) : \mathcal{M}(K) \times \mathcal{M}(K) \to \mathbb{R}_+$ (where $\mathbb{R}_+ = [0, \infty)$) be defined by

$$\mathcal{M}(d)(\mu, \nu) = \sup \left\{ \left| \int_K f \, \mathrm{d}\mu - \int_K f \, \mathrm{d}\nu \right| : \ f \in \mathrm{Contr}(K, \mathbb{R}) \right\},$$

where $\mathrm{Contr}(K, \mathbb{R})$ stands for the family of all d-nonexpansive maps of K into $\mathbb{R}$. Then $\mathcal{M}(d) \in \mathrm{Metr}(\mathcal{M}(K))$ (the metric $\mathcal{M}(d)$ was rediscovered by many mathematicians, e.g., by Kantorovich, Monge, Rubinstein, Wasserstein; in Fractal Geometry it is known as the Hutchinson metric). Observe that

$$\mathcal{M}(d)(\epsilon_a, \epsilon_b) = d(a, b), \qquad a, b \in K. \tag{4}$$

Finally, for a map $\varphi\colon K \to L$ between compact metrizable spaces, we define $\mathcal{M}(\varphi)\colon \mathcal{M}(K) \to \mathcal{M}(L)$ by the formula

$$(\mathcal{M}(\varphi)(\mu))(B) = \mu(\varphi^{-1}(B)), \qquad \mu \in \mathcal{M}(K), \quad B \subset L \text{ is Borel.}$$

Thus $\mathcal{M}(\varphi)(\mu)$ is the transport of the measure μ by the map φ. Observe that if $\lambda = \mathcal{M}(\varphi)(\mu)$, then for each $g \in \mathcal{C}(L, \mathbb{R})$, $\int_L g \, d\lambda = \int_K g \circ \varphi \, d\mu$. This implies that $\mathcal{M}(\varphi) \in \mathcal{C}(\mathcal{M}(K), \mathcal{M}(L))$. Moreover, $\mathcal{M}(\varphi)$ is affine (so its image is a compact convex set) and

$$\mathcal{M}(\varphi)(\gamma_K(a)) = \gamma_L(\varphi(a)), \qquad a \in K. \tag{5}$$

It is easy to check that both $\mathcal{I}$ and $\mathcal{M}$ are functors. Now define a functor $\mathcal{L}$ as their composition; that is, $\mathcal{L} = \mathcal{M} \circ \mathcal{I}$. For transparency, for each $K \in \mathcal{Z}(Q)$ let $\delta_K\colon K \to \gamma_{\mathcal{I}(K)}(K) \subset \mathcal{L}(K)$ be a map obtained by restricting $\gamma_{\mathcal{I}(K)}$. Below we collect most important properties of the functor $\mathcal{L}$.

Lemma 2.2.

Under the notation introduced above, for arbitrary two Z-sets K and L in Q, a map $\varphi\colon K \to L$ and compatible metrics d and ϱ on K and L, respectively, the following conditions hold:

($\mathcal{L}1$) $\mathcal{L}(K) \cong Q$;

($\mathcal{L}2$) δ_K *is a homeomorphism and* $\mathrm{im}(\delta_K) \in \mathcal{Z}(\mathcal{L}(K))$;

($\mathcal{L}3$) $\mathcal{L}(\varphi)(\delta_K(x)) = \delta_L(\varphi(x))$ *for each* $x \in K$;

($\mathcal{L}4$) $\mathcal{L}(d)(\delta_K(x), \delta_K(y)) = d(x, y)$ *for any* $x, y \in K$;

($\mathcal{L}5$) *the function* $(\mathcal{C}(K, L), \varrho_{\sup}) \ni \xi \mapsto \mathcal{L}(\xi) \in \big(\mathcal{C}(\mathcal{L}(K), \mathcal{L}(L)), \mathcal{L}(\varrho)_{\sup}\big)$ *is isometric;*

($\mathcal{L}6$) *the assignment* $\mathcal{C}(K, L) \ni \xi \mapsto \mathcal{L}(\xi) \in \mathcal{C}(\mathcal{L}(K), \mathcal{L}(L))$ *preserves pointwise convergence of sequences;*

