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Multiple Solutions for Nonlinear Coercive Problems with a Nonhomogeneous Differential Operator and a Nonsmooth Potential

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Abstract We consider a nonlinear elliptic problem driven by a nonlinear nonhomogeneous differential operator and a nonsmooth potential. We prove two multiplicity theorems for problems with coercive energy functional. In both theorems we produce three nontrivial smooth solutions. In the second multiplicity theorem, we provide precise sign information for all three solutions (the first positive, the second negative and the third nodal). Out approach is variational, based on the nonsmooth critical point theory. We also prove an auxiliary result relating smooth and Sobolev local minimizer for a large class of locally Lipschitz functionals.

Keywords Locally Lipschitz function · Generalized subdifferential · Palais-Smale condition · Mountain pass theorem · Second deformation theorem · Nodal solutions

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear elliptic problem with a nonsmooth potential (hemivariational inequality):

$$\begin{cases} -\operatorname{div} a\big(\nabla u(z)\big) \in \partial F\big(z, u(z)\big) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

Here $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a C^1-map, which is strictly monotone and satisfies certain other regularity conditions (see hypotheses H_0'). Two important special cases of the map a are the following:

$$a(y) = \|y\|^{p-2}y \quad \forall y \in \mathbb{R}^N$$

which corresponds to the p-Laplace differential operator

$$\Delta_p u = \operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) \quad \forall u \in W_0^{1,p}(\Omega)$$

and

$$a(y) = \|y\|^{p-2}y + \mu\|y\|^{q-2}y \quad \forall y \in \mathbb{R}^N,$$

with $\mu \geqslant 0$, $2 \leqslant q \leqslant p < +\infty$, which corresponds to the (p,q)-differential operator

$$\Delta_p u + \mu\Delta_q u, \quad \text{with } u \in W_0^{1,p}(\Omega).$$

Also $F\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable potential which is only locally Lipschitz and in general nonsmooth in the second variable. By $\partial F(z,\zeta)$ we denote the generalized (Clarke) subdifferential of $\zeta \longmapsto F(z,\zeta)$ (see Section 2).

We are interested in the existence of multiple nontrivial solutions for problem (1.1), when the energy functional of the problem is coercive. We prove two such multiplicity theorems ("three solutions theorems"). In the first, we produce three nontrivial smooth solutions, two of which have constant sign (one positive and the other negative). In the second multiplicity theorem, by strengthening the hypotheses on the potential $F(z,\cdot)$, we show that the third solution is nodal (sign changing). To the best of our knowledge this is the first result (even for smooth problems, i.e., when $F(z,\cdot) \in C^1(\mathbb{R})$), which produces a nodal solution for problems with a nonhomogeneous differential operator.

Our approach is variational based on the nonsmooth critical point theory (see Gasiński-Papageorgiou [18] and Motreanu-Rădulescu [31]). We mention that three solutions theorems for coercive equations were proved by Ambrosetti-Lupo [2], Ambrosetti-Mancini [3], Iannizzotto [24], Struwe [34] for certain parametric semilinear equations (Iannizzotto [24] deals with hemivariational inequalities, while the other consider "smooth" problems) and by Averna-Marano-Motreanu [4], Liu-Liu [27], Liu [28], Papageorgiou-Papageorgiou [33] for problems driven by p-Laplacian (Averna-Marano-Motreanu [4] deal with parametric hemivariational inequalities, while the others examine "smooth" potentials). Our work here is closer to those of Liu-Liu [27] and Liu [28], since no parameter appears in (1.1) and our theorems extend the results of [27] and [28] in many different ways. In the next three papers of Filippakis-Gasiński-Papageorgiou [14] and Gasiński-Papageorgiou [16, 17] we also find multiplicity results for hemivariational inequalities in the case

of various boundary value conditions: Dirichlet, periodic and Neumann. Finally, we mention that hemivariational inequalities arise naturally in problems of nonsmooth mechanics. For several such applications, we refer to the book of Naniewicz-Panagiotopoulos [32].

In the next section, for the convenience of the reader, we recall some basic facts from the nonsmooth critical point theory, which is based on the notion of subdifferential of a locally Lipschitz functional. We also prove an auxiliary result of independent interest relating smooth and Sobolev local minimizers for a large class of nonsmooth locally Lipschitz functions.

2 Mathematical Background—Preliminary Results

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). For a given locally Lipschitz functional $\varphi\colon X \longrightarrow \mathbb{R}$, the generalized directional derivative $\varphi^0(z; h)$ of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x; h) \stackrel{df}{=} \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{\varphi(x' + th) - \varphi(x')}{t}.$$

It is easy to see that the map $x \longmapsto \varphi^0(x; h)$ is sublinear continuous. Therefore, it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$, defined by

$$\partial\varphi(x) \;=\; \left\{x^* \in X^* : \langle x^*, h\rangle \leqslant \varphi^0(x; h) \text{ for all } h \in X\right\}.$$

The multifunction $x \longmapsto \partial\varphi(x)$ is called the generalized (or Clarke) subdifferential of φ. If $\varphi\colon X \longrightarrow \mathbb{R}$ is continuous convex, then φ is locally Lipschitz and the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis, given by

$$\partial_c\varphi(x) \;=\; \left\{x^* \in X^* : \langle x^*, h\rangle \leqslant \varphi(x + h) - \varphi(h) \text{ for all } h \in X\right\}.$$

Moreover, if $\varphi \in C^1(X)$, then φ is locally Lipschitz and $\partial\varphi(x) = \left\{\varphi'(x)\right\}$.

If $\varphi, \psi\colon X \longrightarrow \mathbb{R}$ are locally Lipschitz functionals and $\lambda \in \mathbb{R}$, then

$$\partial(\varphi + \psi)(x) \;\subseteq\; \partial\varphi(x) + \partial\psi(x) \quad \forall x \in X$$

and

$$\partial(\lambda\varphi)(x) \;=\; \lambda\partial\varphi(x) \quad \forall x \in X,\ \lambda \in \mathbb{R}.$$

The generalized subdifferential has a very rich calculus, which extends that of smooth and of continuous convex functionals. For more details, we refer to the book of Clarke [9].

Let $\varphi\colon X \longrightarrow \mathbb{R}$ be a locally Lipschitz functional. We say that $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. If $x \in X$ is a local extremum of φ (i.e., x is either a local minimizer or a local maximizer of φ), then $x \in X$ is a critical point of φ.

For a given locally Lipschitz functional $\varphi\colon X \longrightarrow \mathbb{R}$, we set

$$m_\varphi(x) \;=\; \inf\left\{\|x^*\|_* : x^* \in \partial\varphi(x)\right\}$$

(here $\|\cdot\|_*$ denotes the norm of the dual space X^*). We say that φ satisfies the Palais-Smale condition, if the following holds:

Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is a bounded sequence and

$$m_\varphi(x_n) \longrightarrow 0$$

admits a strongly convergent subsequence.

Using this compactness-type condition, we can have the following nonsmooth extension of the well known mountain pass theorem.

Theorem 2.1 *If X is a Banach space, $\varphi\colon X \longrightarrow \mathbb{R}$ is a locally Lipschitz functional which satisfies the Palais-Smale condition, $x_0, x_1 \in X$ are such that $\|x_1 - x_0\| > r > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} = \eta_0$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geqslant \eta_0$ and c is a critical value of the functional φ.

The nonsmooth critical point theory was initiated with the work of Chang [6]. Detailed presentations of the theory with extensions and generalizations can be found in the books of Gasiński-Papageorgiou [18] and Motreanu-Rădulescu [31].

Let $\varphi\colon X \longrightarrow \mathbb{R}$ be a locally Lipschitz functional and $c \in \mathbb{R}$. We define

$$\dot{\varphi}^c = \{x \in X : \varphi(x) < c\},$$
$$K_\varphi = \{x \in X : 0 \in \partial\varphi(x)\},$$
$$K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

The next result is due to Corvellec [10] and it is the nonsmooth counterpart of the so called second deformation theorem (see Gasiński-Papageorgiou [19, p. 628]).

Theorem 2.2 *If X is a Banach space, $\varphi\colon X \longrightarrow \mathbb{R}$ is a locally Lipschitz functional which satisfies the Palais-Smale condition, $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, $K_\varphi \cap \varphi^{-1}(a,b) = \emptyset$ and K_φ^c is finite and contains only local minimizers of φ, then there exists a continuous deformation $h\colon [0,1] \times \dot{\varphi}^b \longrightarrow \dot{\varphi}^b$, such that:*

(a) $h(t,\cdot)|_{K_\varphi^a} = id|_{K_\varphi^a}$ *for all* $t \in [0,1]$;
(b) $h(1, \dot{\varphi}^b) \subseteq \dot{\varphi}^a \cup K_\varphi^a$
(c) $\varphi(h(t,x)) \leqslant \varphi(x)$ *for all* $(t,x) \in [0,1] \times \dot{\varphi}^b$.

In particular $\dot{\varphi}^a \cup K_\varphi^a$ is a weak deformation retract of $\dot{\varphi}^b$.

