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Multiplicity of positive solutions for eigenvalue problems of $(p, 2)$ -equations

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Abstract

We consider a nonlinear parametric equation driven by the sum of a p -Laplacian ($p > 2$) and a Laplacian (a $(p, 2)$ -equation) with a Carathéodory reaction, which is strictly $(p - 2)$ -sublinear near $+\infty$. Using variational methods coupled with truncation and comparison techniques, we prove a bifurcation-type theorem for the nonlinear eigenvalue problem. So, we show that there is a critical parameter value $\lambda_* > 0$ such that for $\lambda > \lambda_*$ the problem has at least two positive solutions, if $\lambda = \lambda_*$, then the problem has at least one positive solution and for $\lambda \in (0, \lambda_*)$, it has no positive solutions.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper, we study the following nonlinear Dirichlet eigenvalue problem:

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0, \quad \lambda > 0, \quad 2 < p < +\infty. \end{cases} \quad (P)_\lambda$$

Here, by Δ_p we denote the p -Laplace differential operator defined by

$$\Delta_p u(z) = \operatorname{div}(\|\nabla u(z)\|^{p-2} \nabla u(z)) \quad \forall u \in W_0^{1,p}(\Omega)$$

(with $2 < p < +\infty$). In $(P)_\lambda$, $\lambda > 0$ is a parameter and $f(z, \zeta)$ is a Carathéodory function (*i.e.*, for all $\zeta \in \mathbb{R}$, the function $z \mapsto f(z, \zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \mapsto f(z, \zeta)$ is continuous), which exhibits strictly $(p - 2)$ -sublinear growth in the ζ -variable near $+\infty$. The aim of this paper is to determine the precise dependence of the set of positive solutions on the parameter $\lambda > 0$. So, we prove a bifurcation-type theorem, which establishes the existence of a critical parameter value $\lambda_* > 0$ such that for all $\lambda > \lambda_*$, problem $(P)_\lambda$ has at least two nontrivial positive smooth solutions, for $\lambda = \lambda_*$, problem $(P)_\lambda$ has at least one nontrivial positive smooth solution and for $\lambda \in (0, \lambda_*)$, problem $(P)_\lambda$ has no positive solution. Similar nonlinear eigenvalue problems with $(p - 2)$ -sublinear reaction were studied by Maya and Shivaji [1] and Rabinowitz [2] for problems driven by the Laplacian and by Guo [3], Hu and Papageorgiou [4] and Perera [5] for problems driven

by the p -Laplacian. However, none of the aforementioned works produces the precise dependence of the set of positive solutions on the parameter $\lambda > 0$ (i.e., they do not prove a bifurcation-type theorem). We mention that in problem $(P)_\lambda$ the differential operator is not homogeneous in contrast to the case of the Laplacian and p -Laplacian. This fact is the source of difficulties in the study of problem $(P)_\lambda$ which lead to new tools and methods.

We point out that $(p, 2)$ -equations (i.e., equations in which the differential operator is the sum of a p -Laplacian and a Laplacian) are important in quantum physics in the search for solitons. We refer to the work of Benci, D'Avenia-Fortunato and Pisani [6]. More recently, there have been some existence and multiplicity results for such problems; see Cingolani and Degiovanni [7], Sun [8]. Finally, we should mention the recent papers of Marano and Papageorgiou [9, 10]. In [9] the authors deal with parametric p -Laplacian equations in which the reaction exhibits competing nonlinearities (concave-convex nonlinearity). In [10], they study a nonparametric (p, q) -equation with a reaction that has different behavior both at $\pm\infty$ and at 0 from those considered in the present paper, and so the geometry of the problem is different.

Our approach is variational based on the critical point theory, combined with suitable truncation and comparison techniques. In the next section, for the convenience of the reader, we briefly recall the main mathematical tools that we use in this paper.