($\mathcal{L}7$) $\mathcal{L}(\varphi)$ *is (bi-)Lipschitz (resp. nonexpansive, isometric) with respect to* $\mathcal{L}(d)$ *and* $\mathcal{L}(\varrho)$ *iff such is* φ *with respect to* d *and* ϱ;

($\mathcal{L}8$) $\mathrm{im}(\mathcal{L}(\varphi)) = \{\mu \in \mathcal{L}(L) : \mu(\mathrm{im}(\varphi) \oplus \Omega') = 1\}$;

($\mathcal{L}9$) $\mathcal{L}(\varphi)$ *is injective iff* φ *is such.*

Proof. ($\mathcal{L}1$) follows from (1) and (2), while ($\mathcal{L}2$) is a consequence of (3). Further, ($\mathcal{L}3$) and ($\mathcal{L}4$) are implied by (5) and (4), respectively. Let us briefly show conditions ($\mathcal{L}5$)–($\mathcal{L}8$) (point ($\mathcal{L}9$) is left to the reader). ($\mathcal{L}6$) follows from the definition of the topology of $\mathcal{L}(L)$ and Lebesgue's dominated convergence theorem, while ($\mathcal{L}8$) is a consequence of the Kreĭn–Milman theorem: $\mathrm{im}(\mathcal{L}(\varphi))$ is a convex compact subset of $\mathcal{L}(\mathrm{im}(\varphi))$ (if we naturally identify the latter set with the set of all measures on $\mathcal{I}(L)$ which are supported on $\mathcal{I}(\mathrm{im}(\varphi))$) and contains all Dirac's measures concentrated on points of $\mathcal{I}(\mathrm{im}(\varphi))$, which are precisely the extreme points of $\mathcal{L}(\mathrm{im}(\varphi))$. In order to check ($\mathcal{L}5$), take $\varphi, \psi \in \mathcal{C}(K, L)$ and $\mu \in \mathcal{L}(K)$, and put $\mu_\varphi = \mathcal{L}(\varphi)(\mu)$ and $\mu_\psi = \mathcal{L}(\psi)(\mu)$. Note that

$$\begin{aligned}
\mathcal{L}(\varrho)\big(\mathcal{L}(\varphi)(\mu), \mathcal{L}(\psi)(\mu)\big) &= \sup\left\{ \left| \int_{\mathcal{I}(L)} f \, d\mu_\varphi - \int_{\mathcal{I}(L)} f \, d\mu_\psi \right| : f \in \mathrm{Contr}(\mathcal{I}(L), \mathbb{R}) \right\} \\
&= \sup\left\{ \left| \int_{\mathcal{I}(K)} f \circ \mathcal{I}(\varphi) \, d\mu - \int_{\mathcal{I}(K)} f \circ \mathcal{I}(\psi) \, d\mu \right| : f \in \mathrm{Contr}(\mathcal{I}(L), \mathbb{R}) \right\} \\
&\leqslant \sup\left\{ \int_{\mathcal{I}(K)} |f \circ \mathcal{I}(\varphi) - f \circ \mathcal{I}(\psi)| \, d\mu : f \in \mathrm{Contr}(\mathcal{I}(L), \mathbb{R}) \right\} \\
&\leqslant \int_{\mathcal{I}(K)} \varrho\big(\mathcal{I}(\varphi)(x), \mathcal{I}(\psi)(x)\big) \, d\mu(x) \leqslant \varrho_{\sup}(\varphi, \psi).
\end{aligned}$$

This gives $\mathcal{L}(\varrho)_{\sup}(\mathcal{L}(\varphi),\mathcal{L}(\psi)) \leqslant \varrho_{\sup}(\varphi,\psi)$. Since the reverse inequality is immediate (thanks to ($\mathcal{L}3$) and ($\mathcal{L}4$), we see that ($\mathcal{L}5$) is fulfilled.