In the analysis of problem (1.1), in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

For the next auxiliary result, we can be more general and allow the map $a(\cdot)$ to be z-dependent. More precisely, we introduce the following hypotheses:

H_0: $G\colon \overline{\Omega} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a C^1-function, such that $G(z, 0) = 0$, $\nabla_y G(z, y) = a(z, y)$ and $a(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $a \in C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}); \mathbb{R}^N)$;

(ii) there exist $c_0 > 0$ and $\eta \geqslant 0$, such that for every $z \in \overline{\Omega}$ and every $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$c_0(\eta + \|y\|)^{p-2}\|\xi\|^2 \leqslant (\nabla_y a(z, y)\xi, \xi)_{\mathbb{R}^N} \quad \forall \xi \in \mathbb{R}^N;$$

(iii) there exists $c_1 > 0$, such that for every $z \in \overline{\Omega}$ and every $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$\|\nabla_y a(z, y)\| \leqslant c_1(\eta + \|y\|)^{p-2},$$

with η as in (*ii*);

(iv) for every $\varrho > 0$, there exists $c_2 = c_2(\varrho) > 0$, such that

$$|a(z, y) - a(z', y)| \leqslant c_2(1 + \|y\|)^{p-1}\|z - z'\| \quad \forall z \in \Omega,\ z' \in \partial\Omega,\ \|y\| \leqslant \varrho.$$

Example 2.3 The following maps satisfy hypotheses H_0:

(a) Let

$$G_1(z, y) = \frac{1}{p}\vartheta(z)\|y\|^p,$$

with $\vartheta \in C^1(\overline{\Omega})$, $\vartheta(z) > 0$ for all $z \in \overline{\Omega}$ and $1 < p < +\infty$. Then

$$a_1(z, y) = \vartheta(z)\|y\|^{p-2}y.$$

This potential function corresponds to a weighted p-Laplacian differential operator.

(b) Let

$$G_2(z, y) = \frac{\vartheta_1(z)}{p}\|y\|^p + \frac{\vartheta_2(z)}{q}\|y\|^q,$$

with $\vartheta_1, \vartheta_2 \in C^1(\overline{\Omega})$, $\vartheta_1(z) > 0$, $\vartheta_2(z) > 0$ for all $z \in \overline{\Omega}$ and $2 \leqslant q \leqslant p < +\infty$. Then

$$a_2(z, y) = \vartheta_1(z)\|y\|^{p-2}y + \vartheta_2(z)\|y\|^{q-2}y.$$

This potential function corresponds to a weighted (p, q)-differential operator. Problems with such potentials were studied recently by Cingolani-Degiovanni [7], Figueiredo [13], Medeiros-Perera [29].

(c) Let

$$G_3(z, y) = \frac{\vartheta(z)}{p}\big(\|y\|^p + \ln\big(1 + \|y\|^p\big)\big),$$

with $\vartheta \in C^1(\overline{\Omega})$, $\vartheta(z) > 0$ for all $z \in \overline{\Omega}$ and $p \geqslant 2$. Then

$$a_3(z, y) = \vartheta(z)\left(\|y\|^{p-2}y + \frac{\|y\|^{p-2}y}{1 + \|y\|^2}\right).$$

From hypotheses H_0 and using the integral form of the mean value theorem, we obtain the following auxiliary result.

Lemma 2.4 *If hypotheses H_0 hold, then for all $z \in \overline{\Omega}$, $a(z, \cdot)$ is strictly monotone and for all $(z, y) \in \overline{\Omega} \times \mathbb{R}^N$, we have*

$$\big(a(z, y),\ y\big)_{\mathbb{R}^N} \geqslant \frac{c_0}{p-1}\|y\|^p \quad \text{and} \quad \big\|a(z, y)\big\| \leqslant c_1\big(\eta + \|y\|\big)^{p-1}.$$

An easy consequence of this lemma are the following growth estimates for the potential $G(z, \cdot)$.

Corollary 2.5 *If hypotheses H_0 hold, then for all $z \in \overline{\Omega}$, $G(z, \cdot)$ is strictly convex and*

$$\frac{c_0}{p(p-1)}\|y\|^p \leqslant G(z, y) \leqslant \hat{c}_1\big(1 + \|y\|\big)^p \quad \forall (z, y) \in \overline{\Omega} \times \mathbb{R}^N.$$

The next result relates local $C_0^1(\overline{\Omega})$ and $W_0^{1,p}(\Omega)$-minimizers for a large class of locally Lipschitz functionals. Such a result was first proved for

$$G(y) = \frac{1}{2}\|y\|^2$$

and smooth (i.e., C^1) functionals by Brézis-Nirenberg [5]. It was extended to the case

$$G(y) = \frac{1}{p}\|y\|^p,$$

with $1 < p < +\infty$ and smooth functionals by García Azorero-Manfredi-Peral Alonso [15] (see also Guo-Zhang [21], where $p \geqslant 2$). For a nonsmooth version we refer to Gasiński-Papageorgiou [18, p. 655]. The next proposition extends all the aforementioned works. Moreover our proof is simpler.

So, let $F_0 \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function, such that for almost all $z \in \Omega$, the function $\zeta \longmapsto F_0(z, \zeta)$ is locally Lipschitz and

$$|u| \leqslant a(z) + c|\zeta|^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \text{ all } u \in \partial F_0(z, \zeta),$$

with $a \in L^\infty(\Omega)_+$, $c > 0$ and $1 < r < p^*$, where

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geqslant N. \end{cases}$$

Let $\psi_0\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the functional, defined by

$$\psi_0(u) = \int_\Omega G\big(z, \nabla u(z)\big)\,dz - \int_\Omega F_0\big(z, u(z)\big)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Evidently ψ_0 is Lipschitz continuous on bounded sets, hence it is locally Lipschitz.

Proposition 2.6 *If hypotheses H_0 hold and $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of ψ_0, i.e., there exists $\varrho_0 > 0$, such that*

$$\psi_0(u_0) \leqslant \psi_0(u_0 + h) \quad \forall h \in C_0^1(\overline{\Omega}),\ \|h\|_{C_0^1(\overline{\Omega})} \leqslant \varrho_0,$$

then $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and it is also a local $W_0^{1,p}(\Omega)$-minimizer of ψ_0, i.e., there exists $\varrho_1 > 0$, such that

$$\psi_0(u_0) \leqslant \psi_0(u_0 + h) \quad \forall h \in W_0^{1,p}(\Omega),\ \|h\| \leqslant \varrho_1.$$

Proof Let $h \in C_0^1(\overline{\Omega})$ and consider $t > 0$ small. Then by hypothesis

$$\psi_0(u_0) \leqslant \psi_0(u_0 + th),$$

so

$$0 \leqslant \psi_0^0(u_0; h). \tag{2.1}$$

Since $h \in C_0^1(\overline{\Omega})$ is arbitrary, $\psi_0^0(u_0; \cdot)$ is continuous and $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,p}(\Omega)$, from (2.1), we infer that

$$0 \leqslant \psi_0^0(u_0; h) \quad \forall h \in W_0^{1,p}(\Omega),$$

so

$$0 \in \partial\psi_0(u_0)$$

and thus

$$V(u_0) = u^*, \tag{2.2}$$

where $V\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) is the nonlinear map, defined by

$$\big\langle V(u),\ y\big\rangle = \int_\Omega \big(a(z, \nabla u),\ \nabla y\big)_{\mathbb{R}^N}\,dz \quad \forall u, y \in W_0^{1,p}(\Omega)$$

and $u_0^* \in L^{r'}(\Omega)$ (with $\frac{1}{r} + \frac{1}{r'} = 1$), $u_0^*(z) \in \partial F_0\big(z, u_0(z)\big)$ for almost all $z \in \Omega$ (see Clarke [9, p. 83]). From (2.2), it follows that

$$\begin{cases} -\operatorname{div} a\big(z, \nabla u_0(z)\big) = u_0^*(z) \in \partial F_0\big(z, u_0(z)\big) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases} \tag{2.3}$$

Invoking Theorem 7.1 of Ladyzhenskaya-Uraltseva [25, p. 286], we have that $u_0 \in L^\infty(\Omega)$. Then on (2.3) we can use Theorem 1 of Lieberman [26] and conclude that $u_0 \in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

Next we show that u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of ψ_0. We argue by contradiction. So, suppose that u_0 is not a local $W_0^{1,p}(\Omega)$-minimizer of ψ_0. For $\varepsilon > 0$, let

$$\overline{B}_\varepsilon^r = \left\{u \in W_0^{1,p}(\Omega) : \|u\|_r \leqslant \varepsilon\right\}$$

and consider the following minimization problem:

$$\inf_{h \in \overline{B}_\varepsilon^r} \psi_0(u_0 + h) = m_0^\varepsilon > -\infty. \tag{2.4}$$

Since u_0 is not a local $W_0^{1,p}(\Omega)$-minimizer of ψ_0, we have

$$m_0^\varepsilon < \psi_0(u_0). \tag{2.5}$$

Let $\{h_n\}_{n\geqslant 1} \subseteq \overline{B}_\varepsilon^r$ be a minimizing sequence for problem (2.4). Using Corollary 2.5 and the growth hypothesis on $\partial F_0(z, \cdot)$, we see that the sequence $\{h_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, passing to a subsequence if necessary, we may assume that

$$h_n \longrightarrow h_\varepsilon \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{2.6}$$

$$h_n \longrightarrow h_\varepsilon \quad \text{in } L^r(\Omega). \tag{2.7}$$

Clearly ψ_0 is sequentially weakly lower semicontinuous. So, from (2.6), we have

$$\psi_0(u_0 + h_\varepsilon) \leqslant \liminf_{n\to+\infty} \psi_0(u_0 + h_n),$$

hence

$$\psi_0(u_0 + h_\varepsilon) = m_0^\varepsilon$$

and thus $h_\varepsilon \neq 0$ (see (2.5)).

