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SURVEY

Dynamical analytic multifunctions

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Abstract We list different examples of analytic dependence on some parameters of Julia type sets or attractors of (generated) iterated function systems.

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1 Introduction

By multifunctions we mean in this paper only functions whose values are non-empty compact sets. Oka [16] was the first to venture beyond the classical theory of multivalued analytic functions such as branching of analytic functions and Riemann surfaces. He started from the famous Hartogs theorem which can be expressed in the following way: if $f : G \longrightarrow \mathbb{C}$ is a continuous function defined on a domain $G \subset \mathbb{C}$, then f is holomorphic if and only if $(G \times \mathbb{C}) \setminus \text{graph}(f)$ is pseudoconvex (see [15, p. 132]). The idea of Oka was to take a mapping defined in a domain G in $\mathbb{C}$ with values being compact subsets of $\mathbb{C}$ and define its graph so that it is a subset of $\mathbb{C}^2$ and then say that this mapping is analytic if the complement of the graph to $G \times \mathbb{C}$ is pseudoconvex. The idea was nearly forgotten for a long time and then sprang to attention in papers of different researchers starting from around 1980, when other definitions were given and compared. The first crucial application of analytic multifunctions was in uniform algebras – Słodkowski used them to solve the so-called Pełczyński conjecture for $C^\star$-algebras [19]. Then other applications followed, e.g. in the interpolation of Banach spaces. For a more detailed history, a very good introduction to the subject and applications see [2].

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Let us give an idea behind the notion. If we have an analytic multifunction $\lambda \mapsto K(\lambda)$ and we know some properties of a set $K(\lambda_0)$, we may see the sets $K(\lambda)$ for λ close to λ_0 as an analytic perturbation of the set $K(\lambda_0)$ and we may ask whether the properties of the sets were preserved or if not how they were changed.

The word "dynamical" in the title of the paper refers to complex dynamics. From its point of view, the type of dependence of the special sets (namely Julia sets, limit sets, attractors) on the parameters involved in their construction is of interest. Analytic multifunction provide natural tools allowing description of this dependence.

Let us finally also note that some special Julia type sets can be obtained when we iterate multifunctions (see [9]), but we will not present this approach here.

2 What is an analytic multifunction?

Let X and Y be Hausdorff topological spaces and denote by $\kappa(Y)$ the family of all non-empty compact subsets of Y. Any mapping $K : X \longrightarrow \kappa(Y)$ is called a *multifunction*. By its *graph* we mean

$$\operatorname{graph}(K) := \{(x, y) \in X \times Y : \ y \in K(x)\}.$$

We say that the multifunction K is *upper semicontinuous* if for each open subset U of Y the set $\{x \in X : \ K(x) \subset U\}$ is open in X. If (X, d) is a metric space and Y is compact, given a multifunction $K : X \longrightarrow \kappa(Y)$ we define its *upper semicontinuous regularization* $K^\star$ by

$$K^\star(x) := \bigcap_{r>0} \overline{\bigcup_{t \in B(x,r)} K(t)},$$

where $B(x, r) := \{t \in X : \ d(t, x) < r\}$ (see [3]). It is the smallest upper semicontinuous multifunction that contains K.

We list now four definitions of analytic multifunctions.

Definition 2.1 [19] Let Ω be an open subset of a complex Banach space E. An upper semicontinuous multifunction $K : \Omega \longrightarrow \kappa(\mathbb{C}^N)$ is *(weakly) analytic* if for every $a \in \Omega$ and for every plurisubharmonic function u in a neighbourhood of $\{a\} \times K(a)$ the function $z \mapsto \sup u(\{z\} \times K(z))$ is plurisubharmonic.

This definition deals with plurisubharmonic functions therefore the following standard example is natural here.

Example 2.2 [17] (c.f. also [10]) Let Ω be an open subset of a complex Banach space E and let $u : \Omega \to [-\infty, \infty)$ be a function. Consider the mapping $D : \Omega \longrightarrow \kappa(\mathbb{C})$ defined by the formula $D(z) := \{\zeta \in \mathbb{C} : |\zeta| \le \exp(u(z))\}$. Then D is (weakly) analytic if and only if u is plurisubharmonic.