It remains to show ($\mathcal{L}7$). This property is actually well known in Fractal Geometry, but for the reader's convenience, we shall prove it here. Assume that for some $L \in [1,\infty)$ and any $x,y \in K$, $\varrho(\varphi(x),\varphi(y)) \leqslant Ld(x,y)$. Then also $\mathcal{I}(\varrho)\big(\mathcal{I}(\varphi)(x),\mathcal{I}(\varphi)(y)\big) \leqslant L\mathcal{I}(d)(x,y)$ for any $x,y \in \mathcal{I}(K)$. We conclude that $(1/L)f\circ\mathcal{I}(\varphi) \in \mathrm{Contr}(\mathcal{I}(K),\mathbb{R})$ for $f \in \mathrm{Contr}(\mathcal{I}(L),\mathbb{R})$. This simply yields that $\mathcal{L}(\varphi)$ satisfies Lipschitz condition with constant L. Similarly, if $\varrho(\varphi(x),\varphi(y)) \geqslant d(x,y)/L$ for any $x,y \in K$ (where still $L \geqslant 1$), then $\mathcal{I}(\varrho)\big(\mathcal{I}(\varphi)(x),\mathcal{I}(\varphi)(y)\big) \geqslant \mathcal{I}(d)(x,y)/L$ for all $x,y \in \mathcal{I}(K)$ and therefore for any $f \in \mathrm{Contr}(\mathcal{I}(K),\mathbb{R})$ the function

$$\mathrm{im}(\mathcal{I}(\varphi)) \ni y \mapsto \frac{1}{L} f\big(\mathcal{I}(\varphi)^{-1}(y)\big) \in \mathbb{R}$$

(is well defined and) extends to a function $g \in \mathrm{Contr}(\mathcal{I}(L),\mathbb{R})$. This implies that every $f \in \mathrm{Contr}(\mathcal{I}(K),\mathbb{R})$ may be written in the form $f = L\cdot g\circ\mathcal{I}(\varphi)$ with $g \in \mathrm{Contr}(\mathcal{I}(L),\mathbb{R})$ chosen appropriately. Hence, for any $\mu_1,\mu_2 \in \mathcal{L}(K)$, putting $\nu_j = \mathcal{L}(\varrho)(\mu_j)$, $j=1,2$, we obtain

$$\begin{aligned}\mathcal{L}(\varrho)\big(\mathcal{L}(\varphi)(\mu_1),\mathcal{L}(\varphi)(\mu_2)\big) &= \sup\left\{\left|\int_{\mathcal{I}(L)} g\,\mathrm{d}\nu_1 - \int_{\mathcal{I}(L)} g\,\mathrm{d}\nu_2\right| : g \in \mathrm{Contr}(\mathcal{I}(L),\mathbb{R})\right\}\\ &= \sup\left\{\left|\int_{\mathcal{I}(K)} g\circ\mathcal{I}(\varphi)\,\mathrm{d}\mu_1 - \int_{\mathcal{I}(K)} g\circ\mathcal{I}(\varphi)\,\mathrm{d}\mu_2\right| : g \in \mathrm{Contr}(\mathcal{I}(L),\mathbb{R})\right\}\\ &\geqslant \frac{1}{L}\sup\left\{\left|\int_{\mathcal{I}(K)} f\,\mathrm{d}\mu_1 - \int_{\mathcal{I}(K)} f\,\mathrm{d}\mu_2\right| : f \in \mathrm{Contr}(\mathcal{I}(K),\mathbb{R})\right\} = \frac{1}{L}\,\mathcal{L}(d)(\mu_1,\mu_2)\end{aligned}$$

and we are done. □

Now we are ready to give

Proof of Theorem 1.1 for $\Omega = Q$. We continue the notation of the section. By ($\mathcal{L}2$) and Anderson's theorem [2], for every $K \in \mathcal{Z}(Q)$ there exists a homeomorphism $H_K\colon Q \to \mathcal{L}(K)$ which extends δ_K. We define $\mathcal{E}$ by: ($\mathcal{E}(K) = Q$ for $K \in \mathcal{Z}(Q)$ and)

$$\mathcal{E}(\varphi) = H_L^{-1}\circ\mathcal{L}(\varphi)\circ H_K, \qquad \varphi \in \mathcal{C}(K,L), \quad K,L \in \mathcal{Z}(Q).$$