So, the infimum in problem (2.4) is realized at some $h_\varepsilon \in \overline{B}_\varepsilon^r \setminus \{0\}$ (see (2.6)). Invoking the nonsmooth Lagrange multiplier rule of Clarke [8], we can find $\lambda_\varepsilon \leqslant 0$, such that

$$0 \in \partial\psi_0(u_0 + h_\varepsilon) - \lambda_\varepsilon |h_\varepsilon|^{r-2} h_\varepsilon,$$

so

$$V(u_0 + h_\varepsilon) = u_\varepsilon^* + \lambda_\varepsilon |h_\varepsilon|^{r-2} h_\varepsilon,$$

where $u_\varepsilon^* \in L^{r'}(\Omega)$, $u_\varepsilon^*(z) \in \partial F_0\big(z, (u_0 + h_\varepsilon)(z)\big)$ for almost all $z \in \Omega$. Then

$$\begin{cases} -\operatorname{div} a\big(z, \nabla(u_0 + h_\varepsilon)(z)\big) = u_\varepsilon^*(z) + \lambda_\varepsilon \big|h_\varepsilon(z)\big|^{r-2} h_\varepsilon(z) \text{ in } \Omega, \\ h_\varepsilon|_{\partial\Omega} = 0. \end{cases} \tag{2.8}$$

From (2.3) and (2.8), for almost all $z \in \Omega$, we have

$$\begin{aligned} &-\operatorname{div}\big(a\big(z, \nabla(u_0 + h_\varepsilon)(z)\big) - a\big(z, \nabla u_0(z)\big)\big) \\ &\quad = u_\varepsilon^*(z) - u_0^*(z) + \lambda_\varepsilon \big|h_\varepsilon(z)\big|^{r-2} h_\varepsilon(z) \end{aligned} \tag{2.9}$$

Case 1 Suppose that $\lambda_\varepsilon \in [-1, 0]$ for all $\varepsilon \in (0, 1]$.
We set

$$w_\varepsilon(z) = (u_0 + h_\varepsilon)(z),$$
$$\sigma_\varepsilon(z, y) = a(z, y) - a\big(z, \nabla u_0(z)\big).$$

Then from (2.9), for almost all $z \in \Omega$, we have

$$-\operatorname{div} \sigma_\varepsilon\big(z, \nabla w_\varepsilon(z)\big) = u_\varepsilon^*(z) - u_0^*(z) + \lambda_\varepsilon \big|(w_\varepsilon - u_0)(z)\big|^{r-2}(w_\varepsilon - u_0)(z). \tag{2.10}$$

On (2.10) we apply Theorem 7.1 of Ladyzhenskaya-Uraltseva [25, p. 286] and produce $M_1 > 0$, such that

$$\|w_\varepsilon\|_\infty \leqslant M_1 \quad \forall \varepsilon \in (0, 1]. \tag{2.11}$$

Clearly $\sigma_\varepsilon(z, y)$ satisfies hypotheses H_0. This fact and (2.11), permit the use of Theorem 1 of Lieberman [26] and so we can find $\gamma \in (0, 1)$ and $M_2 > 0$, such that

$$w_\varepsilon \in C_0^{1,\gamma}(\overline{\Omega}) \quad \text{and} \quad \|w_\varepsilon\|_{C_0^{1,\gamma}(\overline{\Omega})} \leqslant M_2 \quad \forall \varepsilon \in (0, 1]. \tag{2.12}$$

Case 2 Suppose that $\lambda_{\varepsilon_n} < -1$ for all $n \geqslant 1$, with $\varepsilon_n \searrow 0$, $\varepsilon_n \in (0, 1]$ for all $n \geqslant 1$.
In this case we set

$$w_{\varepsilon_n} = u_0 + h_{\varepsilon_n}$$
$$\widehat{\sigma}_{\varepsilon_n}(z, y) = \frac{1}{|\lambda_{\varepsilon_n}|}\big(a\big(z, \nabla u_0(z) + y\big) - a\big(z, \nabla u_0(z)\big)\big).$$

Then for almost all $z \in \Omega$ and all $n \geqslant 1$, we have

$$-\operatorname{div} \widehat{\sigma}_{\varepsilon_n}\big(z, \nabla h_{\varepsilon_n}(z)\big) = \frac{1}{|\lambda_{\varepsilon_n}|}\big(u_{\varepsilon_n}^*(z) - u_0^*(z)\big) - \big|h_{\varepsilon_n}(z)\big|^{r-2} h_{\varepsilon_n}(z). \tag{2.13}$$

For every $\xi \in W_0^{1,p}(\Omega)$, we have

$$\big\langle V(u_0), \xi \big\rangle = \int_\Omega u_0^* \xi \, dz \tag{2.14}$$

(see (2.3)) and

$$\big\langle V(u_{\varepsilon_n}), \xi \big\rangle = \int_\Omega u_{\varepsilon_n}^* \xi \, dz + \lambda_{\varepsilon_n} \int_\Omega \big|w_{\varepsilon_n} - u_0\big|^{r-2}\big(w_{\varepsilon_n} - u_0\big)\xi \, dz \tag{2.15}$$

(see (2.9)).

Let $\mu \geqslant 1$ and consider the function $\big|w_{\varepsilon_n} - u_0\big|^\mu \big(w_{\varepsilon_n} - u_0\big)$. Then

$$\nabla\big(\big|w_{\varepsilon_n} - u_0\big|^\mu \big(w_{\varepsilon_n} - u_0\big)\big) = (\mu + 1)\big|w_{\varepsilon_n} - u_0\big|^\mu \nabla\big(w_{\varepsilon_n} - u_0\big),$$

so

$$\big|w_{\varepsilon_n} - u_0\big|^\mu \big(w_{\varepsilon_n} - u_0\big) \in W_0^{1,p}(\Omega) \quad \forall n \geqslant 1$$

(recall that $w_{\varepsilon_n}, u_0 \in C_0^1(\overline{\Omega})$).

So, we can use this function as the test function ξ in (2.14) and (2.15). We do this and then subtract (2.14) from (2.15). Using Lemma 2.4, for all $n \geqslant 1$ we obtain

$$\begin{aligned} 0 &\leqslant (\mu+1)\int_\Omega \big(a(z,\nabla w_{\varepsilon_n}) - a(z,\nabla u_0),\ \nabla w_{\varepsilon_n} - \nabla u_0\big)_{\mathbb{R}^N} \big|w_{\varepsilon_n} - u_0\big|^\mu\, dz \\ &= \int_\Omega \big(u^*_{\varepsilon_n} - u^*_0\big)(w_{\varepsilon_n} - u_0)\big|w_{\varepsilon_n} - u_0\big|^\mu dz \\ &\quad + \lambda_{\varepsilon_n}\int_\Omega \big|w_{\varepsilon_n} - u_0\big|^{r+\mu} dz. \end{aligned} \tag{2.16}$$

Because of (2.11), recalling that $u_0 \in C^1_0(\overline{\Omega})$ and using Hölder inequality with exponents $\frac{\mu+r}{\mu+1}$, $\frac{\mu+r}{r-1}$, we have

$$\begin{aligned} &\left|\int_\Omega \big(u^*_{\varepsilon_n} - u^*_0\big)\big(w_{\varepsilon_n} - u_0\big)\big|w_{\varepsilon_n} - u_0\big|^\mu\, dz\right| \\ &\quad \leqslant M_3\int_\Omega \big|w_{\varepsilon_n} - u_0\big|^{\mu+1} dz \\ &\quad \leqslant M_3|\Omega|_N^{\frac{r-1}{r+p}}\big\|w_{\varepsilon_n} - u_0\big\|_{r+\mu}^{\mu-1} \quad \forall n \geqslant 1, \end{aligned} \tag{2.17}$$

for some $M_3 > 0$. Here $|\cdot|_N$ stands for the Lebesgue measure on $\mathbb{R}^N$. We return to (2.16) and use (2.17). Then

$$-\lambda_{\varepsilon_n}\big\|w_{\varepsilon_n} - u_0\big\|_{r+\mu}^{r+\mu} \leqslant M_3|\Omega|_N^{\frac{r-1}{r+\mu}}\big\|w_{\varepsilon_n} - u_0\big\|_{r+\mu}^{\mu-1},$$

so

$$-\lambda_{\varepsilon_n}\big\|w_{\varepsilon_n} - u_0\big\|_{r+\mu}^{r-1} \leqslant M_3|\Omega|_N^{\frac{r-1}{r+\mu}} \quad \forall \mu \geqslant 1,\ n \geqslant 1.$$

We let $\mu \to +\infty$ and obtain

$$-\lambda_{\varepsilon_n}\big\|w_{\varepsilon_n} - u_0\big\|_\infty^{r-1} \leqslant M_3 \quad \forall n \geqslant 1,$$

so

$$\big\|w_{\varepsilon_n} - u_0\big\|_\infty^{r-1} \leqslant \frac{M_3}{|\lambda_{\varepsilon_n}|} \quad \forall n \geqslant 1. \tag{2.18}$$

We return to (2.13) and denote the right hand side by $\eta_{\varepsilon_n}(z,\zeta)$. If $M_4 = \|u_0\| + M_1 > 0$ (see (2.11)), then for almost all $z \in \Omega$ and all $\zeta \in [-M_4, M_4]$, we have

$$\big|\eta_{\varepsilon_n}(z,\zeta)\big| \leqslant \frac{1}{|\lambda_{\varepsilon_n}|}[M_5 + M_3] \quad \forall n \geqslant 1,$$

for some $M_5 > 0$. This fact and since $\widehat{\sigma}_{\varepsilon_n}(z,y)$ satisfies hypotheses H_0, permit the use of Theorem 1 of Lieberman [26] and so we can find $\gamma_0 \in (0,1)$ and $M_6 > 0$, such that

$$h_{\varepsilon_n} \in C_0^{1,\gamma_0}(\overline{\Omega}) \quad \text{and} \quad \|h_{\varepsilon_n}\|_{C_0^{1,\gamma_0}(\overline{\Omega})} \leqslant M_6 \quad \forall n \geqslant 1. \tag{2.19}$$