2 Mathematical background

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X) . Let $\varphi \in C^1(X)$. A point $x_0 \in X$ is a *critical point* of φ if $\varphi'(x_0) = 0$. A number $c \in \mathbb{R}$ is a *critical value* of φ if there exists a critical point $x_0 \in X$ such that $\varphi(x_0) = c$.

We say that $\varphi \in C^1(X)$ satisfies the Palais-Smale condition if the following is true:

'Every sequence $\{x_n\}_{n \geq 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(x_n) \rightarrow 0 \quad \text{in } X^*,$$

admits a strongly convergent subsequence.'

This compactness-type condition is crucial in proving a deformation theorem which in turn leads to the minimax theory of certain critical values of $\varphi \in C^1(X)$ (see, e.g., Gasiński and Papageorgiou [11]). A well-written discussion of this compactness condition and its role in critical point theory can be found in Mawhin and Willem [12]. One of the minimax theorems needed in the sequel is the well-known 'mountain pass theorem'.

Theorem 2.1 *If $\varphi \in C^1(X)$ satisfies the Palais-Smale condition, $x_0, x_1 \in X$, $\|x_1 - x_0\| > r > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} = \eta_r$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t)),$$

where

$$\Gamma = \{ \gamma \in C([0, 1]; X) : \gamma(0) = x_0, \gamma(1) = x_1 \},$$

then $c \geq \eta$, and c is a critical value of φ .

In the analysis of problem $(P)_\lambda$, in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space

$$C_0^1(\overline{\Omega}) = \{ u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0 \}.$$

This is an ordered Banach space with a positive cone:

$$C_+ = \{ u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega \right\},$$

where by $n(\cdot)$ we denote the outward unit normal on $\partial\Omega$.

Let $f_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function with subcritical growth in $\xi \in \mathbb{R}$, i.e.,

$$|f_0(z, \xi)| \leq a_0(z) + c_0 |\xi|^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \xi \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$, $c_0 > 0$ and $1 < r < p^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N \end{cases}$$

(the critical Sobolev exponent).

We set

$$F_0(z, \xi) = \int_0^\xi f_0(z, s) ds$$

and consider the C^1 -functional $\psi_0 : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\psi_0(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega F_0(z, u(z)) dz \quad \forall u \in W_0^{1,p}(\Omega). \quad (2.1)$$

The next proposition is a special case of a more general result proved by Gasiński and Papageorgiou [13]. We mention that the first result of this type was proved by Brezis and Nirenberg [14].

Proposition 2.2 *If ψ_0 is defined by (2.1) and $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$ -minimizer of ψ_0 , i.e., there exists $\varrho_1 > 0$ such that*

$$\psi_0(u_0) \leq \psi_0(u_0 + h) \quad \forall h \in C_0^1(\overline{\Omega}), \|h\|_{C_0^1(\overline{\Omega})} \leq \varrho_1,$$

then $u_0 \in C_1^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$ -minimizer of ψ_0 , i.e., there exists $\varrho_2 > 0$ such that

$$\psi_0(u_0) \leq \psi_0(u_0 + h) \quad \forall h \in W_0^{1,p}(\Omega), \|h\| \leq \varrho_2.$$

Let $g, h \in L^\infty(\Omega)$. We say that $g < h$ if for all compact subsets $K \subseteq \Omega$, we can find $\varepsilon = \varepsilon(K) > 0$ such that

$$g(z) + \varepsilon \leq h(z) \quad \text{for almost all } z \in K.$$

Clearly, if $g, h \in C(\Omega)$ and $g(z) < h(z)$ for all $z \in \Omega$, then $g < h$. A slight modification of the proof of Proposition 2.6 of Arcoya and Ruiz [15] in order to accommodate the presence of the extra linear term $-\Delta u$ leads to the following strong comparison principle.

Proposition 2.3 *If $\xi \geq 0$, $g, h \in L^\infty(\Omega)$, $g < h$ and $u \in C_0^1(\overline{\Omega})$, $v \in \text{int } C_+$ are solutions of the problems*

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) + \xi |u(z)|^{p-2} u(z) = g(z) & \text{in } \Omega, \\ -\Delta_p v(z) - \Delta v(z) + \xi |v(z)|^{p-2} v(z) = h(z) & \text{in } \Omega, \end{cases}$$

then $v - u \in \text{int } C_+$.