It is readily seen that $\mathcal{E}$ is a functor (since $\mathcal{L}$ is). Let us check ($\mathcal{E}0$). If $\varphi \in \mathcal{C}(K,L)$ and $x \in K$, then $H_K(x) = \delta_K(x)$ and thus $\mathcal{E}(\varphi)(x) = \varphi(x)$ by ($\mathcal{L}3$). Finally, observe that conditions ($\mathcal{E}1$) and ($\mathcal{E}3$) immediately follow from ($\mathcal{L}8$)–($\mathcal{L}9$) and ($\mathcal{L}5$)–($\mathcal{L}6$), respectively, while ($\mathcal{E}2$) is trivial in the compact case. □

Proof of Proposition 1.2 for $\Omega = Q$. Point ($\mathcal{E}4$) follows from ($\mathcal{L}8$) and Keller's theorem (Theorem 2.1) and both ($\mathcal{E}5$) and ($\mathcal{E}6$) are consequences of ($\mathcal{L}8$). (Indeed: if X is a proper closed subset of a compact metrizable space Y, then $\{\mu \in \mathcal{M}(Y) : \mu(X) = 1\}$ is a Z-set in $\mathcal{M}(Y)$; and if $\mathcal{E}(\varphi)(x) = y \in L$ for some $x \in Q$, then $\mathcal{L}(\varphi)(\mu) = \epsilon_y$ for $\mu = H_K(x)$, which yields $\mu\big(\mathcal{I}(\varphi)^{-1}(\{y\})\big) = 1$ and therefore $y \in \mathrm{im}(\varphi)$). □

Proof of Theorem 1.3 for $\Omega = Q$. For $d \in \mathrm{Metr}(K)$ (where $K \in \mathcal{Z}(Q)$) and $x,y \in Q$ we put $\mathcal{E}(d)(x,y) = \mathcal{L}(d)\big(H_K(x),H_K(y)\big)$. Since H_K is a homeomorphism between Q and $\mathcal{L}(K)$ and

$$H_K \text{ is an isometry of } (Q,\mathcal{E}(d)) \text{ onto } (\mathcal{L}(K),\mathcal{L}(d)), \tag{6}$$

we conclude that $\mathcal{E}(d) \in \mathrm{Metr}(Q)$. Moreover, ($\mathcal{L}4$) implies ($\mathcal{E}7$). Finally, ($\mathcal{E}8$) and ($\mathcal{E}9$) follow from (6) and, respectively, ($\mathcal{L}5$) and ($\mathcal{L}7$). □

Recall that $\mathrm{Auth}(X)$ is the group of all autohomeomorphisms of a topological space X. Theorem 1.1 has the following consequence.

Corollary 2.3.
Let K be a Z-set in Q. Let $\mathrm{Auth}(Q,K) = \{h \in \mathrm{Auth}(Q) : h(K) = K\}$ *and* $\mathrm{Auth}_0(Q,K) = \{h \in \mathrm{Auth}(Q,K) : h(x) = x$ for $x \in K\}$ *be equipped with the topology of uniform convergence. Then there is a closed subgroup $\mathcal{G}$ of* $\mathrm{Auth}(Q,K)$ *such that the map*

$$\Phi\colon \mathrm{Auth}_0(Q,K) \times \mathcal{G} \ni (u,v) \mapsto u \circ v \in \mathrm{Auth}(Q,K)$$

is a homeomorphism.

Proof. It is enough to put $\mathcal{G} = \{\mathcal{E}(h) : h \in \mathrm{Auth}(K)\}$. Since the map $\Psi\colon \mathrm{Auth}(K) \ni h \mapsto \mathcal{E}(h) \in \mathrm{Auth}(Q)$ is an embedding and a group homomorphism and both $\mathrm{Auth}(K)$ and $\mathrm{Auth}(Q)$ are completely metrizable, therefore $\mathcal{G}$ is closed (see e.g. [12]). Now it remains to notice that

$$\Phi^{-1}(h) = \big(h \circ [\Psi(h\restriction_K)]^{-1}, \Psi(h\restriction_K)\big). \qquad \square$$

It is worth mentioning that the functor $\mathcal{M}$ introduced above in its full generality was investigated by Banakh in [5, 6].