Recall that for every $\gamma' \in (0,1)$ the space $C_0^{1,\gamma'}(\overline{\Omega})$ is embedded compactly in $C_0^1(\overline{\Omega})$. So, from (2.12) and (2.19) and by passing to a suitable subsequence if necessary, we have

$$u_0 + h_{\varepsilon_n} \longrightarrow u_0 \quad \text{in } C_0^1(\overline{\Omega})$$

(recall that $\varepsilon_n \searrow 0$). Because u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of ψ_0, we can find $n_0 \geqslant 1$, such that

$$\psi_0(u_0) \leqslant \psi_0(u_0 + h_{\varepsilon_n}) \quad \forall n \geqslant n_0. \tag{2.20}$$

On the other hand since h_{ε_n} are solutions of (2.4) and because of (2.5), we have

$$\psi_0(u_0 + h_{\varepsilon_n}) < \psi_0(u_0) \quad \forall n \geqslant 1. \tag{2.21}$$

Comparing (2.20) and (2.21), we reach a contradiction. This proves that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of ψ_0. □

Recall (see the above proof) that $V\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ is the nonlinear map, defined by

$$\langle V(u), y\rangle = \int_\Omega \big(a(z,\nabla u), \nabla y\big)\, dz \quad \forall u, y \in W_0^{1,p}(\Omega). \tag{2.22}$$

From Lemma 3.2 of Gasiński-Papageorgiou [20, p. 562], we have the following result.

Proposition 2.7 *If hypotheses H_0 hold and $V\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ is defined by (2.22), then V is continuous, bounded (i.e., maps bounded sets to bounded sets), strictly monotone and of type $(S)_+$, i.e., if $u_n \longrightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty} \langle V(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$.

Let $\widehat{\lambda}_1$ be the first eigenvalue of $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$. We know that $\widehat{\lambda}_1 > 0$ is isolated, simple and

$$\|\nabla u\|_p^p \geqslant \widehat{\lambda}_1 \|u\|_p^p \quad \forall u \in W_0^{1,p}(\Omega)$$

(see Gasiński-Papageorgiou [19]).

From Aizicovici-Papageorgiou-Staicu [1, Lemma 12], we have

Proposition 2.8 *If $\vartheta \in L^p(\Omega)_+$, $\vartheta(z) \leqslant \frac{c_0\widehat{\lambda}_1}{p-1}$ for almost all $z \in \Omega$, $\vartheta \neq \frac{c_0\widehat{\lambda}_1}{p-1}$, then there exists $\xi_0 > 0$, such that*

$$\frac{c_0}{p-1}\|\nabla u\|_p^p - \int_\Omega \vartheta |u|^p\, dz \geqslant \xi_0 \|u\|^p \quad \forall u \in W_0^{1,p}(\Omega).$$

Finally we mention that throughout this work, for every $u \in W_0^{1,p}(\Omega)$, we set

$$\|u\| = \|\nabla u\|_p$$

(by virtue of Poincaré inequality) and

$$u^+ = \max\{0, u\}, \quad u^- = \max\{0, -u\}.$$

We know that $u^+, u^- \in W_0^{1,p}(\Omega)$ and $u = u^+ - u^-$, $|u| = u^+ + u^-$. We mention that the notation $\|\cdot\|$ will also be used to denote the $\mathbb{R}^N$-norm. No confusion is possible, since it will always be clear from the context which norm is used. Also, as indicated in the proof of Proposition 2.6, $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$.

3 First Multiplicity Theorem

In this section we prove a multiplicity theorem, which produces three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative). To do this we need to drop the z-dependence on the map a. For easy reference, we state in detail the hypotheses:

$\underline{H_0'}$: $G: \mathbb{R}^N \longrightarrow \mathbb{R}$ is a C^1-function, such that $G(0) = 0$, $\nabla G(y) = a(y) = a_0(\|y\|)y$, $a_0(t) > 0$, $a(0) = 0$ and

(i) $a \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R}^N)$

(ii) there exist $c_0 > 0$ and $\eta \geqslant 0$, such that for every $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$c_0(\eta + \|y\|)^{p-2}\|\xi\|^2 \leqslant (\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \quad \forall \xi \in \mathbb{R}^N;$$

(iii) there exists $c_1 > 0$, such that for every $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$\|\nabla a(y)\| \leqslant c_1(\eta + \|y\|)^{p-2},$$

with η as in (ii);

(iv) there exists $\tau \in (1, p)$, such that

$$\lim_{y \to 0} \frac{G(y)}{\|y\|^\tau} = 0.$$

Remark 3.1 Clearly, hypotheses H_0' are a particular case of hypotheses H_0. The reason we have dropped the z-dependence is that we need an extension of the nonlinear strong maximal principle of Vázquez [35], valid for the p-Laplacian, to more general nonhomogeneous differential operators, like the one in this paper. The only such result for z-dependent operators is that of Zhang [36], who though requires that $\eta = 0$ in hypotheses $H_0(ii)$ and (iii). Such a condition excludes from consideration (p, q)-differential operators. Note that the examples presented after hypotheses H_0, satisfy hypotheses H_0' (of course with $\vartheta = \vartheta_1 = \vartheta_2 = 1$).

The hypotheses on the nonlinear potential $F(z, \zeta)$ are the following:

$\underline{H_1}$: $F: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function, such that for almost all $z \in \Omega$, we have $F(z, 0) = 0$, $0 \in \partial F(z, 0)$, $F(z, \cdot)$ is locally Lipschitz and

(i) there exist $a \in L^\infty(\Omega)_+$ and $c > 0$, such that

$$|u^*| \leqslant a(z) + c|\zeta|^{p-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \text{ all } u^* \in \partial F(z, \zeta);$$

(ii) there exists $\vartheta \in L^\infty(\Omega)_+$, $\vartheta(z) \leqslant \frac{c_0}{p-1}$ for almost all $z \in \Omega$, $\vartheta \neq \frac{c_0}{p-1}$, such that

$$\limsup_{\zeta \to \pm\infty} \frac{pF(z, \zeta)}{|\zeta|^p} \leqslant \vartheta(z) \quad \text{uniformly for almost all } z \in \Omega;$$

(iii) if $\tau \in (1, p)$ is as in hypothesis $H_0'(iv)$, then there exists $\beta_0 > 0$, such that

$$\liminf_{\zeta \to 0} \frac{\tau F(z, \zeta)}{|\zeta|^\tau} \geqslant \beta_0 \quad \text{uniformly for almost all } z \in \Omega:$$

(iv) for every $\varrho > 0$, there exists $\gamma_\varrho > 0$, such that, if

$$\sigma(z, \zeta) = \min\{u^* : u^* \in \partial F(z, \zeta)\},$$

then

$$\sigma(z, \zeta) + \gamma_\varrho |\zeta|^{p-2}\zeta \geqslant 0 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in [-\varrho, \varrho].$$

Remark 3.2 Hypothesis $H_1(ii)$ implies that for almost all $z \in \Omega$, the function $F(z, \cdot)$ is p-(sub)linear near $\pm\infty$. Hypothesis $H_1(iii)$ implies the presence of a "concave" nonlinearity near the origin. We stress that no sign condition is imposed on the elements of $\partial F(z, \zeta)$. Instead, we impose the weaker condition $H_1(iv)$.

Example 3.3 The following potential function $F(\zeta)$ satisfies hypotheses H_1 (for the sake of simplicity, we drop the z-dependence):

$$F(\zeta) = \begin{cases} \dfrac{c}{\tau}|\zeta|^\tau & \text{if } |\zeta| \leqslant 1, \\ \dfrac{\vartheta}{p}|\zeta|^p & \text{if } |\zeta| > 1, \end{cases}$$

with $1 < \tau < p$, $c = \frac{\vartheta\tau}{p}$. Note that F is not a C^1-function.

Let $\varphi\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem (1.1), defined by

$$\varphi(u) = \int_\Omega G(\nabla u(z))\, dz - \int_\Omega F(z, u(z))\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Evidently φ is locally Lipschitz.

Theorem 3.4 *If hypotheses H_0' and H_1 hold, then problem (1.1) has at least three nontrivial smooth solutions:*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+, \quad \text{and} \quad y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$$

and u_0, v_0 are local minimizers of φ.