Let $r \in (1, +\infty)$ and let $A_r: W_0^{1,r}(\Omega) \rightarrow W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$ (where $\frac{1}{r} + \frac{1}{r'} = 1$) be a nonlinear map defined by

$$\langle A_r(u), y \rangle = \int_{\Omega} \|\nabla u\|^{r-2} (\nabla u, \nabla y)_{\mathbb{R}^N} dz \quad \forall u, y \in W_0^{1,r}(\Omega). \quad (2.2)$$

The next proposition can be found in Dinca, Jebelean and Mawhin [16] and Gasiński and Papageorgiou [11].

Proposition 2.4 *If $A_r: W_0^{1,r}(\Omega) \rightarrow W^{-1,r'}(\Omega)$ (where $1 < r < +\infty$) is defined by (2.2), then A_r is continuous, strictly monotone (hence maximal monotone too), bounded and of type $(S)_+$, i.e., if $u_n \rightarrow u$ weakly in $W_0^{1,r}(\Omega)$ and*

$$\limsup_{n \rightarrow +\infty} \langle A_r(u_n), u_n - u \rangle \leq 0,$$

then $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$.

If $r = 2$, then we write $A_2 = A \in \mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))$.

In what follows, by $\widehat{\lambda}_1(p)$ we denote the first eigenvalue of the negative Dirichlet p -Laplacian $(-\Delta_p, W_0^{1,p}(\Omega))$. We know that $\widehat{\lambda}_1(p) > 0$ and it admits the following variational characterization:

$$\widehat{\lambda}_1(p) = \inf \left\{ \frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0 \right\}. \quad (2.3)$$

Finally, throughout this work, by $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$\|u\| = \|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

The notation $\|\cdot\|$ will also be used to denote the norm of \mathbb{R}^N . No confusion is possible since it will always be clear from the context which norm is used. For $\zeta \in \mathbb{R}$, we set $\zeta^\pm = \max\{\pm\zeta, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^- \quad \forall u \in W_0^{1,p}(\Omega).$$

If $h: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is superpositionally measurable (for example, a Carathéodory function), then we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \forall u \in W_0^{1,p}(\Omega).$$

By $|\cdot|_N$ we denote the Lebesgue measure on \mathbb{R}^N .

3 Positive solutions

The hypotheses on the reaction f are the following.

H: $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$,

$f(z, \zeta) \geq 0$ for almost all $z \in \Omega$ and all $\zeta \geq 0$ and

(i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)_+$ such that

$$f(z, \zeta) \leq a_\varrho(z) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in [0, \varrho];$$

(ii) $\lim_{\zeta \rightarrow +\infty} \frac{f(z, \zeta)}{\zeta^{p-1}} = 0$ uniformly for almost all $z \in \Omega$;

(iii) $\lim_{\zeta \rightarrow 0^+} \frac{f(z, \zeta)}{\zeta^{p-1}} = 0$ uniformly for almost all $z \in \Omega$;

(iv) for every $\varrho > 0$, there exists $\xi_\varrho > 0$ such that for almost all $z \in \Omega$, the map

$$\zeta \mapsto f(z, \zeta) + \xi_\varrho \zeta^{p-1} \text{ is nondecreasing on } [0, \varrho];$$

(v) if

$$F(z, \zeta) = \int_0^\zeta f(z, s) ds,$$

then there exists $c \in \mathbb{R}$ such that

$$F(z, c) > 0 \quad \text{for almost all } z \in \Omega.$$

Remark 3.1 Since we are looking for positive solutions and hypotheses H concern only the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may and will assume that $f(z, \zeta) = 0$ for almost all $z \in \Omega$ and all $\zeta \leq 0$. Hypothesis H(ii) implies that for almost all $z \in \Omega$, the map $f(z, \cdot)$ is strictly $(p-2)$ -sublinear near $+\infty$. Hypothesis H(iv) is much weaker than assuming the monotonicity of $f(z, \cdot)$ for almost all $z \in \Omega$.