3. Hilbert space

In this section we assume that $\Omega = \ell_2$. The proof in that case goes similarly. The main difference is that we shall change the functor $\mathcal{M}$. (Actually, the same functor $\mathcal{M}$ as used in Section 2, combined with the functor $\mathcal{I}$ described below, leads us, in the same way as before, to the functor of extension. However, the author is unable to resolve whether in this way one obtains a functor with all desired properties.) Moreover, the lack of compactness of the space Ω makes the details more complicated. Instead of Keller's theorem, which was used in the previous part, here we need a theorem of Bessaga and Pełczyński [7]. In order to state their result, we have to describe *spaces of measurable functions*.

Let X be a separable nonempty metrizable space and let $\mathcal{M}(X)$ be the space of all Lebesgue measurable functions of $[0,1]$ into X up to almost everywhere equality. For $d \in \mathrm{Metr}(X)$ the function

$$\mathcal{M}(d)\colon \mathcal{M}(X) \times \mathcal{M}(X) \ni (f,g) \mapsto \int_0^1 d(f(t),g(t))\,dt \in \mathbb{R}_+$$

is a bounded metric on $\mathcal{M}(X)$, $(\mathcal{M}(X),\mathcal{M}(d))$ is separable; and $\mathcal{M}(d)$ is complete iff d is so. The topology on $\mathcal{M}(X)$ induced by $\mathcal{M}(d)$ is independent of the choice of $d \in \mathrm{Metr}(X)$, and functions $f_1, f_2, \ldots \in \mathcal{M}(X)$ converge to $f \in \mathcal{M}(X)$ iff they converge to f *in measure* in the sense e.g. of Halmos (see [10, § 2]). Equivalently,

$$\lim_{n\to\infty} f_n = f \iff \text{every subsequence of } (f_n)_{n=1}^{\infty} \text{ has a subsequence converging to } f \text{ pointwise almost everywhere.} \qquad (7)$$

For us the most important property of $\mathcal{M}(X)$ is the following theorem of Bessaga and Pełczyński [7] (see also [8, Theorem VI.7.1]; for generalizations consult [14]).

Theorem 3.1.
If X is a separable metrizable space, then the space $\mathcal{M}(X)$ is homeomorphic to ℓ_2 iff X is completely metrizable and has more than one point.

Fix for a moment a separable metrizable space X. For $x \in X$ let $\epsilon_x \equiv x$. Put $\gamma_X\colon X \ni x \mapsto \epsilon_x \in \mathrm{im}(\gamma_X) \subset \mathcal{M}(X)$. Clearly, γ_X is a homeomorphism. What is more,

$$\mathrm{card}(X) > 1 \implies \mathrm{im}(\gamma_X) \in \mathcal{Z}(\mathcal{M}(X)). \qquad (8)$$

As in Section 2, observe that

$$\mathcal{M}(d)(\epsilon_x, \epsilon_y) = d(x, y), \qquad x, y \in X, \quad d \in \mathrm{Metr}(X). \tag{9}$$

If A is a subset of X, $\mathcal{M}(A)$ naturally embeds in $\mathcal{M}(X)$ and therefore we shall consider $\mathcal{M}(A)$ as a subset of $\mathcal{M}(X)$. Under such an agreement one has $\overline{\mathcal{M}(A)} = \mathcal{M}(\overline{A})$.

Now let Y be another separable metrizable space and $f \in \mathcal{C}(X, Y)$. We define $\mathcal{M}(f)\colon \mathcal{M}(X) \to \mathcal{M}(Y)$ by the formula $(\mathcal{M}(f))(u) = f \circ u$. It is easy to verify that $\mathcal{M}(f) \in \mathcal{C}(\mathcal{M}(X), \mathcal{M}(Y))$ and $\mathcal{M}$ is a functor in the category of separable metrizable spaces. We collect further properties of the above functor $\mathcal{M}$ in the following lemma.

Lemma 3.2.