Proof Let

$$F_{\pm}(z,\zeta) \;=\; F(z,\pm\zeta^{\pm}) \quad \forall (z,\zeta)\in\Omega\times\mathbb{R}$$

and let $\varphi_{\pm}\colon W_0^{1,p}(\Omega)\longrightarrow\mathbb{R}$ be the locally Lipschitz functionals, defined by

$$\varphi_{\pm}(u) \;=\; \int_{\Omega} G\big(\nabla u(z)\big)\,dz - \int_{\Omega} F_{\pm}\big(z,u(z)\big)\,dz \quad \forall u\in W_0^{1,p}(\Omega).$$

By virtue of hypotheses $H_1(i)$ and (ii) and Lebourg mean value theorem for locally Lipschitz functionals (see e.g., Clarke [9, p. 41]), for a given $\varepsilon>0$, we can find $c_3 = c_3(\varepsilon)>0$, such that

$$F(z,\zeta) \;\leqslant\; \frac{1}{p}\big(\vartheta(z)+\varepsilon\big)|\zeta|^p + c_3 \quad \text{for almost all } z\in\Omega,\ \text{all } \zeta\in\mathbb{R}. \tag{3.1}$$

Then using Corollary 2.5, estimate (3.1) and Proposition 2.8, we have

$$\begin{aligned}\varphi_+(u) &= \int_{\Omega} G(\nabla u)\,dz - \int_{\Omega} F(z,u)\,dz\\ &\geqslant \frac{c_0}{p(p-1)}\|\nabla u\|_p^p - \frac{1}{p}\int_{\Omega}\vartheta|u|^p\,dz - \frac{\varepsilon}{p}\|u\|^p - c_4\\ &\geqslant \frac{1}{p}\left(\xi_0 - \frac{\varepsilon}{\widehat{\lambda}_1}\right)\|u\|^p - c_4 \quad \forall u\in W_0^{1,p}(\Omega).\end{aligned}$$

Choosing $\varepsilon\in(0,\xi_0\widehat{\lambda}_1)$, we infer that φ_+ is coercive. Also, exploiting the compactness of the embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$, we can easily check that φ_+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0\in W_0^{1,p}(\Omega)$, such that

$$\varphi_+(u_0) \;=\; \inf_{u\in W_0^{1,p}(\Omega)} \varphi_+(u) \;=\; m_+. \tag{3.2}$$

By virtue of hypotheses $H_0'(iv)$ and $H_1(iii)$, for a given $\varepsilon>0$, we can find $\delta=\delta(\varepsilon)>0$ and $\beta_1=\beta_1(\varepsilon)>0$, such that for almost all $z\in\Omega$ and all $y\in\mathbb{R}^N$, $\zeta\in\mathbb{R}$ with $\|y\|<\delta$ and $|\zeta|<\delta$, we have

$$G(y) \;\leqslant\; \frac{\varepsilon}{\tau}\|y\|^{\tau} \quad \text{and} \quad F(z,\zeta) \;\geqslant\; \frac{\beta_1}{\tau}|\zeta|^{\tau}. \tag{3.3}$$

Let $u\in\operatorname{int}C_+$ and let $t\in(0,1)$ be small, such that

$$tu(z) \;\in\; [0,\delta] \quad \text{and} \quad \big\|\nabla(tu)(z)\big\| \in [0,\delta] \quad \forall z\in\overline{\Omega}.$$

Then, using (3.3), we have

$$\begin{aligned}\varphi_+(tu) &= \int_{\Omega} G\big(\nabla(tu)\big)\,dz - \int_{\Omega} F(z,tu)\,dz\\ &\leqslant \frac{\varepsilon t^{\tau}}{\tau}\|\nabla u\|_{\tau}^{\tau} - \frac{\beta_1 t^{\tau}}{\tau}\|u\|_{\tau}^{\tau}\\ &= \frac{t^{\tau}}{\tau}\big(\varepsilon\|u\|^{\tau} - \beta_1\|u\|_{\tau}^{\tau}\big).\end{aligned}$$

Choosing $\varepsilon\in\big(0,\frac{\beta_1\|u\|_{\tau}^{\tau}}{\|u\|^{\tau}}\big)$, we see that

$$\varphi_+(tu) \;<\; 0,$$

so

$$\varphi_+(u_0) = m_+ < 0 = \varphi_+(0)$$

(see (3.2)), i.e., $u_0 \neq 0$.

From (3.2), we have

$$0 \in \partial\varphi_+(u_0),$$

so

$$V(u_0) = u_0^*, \tag{3.4}$$

where $u_0^* \in L^{p'}(\Omega)$, $u_0^* \in \partial F_+\big(z, u_0(z)\big)$ for almost all $z \in \Omega$.

From the nonsmooth chain rule (see Clarke [9, p. 42]), we have

$$\partial F_+(z,\zeta) \subseteq \begin{cases} 0 & \text{if } \zeta < 0, \\ \big\{\xi\partial F(z,0) : \xi \in [0,1]\big\} & \text{if } 0 \leqslant \zeta \leqslant 1, \\ \partial F(z,\zeta) & \text{if } 1 < \zeta. \end{cases} \tag{3.5}$$

On (3.3) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and using (3.5) and Lemma 2.4, we obtain

$$\frac{c_0}{p-1}\|\nabla u_0^-\|_p^p \leqslant 0,$$

i.e., $u_0 \geqslant 0$, $u_0 \neq 0$.

From (3.4), we have

$$\begin{cases} -\operatorname{div} a\big(\nabla u_0(z)\big) = u_0^*(z) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

Since

$$\nabla u_0(z) = 0 \quad \text{on } \{u_0 = 0\}$$

(Stampacchia theorem; see e.g., Gasiński-Papageorgiou [19, p. 195]), we infer that

$$u_0^*(z) \in \partial F(z, u_0(z)) \quad \text{for almost all } z \in \Omega$$

(see (3.5)). So, u_0 is a nontrivial positive solution of problem (1.1). Moreover, as before (see the proof of Proposition 2.6), from the nonlinear regularity theory (see Ladyzhenskaya-Uraltseva [25] and Lieberman [26]), we have that $u_0 \in C_+ \setminus \{0\}$. Let $\varrho = \|u_0\|_\infty$ and let $\gamma_\varrho > 0$ be as postulated by hypothesis $H_1(iv)$. Then

$$-\operatorname{div} a\big(\nabla u_0(z)\big) + \gamma_\varrho u_0(z)^{p-1} = u_0^*(z) + \gamma_\varrho u_0(z)^{p-1} \geqslant 0 \quad \text{for almost all } z \in \Omega,$$

so

$$\operatorname{div} a\big(\nabla u_0(z)\big) \leqslant \gamma_\varrho u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega$$

and thus $u_0 \in \operatorname{int} C_+$ (see Montenegro [30, Theorem 6]).

If

$$W_+ = \big\{u \in W_0^{1,p}(\Omega) : u(z) \geqslant 0 \text{ for almost all } z \in \Omega\big\},$$

then clearly

$$\varphi|_{W_+} = \varphi_+|_{W_+}.$$

So, u_0 is a local $C_0^1(\overline{\Omega})$-minimizer of φ. Invoking Proposition 2.6, we infer that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of φ.

Similarly, working with the functional φ_-, we produce one more constant sign smooth solution $v_0 \in -\operatorname{int} C_+$ of problem (1.1), which is a local minimizer of the functional φ.

Without any loss of generality, we may assume that

$$\varphi(v_0) \leqslant \varphi(u_0)$$

(the analysis is similar, if the opposite inequality is true) and that the set K_φ is finite (otherwise, we already have infinity solutions for problem (1.1)). Reasoning as in Aizicovici-Papageorgiou-Staicu [1, proof of Proposition 29] (see also Gasiński-Papageorgiou [20, Theorem 3.4]), we can find $\varrho \in (0, 1)$ small, such that

$$\varphi(v_0) \leqslant \varphi(u_0) < \inf\big\{\varphi(u) : \|u - u_0\| = \varrho\big\} = \eta_\varrho, \quad \|v_0 - u_0\| > \varrho. \tag{3.6}$$

As we did for φ_+, in a similar way, using hypothesis $H_1(ii)$, we can check that φ is coercive and so it satisfies the Palais-Smale condition. This fact, together with (3.6) permit the use of the nonsmooth mountain pass theorem (see Theorem 2.1). So, we can find $y_0 \in W_0^{1,p}(\Omega)$, such that

$$\varphi(v_0) \leqslant \varphi(u_0) < \eta_\varrho \leqslant \varphi(y_0) \tag{3.7}$$

and

$$0 \in \partial\varphi(y_0). \tag{3.8}$$

From (3.7) it is clear that $y_0 \notin \{v_0, u_0\}$, while from (3.8), we have

$$V(y_0) = \widehat{u}_0^*,$$

where $\widehat{u}_0^* \in L^{p'}(\Omega)$, $\widehat{u}_0^*(z) \in \partial F(z, y_0(z))$ for almost all $z \in \Omega$. Hence y_0 is a solution of (1.1) and the nonlinear regularity theory implies that $y_0 \in C_0^1(\overline{\Omega})$. It remains to show that $y_0 \neq 0$. From Theorem 2.1, we have

$$c = \varphi(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} \varphi\big(\gamma(t)\big), \tag{3.9}$$

where

$$\Gamma = \big\{\gamma \in C\big([0, 1]; W_0^{1,p}(\Omega)\big) : \gamma(0) = v_0, \ \gamma(1) = u_0\big\}.$$

From (3.9), we see that, if we can find $\gamma_* \in \Gamma$, such that

$$\varphi\big(\gamma_*(t)\big) < 0 \quad \forall t \in T,$$

then

$$c = \varphi(y_0) < 0 = \varphi(0)$$

and so $y_0 \neq 0$. Hence our effort is on producing such a path $\gamma_* \in \Gamma$.

