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NOISE-INDUCED SYNCHRONIZATION IN THE FAHY–HAMANN MODEL*

Michał Cieśla, Lech Longa

M. Smoluchowski Institute of Physics, Department of Statistical Physics
and Mark Kac Complex Systems Research Center, Jagellonian University
Reymonta 4, 31-059 Kraków, Poland

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We study the noise-induced synchronization in a system of particles moving in Fahy–Hamann potential [S. Fahy, D.R. Hamann, *Phys. Rev. Lett.* **69**, 761 (1992)] and subjected to generalized Langevin forces. We investigate the synchronization dependence on system's parameters and on memory range. The results show that while in general memory acts against synchronization, for intermediate memory ranges the opposite effect can be observed. Generally the synchronization transition is found to depend on memory range, temperature and dissipation in the system.

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1. Introduction

Chaotic systems shows extreme sensitivity to even a minor perturbation of initial conditions. Two trajectories very close to each other usually diverge exponentially in time, which makes long-time evolution unpredictable. That is, identical systems evolving with different initial conditions will not synchronize their trajectories. The situation may change, however, when the systems get coupled *e.g.* by a common signal. Even a random resetting of velocities in an ensemble of particles with different initial conditions can force synchronization, as shown by Fahy and Hamann (FH) in 1992 [1].

Since then coherence of dynamical processes has become an active field of research with examples found in physical, biological, chemical and social systems [2–9]. At least four types of synchronization scenarios have been identified [10–13] of which the synchronization of identical systems coupled by a common noise has received much attention due to its relative simplicity and importance [13, 14].

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Usually we describe chaotic systems by Markovian-type of equations, but realistic time evolution, due to collective effects is, in general, non-Markovian, which inevitably requires memory (time-delayed interactions) to be taken into account [15–18]. Systematic studies of isolated, time-delayed interactions were initialized by Schuster and Wagner [19]. They considered two coupled phase oscillators and found multi-stability of synchronized solutions. Since then, delayed interactions have been analyzed in different contexts [20–23]. The most interesting recent development in this regard are studies showing *(a)* enhancement of neural synchrony in a network of coupled oscillators involving time delays [21] and *(b)* demonstration of synchronization in an ensemble of coupled, chaotic logistic maps with random delay times [23].

In this paper we demonstrate a constructive influence of memory on the noise-induced synchronization. To this end we consider generalized Langevin dynamics of Fahy–Hamann particles, for which the limit of vanishing memory is well understood.

2. Model

We explore influence of memory on the noise-induced synchronization by generalizing the Fahy–Hamann studies [1]. Our system comprises of two identical, independent particles in a two dimensional potential well given by

$$V(x_1,x_2) = \frac{\sin 2\pi x_1}{2\pi x_1} + \frac{\sin 2\pi x_2}{2\pi x_2} + \frac{(x_1^2+x_2^2)^2}{16\pi^2}\,, \tag{1}$$

which is sketched in Fig. 1. The motion of the i-th particle ($i=1,2$) is governed by the generalized Langevin equation:

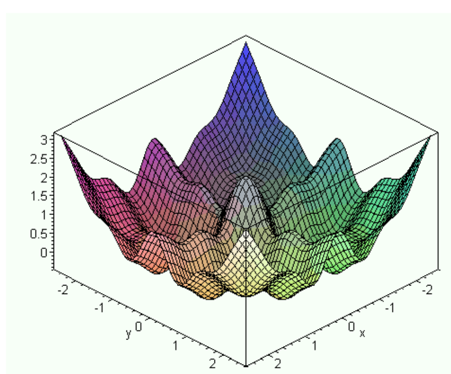

Fig. 1. Fahy–Hamann potential.

$$m\ddot{x}^i_\alpha(t) = -\frac{\partial V(x^i_1, x^i_2)}{\partial x^i_\alpha} - m\gamma \int^t \Gamma(t-t')\,\dot{x}^i_\alpha(t')\,dt' + \xi(t)\,, \tag{2}$$

where m is the particle's mass, γ is the friction constant, also playing the role of memory intensity parameter, T is the absolute temperature and k_{B} is the Boltzmann constant. The particles evolve subject to different initial conditions. They are coupled by the common noise ξ, which is a Γ-correlated stochastic force with zero mean and correlations obeying fluctuation-dissipation theorem

$$\langle\langle\, \xi(t)\xi(t')\,\rangle\rangle = 2m\gamma k_{\mathrm{B}} T\ \Gamma(t-t')\,. \tag{3}$$

In what follows we restrict ourselves to the exponentially correlated noise by choosing $\Gamma(t-t') = e^{-\lambda(t-t')}$, where $1/\lambda$ is the memory range. Double angular brackets denote averaging over a noise realization.