Definition 2.1 was given by Słodkowski [19] who then went to make another stronger definition of analytic dependence for multifunctions.

Definition 2.3 [20] We say that a subset S of a complex Banach space E has the *local maximum property* if there is no holomorphic function $f : W \longrightarrow \mathbb{C}$ (where $W \subset E$ is open) such that $|f|$ restricted to $W \cap S$ has a strict local maximum.

Let Ω be an open subset of a complex Banach space E. An upper semicontinuous multifunction $K : \Omega \longrightarrow \kappa(\mathbb{C}^N)$ is said to be *strongly analytic* if for any $(N + 1)$-dimensional complex affine subspace L of $E \times \mathbb{C}^N$ the set $L \cap \operatorname{graph}(K)$ has the local maximum property.

Before we give an example, we need some notations. Let $d \geq 2$, $N \geq 1$ and put $\mathcal{P}_d := \{P : \mathbb{C}^N \to \mathbb{C}^N \mid P$ is a polynomial mapping and $\deg P \leq d\}$. This can be viewed as a complex Banach space (of finite dimension). Denote by $\tilde{P}$ the homogeneous part of P of degree d. Put

$$\Omega := \{P \in \mathcal{P}_d : \tilde{P}^{-1}(\{0\}) = \{0\}\}.$$

This set is open in $\mathcal{P}_d$ (see also the discussion of it in Sect. 5).

Example 2.4 [10, Remark 2] Let $\zeta \in \mathbb{C}^N$. Then $Z_\zeta : \Omega \ni P \mapsto P^{-1}(\zeta)$ is strongly analytic.

In particular, it may be deduced from Example 2.4 that *algebroid multifunctions*, i.e. multifunctions of the form

$$U \ni \lambda \mapsto K(\lambda) := \{z \in \mathbb{C} : z^n + a_1(\lambda)z^{n-1} + \cdots + a_n(\lambda) = 0\}$$

(where U is an open set in $\mathbb{C}^N$ and $a_j : U \to \mathbb{C}$ are holomorphic) are strongly analytic. For a proof of this implication one needs a composition theorem.

If $E = \mathbb{C}$, the notions of strong and weak analyticity are identical (see [19,21]). If the dimension of the space is higher than 1, the strong analyticity implies the weak one but not vice versa, which is shown by the following example.

Example 2.5 [21] The multifunction

$$S : \mathbb{C}^2 \ni z \mapsto \begin{cases} \{\zeta \in \mathbb{C} : |\zeta| = 1\}, & z \neq 0 \\ \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}, & z = 0 \end{cases}$$

is weakly analytic but is not strongly analytic.

Let us go to a more special case yet.

Definition 2.6 A multifunction is *trivially analytic* if its graph is the union of the graphs of a family of holomorphic functions.

Each trivially analytic multifunction is strongly analytic. The significance of such set-valued mappings is shown in Słodkowski's theorem stating that any strongly analytic multifunction can be approximated by a decreasing sequence of locally trivially analytic multifunctions [22].

Let us define another notion, which has been intensively studied since [13].

Definition 2.7 (see c.f. [1] or [4]) Let A be a subset of $\overline{\mathbb{C}}$, Λ an open subset of $\mathbb{C}$ and $\lambda_0 \in \Lambda$. A *holomorphic motion* of A (parametrized by Λ and λ_0) is a map $\Phi : \Lambda \times A \longrightarrow \overline{\mathbb{C}}$ such that

(i) $\forall a \in A :$ the map $\Phi(\cdot, a) : \Lambda \ni \lambda \mapsto \Phi(\lambda, a) \in \overline{\mathbb{C}}$ is holomorphic;
(ii) $\forall \lambda \in \Lambda :$ the map $\Phi_\lambda := \Phi(\lambda, \cdot) : A \ni a \mapsto \Phi(\lambda, a) \in \overline{\mathbb{C}}$ is injective;
(iii) the map Φ_{λ_0} is the identity on A.

