Shift-Type Properties of Commuting, Completely Non Doubly Commuting Pairs of Isometries

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Abstract. Pairs (V, V') of commuting, completely non doubly commuting isometries are studied. We show, that the space of the minimal unitary extension of V (denoted by U) is a closed linear span of subspaces reducing U to bilateral shifts. Moreover, the restriction of V' to the maximal subspace reducing V to a unitary operator is a unilateral shift. We also get a new hyperreducing decomposition of a single isometry with respect to its wandering vectors which strongly corresponds with Lebesgue decomposition.

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1. Introduction and Preliminaries

Let $L(H)$ denote the algebra of all bounded linear operators acting on a complex Hilbert space H. For an operator $T \in L(H)$ by its negative power T^n we understand $T^{*|n|}$. Recall that a subspace $L \subset H$ reduces $T \in L(H)$ if and only if T commutes with the orthogonal projection P_L onto L. By the span of $E \subset H$, we always mean the minimal closed linear subspace containing E.

Recall the classical result of von Neumann-Wold [18]:

Theorem 1.1. *Let $V \in L(H)$ be an isometry. There is a unique decomposition of H into a sum of two orthogonal, reducing for V subspaces H_u, H_s, such that $V|_{H_u}$ is a unitary operator and $V|_{H_s}$ is a unilateral shift. Moreover,*

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$$H_u = \bigcap_{n \geq 0} V^n H, \quad H_s = \bigoplus_{n \geq 0} V^n (\ker V^*). \tag{1.1}$$

$\square$

For a given isometry $V \in L(H)$ by H_u, H_s we always mean the subspaces in the decomposition (1.1). The restrictions $V|_{H_u}, V|_{H_s}$ are referred as the unitary part and the shift part of the considered isometry. A natural question arises about generalizations for pairs or families of operators. The most natural generalization, which following [7] is proposed to be called a *multiple canonical von Neumann-Wold decomposition*, has been achieved only in some special cases ([4,16]). In the general case various von Neumann-Wold type decompositions or models were established ([1–3,5,9,10,14,17]). Recall that operators $T_1, T_2 \in L(H)$ doubly commute if they commute and $T_1^* T_2 = T_2 T_1^*$. Consider a pair of isometries (V_1, V_2) on H. One can find a unique maximal subspace reducing it to a doubly commuting pair. In [16] a multiple von Neumann-Wold decomposition in the case of doubly commuting pairs is constructed along with a model for pairs of doubly commuting unilateral shifts. Therefore, we consider only completely non doubly commuting pairs (i.e. such that the only subspace of H reducing (V_1, V_2) to a doubly commuting pair is $\{0\}$). Examples of such pairs are: non doubly commuting unilateral shifts or the so called modified bi-shifts (see [14]). Note that if operators commute and one of them is unitary, then they doubly commute. Thus, in completely non doubly commuting pairs both of the isometries have nontrivial unilateral shift parts and restrictions to any nontrivial subspace reducing both operators also have a nontrivial unilateral shift part. However, the unitary part may be, but need not to be trivial.

The property of being bilateral shift is not hereditary (i.e. the restriction of a bilateral shift to some reducing subspace may be not a bilateral shift). Therefore, usually there cannot be found the largest subspace reducing a given isometry to a bilateral shift or a span of bilateral shifts (see Definition 4.8). This means that usually it cannot be constructed a canonical decomposition of an isometry into a bilateral shift (or a span of bilateral shifts) and a completely non bilateral shift operator. However, in Theorem 3.10 we are able to construct a decomposition of a single isometry with respect to its wandering vectors. One of the summands of the constructed decomposition contains all the bilateral shifts.

There are known examples of undecomposable pairs where the unitary part of any isometry is a bilateral shift (the aforementioned modified bi-shift). We describe the unitary part and the minimal unitary extension of an isometry which commutes but completely non doubly commutes with some other isometry. One of our main results contained in Theorem 4.5 and Corollary 4.9 says that the unitary extension of any member of a completely non doubly commuting pair is a span of bilateral shifts. Since being a span of bilateral shifts is not a hereditary property, it does not mean that the unitary part of a considered isometry is a span of bilateral shifts. In Sect. 5 we show that the unitary part of an isometry may not contain any subspace reducing it to a bilateral shift but its unitary extension can be a span of bilateral shifts

or even a bilateral shift. Denote by H_{u1} the maximal subspace reducing the isometry V_1 to the unitary operator and by V_2 an isometry commuting with V_1. It is known that H_{u1} is a hyperinvariant subspace for V_1. Theorem 4.11, says that $V_2|_{H_{u1}}$ is a unilateral shift. Moreover it shifts between subspaces reducing V_1. Such a description seems to be useful for building models for commuting pairs of isometries.

2. Multiple von Neumann-Wold Decomposition for Pairs of Isometries

Let us recall the notion of multiple canonical von Neumann-Wold decompositions introduced in [7] in the general case, here taking a simplified form in the case of a pair of commuting isometries:

Definition 2.1. Suppose (V_1, V_2) is a pair of isometries on H. The multiple canonical von Neumann-Wold decomposition is given by a decomposition of the Hilbert space

$$H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss},$$

where H_{uu}, H_{us}, H_{su}, H_{ss} are reducing subspaces for V_1 and V_2 such that

$V_1|_{H_{uu}}$, $V_2|_{H_{uu}}$ are unitary operators,
$V_1|_{H_{us}}$ is a unitary operator, $V_2|_{H_{us}}$ is a unilateral shift,
$V_1|_{H_{su}}$ is a unilateral shift, $V_2|_{H_{su}}$ is a unitary operator,
$V_1|_{H_{ss}}$, $V_2|_{H_{ss}}$ are unilateral shifts.