Let X and Y be separable metrizable spaces, $f\colon X \to Y$ be a map and let d and ϱ be any bounded compatible metrics on X and Y, respectively.

($\mathcal{M}$1) *If A is a closed **proper** subset of X, then $\mathcal{M}(A)$ is a Z-set in $\mathcal{M}(X)$.*

($\mathcal{M}$2) *$\mathcal{M}(f)(\gamma_X(x)) = \gamma_Y(f(x))$ for any $x \in X$.*

($\mathcal{M}$3) *$\mathcal{M}(f)$ is an injection or an embedding iff so is f.*

($\mathcal{M}$4) *If $f_1, f_2, f_3, \ldots \in \mathcal{C}(X, Y)$ converge pointwise or uniformly on compact sets to f, then $\mathcal{M}(f_n)$ converge so to $\mathcal{M}(f)$.*

($\mathcal{M}$5) *The map $(\mathcal{C}(X, Y), \varrho_{\sup}) \ni u \mapsto \mathcal{M}(u) \in \big(\mathcal{C}(\mathcal{M}(X), \mathcal{M}(Y)), \mathcal{M}(\varrho)_{\sup}\big)$ is isometric.*

($\mathcal{M}$6) *$\mathcal{M}(f)$ is uniformly continuous (resp. a uniform embedding, (bi-)Lipschitz, nonexpansive, isometric) with respect to $\mathcal{M}(d)$ and $\mathcal{M}(\varrho)$ iff so is f with respect to d and ϱ.*

Proof. To see ($\mathcal{M}$1), take $a \in X \setminus A$ and observe that the maps

$$\Phi_n\colon \mathcal{M}(X) \ni f \;\mapsto\; \epsilon_a\restriction_{[0,1/n)} \cup f\restriction_{[1/n,1]} \in \mathcal{M}(X)$$

converge uniformly on compact subsets of $\mathcal{M}(X)$ to $\mathrm{id}_{\mathcal{M}(X)}$ and their images are disjoint from $\mathcal{M}(A)$ (and, of course, $\mathcal{M}(A)$ is closed in $\mathcal{M}(X)$).

Items ($\mathcal{M}$2), ($\mathcal{M}$5) and first claims of ($\mathcal{M}$3) and ($\mathcal{M}$4) are quite easy and we leave them to the reader. Also the part of ($\mathcal{M}$6) concerning (bi-)Lipschitz, nonexpansive and isometric maps is immediate. Let us show the second claim of ($\mathcal{M}$4). Below we involve criterion (7). Assume $f_1, f_2, f_3, \ldots \in \mathcal{C}(X, Y)$ converge uniformly on compact sets to $f \in \mathcal{C}(X, Y)$. Let $(u_n)_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{M}(X)$ which is convergent to $u \in \mathcal{M}(X)$. We have to prove that $(\mathcal{M}(f_n)(u_n))_{n=1}^{\infty}$ converges to $\mathcal{M}(f)(u)$. For an arbitrary subsequence of $(u_n)_{n=1}^{\infty}$ take its subsequence $(u_{\nu_n})_{n=1}^{\infty}$ such that the set $T = \{t \in [0,1] : u_{\nu_n}(t) \to u(t)\}$ has Lebesgue measure equal to 1. Observe that $f_{\nu_n}(u_{\nu_n}(t)) \to f(u(t))$ for $t \in T$. But this means that $(\mathcal{M}(f_n)(u_n))_{n=1}^{\infty}$ tends to $\mathcal{M}(f)(u)$ in the topology of $\mathcal{M}(Y)$ and we are done. In a similar manner one checks that $\mathcal{M}(f)$ is an embedding provided f is so.