It is noteworthy that every motion defined above extends to a holomorphic motion of $\Lambda \times \overline{A}$ (see [1,23]). Therefore we can restrict our attention to holomorphic motions of compact sets.

It follows directly from Definitions 2.7 and 2.6 that if Φ is a holomorphic motion of a compact set $A \subset \mathbb{C}$ and with values in $\mathbb{C}$, then

$$\Lambda \ni \lambda \mapsto \Phi_\lambda(A) \in \kappa(\mathbb{C})$$

is a trivially analytic multifunction: thus analytic multifunctions can be viewed as generalizations of such holomorphic motions.

Now we turn to quite another type of definition. It was motivated by holomorphic motions on the one hand and by some elementary properties of analytic multifunctions on the other. Namely, we consider now a concept of analyticity of functions defined on open subsets of $\overline{\mathbb{C}}$ and with compact values included in $\overline{\mathbb{C}}$. Before we can do this, however, we must first make another definition, that of a *multigauge*.

Definition 2.8 [18] Let X and Y be Hausdorff topological spaces, $K : X \longrightarrow \kappa(Y)$ be an upper semicontinuous multifunction and $\mathcal{M}$ and $\mathcal{L}$ be families of upper semicontinuous multifunctions. We write

(i) $K \in \mathcal{M}^{\downarrow}$ if there exists a decreasing sequence (K_n) in $\mathcal{M}$ such that $\forall x \in X : \ K(x) = \bigcap_n K_n(x)$;
(ii) $K \in \mathcal{M}^{\uparrow}$ if $\forall x_0 \in X \ \forall y_0 \in \partial K(x_0) \ \exists U_0$ a neighbourhood of x_0 and $L \in \mathcal{M}$ such that $L(x) \subset K(x), x \in U_0$ and $y_0 \in L(x_0)$.

The family $\mathcal{M}$ is called a *multigauge* if $\mathcal{M}^{\downarrow} = \mathcal{M}$ and $\mathcal{M}^{\uparrow} = \mathcal{M}$.

The *multigauge generated* by $\mathcal{L}$ is the smallest multigauge containing $\mathcal{L}$ (i.e. the intersection of all multigauges containing $\mathcal{L}$).

Now we can define an analytic multifunction.

Definition 2.9 [18] Let Ω be an open subset of $\overline{\mathbb{C}}$. Put

$$\mathcal{R}(\Omega) := \left\{\Omega \ni z \mapsto \{q(z)\} \in \kappa(\overline{\mathbb{C}}) \mid q : \Omega \longrightarrow \overline{\mathbb{C}} \text{ is a rational function}\right\}$$

(we can take above also $q \equiv \infty$). Let $\mathcal{A}(\Omega)$ be the multigauge generated by $\mathcal{R}(\Omega)$. We say that a multifunction $K : \Omega \longrightarrow \kappa(\overline{\mathbb{C}})$ is *analytic* in Ω if $K \in \mathcal{A}(\Omega)$.

Ransford proposed this definition in [18] and exhibited many properties of the obtained family. This approach allowed him to tackle the Julia sets of rational and entire functions, which are not compact in $\mathbb{C}$. Previously, for this special case, meromorphic multifunctions were defined in [3], but in [18] they are viewed as analytic functions with compact values in $\kappa(\overline{\mathbb{C}})$. As should be expected, Ransford showed also that for upper semicontinuous multifunctions $K : \Omega \longrightarrow \kappa(\mathbb{C})$ with $\Omega \subset \mathbb{C}$ the notions of analyticity from Definitions 2.1 and 2.9 are identical.