By [4,16] there are multiple canonical von Neumann-Wold decompositions in either of the two cases: for doubly commuting pairs of isometries and for pairs satisfying the conditions $\dim(\ker V_1^*) < \infty$ and $\dim(\ker V_2^*) < \infty$. Moreover, in the case of doubly commuting isometries, the subspace $\ker V_1^* \cap \ker V_2^*$ is wandering for the semigroup generated by V_1, V_2. However in the general case we have only a weaker result. Recall a definition from [14].

Definition 2.2. A pair (V_1, V_2) of isometries is called a weak bi-shift if all the isometries $V_1|_{\bigcap_{i \geq 0} \ker V_2^* V_1^i}$, $V_2|_{\bigcap_{i \geq 0} \ker V_1^* V_2^i}$ and $V_1 V_2$ are shifts.

The following general decomposition of pairs of commuting isometries obtained in [14] is not necessarily a canonical one.

Theorem 2.3. *For any pair of commuting isometries (V_1, V_2) on H there is a unique decomposition*

$$H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ws}, \tag{2.1}$$

such that H_{uu}, H_{us}, H_{su}, H_{ws} reduce V_1 and V_2 and

$V_1|_{H_{uu}}, V_2|_{H_{uu}}$ *are unitary operators,*
$V_1|_{H_{us}}$ *is a unitary operator, $V_2|_{H_{us}}$ is a unilateral shift,*
$V_1|_{H_{su}}$ *is a unilateral shift, $V_2|_{H_{su}}$ is a unitary operator,*
$(V_1|_{H_{ws}}, V_2|_{H_{ws}})$ *is a weak bi-shift.* $\square$

We are going to focus on the weak bi-shift part. Precisely, we consider a pair of isometries whose decomposition (2.1), trivializes to the weak bi-shift subspace. In such a case the subspace reducing our isometries to a doubly commuting pair is either trivial or reduces the isometries to a pair of unilateral shifts. Indeed, in the other case the decomposition of the restriction to a doubly commuting pair of isometries would give a non trivial subspace orthogonal to H_{ws}. By [14] there can be found a maximal subspace of H_{ws} which reduces these isometries to a doubly commuting pair of unilateral shifts. Their model can be found in [16]. Therefore we reduce our attention to completely non doubly commuting pairs of isometries. Such pairs are a special case of a weak bi-shift class whose finer, but not fully satisfying decomposition has been described in [5].

3. Decomposition for Single Isometries

A unitary part of an isometry in a modified bi-shift is a bilateral shift. Our aim in the present section is to construct a decomposition with respect to a property which is close to the property of being a span of all the shifts (bilateral or unilateral). The following example shows that "being a bilateral shift" is not a hereditary property.

Example 3.1. Let $H = L^2(m)$, where m denotes the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Let V be the operator of multiplication by the independent variable, $(Vf)(z) = zf(z)$ for $f \in L^2(m)$. Then its spectral measure F satisfies $F(\alpha)f = \chi_\alpha f$ for $f \in L^2(m)$ and all Borel subsets α of $\mathbb{T}$. Let α be a proper subarc of $\mathbb{T}$. Let V_α be the restriction of V to its reducing subspace $F(\alpha)H$. Since α is not of total measure in the circle, the operator V_α is not a bilateral shift.

Note that a canonical decomposition of an operator $T \in L(H)$ with respect to some property is in fact a construction of a unique maximal reducing subspace $H_p \subset H$ such that the restriction $T|_{H_p}$ has the considered property. Since being a bilateral shift is not a hereditary property, there is a problem with construction of a maximal subspace reducing operator to a bilateral shift. Example 5.2 in the last section will show that such a maximal subspace is not unique. The construction of any maximal bilateral shift subspace can be done by considering a maximal wandering subspace. We follow the idea of wandering vectors from [5]. Let G be a semigroup and $\{T_g\}_{g \in G}$ be a semigroup of isometries on H. The vector $x \in H$ is called a *wandering vector* (for a given semigroup of isometries) if for any $g_1 \neq g_2$ we have $\langle T_{g_1}x, T_{g_2}x \rangle = 0$. For a semigroup generated by a single isometry we obtain the following definition of a wandering vector.

Definition 3.2. A vector $x \in H$ is a wandering vector of isometry $V \in L(H)$ if $V^n x \perp x$ for every positive n.

Note that for any wandering vector x the vector $x + Vx$ is not wandering. Indeed, since x is wandering, so is Vx. Therefore $\langle x + Vx, V(x + Vx) \rangle = \langle Vx, Vx \rangle$. Thus the only linear V – invariant subspace of wandering vectors is

the trivial one. Since we are interested in reducing subspaces, there is no point in considering subspaces of wandering vectors. Instead we consider subspaces generated by wandering vectors. Note that for a wandering vector x we have $V^n x \perp V^m x$ for any pair of distinct positive indices n, m. Let $H = H_u \oplus H_s$ be von Neumann-Wold decomposition of a given isometry $V \in L(H)$. If a wandering vector $x \in H_u$ is chosen, then $V^n x \perp V^m x$ for $n, m \in \mathbb{Z}, n \neq m$. However for $x \in H_s$ this is not so clear. On the other hand, every vector in the set $\bigcup_{n \geq 0} V^n(\ker V^*)$ is wandering, fulfills the orthogonality condition also for the negative powers and generates the whole H_s. Therefore, despite Definition 3.2 is "weaker" it seems to be sufficient.

Theorem 3.3. *For any isometry $V \in L(H)$ there is a unique decomposition:*

$$H = H_0 \oplus H_w,$$

reducing operator V such, that

- H_w *is a span of all wandering vectors,*
- $H_0 \subset H_u$.