To this end, let

$$\Gamma_c = \big\{\gamma \in C\big([0, 1]; C_0^1(\overline{\Omega})\big) : \gamma(0) = v_0, \ \gamma(1) = u_0\big\}.$$

By virtue of the density of the embedding of $C_0^1(\overline{\Omega})$ into $W_0^{1,p}(\Omega)$, we see that Γ_c is dense in Γ. We can find $\widehat{\gamma} \in \Gamma_c$, such that $0 \notin \widehat{\gamma}([0,1])$. Since

$$\widehat{\gamma}([0,1]) \subseteq C_0^1(\overline{\Omega}) \quad \text{and} \quad 0 \notin \widehat{\gamma}([0,1]),$$

we can find $\lambda \in (0,1)$ small, such that

$$\lambda \|\nabla u(z)\| \leqslant \delta, \quad \text{and} \quad \lambda |u(z)| \leqslant \delta \quad \forall z \in \overline{\Omega},\ u \in \widehat{\gamma}([0,1]) \tag{3.10}$$

(where δ is as in (3.3)) and

$$\inf_{u \in \widehat{\gamma}([0,1])} \|u\|_\tau^\tau = \widehat{m} > 0. \tag{3.11}$$

For all $u \in \widehat{\gamma}([0,1])$, we have

$$\begin{aligned} \varphi(\lambda u) &= \int_\Omega G(\lambda \nabla u)\, dz - \int_\Omega F(z, \lambda u)\, dz \\ &\leqslant \lambda^\tau \varepsilon \|\nabla u\|_\tau^\tau - \beta_1 \lambda^\tau \|u\|_\tau^\tau \\ &\leqslant \lambda^\tau (\varepsilon c_5 - \beta_1 \widehat{m}) \end{aligned} \tag{3.12}$$

for some $c_5 > 0$ (see (3.3), (3.10), (3.11) and note that $\widehat{\gamma}([0,1])$ is compact in $W_0^{1,p}(\Omega)$). Choosing $\varepsilon \in (0, \frac{\beta_1 \widehat{m}}{c_5})$ and setting $\widetilde{\gamma} = \lambda \widehat{\gamma}$, from (3.12), we see that

$$\varphi|_{\widetilde{\gamma}} < 0 \tag{3.13}$$

and $\widetilde{\gamma}$ is a continuous path in $W_0^{1,p}(\Omega)$ which connects λv_0 and λu_0.

Next, we will produce a continuous path in $W_0^{1,p}(\Omega)$ which connects λu_0 and u_0 and along which φ is strictly negative. To this end, recall that

$$m_+ = \inf_{u \in W_0^{1,p}(\Omega)} \varphi_+(u) < 0 = \varphi_+(0).$$

Also, we may assume that $K_{\varphi_+}^{m_+} = \{u_0\}$ or otherwise we already have a second positive solution (note that by virtue of (3.5) and the nonlinear regularity theory, $K_{\varphi_+} \subseteq C_+$). Invoking Theorem 2.2, we can find a continuous deformation $h\colon [0,1] \times \dot{\varphi}_+^0 \longrightarrow \dot{\varphi}_+^0$, such that

$$h(1, \dot{\varphi}_+^0) \subseteq \dot{\varphi}_+^{m_+} \cup K_{\varphi_+}^{m_+} = \dot{\varphi}_+^{m_+} \cup \{u_0\} = \{u_0\} \tag{3.14}$$

(since $\dot{\varphi}_+^{m_+} = \emptyset$) and

$$\varphi_+(h(t,u)) \leqslant \varphi_+(u) \quad \forall t \in [0,1],\ u \in \dot{\varphi}_+^0. \tag{3.15}$$

Consider the continuous path $\gamma_+\colon [0,1] \longrightarrow W_0^{1,p}(\Omega)$, defined by

$$\gamma_+(t) = h(t, \lambda u_0)^+ \quad \forall t \in [0,1].$$

Then

$$\gamma_+(0) = h(0, \lambda u_0)^+ = (\lambda u_0)^+ = \lambda u_0.$$

Also

$$\gamma_+(1) = h(1, \lambda u_0)^+ = u_0$$

(see (3.14)). Hence γ_+ is a continuous path in $W_0^{1,p}(\Omega)$ which connects λu_0 and u_0. Moreover, from (3.15) and since $\varphi|_{W_+} = \varphi_+|_{W_+}$, we have

$$\begin{aligned}\varphi\big(\gamma_+(t)\big) = \varphi\big(h(t, \lambda u_0)^+\big) &= \varphi_+\big(h(t, \lambda u_0)^+\big) \\ \leqslant \varphi_+(\lambda u_0) &= \varphi(\lambda u_0) < 0 \qquad \forall t \in [0, 1]\end{aligned}$$

(see (3.13)), so

$$\varphi|_{\gamma_+} < 0. \tag{3.16}$$

In a similar fashion, we produce a continuous path γ_- in $W_0^{1,p}(\Omega)$ which connects λv_0 and v_0 and

$$\varphi|_{\gamma_-} < 0. \tag{3.17}$$

We concatenate γ_-, $\widetilde{\gamma}$ and γ_+ and produce $\gamma_* \in \Gamma$, such that

$$\varphi|_{\gamma_*} < 0$$

(see (3.13), (3.16) and (3.17)), so $y_0 \neq 0$ (see (3.9)).

So $y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$ is the third nontrivial smooth solution of (1.1). □

4 Second Multiplicity Theorem. Nodal Solutions

In this section, we look for nodal solutions. To the best of our knowledge, there has been no previous work producing nodal solutions for equations driven by a nonhomogeneous differential operator. To do this, we need to strengthen the hypotheses on the nonsmooth potential $F(z, \zeta)$. For this purpose, let us first introduce some notation. Consider a measurable function $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, such that for every $r > 0$, we can find $a_r \in L^\infty(\Omega)_+$ for which we have

$$\big|f(z, \zeta)\big| \leqslant a_r(z) \quad \text{for almost all } z \in \Omega \text{ and all } |\zeta| \leqslant r.$$

We allow $f(z, \cdot)$ to have jump discontinuities and define

$$f_l(z, \zeta) = \liminf_{\zeta' \to \zeta} f(z, \zeta') \quad \text{and} \quad f_u(z, \zeta) = \limsup_{\zeta' \to \zeta} f(z, \zeta'). \tag{4.1}$$

For almost all $z \in \Omega$, these limits are finite. We assume that both f_l and f_u are superpositionally measurable. This means that, if $u\colon \Omega \longrightarrow \mathbb{R}$ is a measurable function, then so are functions $z \longmapsto f_l\big(z, u(z)\big)$ and $z \longmapsto f_u\big(z, u(z)\big)$.

We set

$$F(z, \zeta) = \int_0^\zeta f(z, s)\, ds.$$

Then for almost all $z \in \Omega$, the function $F(z, \cdot)$ is locally Lipschitz and

$$\partial F(z, \zeta) = \big[f_l(z, \zeta), f_u(z, \zeta)\big]$$

(see e.g., Chang [6] or Clarke [9]). We have $F(z, 0) = 0$ for almost all $z \in \Omega$.

So, now we can state precisely the new stronger conditions on the nonsmooth potential.

$\underline{H_2}$: $F\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that

$$F(z,\zeta) = \int_0^{\zeta} f(z,s)\,ds,$$

where $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function satisfying:

(i) there exist $a \in L^{\infty}(\Omega)_+$ and $c > 0$, such that

$$\left|f(z,\zeta)\right| \leqslant a(z) + c|\zeta|^{p-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R};$$

the functions $f_l(z,\zeta)$ and $f_u(z,\zeta)$ defined by (4.1) are superpositionally measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta)$ is continuous at $\zeta = 0$;

(ii) there exists a function $\vartheta \in L^{\infty}(\Omega)_+$, $\vartheta(z) \leqslant \frac{c_0}{p-1}$ for almost all $z \in \Omega$, $\vartheta \neq \frac{c_0}{p-1}$, such that

$$\limsup_{\zeta \to \pm\infty} \frac{pF(z,\zeta)}{|\zeta|^p} \leqslant \vartheta(z) \quad \text{uniformly for almost all } z \in \Omega;$$

(iii) there exist $\tau \in (1,p)$ and $c_6 > 0$, such that

$$f_l(z,\zeta)\zeta \geqslant c_6|\zeta|^{\tau} \quad \text{for almost all } z \in \Omega \text{ and all } \zeta \in \mathbb{R}.$$

Remark 4.1 Note that the above hypotheses imply that

$$\partial F(z,0) = \{0\} \quad \text{for almost all } z \in \Omega.$$

Also note that

$$u^*\zeta \geqslant 0 \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R} \text{ and all } u^* \in \partial F(z,\zeta)$$

(sign condition).

Example 4.2 The following potential function F satisfies hypotheses H_2:

$$F(\zeta) = \frac{\vartheta}{p}|\zeta|^p + \max\left\{\frac{1}{\tau}|\zeta|^{\tau}, \frac{1}{q}|\zeta|^q\right\} + c|\zeta|,$$

where $\vartheta < \frac{c_0}{p-1}$, $\tau, q \in (0,p)$ and $c \geqslant 0$. If $\tau \neq q$ and $c > 0$, then F is not a C^1-function.

First we show that problem (1.1) has extremal constant sign smooth solutions, i.e., there exists a smallest nontrivial positive solution and a biggest nontrivial negative solution.

$\underline{H_0''}$: Hypotheses H_0' hold and there exists $q \in (\tau, p)$, such that the map $t \mapsto G_0\left(t^{\frac{1}{q}}\right)$ is convex, with $G_0(t) = \int_0^t a_0(s)s\,ds$.

Proposition 4.3 *If hypotheses H_0' and H_2 hold, then problem (1.1) has a smallest nontrivial positive solution $u_+ \in \operatorname{int} C_+$ and a biggest nontrivial negative solution $v_- \in -\operatorname{int} C_+$.*

Proof We consider the following auxiliary Dirichlet problem:

$$\begin{cases} -\operatorname{div} a\big(\nabla u(z)\big) = c_6\big|u(z)\big|^{\tau-2}u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4.2}$$

Claim Problem (4.2) has a unique nontrivial positive solution $\underline{u} \in \operatorname{int} C_+$ and a unique nontrivial negative solution $\overline{v} \in -\operatorname{int} C_+$.

Let $\psi_+ : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional, defined by

$$\psi_+(u) = \int_\Omega G\big(\nabla u(z)\big)\,dz - \frac{c_6}{\tau}\|u^+\|_\tau^\tau \quad \forall u \in W_0^{1,p}(\Omega).$$

Using Corollary 2.5, we have

$$\psi_+(u) \geqslant \frac{c_0}{p(p-1)}\|u\|^p - c_7\|u\|^\tau \quad \forall u \in W_0^{1,p}(\Omega), \tag{4.3}$$

for some $c_7 > 0$.