3 Julia sets

Let R be a non-constant rational function of degree at least 2. The *Fatou set* of R is the maximal open subset of $\overline{\mathbb{C}}$ on which the family of iterates $\{R^n : n \in \mathbb{N}\}$ is equicontinuous. The *Julia set* of R, denoted by $J(R)$, is the complement of the Fatou set in $\overline{\mathbb{C}}$ (see [5]). Note that the Julia set is always compact in $\overline{\mathbb{C}}$, it is also non-empty.

We want to underline a special case: if $R = p$ is a polynomial of degree $d \geq 2$, then the *filled-in Julia set* of p is the set

$$K[p] := \{z \in \mathbb{C} : (p^n(z))_{n=1}^{\infty} \text{ is bounded}\}.$$

Then $J(p) = \partial K[p]$ and on the other hand $K[p]$ is the polynomially convex hull of $J(p)$. Both sets $J(p)$ and $K[p]$ are compact in $\mathbb{C}$.

And now let us discuss a variant of these definitions. The Fatou set of a non-constant entire function f is the maximal open subset of $\mathbb{C}$ (sic!) on which the family of iterates $\{f^n : n \in \mathbb{N}\}$ is equicontinuous. The Julia set $J(f)$ is then defined as the complement of the

Fatou set relative to $\mathbb{C}$. It is closed, but in general not bounded. Let $\overline{J(f)}$ denote its closure in $\overline{\mathbb{C}}$.

We start with an example of a holomorphic motion: the very first one given by Mañé, Sad and Sullivan.

Theorem 3.1 [13] *Let Λ be a domain in $\mathbb{C}$ and let $\{R_\lambda\}$ be a family of rational functions $R_\lambda : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ depending analytically on the parameter $\lambda \in \Lambda$. Then there exists an open dense subset Λ' of Λ such that for every $\lambda_0 \in \Lambda'$ there exists a neighbourhood Λ_0 and a holomorphic motion $h : \Lambda_0 \times J(R_{\lambda_0}) \longrightarrow \overline{\mathbb{C}}$ such that $\forall \lambda \in \Lambda_0 : h_\lambda(J(R_{\lambda_0})) = J(R_\lambda)$.*

In the assertion of this theorem there appears the dense open subset λ of the parameter domain Λ, which usually is different from the whole set. Baribeau and Ransford addressed this inconvenience and proved

Theorem 3.2 [3] (c.f. also [18]) *Let Λ be a domain in $\mathbb{C}$.*

(1) *Let $\{R_\lambda\}$ be a family of rational maps of degree at least 2 depending analytically on the parameter λ. Then*

$$J^\star : \Lambda \ni \lambda \mapsto J(R_\lambda)^\star \in \kappa(\overline{\mathbb{C}})$$

is analytic.

(2) *Let $\{f_\lambda\}$ be a family of non-constant entire functions depending analytically on the parameter λ. Then*

$$\overline{J}^\star : \Lambda \ni \lambda \mapsto \overline{J(f_\lambda)}^\star \in \kappa(\overline{\mathbb{C}})$$

is analytic.

Let us note a consequence.

Corollary 3.3 *Let Λ be a domain in $\mathbb{C}$. If $\{p_\lambda\}$ is a family of polynomials of degree at least 2 depending analytically on the parameter λ, then*

$$K : \Lambda \ni \lambda \mapsto K[p_\lambda] \in \kappa(\mathbb{C})$$

is analytic.

Let us now move to higher dimensions. Our attention will be restricted here only to polynomial mappings. Recall the notation of $\mathcal{P}_d$ and its open subset Ω which we used in the previous section (just before Example 2.4). For $P \in \Omega$ we can define the filled-in Julia set $K[P]$ for a polynomial mapping P in the same way as it was done for the polynomials on the complex plane. Then $K[P]$ is a non-empty polynomially convex compact subset of $\mathbb{C}^N$.

Now we can state a generalization and in the same time a strengthening of Corollary 3.3.

Theorem 3.4 [10] *The multifunction*

$$K : \Omega \ni P \mapsto K[P] \in \kappa(\mathbb{C}^N)$$

is strongly analytic.

4 Attractors of IFSs

Recall the definition of an *iterated function system* (IFS for short). It is a finite family of contracting mappings on a complete metric space (see [8]).