Proof. Since $H_s \subset H_w$ we need to show only that H_w is V-reducing. Obviously, H_w is V-invariant. Note that for w wandering we have $w, P_{H_s} w \in H_w$ and consequently also $P_{H_u} w = w - P_{H_s} w \in H_w$. Note also that $P_{H_u} H_w = P_{H_u}(H_w \ominus H_s) = H_w \ominus H_s$. Thus $x \in H_w$ if and only if $P_{H_u} x \in H_w$. Let us show that $P_{H_u} V^* w \in H_w$ for an arbitrary wandering vector w. For any vector $w \in H$ let $w_u = P_{H_u} w, w_s = P_{H_s} w$. By $\langle V^n w, w \rangle = \langle V^n w_u, w_u \rangle + \langle V^n w_s, w_s \rangle$ the vector w is wandering if and only if $\langle V^n w_u, w_u \rangle = -\langle V^n w_s, w_s \rangle$ for every positive n. On the other hand $\langle V^n V^* w_u, V^* w_u \rangle = \langle V^n w_u, w_u \rangle = -\langle V^n w_s, w_s \rangle$. Thus if $w = w_u \oplus w_s$ is wandering, then $\tilde{w} = (V^* w_u) \oplus w_s$ is wandering as well. Moreover, $P_{H_u} V^* w = V^* P_{H_u} w = P_{H_u} \tilde{w}$. On the other hand, since $\tilde{w}$ is wandering, by the previous argumentation we have $P_{H_u} \tilde{w} \in H_w$. Consequently, $P_{H_u} V^* w \in H_w$. Since w was an arbitrary wandering vector and H_w is spanned by wandering vectors, we get $V^* P_{H_u} H_w \subset H_w$. By the just showed inclusion and inclusion $H_s \subset H_w$ we get $V^* H_w \subset H_w$. Consequently, H_w reduces V. $\qquad\square$

Remark 3.4. Note, that if $L \subset H$ reduces V to a shift (unilateral or bilateral), then $L \subset H_w$.

Remark 3.5. Every invariant subspace of H_0 is reducing.

Proof. Let L be an invariant subspace of H_0, then the set $L \ominus VL$ consists of wandering vectors. Thus $L \ominus VL = \{0\}$. $\qquad\square$

For H^2, the Hardy space on the unit circle, the following result is well known (see [11] p. 53).

Proposition 3.6. *Let h be a non-negative Lebesgue integrable function on the circle. A necessary and sufficient condition that h be of the form $h = |f|^2$, with f a non-zero function in H^2, is that $\log h$ is integrable.*

Corollary 3.7. *Let μ be a non-negative absolutely continuous measure (with respect to Lebesgue measure m on the unit circle). Let us denote $\sigma := \operatorname{supp} \mu$.*

Assume that the logarithm of the Radon-Nikodym derivative $\frac{d\mu}{dm}$ is integrable on σ.

Then the operator M_μ of the multiplication by independent variable $"z"$ on the space $L^2(\sigma, \mu)$, is unitary equivalent to M_z - the operator of multiplication by independent variable $"z"$ on the space $L^2(\sigma, m)$.

Proof. Put $h := \frac{d\mu}{dm} + \chi_{\mathbb{T}\backslash\sigma}$. Obviously, h is positive almost everywhere on the unite circle $\mathbb{T}$. Since $\log h(z) = 0$ for $z \in \mathbb{T}\backslash\sigma$ and $\log\frac{d\mu}{dm}$ is Lebesgue integrable on σ then $\log h$ is Lebesgue integrable on $\mathbb{T}$.

Consequently, by Proposition 3.6 we have $h = |f|^2$ for some $f \in H^2$. Consider the following operator:

$$U_f : L^2(\sigma, \mu) \ni u \rightarrow uf \in L^2(\mathbb{T}, m),$$

where u is extended to the whole $\mathbb{T}$ as usual by taking its value 0 on $\mathbb{T}\backslash\sigma$. Since $|f|^2 = \frac{d\mu}{dm}$ on σ and $uf = 0$ on $\mathbb{T}\backslash\sigma$ then operator U_f preserves the scalar product (it is an isometry). It follows also that uf belongs to $L^2(\mathbb{T}, m)$ if and only if $u \in L^2(\sigma, \mu)$. The range space of U_f can be understand as a subspace of $L^2(\mathbb{T}, m)$. We will show that $\mathcal{R}(U_f) = L^2(\sigma, m)$ and consequently that U_f is a unitary mapping onto its range. Since $\frac{d\mu}{dm}$ is positive almost everywhere on σ then f is not equal 0 almost everywhere on σ. Thus for $u \in L^2(\sigma, m)$ we can define $\frac{u}{f}$. Since $u \in L^2(\sigma, m)$ then $\int_\sigma |u|^2 dm$ exists and is finite. On the other hand,

$$\int_\sigma |u|^2 dm = \int_\sigma \left|\frac{u}{f}f\right|^2 dm = \int_\sigma \left|\frac{u}{f}\right|^2 |f|^2 dm = \int_\sigma \left|\frac{u}{f}\right|^2 d\mu$$

and integers exists simultaneously. Thus $u \in L^2(\sigma, m)$ implies $\frac{u}{f} \in L^2(\sigma, \mu)$. Moreover, $M_\mu = U_f^{-1} M_z U_f$. Consequently, U_f gives the unitary equivalence between M_μ and M_z. $\qquad\square$

Any isometry V acting on a Hilbert space H has Lebesgue decomposition which combined with von Neumann-Wold decomposition gives us the equality:

$$H = H_s \oplus H_{ac} \oplus H_{sing}, \qquad (3.1)$$

where the subspaces H_s, H_{ac}, H_{sing} reduce V, the operator $V|_{H_s}$ is a unilateral shift, the operator $V|_{H_{sing}}$ is unitary singular, i.e. its spectral measure is singular to the Lebesgue measure on the unit circle and $V|_{H_{ac}}$ is unitary absolutely continuous, i.e. its spectral measure is absolutely continuous with respect to the Lebesgue measure on the unit circle.

Consider unitary extension $U \in L(K)$ of V. Then $K_{sing} = H_{sing}$. Every wandering vector of a unitary operator generates a bilateral shift. Thus such a vector is contained in K_{ac}. Consequently K_w is orthogonal to K_{sing}. On the other hand a wandering vector of V is wandering for U which means $H_w \subset K_w$. Summing up $H_w \subset K_w \perp K_{sing} = H_{sing}$.