Now assume f is uniformly continuous with respect to d and ϱ. Since ϱ is bounded, we conclude that there exists a bounded continuous monotone concave function $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ vanishing at 0 such that

$$\varrho(f(x), f(y)) \leqslant \omega(d(x, y)), \qquad x, y \in X \tag{10}$$

(see e.g. [4]). Now Jensen's inequality (applied for a convex function $t \mapsto M - \omega(t)$ where M is an upper bound of ω) combined with (10) yields that, for any $u, v \in \mathcal{M}(X)$,

$$\mathcal{M}(\varrho)\big(\mathcal{M}(f)(u), \mathcal{M}(f)(v)\big) = \int_0^1 \varrho(f(u(t)), f(v(t)))\,\mathrm{d}t \leqslant \int_0^1 \omega(d(u(t), v(t)))\,\mathrm{d}t \leqslant \omega\left(\int_0^1 d(u(t), v(t))\,\mathrm{d}t\right) = \omega(\mathcal{M}(d)(u, v))$$

and thus $\mathcal{M}(f)$ is uniformly continuous. Conversely, if $\mathcal{M}(f)$ is uniformly continuous, then f is so, by ($\mathcal{M}$2) and (9).

Finally, if f is a uniform embedding, we may repeat the above argument, starting from a bounded continuous monotone concave function $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ vanishing at 0 such that $d(x, y) \leqslant \tau(\varrho(f(x), f(y)))$ for any $x, y \in X$, and finishing with $\mathcal{M}(d)(u, v) \leqslant \tau\big(\mathcal{M}(\varrho)(\mathcal{M}(f)(u), \mathcal{M}(f)(v))\big)$ for all $u, v \in \mathcal{M}(X)$. $\square$

Our last step is to prove that $\mathrm{im}(\mathcal{M}(f)) = \mathcal{M}(\mathrm{im}(f))$. It is however not as simple as it looks. To show this, we shall apply two theorems of the descriptive set theory and we have to introduce the terminology.

A *Souslin space* is the empty topological space or a metrizable space which is a continuous image of the space $\mathbb{R} \setminus \mathbb{Q}$. We shall need the following three properties of Souslin spaces (for proofs and more information see e.g. [13, Chapter XIII] or [15, Appendix]):

(S1) a continuous image of a Borel subset of a separable completely metrizable space is a Souslin space;

(S2) the inverse image of a Souslin space under a Borel function (between Borel subsets of separable completely metrizable spaces) is Souslin as well;

(S3) every Souslin subspace of the interval $[0,1]$ is Lebesgue measurable ([13, Theorem XIII.4.1] or [15, Theorem A.13]).

The main tool used in the next result is the following theorem.

Theorem 3.3.

Let $Y \neq \varnothing$ be a separable completely metrizable space; let $X \neq \varnothing$ be any set and let $\mathcal{R}$ be a σ-algebra of subsets of X. If a function $F\colon X \to 2^Y$ satisfies the following two conditions:

(i) *$F(x)$ is a nonempty and closed subset of Y for any $x \in X$,*

(ii) *$\{x \in X : F(x) \cap U \neq \varnothing\} \in \mathcal{R}$ for any open subset U of the space Y,*

then there exists a function $f\colon X \to Y$ such that $f(x) \in F(x)$ for every $x \in X$ and f is $\mathcal{R}$-measurable, that is, $f^{-1}(U) \in \mathcal{R}$ for all open sets $U \subset Y$.

For a proof and a discussion, consult [13, Theorem XIV.1.1].

Proposition 3.4.

If X and Y are two separable metrizable spaces and $f \in \mathcal{C}(X,Y)$, then $\mathrm{im}(\mathcal{M}(f)) = \mathcal{M}(\mathrm{im}(f))$, provided X is completely metrizable.

Proof. The inclusion "$\subset$" easily follows from the relation $\mathrm{im}(\mathcal{M}(f)(u)) \subset \mathrm{im}(f)$. To prove the reverse one, take a Borel function $v\colon [0,1] \to Y$ such that $v([0,1]) \subset \mathrm{im}(f)$. Let $\mathcal{L}$ denote the σ-algebra of all Lebesgue measurable subsets of $[0,1]$. Define $F\colon [0,1] \to 2^X$ by the formula $F(t) = f^{-1}(\{v(t)\})$. Clearly, $F(t)$ is nonempty and closed in X for any $t \in [0,1]$. What is more, if U is open in X, then, by (S1), $f(U)$ is a Souslin space and hence, by (S2), so is the set $v^{-1}(f(U))$ and therefore it is Lebesgue measurable (by (S3)). But $v^{-1}(f(U)) = \{t \in [0,1] : F(t) \cap U \neq \varnothing\}$, so $\{t \in [0,1] : F(t) \cap U \neq \varnothing\} \in \mathcal{L}$. Now Theorem 3.3 gives us a Lebesgue measurable function $u\colon [0,1] \to X$ such that $u(t) \in F(t)$ for any $t \in [0,1]$. This means that $u \in \mathcal{M}(X)$ and $(\mathcal{M}(f))(u) = v$. □