3. Simulations

We analyze trajectories of the particles by integrating numerically the equations of motion (2) with the help of stochastic version of the Euler algorithm. Discretization of the equations (2) entails re-scaling the noise strength by a factor $1/\sqrt{\Delta t}$, where Δt is the time step. The exponentially correlated noise is generated from uniformly distributed random numbers through Ornstein–Uhlenbeck process. Finally, the integral present in (2) is discretized according the scheme

$$\int^t dt'\ \dot{x}^i_\alpha(t')\ \exp\left[-\lambda(t-t')\right] = \sum_{k=0}^{N} \dot{x}^i_\alpha(t-k\Delta t)\ \exp\left(-\lambda k \Delta t\right). \tag{4}$$

The number N of stored past velocities was chosen equal to

$$N = \left\lfloor \frac{-\ln p}{\lambda \Delta t} \right\rfloor\,, \tag{5}$$

where $p = 10^{-5}$ is the assumed accuracy of the approximation used in evaluating integral (4). In the limit of $\lambda \to \infty$ the dynamics becomes reduced to the ordinary Langevin dynamics without memory ($N = 0$).

In simulations we determined the maximal Lyapunov exponent, Λ, as function of memory range ($1/\lambda$) and γ. Λ was estimated from relative changes in the distance between the two initially nearby trajectories

$$\Lambda = \left\langle \frac{1}{t-t_0} \ln \frac{d(t)}{d(t_0)} \right\rangle\,, \tag{6}$$

where d stands for distance at time t and $\langle \ldots \rangle$ is the averaging over trajectory. We used a natural system of units provided by the potential parameters: energy $\varepsilon_{\mathrm{u}} = V(0,0) - V_{\min} \approx 2.41$, time $t_{\mathrm{u}} = \sqrt{3}$, measuring curvature of the potential at origin and length $l_{\mathrm{u}} = 1$ giving the period of the oscillating part of the potential. Also we set the particle's mass to $m = 1$.

Simulations were carried out according to the following scheme: first of all we selected initial conditions for a particle at random following Boltzmann distribution at a given temperature T. Then, the particle was thermalized for about $t = 500$ time steps. Initial position of the second particle was chosen at random, close to the thermalized position of the first particle. The distance in positions between the particles did not exceed 10^{-6} at start, and in velocities was limited by $10^6\, t_{\mathrm{u}}$. Averaging procedure was performed along a trajectory over 10^3 subsequent time steps for 10^4 different trajectories. Further increase in the number of the time steps and the trajectories did not lead to systematic changes in Λ.

4. Results

In the absence of memory the system was originally studied by Fahy and Hamann [1] using Andersen thermostat. The results unambiguously showed that trajectories were exponentially convergent to a common, *master* trajectory after a transient period. The same phenomenon has been reported for an ordinary Langevin dynamics (without memory) applied to a one-dimensional Lennard–Jones chain [27]. Recently Uberuaga and co-workers [9] re-analyzed the synchronization in Andersen and Langevin thermostats and suggested that this synchronization should occur for a wide range of potentials and temperatures. Their observation agree with our understanding of the noise-induced synchronization as being an averaged effect of contraction and expansion of the distances between nearby trajectories during their time evolution [14, 25, 27, 28]. The questions arise as whether Andersen thermostat and Langevin dynamics lead to the same predictions for our model and how synchronization is affected by the range of memory.

A typical simulation for the memoryless system shows, as expected, synchronization of the trajectories ($\Lambda < 0$, Fig. 2). Generally the distance between two trajectories decays exponentially after the transient period until fluctuations of the order of numerical precision are reached. Inset shows the distance behavior at short times. The oscillations arise as a result of motion around local potential minimum. A systematic study shows that Λ changes with temperature and friction constant γ, as shown in Fig. 3. The results are summarized in Fig. 4, which shows regions of synchronization ($\Lambda < 0$) and chaos ($\Lambda > 0$) in (temperature, γ) plane.

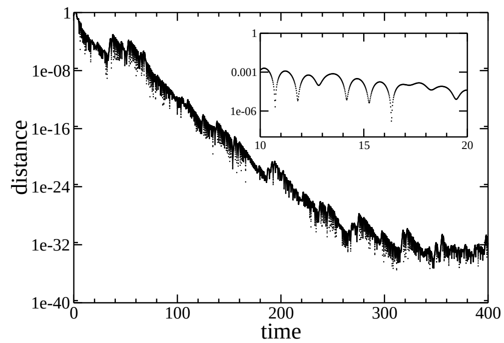

Fig. 2. Time evolution of the distance between two trajectories. System parameters are: $T = 0.1$, $\lambda \to \infty$ and $\gamma = 1.0$. Inset shows the evolution at short times.