Theorem 3.8. *Let $V \in L(H)$ be a non unitary isometry. Then using the notation of (3.1) we get*

$$H_w = H_{ac} \oplus H_s,$$

where H_w is a linear subspace generated by all wandering vectors of V.

Proof. Since the singular part subspace H_{sing} is orthogonal to H_w and reducing for V we can assume that $H = H_{ac} \oplus H_s$. Consider a vector x orthogonal to H_w and assume $x \neq 0$. Define a subspace $H_x := \text{Span}\{V^n x : n \geq 0\}$. Since $H_x \subset H_0$ is an invariant subspace, then by Remark 3.5, it is reducing. Moreover, x is a cyclic vector for the restriction $V|_{H_x}$.

Therefore, by Theorem IX.3.4 of [8] the operator $V|_{H_x}$ can be represented as multiplication by $''z''$ on the space $L^2(\text{supp}\,\mu, \mu)$, where μ is a measure absolutely continuous with respect to the Lebesgue measure on the unit circle $\mathbb{T}$. Since the measure μ is different from 0, we can find $\epsilon > 0$ such that the set $\sigma := \{z \in \mathbb{T} : \frac{d\mu}{dm}(z) > \epsilon\}$ has positive measure. Then $\log \frac{d\mu}{dm}$ is integrable on σ. By Corollary 3.7 the operator $V|_{L^2(\sigma, \mu)}$ is unitarily equivalent to multiplication by the independent variable on $L^2(\sigma, m)$. Since $L^2(\sigma, m) \subset H_x$ is orthogonal to H_w, it can not be a bilateral shift. Therefore $\mathbb{T} \backslash \sigma$ has positive Lebesgue measure. Take $h(z) = \frac{1}{2}$ for $z \in \sigma$ and $h(z) = 1$ for $z \in \mathbb{T} \backslash \sigma$. By Proposition 3.6 there is $f \in H^2$ such that $|f|^2 = h$. Since we assumed isometry V to be non unitary, there is a subspace of H_s reducing V which is identified with multiplication by $''z''$ on H^2. In other words we can assume that $f \in H_s$. Take $g = \frac{\chi_\sigma}{\sqrt{2}}$. Note that $|f|^2 + |g|^2 = 1$. (To be precise function $g \in L^2(\sigma, m) \subset L^2(\mathbb{T}, m)$ can be understand as defined on the unit circle). Since $f \in H_s \subset H_w$ it is orthogonal to g. One can check that

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2 = \frac{1}{2\pi}\int_{\mathbb{T}} |f|^2 dm + \frac{1}{2\pi}\int_{\mathbb{T}} |g|^2 dm = \frac{1}{2\pi}\int_{\mathbb{T}} 1 dm = 1.$$

Similarly,

$$\langle z^n(f+g), f+g \rangle = \langle z^n f, f \rangle + \langle z^n g, g \rangle$$
$$= \frac{1}{2\pi}\int_{\mathbb{T}} z^n |f|^2 dm + \frac{1}{2\pi}\int_{\mathbb{T}} z^n |g|^2 dm = \frac{1}{2\pi}\int_{\mathbb{T}} z^n dm = 0,$$

for positive n. It follows that $f + g$ is a non trivial wandering vector. Since g was orthogonal to H_w we have $\frac{m(\sigma)}{2} = \|g\|^2 = \langle f + g, g \rangle = 0$. Thus σ is a set of measure 0 which contradicts the hypothesis $x \neq 0$. Consequently, the orthogonal complement of H_w is trivial. $\square$

Corollary 3.9. *Let $V \in L(H)$ be a unitary operator, such that $H_w \neq \{0\}$. Then*

$$H_w = H_{ac},$$

where H_w is a linear subspace generated by wandering vectors and $V|_{H_{ac}}$ is the whole absolutely continuous part of V in decomposition (3.1).

Proof. Let $v \in H_w$ be a nonzero wandering vector. Denote

$$L := \text{Span}\{v, Vv, V^2 v, \cdots\}, \ M := \{\cdots, V^* v, v, Vv, V^2 v, \cdots\}^\perp,$$
$$K := L \oplus M.$$

The operator $V|_L$ is a unilateral shift and $V|_K$ is a non unitary isometry. Thus by Theorem 3.8 we get $K_w = L \oplus K_{ac}$, where the subspaces are denoted as in Theorem 3.8. Similarly, for any $K^n := V^{*n} L \oplus M$ we have $K_w^n = V^{*n} L \oplus K_{ac}$. Since every wandering vector is wandering for operator extended to some superspace then $K_w^n \subset H_w$ for every $n \geq 0$. Finally, $H_{ac} \subset \bigvee_{n \in \mathbb{N}} V^{*n} L \oplus K_{ac} \subset H_w \subset H_{ac}$. $\qquad \square$

Lebesgue decomposition is hyperreducing (see Thm. 2.1 [13] or Thm. 2.2 [12]). It means that the subspaces $H_s \oplus H_{ac}$ and H_{sing} in (3.1) are hyperreducing. Summing up Theorems 3.3, 3.8, Corollary 3.9 and Remark 3.5, we get

Theorem 3.10. *For any isometry $V \in L(H)$ there is a unique decomposition:*

$$H = H_0 \oplus H_w,$$

where H_w is a span of all wandering vectors. Moreover:
- *the subspaces H_0, H_w are hyperreducing for V,*
- *$H_0 \subset H_u$, and every V invariant subspace of H_0 is V reducing.*

In comparison with decomposition (3.1), *we have*
- *if $H_w = \{0\}$ then $H = H_0 = H_{ac} \oplus H_{sing}$,*
- *if $H_w \neq \{0\}$ then $H_0 = H_{sing}$, $H_w = H_{ac} \oplus H_s$.*

Corollary 3.11. *Let $V \in L(H)$ be an isometry, $H = H_0 \oplus H_w$ a decomposition like in Theorem 3.10 and $L \subset H_w$ a reducing subspace. Then the decomposition of Theorem 3.10 for $V|_L$ is trivial (i.e. $L = L_0$ or $L = L_w$.)*

Proof. Note that H_w is orthogonal to H_{sing}. It follows $L_{sing} = \{0\}$. Thus, by Theorem 3.10 either $L_w = \{0\}$ or $L_w = L_{ac} \oplus L_s = L$. $\qquad \square$

4. Decomposition for Pairs of Isometries

In this section we take the advantage of the decomposition obtained in Theorem 3.3 and construct a decomposition for pairs of isometries. Despite being spanned by wandering vectors is not a hereditary property it has a multiple canonical decomposition.