Proof of Theorem 1.1 for $\Omega = \ell_2$. As in Section 2, we fix a homeomorphic copy Ω' of ℓ_2 and a complete metric $d' \in \mathrm{Metr}(\Omega')$ such that $\mathrm{diam}(\Omega', d') = 1$. Now let $\mathcal{I}$ be a functor built in the same way as in the previous part of the paper:

- $\mathcal{I}$ assigns to each Z-set K in ℓ_2 the space $K \oplus \ell_2$;
- for $K, L \in \mathcal{Z}(\ell_2)$ and a map $f\colon K \to L$, $\mathcal{I}(f) \in \mathcal{C}(\mathcal{I}(K), \mathcal{I}(L))$ coincides with f on K and with $\mathrm{id}_{\Omega'}$ on Ω';
- for $K \in \mathcal{Z}(\ell_2)$ and $d \in \mathrm{Metr}(K)$, $\mathcal{I}(d) \in \mathrm{Metr}(\mathcal{I}(K))$ coincides with d on $K \times K$, with d' on $\Omega' \times \Omega'$ and $\mathcal{I}(d)(x,y) = \max(\mathrm{diam}(K,d), 1)$ otherwise.

Now we mimic the proof of the theorem for $\Omega = Q$. We define a functor $\mathcal{L}$ by $\mathcal{L} = \mathcal{M} \circ \mathcal{I}$, and for any $K \in \mathcal{Z}(\ell_2)$ denote by $\delta_K\colon K \to \gamma_{\mathcal{I}(K)}(K)$ the restriction of $\gamma_{\mathcal{I}(K)}$ and take a homeomorphism $H_K\colon \ell_2 \to \mathcal{L}(K)$ which extends δ_K, based on Theorem 3.1 and (8). Finally, for $\varphi \in \mathcal{C}(K,L)$ (with $K, L \in \mathcal{Z}(\ell_2)$) we put $\mathcal{E}(\varphi) = H_L^{-1} \circ \mathcal{L}(\varphi) \circ H_K$. Note that, by Proposition 3.4,

$$\mathrm{im}(\mathcal{M}(\varphi)) = \mathcal{M}(\mathrm{im}(\varphi) \oplus \Omega'). \tag{11}$$

Now in the same way as in Section 2 one checks ($\mathcal{E}0$), ($\mathcal{E}1$)–($\mathcal{E}2$) (use ($\mathcal{M}3$) and (11)) and ($\mathcal{E}3$) (apply ($\mathcal{M}4$)). The details are left to the reader. □

Proof of Proposition 1.2 for $\Omega = \ell_2$. It is readily seen that ($\mathcal{E}4$) follows from Theorem 3.1 and (11), while ($\mathcal{E}5$) from ($\mathcal{M}1$) and (11). Finally, ($\mathcal{E}6$) may briefly be deduced from (11) and the formula for $\mathcal{E}(\varphi)$ (cf. the proof of the proposition for $\Omega = Q$). $\square$

Proof of Theorem 1.3 for $\Omega = \ell_2$. As in Section 2, for $d \in \mathrm{Metr}(K)$ (where $K \in \mathcal{Z}(\ell_2)$) define $\mathcal{E}(d) \in \mathrm{Metr}(\ell_2)$ by $\mathcal{E}(d)(x, y) = \mathcal{L}(d)\big(H_K(x), H_K(y)\big)$. Now it suffices to repeat the proof from the previous case, involving ($\mathcal{M}5$) and ($\mathcal{M}6$). $\square$

We end the paper with a note that we do not know if there exists an analogous functor of extension of mappings between Z-sets of ℓ_2 which is continuous in the limitation topologies.

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