Example 3.2 The following functions satisfy hypotheses H (for the sake of simplicity, we drop the z -dependence):

$$f_1(\xi) = \begin{cases} \xi^{\tau-1} & \text{if } \xi \in [0, 1], \\ \xi^{q-1} & \text{if } 1 < \xi, \end{cases}$$

$$f_2(\xi) = \begin{cases} \xi^{\tau-1} - \xi^{\eta-1} & \text{if } \xi \in [0, 1], \\ \xi^{q-1} \ln \xi & \text{if } 1 < \xi, \end{cases}$$

with $1 < q < p < \tau < \eta$. Clearly f_2 is not monotone.

Let

$$\mathcal{Y} = \{ \lambda > 0 : \text{problem } (P)_\lambda \text{ has a nontrivial positive solution} \}$$

and let $S(\lambda)$ be the set of solutions of $(P)_\lambda$. We set

$$\lambda_* = \inf \mathcal{Y}$$

(if $\mathcal{Y} = \emptyset$, then $\lambda_* = +\infty$).

Proposition 3.3 *If hypotheses H hold, then*

$$S(\lambda) \subseteq \text{int } C_+ \quad \text{and} \quad \lambda_* > 0.$$

Proof Clearly, the result is true if $\mathcal{Y} = \emptyset$. So, suppose that $\mathcal{Y} \neq \emptyset$ and let $\lambda \in \mathcal{Y}$. So, we can find $u \in S(\lambda) \cap W_0^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

From Ladyzhenskaya and Uraltseva [17, p.286], we have that $u \in L^\infty(\Omega)$. Then we can apply Theorem 1 of Lieberman [18] and have that $u \in \text{int } C_+ \setminus \{0\}$. Let $\varrho = \|u\|_\infty$ and let $\xi_\varrho > 0$ be as postulated by hypothesis H(iv). Then

$$-\Delta_p u(z) - \Delta u(z) + \xi_\varrho u(z)^{p-1} \geq 0 \quad \text{for almost all } z \in \Omega,$$

so

$$\Delta_p u(z) + \Delta u(z) \leq \xi_\varrho u(z)^{p-1} \quad \text{for almost all } z \in \Omega.$$

From the strong maximum principle of Pucci and Serrin [19, p.34], we have that

$$u(z) > 0 \quad \forall z \in \Omega.$$

So, we can apply the boundary point theorem of Pucci and Serrin [19, p.120] and have that $u \in \text{int } C_+$. Therefore, $S(\lambda) \subseteq \text{int } C_+$.

By virtue of hypotheses H(ii) and (iii), we see that we can find $c_1 > 0$ such that

$$f(z, \zeta) \leq c_1 \zeta^{p-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geq 0. \tag{3.1}$$

Let $\lambda_0 \in (0, \frac{\widehat{\lambda}_1(p)}{c_1})$ and $\vartheta \in (0, \lambda_0]$. Suppose that $\vartheta \in \mathcal{Y}$. Then from the first part of the proof, we know that we can find $u_\vartheta \in S(\vartheta) \subseteq \text{int } C_+$. We have

$$A_p(u_\vartheta) + A(u_\vartheta) = \vartheta N_f(u_\vartheta),$$

so

$$\|\nabla u_\vartheta\|_p^p \leq \int_\Omega \vartheta f(z, u_\vartheta) u_\vartheta \, dz \leq \vartheta c_1 \|u_\vartheta\|_p^p \leq \lambda_0 c_1 \|u_\vartheta\|_p^p < \widehat{\lambda}_1(p) \|u_\vartheta\|_p^p$$

(see (3.1) and recall that $\vartheta \leq \lambda_0 < \frac{\widehat{\lambda}_1(p)}{c_1}$), which contradicts (2.3). Therefore, $\lambda_* \geq \lambda_0 > 0$. \square

For $\lambda > 0$, let $\varphi_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the energy functional for problem $(P)_\lambda$ defined by

$$\varphi_\lambda(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \lambda \int_\Omega F(z, u) \, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Evidently, $\varphi_\lambda \in C(W_0^{1,p}(\Omega))$.