This section is based on [4], where a special form of IFSs will be considered. The definition of an IFS is generalized: the family is allowed to be countable. On the other hand, we restrict our attention to the complex plane. Our setting is therefore as follows.

Let $S = \{f_\iota\}_{\iota \in I}$ be a finite or countable family of contractions of $\mathbb{C}$ with contraction ratios $\{c_\iota\}$ satisfying

$$C := \sup_{\iota \in I} c_\iota < 1 \quad \text{and} \quad B := \sup_{\iota \in I} |f_\iota(z_0)| < \infty \quad \text{for some } z_0 \in \mathbb{C}.$$

The limit set of the IFS (for definition see [4]) may fail to be compact if I is infinite. But its closure is always compact. Since the object of study of this paper are compact-valued functions we will speak here only about the closure of the limit set, which we may define (c.f. [4, Lemma 1]) as the unique fixed point of the map

$$\kappa(\mathbb{C}) \ni K \mapsto \overline{\bigcup_{\iota \in I} f_\iota(K)} \in \kappa(\mathbb{C})$$

(here $\kappa(\mathbb{C})$ is equipped with the Hausdorff metric) and which we denote by $A(S)$ and call the *attractor* of the IFS S. We say (see [8]) that S satisfies the *open set condition* (OSC for short) if there exists a non-empty open set U such that

$$f_\iota(U) \subset U, \ \iota \in I, \quad \text{and} \quad \iota \neq j \Longrightarrow f_\iota(U) \cap f_j(U) = \emptyset.$$

We say (see [4]) that S satisfies the *closed open set condition* (COSC) if there exists a non-empty open set U such that

$$f_\iota(U) \subset U, \ \iota \in I, \quad \text{and} \quad \iota \neq j \Longrightarrow f_\iota(\overline{U}) \cap f_j(\overline{U}) = \emptyset.$$

We are ready to state the results. The situation is easier if we assume COSC.

Theorem 4.1 [4] *Let Λ be an open subset of $\mathbb{C}$ and let $\{S_\lambda\}$ be a family of IFSs satisfying COSC and depending analytically on the parameter $\lambda \in \Lambda$. Then for every λ_0 there exists a holomorphic motion $\Phi : \Lambda \times A(S_{\lambda_0}) \longrightarrow \mathbb{C}$ such that $\Phi_\lambda(A(S_{\lambda_0})) = A(S_\lambda)$, $\lambda \in \Lambda$.*

The next result holds under the weaker assumption of OSC.

Theorem 4.2 [4] *Let Λ be an open subset of $\mathbb{C}$ and $\{S_\lambda\}$ be a family of IFSs of injective contractions satisfying OSC where S_λ is of the form $S_\lambda = \{f_{\iota,\lambda}\}$. For each $\lambda \in \Lambda$ let $U(\lambda)$ be the set for S_λ which arises from OSC. We assume that all of the functions $(\lambda, z) \mapsto f_{\iota,\lambda}(z)$ are holomorphic in two variables and that there exists a holomorphic function $g : \Lambda \longrightarrow \mathbb{C}$ such that $g(\lambda_0) \in U(\lambda_0) \backslash A(S_{\lambda_0})$ for some λ_0 and $g(\lambda) \in U(\lambda)$, $\lambda \in \Lambda$. Then there exists a neighbourhood $\Lambda_0 \subset \Lambda$ of λ_0 and a holomorphic motion $\Phi : \Lambda_0 \times A(S_{\lambda_0}) \longrightarrow \mathbb{C}$ such that $\Phi_\lambda(A(S_{\lambda_0})) = A(S_\lambda)$, $\lambda \in \Lambda_0$.*

Let us recall that the originals of these theorems from [4] deal also with the limit set but we omit here this aspect.

The final result of this section does not require COSC nor OSC and does not deal with the limit set in the original version.