Since $\tau < p$, from (4.3) we infer that ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\underline{u} \in W_0^{1,p}(\Omega)$, such that

$$\psi_+(\underline{u}) = \inf_{u \in W_0^{1,p}(\Omega)} \psi_+(u) = \overline{m}_+. \tag{4.4}$$

As in the proof of Theorem 3.4, using hypotheses $H_0^1(iv)$ and $H_2(iii)$ and since $\tau < p$, we obtain

$$\psi_+(\underline{u}) = \overline{m}_+ < 0 = \psi_+(0)$$

(see (4.4)), so

$$\underline{u} \neq 0.$$

From (4.4), we have

$$\psi_+'(\underline{u}) = 0,$$

thus

$$V(\underline{u}) = c_6(\underline{u}^+)^{\tau-1}. \tag{4.5}$$

Acting on (4.5) with $-\underline{u}^- \in W_0^{1,p}(\Omega)$ and using Lemma 2.4, we obtain $\underline{u} \geqslant 0$, $\underline{u} \neq 0$. So, from (4.5), we have that $\underline{u}$ solves (4.2) and in fact the nonlinear regularity theory (see Ladyzhenskaya-Uraltseva [25] and Lieberman [26]) and the nonlinear maximum principle of Montenegro [30, Theorem 6], imply that $\underline{u} \in \operatorname{int} C_+$.

To show the uniqueness of the solution $\underline{u} \in \operatorname{int} C_+$, inspired by Diaz-Saa [11], we introduce the integral functional $\xi : W_0^{1,p}(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, defined by

$$\xi(u) = \begin{cases} \int_\Omega G(\nabla u^{\frac{1}{q}})\,dz & \text{if } u \geqslant 0,\ u^{\frac{1}{q}} \in W_0^{1,q}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly ξ is convex, lower semicontinuous and it is not identical $+\infty$.

Let $u \in W_0^{1,p}(\Omega)$ be a nontrivial positive solution of the auxiliary problem (4.2). The nonlinear regularity theory (see [25, 26]) and the nonlinear maximum principle

(see [30]) imply that $u \in \operatorname{int} C_+$. Note that $u^q \geqslant 0$ and $(u^q)^{\frac{1}{q}} \in W_0^{1,q}(\Omega)$. So, u^q is in the effective domain of the $\overline{\mathbb{R}}$-valued functional ξ. For $h \in C_0^1(\overline{\Omega})$ and $t > 0$ small, we have $u^q + th \in C_+$ and so the directional derivative of ξ at u^q in the direction h exists. Moreover, using the chain rule, we have

$$\xi'(u^q)(h) = \int_\Omega \frac{(G \circ D)'(u)}{u^{q-1}} h\, dz = \int_\Omega \frac{-\operatorname{div} a(\nabla u)}{u^{q-1}} h\, dz. \tag{4.6}$$

Let $\underline{w}$ be any nontrivial positive solution of (4.2). As above, we have that $\underline{w} \in \operatorname{int} C_+$. By virtue of the convexity of ξ and (4.6), we have

$$0 \leqslant \int_\Omega \left(-\frac{\operatorname{div} a(\nabla \underline{u})}{\underline{u}^{q-1}} + \frac{\operatorname{div} a(\nabla \underline{w})}{\underline{w}^{q-1}} \right) (\underline{u}^q - \underline{w}^q)\, dz,$$

so

$$0 \leqslant c_6 \int_\Omega \left(\frac{1}{\underline{u}^{q-\tau}} - \frac{1}{\underline{w}^{q-\tau}} \right) (\underline{u}^q - \underline{w}^q)\, dz, \leqslant 0$$

(see (4.2) and recall that $\tau < q$) and thus

$$\underline{u} = \underline{w}.$$

Similarly, we establish the uniqueness of the nontrivial negative solution $\overline{v} \in -\operatorname{int} C_+$. This proves the Claim.

Now, let u be a nontrivial positive solution for problem (1.1). Such a solution exists by virtue of Theorem 3.4 and

$$-\operatorname{div} a\big(\nabla u(z)\big) = u^*(z) \quad \text{for almost all } z \in \Omega, \tag{4.7}$$

where $u^* \in L^{p'}(\Omega)$, $u^*(z) \in \partial F\big(z, u(z)\big)$ for almost all $z \in \Omega$. It follows that $u \in \operatorname{int} C_+$ (see [25, 26]). Let

$$h_+(z, \zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ c_6 \zeta^{\tau-1} & \text{if } 0 \leqslant \zeta \leqslant u(z), \\ c_6 u(z)^{\tau-1} & \text{if } u(z) < \zeta. \end{cases} \tag{4.8}$$

This is a Carathéodory function (i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto h_+(z, \zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto h_+(z, \zeta)$ is continuous). We set

$$H_+(z, \zeta) = \int_0^\zeta h_+(z, s)\, ds$$

and consider the C^1-functional $\widehat{\psi}_+ : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\psi}_+(u) = \int_\Omega G\big(\nabla u(z)\big)\, dz - \int_\Omega H_+\big(z, u(z)\big)\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

It is clear from (4.8) and Corollary 2.5 that $\widehat{\psi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{\underline{u}} \in W_0^{1,p}(\Omega)$, such that

$$\widehat{\psi}_+(\widehat{\underline{u}}) = \inf_{u \in W_0^{1,p}(\Omega)} \widehat{\psi}_+(u) = \widehat{m}_+. \tag{4.9}$$

As before, since $\tau < p$, we have that

$$\widehat{\psi}_+(\underline{\widehat{u}}) = \widehat{m}_+ < 0 = \widehat{\psi}_+(0)$$

(see (4.9)), i.e., $\underline{\widehat{u}} \neq 0$.

From (4.9), we have

$$\widehat{\psi}'_+(\underline{\widehat{u}}) = 0,$$

so

$$V(\underline{\widehat{u}}) = N_{h_+}(\underline{\widehat{u}}), \tag{4.10}$$

where

$$N_{h_+}(u)(\cdot) = h_+\big(\cdot, u(\cdot)\big) \quad \forall u \in W_0^{1,p}(\Omega).$$

Acting on (4.10) with $-\underline{u}^- \in W_0^{1,p}(\Omega)$, we obtain $\underline{\widehat{u}} \geqslant 0$, $\underline{\widehat{u}} \neq 0$. On (4.10) we act also with $(\underline{\widehat{u}} - u)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\big\langle V(\underline{\widehat{u}}),\ (\underline{\widehat{u}} - u)^+\big\rangle &= \int_\Omega h_+(z, \underline{\widehat{u}})(\underline{\widehat{u}} - u)^+ dz \\ &= \int_\Omega c_6 u(z)^{\tau-1}(\underline{\widehat{u}} - u)^+ dz \\ &\leqslant \int_\Omega u^*(\underline{\widehat{u}} - u)^+ dz \\ &= \big\langle V(u),\ (\underline{\widehat{u}} - u)^+\big\rangle\end{aligned}$$

(see (4.7) and hypothesis $H_2(iii)$), so

$$\int_{\{\underline{\widehat{u}} > u\}} \big(a(\nabla \underline{\widehat{u}}) - a(\nabla u),\ \nabla \underline{\widehat{u}} - \nabla u\big)_{\mathbb{R}^N} dz \leqslant 0,$$

thus

$$\big|\{\underline{\widehat{u}} > u\}\big|_N = 0$$

(see Lemma 2.4) and we obtain

$$\underline{\widehat{u}} \leqslant u.$$

So

$$\underline{\widehat{u}} \in [0, u], \quad \underline{\widehat{u}} \neq 0,$$

where

$$[0, u] = \big\{w \in W_0^{1,p}(\Omega) :\ 0 \leqslant w(z) \leqslant u(z) \text{ for almost all } z \in \Omega\big\}.$$

Then (4.10) becomes

$$V(\underline{\widehat{u}}) = c_6 \underline{\widehat{u}}^{\tau-1}$$

(see (4.8)) and so

$$\underline{\widehat{u}} = \underline{u} \in \operatorname{int} C_+$$

(see the Claim).