Theorem 4.1. *Let (V_1, V_2) be a pair of commuting isometries on the Hilbert space H. There is a decomposition*

$$H = H_{00} \oplus H_{0w} \oplus H_{w0} \oplus H_{ww},$$

where the subspaces $H_{00}, H_{w0}, H_{0w}, H_{ww}$ are such, that:
1. *H_{00} is of type H_0 for both operators,*
2. *H_{0w} is of type H_0 for operator V_1 and spanned by wandering vectors for operator V_2,*
3. *H_{w0} is spanned by wandering vectors for operator V_1 and of type H_0 for operator V_2,*
4. *there are sets W_1, W_2 of vectors wandering for V_1, V_2 respectively, each spanning H_{ww}.*

Proof. Let $H = H_0 \oplus H_w$ be a decomposition for V_1. By Theorem 3.10 the subspaces H_0 and H_w are reducing for V_2. Let us decompose H_0 and H_w with respect to V_2 into $H_0 = H_{00}' \oplus H_{0w}'$ and $H_w = H_{w0}' \oplus H_{ww}'$. If $H_w = \{0\}$ then H_{00}' and H_{0w}' are of type H_0 with respect to V_1. If $H_w \neq \{0\}$ then the operator $V_1|_{H_0}$ is singular. Consequently $V_1|_{H_{00}'}$ and $V_1|_{H_{0w}'}$ are singular. In both cases $V_1|_{H_{00}'}$ and $V_1|_{H_{0w}'}$ are of type H_0.

By corollary 3.11 either $V_1|_{H_{w0}'}$ is of type H_0 or H_{w0}' is linearly spanned by vectors wandering for V_1. In the first case we take $H_{00} := H_{00}' \oplus H_{w0}'$ and $H_{w0} = \{0\}$. In the second case we take $H_{00} := H_{00}'$ and $H_{w0} = H_{w0}'$.

Similarly if $V_1|_{H_{ww}'}$ is of type H_0 we take $H_{0w} := H_{0w}' \oplus H_{ww}'$ and $H_{ww} = \{0\}$. If H_{ww}' is spanned by vectors wandering for V_1 we take $H_{0w} := H_{0w}'$ and $H_{ww} := H_{ww}'$. $\qquad\square$

Remark 4.2. The above decomposition is not unique.

Indeed, we have the following example.

Example 4.3. Denote by $\mathbb{T}$ the unit circle and

$$\mathbb{T}_+ := \{z \in \mathbb{T} : \text{im } z \geq 0\}, \quad \mathbb{T}_- := \{z \in \mathbb{T} : \text{im } z < 0\}.$$

Let us consider the space $K = L^2(\mathbb{T}, m) \oplus \bigoplus_{n=1}^\infty K_n$, where $K_n = L^2(\mathbb{T}_+, m)$. We can decompose $K = L^2(\mathbb{T}_-, m) \oplus L^2(\mathbb{T}_+, m) \oplus \bigoplus_{n=1}^\infty K_n$. Let V_1 be the multiplication by independent variable on K. Let V_2 be the multiplication by independent variable on $L^2(\mathbb{T}, m)$ and be a unilateral shift on $\bigoplus_{n=1}^\infty K_n$ such that the wandering space is K_1. The isometry V_1 is unitary absolutely continuous and contains a bilateral shift, hence, by Corollary 3.9, its wandering vectors span K. The isometry V_2 is an orthogonal sum of a unilateral shift and unitary absolutely continuous operator, hence its wandering vectors also span K (see Theorem 3.10). Thus we have $H_{ww} = K$.

Unfortunately the decomposition is not unique. For example take $H_{00} = L^2(\mathbb{T}_-, m)$ and $H_{0w} := L^2(\mathbb{T}_+, m) \oplus \bigoplus_{n=1}^\infty K_n$.

Recall from [6], that a pair of commuting contractions (T_1, T_2) is called *strongly completely non unitary* if there is no proper subspace reducing T_1, T_2 and at least one of them to a unitary operator. Moreover, there is a decomposition theorem ([6], Thm. 2.1):

Theorem 4.4. *Let (T_1, T_2) be a pair of commuting contractions on a Hilbert space H. There is a unique decomposition*

$$H = H_{uu} \oplus H_{u\neg u} \oplus H_{\neg uu} \oplus H_{\neg(uu)},$$

where the subspaces $H_{uu}, H_{u\neg u}, H_{\neg uu}, H_{\neg(uu)}$ are maximal such, that:

$T_1|_{H_{uu}}, T_2|_{H_{uu}}$ *are unitary operators,*
$T_1|_{H_{u\neg u}}$ *is a unitary operator, $T_2|_{H_{u\neg u}}$ is a completely non unitary operator,*
$T_1|_{H_{\neg uu}}$ *is a completely non unitary operator, $T_2|_{H_{\neg uu}}$ is a unitary operator,*
$(T_1|_{H_{\neg(uu)}}, T_2|_{H_{\neg(uu)}})$ *is a strongly completely non unitary pair of contractions.*

The above theorem in the case of pairs of commuting isometries appears also in [2] or more general in [9]. Can be found also in [14].