Theorem 4.3 [4] *Let Λ be an open subset of $\mathbb{C}$ and $\{S_\lambda\}$ be a family of IFSs where S_λ is of the form $S_\lambda = \{f_{\iota,\lambda}\}$. We assume that all of the functions $(\lambda, z) \mapsto f_{\iota,\lambda}(z)$ are holomorphic in two variables. Then the multifunction*

$$A : \Lambda \ni \lambda \mapsto A(S_\lambda) \in \kappa(\mathbb{C})$$

is analytic.

5 Some generalizations

We list here only three generalizations of the results from the previous two sections, namely of the theorems concerning analytic multifunctions. For some generalizations of those dealing with holomorphic motions see e.g. [6,7,14].

The first part of this section deals with Julia type sets and is based on [11]. Let $d \geq 2$ and $N \geq 1$. Fix any norm $||\cdot||$ on $\mathcal{P}_d$ and put

$$\lfloor P \rfloor := \inf_{|z|=1} |\tilde{P}(z)|, \quad P \in \mathcal{P}_d.$$

It is easy to check that $\lfloor P \rfloor > 0 \iff \tilde{P}^{-1}(\{0\}) = \{0\}$. We can rewrite therefore

$$\Omega = \{P \in \mathcal{P}_d : \lfloor P \rfloor > 0\}.$$

Take a sequence (P_n) of polynomial mappings lying in Ω. It appears that the natural generalization of the filled-in Julia set of a polynomial mapping is given by

$$K[(P_n)] := \{z \in \mathbb{C}^N : ((P_n \circ \cdots \circ P_2 \circ P_1)(z))_{n=1}^{\infty} \text{ is bounded}\},$$

which is non-empty, polynomially convex and compact. We call this set the *filled-in Julia set* of the sequence (P_n).

Take now a function $\varrho : \mathbb{N} \to \mathbb{N}$ and define the set

$$\mathbb{N}_\varrho := \{(n, j) \in \mathbb{N}^2 : j \leq \varrho(n)\}.$$

Consider

$$E_\varrho := \left\{ P = [P_{n,j}]_{(n,j)\in\mathbb{N}_\varrho} : P_{n,j} \in \mathcal{P}_d, (n, j) \in \mathbb{N}_\varrho, \sup_{(n,j)\in\mathbb{N}_\varrho} \|P_{n,j}\| < \infty \right\}$$

and for $P \in E_\varrho$, define

$$\|P\| := \sup_{(n,j)\in\mathbb{N}_\varrho} \|P_{n,j}\|.$$

Then $(E_\varrho, \|\cdot\|)$ is a complex Banach space. We are interested in the open set

$$\Omega_\varrho := \left\{ P \in E_\varrho \mid \inf_{(n,j)\in\mathbb{N}_\varrho} \lfloor P_{n,j} \rfloor > 0 \right\}.$$

Note that if $\varrho = \mathbf{1} \equiv 1$, then $E_{\mathbf{1}}$ is a space of sequences (of polynomial mappings). We may state the first generalization of Theorem 3.4

Theorem 5.1 [11] *The multifunction*

$$K : \Omega_{\mathbf{1}} \in (P_n) \mapsto K[(P_n)] \in \kappa(\mathbb{C}^N)$$

is strongly analytic.

Put $\Sigma_\varrho := \{\sigma : \mathbb{N} \longrightarrow \mathbb{N} \mid \sigma \leq \varrho\}$. For $P = [P_{n,j}] \in \Omega_\varrho$ and $\sigma \in \Sigma_\varrho$ take the sequence $(P_{1,\sigma(1)}, P_{2,\sigma(2)}, \ldots) = (P_{n,\sigma(n)}) \in \Omega_{\mathbf{1}}$, We define

$$k[P] := \bigcup_{\sigma\in\Sigma_\varrho} K[(P_{n,\sigma(n)})], \qquad P \in \Omega_\varrho.$$

This set is compact and non-empty, but in general not polynomially convex. Therefore we call it the *partly filled-in composite Julia set* generated by P. Its polynomially convex hull is denoted by $K[P]$ and called the *composite (filled-in) Julia set* associated with P. Both sets can be viewed as generalizations of the filled-in Julia sets for polynomial mappings.