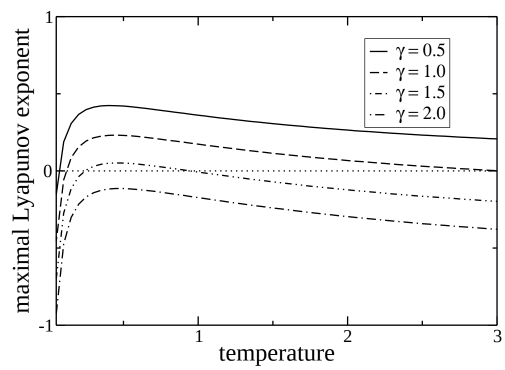

Fig. 3. Maximal Lyapunov exponent Λ *versus* temperature T at different friction constant γ for memoryless system ($\lambda \to \infty$).

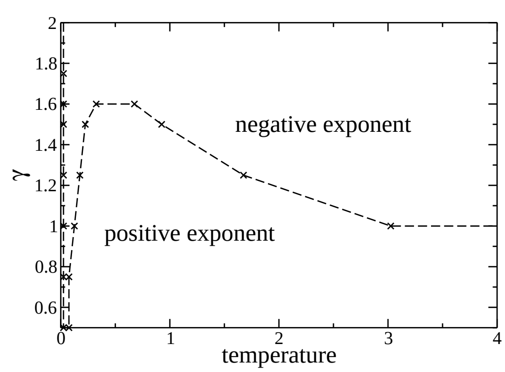

Fig. 4. Regions of positive and negative maximal Lyapunov exponent for Fahy–Hamann system without memory ($\lambda \to \infty$).

Generally synchronization is observed for large frictions, which stays in agreement with our earlier results for a different system [27]. Temperature dependence shows that Λ reaches it's maximum at mid-temperature range. In this regime particle trajectories easily penetrate local maxima of the potential, whereas at low temperatures they stay for a long time in local minima. On the other hand, for high temperatures the local harmonic structure can be neglected in comparison with the harmonic $(x_1^2 + x_2^2)^2$ boundaries.

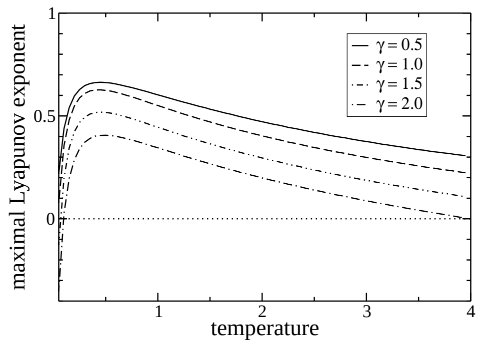

Fig. 5. Maximal Lyapunov exponent Λ *versus* temperature T for different friction constant γ and short memory $(1/\lambda = 0.01)$.

Finally, in Fig. 5 we show maximal Lyapunov exponent against temperature for different values of friction constant and for memory range $1/\lambda = 0.01$ $(N = 5)$. It is clearly seen that for this memory range no synchronization occurs. A simple intuitive explanation would be that memory adds additional degrees of freedom, which should make harder for the system to synchronize. Unexpectedly this argumentation fails for longer memory. Calculations of Λ as function of $N \sim 1/\lambda$ show at least three regimes, Fig. 6. In the first regime, corresponding to a short memory range, we observe

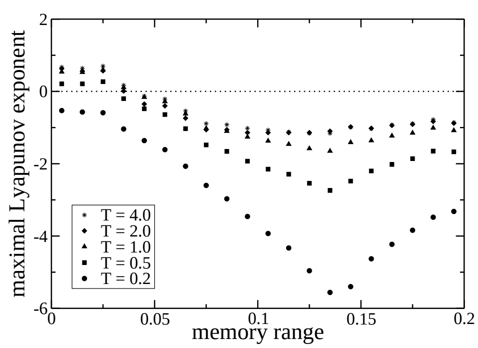

Fig. 6. Maximal Lyapunov exponent Λ against memory $1/\lambda \leq 0.2$ $(N \leq 100)$ for different temperatures with friction constants $\gamma = 1.0$.

de-synchronization of the system by memory. However, after reaching maximum Λ drops down and for intermediate memory range synchronization is considerably enhanced. The strongest synchronization conditions are met for $1/\lambda \approx 0.14$, where Λ approaches minimum. Interestingly, the positions of the extremes are practically independent on γ. Further details are given elsewhere [30].

5. Summary

Summarizing, our results show a possibility of having the *constructive* influence of memory on the noise-induced synchronization. To our knowledge the only other observation pointing to the same conclusion is an enhancement of neural synchrony in the Hindmarsh–Rose neural network with time-delays [21]. Please note that the effect is quite counterintuitive for, in the first place, we would expect that memory by introducing extra dimensions [29] should act in just the opposite way, *i.e.* make synchronization more difficult [22,23]. Though the proposed models are relatively simple, our observations to date suggest that this enhancement of synchrony by memory is a general phenomenon, occurring for a wide class of memory profiles.

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