Theorem 4.5. *Let* (V_1, V_2) *be a pair of commuting isometries on a Hilbert space* H. *There is a decomposition*

$$H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_S,$$

where

1. H_{uu} *is a maximal subspace reducing* V_1, V_2 *to a pair of unitary operators,*
2. H_{us} *is a maximal subspace reducing* V_1 *to a unitary operator and* V_2 *to a unilateral shift,*
3. H_{su} *is a maximal subspace reducing* V_1 *to a unilateral shift and* V_2 *to a unitary operator,*
4. H_S *reduces* V_1, V_2 *and there are sets* W_1, W_2 *of vectors wandering for* V_1, V_2 *respectively, each spanning* H_S.

Proof. Since any completely non unitary isometry is just a unilateral shift, Theorem 4.4 applied to isometries gives a decomposition $H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{\neg(uu)}$. We need to show that $H_S = H_{\neg(uu)}$ has suitable properties. We prove it for the operator V_1. Let $H_{\neg(uu)} = H_0 \oplus H_w$ be the decomposition of $V_1|_{H_{\neg(uu)}}$ given by Theorem 3.3. By Theorem 3.10 the subspace H_0 reduces V_1 to a unitary operator and reduces V_2. Consequently $V_1|_{H_0}, V_2|_{H_0}$ doubly commute and $H_0 \subset H_{\neg(uu)}$. On the other hand, $\left(V_1|_{H_{\neg(uu)}}, V_2|_{H_{\neg(uu)}}\right)$ is a strongly completely nonunitary pair. Thus $H_0 = \{0\}$. Hence for V_1 we have $H_{\neg(uu)} = H_w$ which means that $H_{\neg(uu)}$ is spanned by vectors wandering for V_1. $\qquad\square$

Note that by the decomposition in the last theorem the subspace H_S reduces V_1, V_2 to such a pair, that there is no subspace reducing both isometries and at least one of them to a unitary operator. As we recalled earlier such a pair is called strongly completely non unitary. Using more general language of [2] such a pair is $\{1\}$ - pure and $\{2\}$ - pure. An immediate consequence is the following:

Proposition 4.6. *Let* (V_1, V_2) *be a pair of commuting, strongly completely non unitary pair of isometries on the Hilbert space* H. *There are sets* W_1, W_2 *of vectors wandering for* V_1, V_2 *respectively, each spanning* H.

By the proof of Theorem 3.3, the projection of a wandering vector onto the unitary part subspace of the isometry is in the span of wandering vectors but may not be wandering.

Remark 4.7. Let $V \in L(H)$ be an isometry with a wandering vector w. Note that for every wandering vector in H_u the equality $\langle V^n w, V^m w \rangle = 0$ holds true for every $n, m \in \mathbb{Z}, n \neq m$. Thus the minimal V-reducing subspace generated by w is $\bigoplus_{n \in \mathbb{Z}} V^n (\mathbb{C} w)$. In other words, the minimal V-reducing subspace generated by a wandering vector in H_u reduces V to a bilateral shift.

Let us introduce a definition of the following class of operators.

Definition 4.8. We call an operator $V \in L(H)$ *a span of bilateral shifts* if there are subspaces $\{H_\iota\}_{\iota \in I}$ such that H is spanned by $\bigcup_{\iota \in I} H_\iota$ and $V|_{H_\iota}$ is a bilateral shift for every $\iota \in I$.

Note that a span of bilateral shifts is a unitary operator.

Corollary 4.9. *Let $V \in L(H)$ be an isometry and $U \in L(K)$ its minimal unitary extension. Space H is a span of V-wandering vectors if and only if K is a span of bilateral shifts.*

Proof. If V is unitary the corollary is obvious. Thus we may assume that V is not unitary and consequently that $H_w \neq \{0\}$. By Theorem 3.10 we get $H_w = H_{ac} \oplus H_s$.

If U is a span of bilateral shifts then $K_{sing} = \{0\}$. On the other hand $H_{sing} = K_{sing}$. Thus $H = H_{ac} \oplus H_s = H_w$.

For the reverse implication note that every V wandering vector $w \in H$ is U wandering. On the other hand, U is unitary. According to Remark 4.7 the subspace $L_w := \bigoplus_{n \in \mathbb{Z}} V^n (\mathbb{C}w)$ is the minimal U-reducing subspace generated by w. Since H is spanned by wandering vectors, H is contained in the span of $\{L_w : w \in W\}$ taken over the set W of all V-wandering vectors. Since $L_w \subset K$ for $w \in W$, by minimality of U as a unitary extension, K equals to the latter span. On the other hand, $U|_{L_w}$ is a bilateral shift. This finishes the proof. $\qquad\square$

As a corollary of Proposition 4.6 and Corollary 4.9 we obtain the following result.

Theorem 4.10. *Let (V_1, V_2) be a pair of commuting, completely non doubly commuting isometries on the Hilbert space H. The unitary extension of each isometry is a span of bilateral shifts.*

The following result shows some geometry of completely non-doubly commuting pairs of isometries.

Theorem 4.11. *Let (V_1, V_2) be a completely non doubly commuting pair of commuting isometries on a Hilbert space H. Denote by $H = H_{ui} \oplus H_{si}$ the von Neumann-Wold decomposition for the isometry V_i. Then there are subspaces $W_i \subset H_{ui}$ reducing V_i for $i = 1, 2$, such that*

$$H_{u1} = \bigoplus_{n \geq 0} V_2^n(W_1), \quad H_{u2} = \bigoplus_{n \geq 0} V_1^n(W_2).$$

Proof. We make the proof for $i = 1$. By (1.1) we conclude that the subspace H_{u1} is hyperinvariant. Thus we can consider the operators

$$U_1 := V_1|_{H_{u1}}, \widetilde{V}_2 := V_2|_{H_{u1}} \in L(H_{u1}).$$

Since the pair (V_1, V_2) is completely non doubly commuting, it is a weak bi-shift. Consequently, the product $V_1 V_2$ is a unilateral shift. It implies that $U_1 \widetilde{V}_2$ is a unilateral shift and

$$H_{u1} = \bigoplus_{n \geq 0} (U_1 \widetilde{V}_2)^n (\ker(U_1 \widetilde{V}_2)^*).$$