The above argument shows that

$$\text{every nontrivial positive solution } u \in W_0^{1,p}(\Omega) \text{ of (1.1) satisfies } \underline{u} \leqslant u. \tag{4.11}$$

A similar argument, using this time $\overline{v} \in -\text{int}\, C_+$ (see the Claim), shows that

$$\text{every nontrivial negative solution } v \in W_0^{1,p}(\Omega) \text{ of (1.1) satisfies } v \leqslant \overline{v}. \tag{4.12}$$

Now we are ready to establish the existence of extremal nontrivial constant sign solutions of (1.1). So, let S_+ be the set of nontrivial positive solutions of (1.1). From Theorem 3.4 we know that $S_+ \neq \emptyset$. Let $C \subseteq S_+$ be a chain (i.e., a total ordered subset of S_+). From Dunford-Schwartz [12, p. 336], we know that there exists a sequence $\{u_n\}_{n \geqslant 1} \subseteq C$, such that

$$\inf_{n \geqslant 1} u_n = \inf C.$$

Moreover, we can have the sequence $\{u_n\}_{n \geqslant 1}$ decreasing (see Heikkilä-Lakshmikantham [22, Lemma 1.1.5; p. 15]). We have

$$V(u_n) = u_n^* \quad \forall n \geqslant 1, \tag{4.13}$$

where $u_n^* \in L^{p'}(\Omega)$ and $u_n^*(z) \in \partial F\big(z, u_n(z)\big)$ for almost all $z \in \Omega$. Evidently the sequence $\{u_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded (see (4.13) and recall that $u_n \leqslant u_1$ for all $n \geqslant 1$). So, we may assume that

$$u_n \longrightarrow \widetilde{u} \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{4.14}$$

$$u_n \longrightarrow \widetilde{u} \quad \text{in } L^p(\Omega). \tag{4.15}$$

On (4.13) we act with $u_n - \widetilde{u}$, pass to the limit as $n \to +\infty$ and use (4.14). Then

$$\lim_{n \to +\infty} \big\langle V(u_n),\ u_n - \widetilde{u} \big\rangle = 0$$

and so

$$u_n \longrightarrow \widetilde{u} \quad \text{in } W_0^{1,p}(\Omega) \tag{4.16}$$

(see Proposition 2.7). By virtue of hypothesis $H_2(i)$, the sequence $\{u_n^*\}_{n \geqslant 1} \subseteq L^{p'}(\Omega)$ is bounded and so, passing to a subsequence if necessary, we may assume that

$$u_n^* \longrightarrow \widetilde{u}^* \quad \text{weakly in } L^{p'}(\Omega).$$

Invoking Proposition 3.9 of Hu-Papageorgiou [23, p. 694], we have

$$\widetilde{u}^*(z) \subseteq \text{conv} \limsup_{n \to +\infty} \partial F\big(z, u_n(z)\big) \subseteq \partial F\big(z, \widetilde{u}(z)\big) \quad \text{for almost all } z \in \Omega.$$

So, if in (4.13) we pass to the limit as $n \to +\infty$ and use (4.16), we have

$$V(\widetilde{u}) = \widetilde{u}^*,$$

where $\widetilde{u}^* \in L^{p'}(\Omega)$, $\widetilde{u}^*(z) \in \partial F\big(z, \widetilde{u}(z)\big)$ for almost all $z \in \Omega$ and

$$\widetilde{u} \geqslant \underline{u}$$

(see (4.11)).

Therefore, we infer that $\widetilde{u} \in S_+$ and $\widetilde{u} \in \inf C$. Since C is an arbitrary chain, from the Kuratowski-Zorn lemma, we know that S_+ has a minimal element $u_+ \geqslant \underline{u}$, $u_+ \in \operatorname{int} C_+$ (from the nonlinear regularity theory). Using Lemma 3.2 of Gasiński-Papageorgiou [20] (the lemma remains valid if the p-Laplacian is replaced by the more general differential operator V, since all we need is the monotonicity of V), we have that S_+ is downward directed (i.e., if $u, w \in S_+$, then there exists $y \in S_+$, such that $y \leqslant \min\{u, w\}$). Therefore $u_+ \in \operatorname{int} C_+$ is the smallest positive solution of (1.1).

Similarly, we produce the biggest nontrivial negative solution $v_- \in -\operatorname{int} C_+$ of problem (1.1) with $v_- \leqslant \overline{v}$ (see (4.12)) using this time Lemma 3.3 of Gasiński-Papageorgiou [20]. □

Having these extremal solutions we can produce a nodal solution and have the second multiplicity theorem, which provides precise sign information for all three solutions.

Theorem 4.4 *If hypotheses H_0'' and H_2 hold, then problem (1.1) has at least three nontrivial smooth solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+, \quad \text{and} \quad y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\} \textit{ nodal.}$$

Proof From Theorem 3.4 we already have two constant sign solutions

$$u_0 \in \operatorname{int} C_+ \quad \text{and} \quad v_0 \in -\operatorname{int} C_+.$$

Let $u_+ \in \operatorname{int} C_+$ and $v_- \in -\operatorname{int} C_+$ be the two extremal constant sign solutions of problem (1.1) postulated in Proposition 4.3. We have

$$-\operatorname{div}\big(\nabla u_+(z)\big) = u_+^*(z) \quad \text{for almost all } z \in \Omega,$$

where $u_+^* \in L^{p'}(\Omega)$, $u_+^*(z) \in \partial F\big(z, u_+(z)\big)$ for almost all $z \in \Omega$, and

$$-\operatorname{div}\big(\nabla v_-(z)\big) = v_-^*(z) \quad \text{for almost all } z \in \Omega,$$

where $v_-^* \in L^{p'}(\Omega)$, $v_-^*(z) \in \partial F\big(z, v_-(z)\big)$ for almost all $z \in \Omega$.

We consider the following truncation of $f(z, \cdot)$:

$$\widehat{f}(z, \zeta) = \begin{cases} v_-^*(z) & \text{if } \zeta < v_-(z), \\ f(z, \zeta) & \text{if } v_-(z) \leqslant \zeta \leqslant u_+(z), \\ u_+^*(z) & \text{if } u_+(z) < \zeta. \end{cases} \tag{4.17}$$

Also, let

$$\widehat{f}_\pm(z, \zeta) = \widehat{f}(z, \pm\zeta^\pm).$$

We set

$$\widehat{F}(z, \zeta) = \int_0^\zeta \widehat{f}(z, s)\, ds, \quad \widehat{F}_\pm(z, \zeta) = \int_0^\zeta \widehat{f}_\pm(z, s)\, ds$$

and consider the locally Lipschitz functionals $\widehat{\varphi}, \widehat{\varphi}_{\pm} : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\varphi}(u) = \int_\Omega G\big(\nabla u(z)\big)\,dz - \int_\Omega \widehat{F}\big(z, u(z)\big)\,dz \quad \forall u \in W_0^{1,p}(\Omega),$$

$$\widehat{\varphi}_{\pm}(u) = \int_\Omega G\big(\nabla u(z)\big)\,dz - \int_\Omega \widehat{F}_{\pm}\big(z, u(z)\big)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

As in the proof of Proposition 4.3, we show that

$$K_{\widehat{\varphi}} \subseteq [v_-, u_+], \quad K_{\widehat{\varphi}_+} = \{0, u_+\}, \quad K_{\widehat{\varphi}_-} = \{v_-, 0\}. \tag{4.18}$$

Claim u_+ and v_- are local minimizers of the functional $\widehat{\varphi}$.

Evidently $\widehat{\varphi}_+$ is coercive (see (4.17)) and sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_+ \in W_0^{1,p}(\Omega)$, such that

$$\widehat{\varphi}_+(\widehat{u}_+) = \inf_{u \in W_0^{1,p}(\Omega)} \widehat{\varphi}_+(u) = \widehat{m}_+. \tag{4.19}$$

From hypothesis $H_2(iii)$ and since $\tau < p$, we have

$$\widehat{\varphi}_+(\widehat{u}_+) = \widehat{m}_+ < 0 = \widehat{\varphi}_+(0)$$

(see (4.19)), i.e., $\widehat{u}_+ \neq 0$, so

$$\widehat{u}_+ = u_+$$

(see (4.18) and (4.19)).

But $u_+ \in \operatorname{int} C_+$ and $\widehat{\varphi}|_{W_+} = \widehat{\varphi}_+|_{W_+}$. Hence u_+ is a local $C_0^1(\overline{\Omega})$-minimizer of $\widehat{\varphi}$, thus by virtue of Proposition 2.6, it is also a local $W_0^{1,p}(\Omega)$-minimizer of $\widehat{\varphi}$. Similarly, for $v_- \in -\operatorname{int} C_+$, using this time the functional $\widehat{\varphi}_-$. This proves the Claim.

As before (see the proof of Theorem 3.4), without any loss of generality, we may assume that $\widehat{\varphi}(v_-) \leqslant \widehat{\varphi}(u_+)$ and because of the Claim, we can find $\varrho \in (0, 1)$ small, such that

$$\widehat{\varphi}(v_-) \leqslant \widehat{\varphi}(u_+) < \inf\big\{\widehat{\varphi}(u) : \|u - u_+\| = \varrho\big\} = \widehat{\eta}_+, \quad \|v_- - u_+\| > \varrho. \tag{4.20}$$

(see Aizicovici-Papageorgiou-Staicu [1, proof of Proposition 29] or Gasiński-Papageorgiou [20, Theorem 3.4]).

Since $\widehat{\varphi}$ is coercive (see (4.17)), it satisfies the Palais-Smale condition. This fact and (4.20), permit the use of the mountain pass theorem (see Theorem 2.1). So, we can find a solution $y_0 \in C_0^1(\overline{\Omega}) \setminus \{u_+, v_-\}$ of problem (1.1) (see (4.19) and (4.18)). Moreover, as in the proof of Theorem 3.4, using Theorem 2.2, we show that $y_0 \neq 0$. Since $y_0 \in [v_-, u_+] \cap C_0^1(\overline{\Omega})$ (see (4.18) and use the nonlinear regularity theory), $y_0 \notin \{u_+, v_-\}$ and given the extremality of u_+ and v_-, we conclude that $y_0 \in C_0^1(\overline{\Omega}) \setminus \{0\}$ must be nodal. □

Remark 4.5 Compared with the results of Liu-Liu [27, Theorem 1.1] and Liu [28, Theorem 1.2], our work here is more general in many respects. In both the aforementioned works, the differential operator is the p-Laplacian (i.e., $G(y) = \frac{1}{p}\|y\|^p$), $F(z, \cdot) \in C^1$, asymptotically at $\pm\infty$ no interaction is allowed with $\widehat{\lambda}_1$ and they do not provide sign information for the third solution. However, their condition on $f(z, \cdot)$ near the origin is a little more general than ours, since $f(z, \cdot)$ can be $(p-1)$-linear

near zero. Here we are forced to assume a "concave" nonlinearity near the origin in order to overcome the nonhomogeneity of the differential operator and produce extremal constant sign solutions and through them produce a nodal solution.

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