Theorem 5.2 [11] *Let $\varrho : \mathbb{N} \to \mathbb{N}$ be a function. Then the multifunction*

$$k : \Omega_\varrho \ni P \mapsto k[P] \in \kappa(\mathbb{C}^N)$$

is strongly analytic and the multifunction

$$K : \Omega_\varrho \ni P \mapsto K[P] \in \kappa(\mathbb{C}^N)$$

is (weakly) analytic.

It could also be the case that this last multifunction is in fact strongly analytic too, but this is not known.

The last part of this article is about a generalization of IFSs and is based on [12]. Let $\mathcal{L}(\mathbb{C}^N)$ denote the space of all bounded linear operators on $\mathbb{C}^N$, furnished with the usual operator norm $\|\cdot\|$. Let $\mathcal{F}(\mathbb{C}^N)$ be the space of all continuous affine operators on $\mathbb{C}^N$. Every operator $T : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ in $\mathcal{F}(\mathbb{C}^N)$ has the natural decomposition $T = \tilde{T} + T(0)$ with $\tilde{T} \in \mathcal{L}(\mathbb{C}^N)$. Hence $\mathcal{F}(\mathbb{C}^N) = \mathcal{L}(\mathbb{C}^N) \oplus \mathbb{C}^N$ and the natural norm in $\mathcal{F}(\mathbb{C}^N)$ is given by the formula $\|T\| = \|\tilde{T}\| + |T(0)|$.

Take now a function $\varrho : \mathbb{N} \to \mathbb{N}$. In a similar way as before we put

$$E_\varrho := \left\{ T = [T_{n,j}]_{(n,j)\in\mathbb{N}_\varrho} : \ T_{n,j} \in \mathcal{F}(\mathbb{C}^N), (n,j) \in \mathbb{N}_\varrho, \ \sup_{(n,j)\in\mathbb{N}_\varrho} \|T_{n,j}\| < \infty \right\}$$

and for $T \in E_\varrho$, define

$$\|T\| := \sup_{(n,j)\in\mathbb{N}_\varrho} \|T_{n,j}\|.$$

It can be shown that $(E_\varrho, \|\cdot\|)$ is a complex Banach space. For $T \in E_\varrho$ we put $\tilde{T} := [\tilde{T}_{n,j}]$. We are interested in the open set

$$\Omega_\varrho := \left\{ T \in E_\varrho \mid \ \|\tilde{T}\| < 1 \right\}.$$

Note again that $E_{\mathbf{1}}$ is a space of sequences (of affine operators). Fix a sequence $(T_n) \in \Omega_{\mathbf{1}}$. Then for each n the mapping $T_1 \circ \cdots \circ T_n$ is a contraction in $\mathbb{C}^N$ and hence by the Banach contraction principle it has the unique fixed point $b[T_1 \circ \cdots \circ T_n] \in \mathbb{C}^N$. It can be shown that the limit

$$a[(T_n)] := \lim_{n\to\infty} b[T_1 \circ \cdots \circ T_n]$$

exists. Now for $\sigma \in \Sigma_\varrho$ and $T \in \Omega_\varrho$, in a similar way as before, we consider the sequence $(T_{n,\sigma(n)}) \in \Omega_{\mathbf{1}}$. We define finally for $T \in \Omega_\varrho$ the set

$$A(T) := \{a[(T_{n,\sigma(n)})] : \sigma \in \Sigma_\varrho\}.$$

This set is compact and can be viewed as a generalization of some attractors defined in the previous section. Hence we call it the *attractor* of T. We are ready to state the last theorem of this article.

Theorem 5.3 [12] *Fix a function $\varrho : \mathbb{N} \to \mathbb{N}$. Then the multifunction*

$$A : \Omega_\varrho \ni T \mapsto A(T) \in \kappa(\mathbb{C}^N)$$

is trivially analytic.

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