Since U_1 is unitary, $\ker(U_1\widetilde{V}_2)^* = \ker\widetilde{V}_2^*$. By a similar argument applied for the operators $U_1^n, \widetilde{V}_2^n$ we obtain $\ker(U_1^n\widetilde{V}_2^n)^* = \ker\widetilde{V}_2^{*n}$ for any n. Note that

$$
\begin{aligned}
(U_1\widetilde{V}_2)^n(\ker(U_1\widetilde{V}_2)^*) &= \ker(U_1\widetilde{V}_2)^{*(n+1)} \ominus \ker(U_1\widetilde{V}_2)^{*n} \\
&= \ker\widetilde{V}_2^{*(n+1)} \ominus \ker\widetilde{V}_2^{*n} = \widetilde{V}_2^n(\ker\widetilde{V}_2^*).
\end{aligned}
$$

Consequently

$$
H_{u1} = \bigoplus_{n\geq 0}\widetilde{V}_2^n(\ker\widetilde{V}_2^*) = \bigoplus_{n\geq 0}V_2^n(\ker\widetilde{V}_2^*).
$$

Note also, that $\ker\widetilde{V}_2^*$ is U_1^* – invariant. Since U_1 commutes with $\widetilde{V}_2$ and is unitary, they doubly commute. Consequently, $\widetilde{V}_2^n(\ker\widetilde{V}_2^*)$ is U_1^* – invariant. Since $H_{u1} \ominus \widetilde{V}_2^n(\ker\widetilde{V}_2^*) = \bigoplus_{k\geq 0, k\neq n}V_2^k(\ker\widetilde{V}_2^*)$ is also U_1^* – invariant then the subspace $\widetilde{V}_2^n(\ker\widetilde{V}_2^*)$ is U_1 – reducing for every n. We have showed the theorem with $W_1 = \ker\widetilde{V}_2^*$. $\qquad\square$

A similar result is known for a normal operator and a unilateral shift (see [15] Proposition 9). Consequently we have:

Corollary 4.12. *The conclusion of the Theorem* 4.11 *holds true also for completely non unitary pairs of isometries.*

Proof. Any pair of commuting isometries can be decomposed into a doubly commuting pair and a completely non doubly commuting pair. In the case of doubly commuting pair we need to consider only pairs consisting of a unitary operator and a unilateral shift for which the result is trivial. It can be deduced from the mentioned result ([15] Proposition 9) or from the model in [16]. $\quad\square$

Theorem 4.11 can be deduced also from [2]. By von Neumann-Wold decomposition for the operator V_1 and by Theorem 4.11 we get

Corollary 4.13. *Let* (V_1, V_2) *be a completely non doubly commuting pair of commuting isometries on the Hilbert space* H. *Then there exist a subspace* W *which is wandering for* V_2 *such that*

$$
H = \bigoplus_{n\geq 0}V_1^n(\ker V_1^*) \oplus \bigoplus_{n\geq 0}V_2^n(W).
$$

In the corollary above we get an orthogonal decomposition of H into two orthogonal sums of subspaces wandering for V_1 and V_2 respectively. In Proposition 4.6 for each of the isometries V_1, V_2 we can find a collection of wandering vectors spanning H.

Summing up, Corollary 4.13 is stronger with respect to the orthogonality of wandering subspaces. On the other hand, Proposition 4.6 is stronger with respect to the fact that the space H is spanned by wandering vectors of a single arbitrarily chosen isometry.

5. Examples

We are going to conclude by having a closer look at several examples. The first is an example of an isometry $V \in L(H)$ such that $H_s \neq \{0\}$ and H is not spanned by wandering vectors. It is trivial example where the unitary part is singular according to the Lebesgue decomposition. We give it for the sake of completeness.

Example 5.1. Let us consider an orthonormal collection of the form $\{f\} \cup \{e_i : i \in \mathbb{Z}_+\}$ in some Hilbert space. Define a new Hilbert space $H := \mathbb{C}f \oplus \bigoplus_{i \in \mathbb{Z}_+} \mathbb{C}e_i$ and the isometry $V \in L(H)$ by $Vf = f, Ve_i = e_{i+1}$ for $i \in \mathbb{Z}_+$. Assume that H is spanned by vectors wandering for V. Then there is a wandering vector w, such that $P_{H_u}w \neq 0$. Obviously, $H_u = \mathbb{C}f$. Assume for the convenience that $w = f + v$ where $v \in H_s = \bigoplus_{i \in \mathbb{Z}_+} \mathbb{C}e_i$. By the proof of Theorem 3.3, since w is wandering we have $\langle V^n f, f \rangle = -\langle V^n v, v \rangle$. By the definition of V we obtain $\|f\|^2 = -\langle V^n v, v \rangle = -\langle v, V^{*n}v \rangle$. Since $v \in H_s$, the sequence $V^{*n}v$ converges to zero. Consequently we obtain a contradiction $1 = \|f\|^2 = \lim_{n \to \infty} -\langle v, V^{*n}v \rangle = 0$. Thus H can not be spanned by V – wandering vectors.

The next is an example of a span of bilateral shifts which is not a bilateral shift. We would like to thank Professor László Kérchy for this example. Denote by $\mathbb{T}$ the unit circle in the complex plane.

Example 5.2. Denote $\alpha := \{z \in \mathbb{T} : \arg z \in [\frac{2}{3}\pi, \frac{4}{3}\pi]\}$. Then $\alpha \cup \alpha^2 = \mathbb{T}$. Let $H = L^2(\alpha) \oplus L^2(\alpha^2) \oplus L^2(\alpha)$ and $U \in L(H)$ be multiplying by $''z''$. Then $H_0 = \{0\}$. If the operator would be unitarily equivalent to some bilateral shift then their spectral multiplicities would be equal. However, the spectral multiplicity of a bilateral shift is constant, while in our example it is not.

It is clear that wandering vectors of an isometry $V \in L(H)$ span the whole subspace H_s. On the other hand by Remark 4.7 any wandering vector in a subspace H_u fulfills the orthogonality $V^n w \perp V^m w$ for every distinct integer powers. The natural question is what will be changed if we make a definition of wandering vectors stronger in the following sense. We call a wandering vector w *strongly wandering* if it fulfills the condition $V^n w \perp V^m w$ for every $n, m \in \mathbb{Z}, n \neq m$. Denote by H_{ws} the minimal subspace spanned by strongly wandering vectors and by H_w the subspace spanned by wandering vectors. Obviously, $H_{ws} \subset H_w$ and both subspaces are reducing for our isometry V. As we know, the subspace H_w is invariant for every isometry commuting with V, but H_{ws} does not need to be. We have the following lemma:

Lemma 5.3. *Let $H = H_u \oplus H_s$ be the von Neumann-Wold decomposition of an isometry $V \in L(H)$. Then the following conditions hold:*

1. *for x a strongly wandering vector, also $x_u := P_{H_u}x, x_s := P_{H_s}x$ are strongly wandering vectors,*
2. *$W = H_s \oplus W_u$ where W, W_u denote subspaces generated by strongly wandering vectors for V and $V|_{H_u}$ respectively.*

Proof. Since H_u, H_s reduce V and x is wandering we have $\langle V^n x_u, V^m x_u \rangle = -\langle V^n x_s, V^m x_s \rangle$ for every $m, n \in \mathbb{Z}, m \neq n$. On the other hand

$$\langle V^n x_u, V^m x_u \rangle = \langle V^{*k} V^n x_u, V^{*k} V^m x_u \rangle = \lim_{k \to \infty} \langle V^{n-k} x_u, V^{m-k} x_u \rangle = \dots$$

for $n \neq m$ as well as for $n - k \neq m - k$ and consequently

$$\dots = \lim_{k \to \infty} -\langle V^{n-k} x_s, V^{m-k} x_s \rangle = 0.$$

Thus x_u is a wandering vector and by $\langle V^n x_u, V^m x_u \rangle = -\langle V^n x_s, V^m x_s \rangle$, also x_s is wandering.

For the second part, note that $V^n(\ker V^*)$ for every $n \geq 0$ is a set of V – strongly wandering vectors. Thus $H_s \subset W$. Since H_u reduces V, every vector wandering for $V|_{H_u}$ is wandering for V. Thus $W_u \subset W$. By the first part of the lemma, the reverse inclusion $W \subset H_s \oplus W_u$ follows. □

We want to show that unitary extension can be a span of bilateral shifts and Hilbert space H is not spanned by strongly wandering vectors. This follows from the next example:

Example 5.4. Consider Example 5.2, put $K = L^2(\alpha) \oplus L^2(\alpha^2) \oplus L^2(\alpha)$ and denote by U the operator of multiplication by $''z''$. Find a wandering subspace W for the bilateral shift in $L^2(\alpha^2) \oplus L^2(\alpha)$ such that $L^2(\alpha^2) \oplus L^2(\alpha) = \bigoplus_{n \in \mathbb{Z}} U^n W$. Then take $H = L^2(\alpha) \oplus \bigoplus_{n \geq 0} U^n(W)$. The restriction $U|_H$ is an isometry with its unitary part acting on $L^2(\alpha)$, and U is its minimal unitary extension. Since $\sigma(T|_{H_u})$ does not contain the unit circle, there is no subspace reducing it to a bilateral shift. Consequently, H_u does not contain any wandering vector and $H_{ws} = H_s = \bigoplus_{n \geq 0} U^n W$. On the other hand, the unitary extension U of $U|_H$ is a span of bilateral shifts.

We follow the same idea to show an example of an isometry having unitary part of H_0 type, but whose unitary extension is a bilateral shift.

Example 5.5. Denote

$$\mathbb{T}_+ := \{z \in \mathbb{T} : \text{im } z \geq 0\}, \quad \mathbb{T}_- := \{z \in \mathbb{T} : \text{im } z < 0\}.$$

Let $L^2(\mathbb{T}), L^2(\mathbb{T}_+), L^2(\mathbb{T}_-)$ be the subspaces of functions on $\mathbb{T}, \mathbb{T}_+, \mathbb{T}_-$ respectively, which moduli are square summable with respect to Lebesgue measure and let $H^2(\mathbb{T})$ be the Hardy subspace of $L^2(\mathbb{T})$. Consider $H = L^2(\mathbb{T}_-) \oplus \bigoplus_{n \geq 0} H^2(\mathbb{T})$ and the isometric operator M_z of multiplication by $''z''$. Then the unitary part of M_z acts on $L^2(\mathbb{T}_-)$. Since the spectrum of M_z restricted to $L^2(\mathbb{T}_-)$ does not contain $\mathbb{T}$ it is not a span of bilateral shifts. On the other hand M_z extends to a unitary operator of multiplication by $''z''$ on the space $K = L^2(\mathbb{T}_-) \oplus \bigoplus_{n \geq 0} L^2(\mathbb{T})$. Since we have the decomposition $L^2(\mathbb{T}) = L^2(\mathbb{T}_+) \oplus L^2(\mathbb{T}_-)$ we obtain $K = L^2(\mathbb{T}_-) \oplus L^2(\mathbb{T}_+) \oplus L^2(\mathbb{T}_-) \oplus L^2(\mathbb{T}_+) \oplus L^2(\mathbb{T}_-) \oplus \dots = \bigoplus_{n \geq 0} L^2(\mathbb{T})$. Thus the unitary extension is a bilateral shift.

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