Proposition 3.4 *If hypotheses H hold, then $\mathcal{Y} \neq \emptyset$.*

Proof By virtue of hypotheses H(i) and (ii), for a given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z, \zeta) \leq \frac{\varepsilon}{p} \zeta^p + c_\varepsilon \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geq 0. \tag{3.2}$$

Then for $u \in W_0^{1,p}(\Omega)$ and $\lambda > 0$, we have

$$\begin{aligned} \varphi_\lambda(u) &= \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \lambda \int_\Omega F(z, u) \, dz \\ &\geq \frac{1}{p} \|\nabla u\|_p^p - \frac{\lambda \varepsilon}{p} \|u^+\|_p^p - \lambda c_\varepsilon |\Omega|_N \\ &= \frac{1}{p} \left(1 - \frac{\lambda \varepsilon}{\widehat{\lambda}_1(p)} \right) \|u\|_p^p - \lambda c_\varepsilon |\Omega|_N \end{aligned} \tag{3.3}$$

(see (3.2) and (2.3)).

Let $\varepsilon \in (0, \frac{\widehat{\lambda}_1(p)}{\lambda})$. Then from (3.3) it follows that φ_λ is coercive. Also, exploiting the compactness of the embedding $W_0^{1,p}(\Omega) \subseteq L^p(\Omega)$ (by the Sobolev embedding theorem), we see that φ_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_0) = \inf_{u \in W_0^{1,p}(\Omega)} \varphi_\lambda(u). \tag{3.4}$$

Consider the integral functional $K: L^p(\Omega) \rightarrow \mathbb{R}$ defined by

$$K(u) = \int_{\Omega} F(z, u(z)) \, dz \quad \forall u \in L^p(\Omega).$$

Hypothesis H(v) implies that $K(c) > 0$ and since $F(z, \zeta) = 0$ for almost all $z \in \Omega$, all $\zeta \leq 0$, we may assume that $c > 0$. Since $W_0^{1,p}(\Omega)$ is dense in $L^p(\Omega)$ and $c > 0$, we can find $\hat{v} \in W_0^{1,p}(\Omega)$, $\hat{v} \geq 0$, such that $K(\hat{v}) > 0$. Then for $\lambda > 0$ large, we have

$$\lambda K(\hat{v}) > \frac{1}{p} \|\nabla \hat{v}\|_p^p + \frac{1}{2} \|\nabla \hat{v}\|_2^2,$$

so

$$\varphi_{\lambda}(\hat{v}) < 0 \quad \text{for } \lambda > 0 \text{ large}$$

and thus

$$\varphi_{\lambda}(u_0) < 0 = \varphi_{\lambda}(0)$$

(see (3.4)), hence $u_0 \neq 0$. From (3.4), we have

$$\varphi'_{\lambda}(u_0) = 0,$$

so

$$A_p(u_0) + A(u_0) = \lambda N_f(u_0). \tag{3.5}$$

On (3.5), we act with $-u_0^- \in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_0^-\|_p^p + \|\nabla u_0^-\|_2^2 = 0,$$

hence $u_0 \geq 0$, $u_0 \neq 0$.

From (3.5), we have

$$\begin{cases} -\Delta_p u_0(z) - \Delta u_0(z) = \lambda f(z, u_0(z)) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0, \quad u_0 \geq 0, \quad u_0 \neq 0, \end{cases}$$

so $u_0 \in S(\lambda) \subseteq \text{int } C_+$ (see Proposition 3.3).

So, for $\lambda \geq \lambda_*$ big, we have $\lambda \in \mathcal{Y}$ and so $\mathcal{Y} \neq \emptyset$. □

Proposition 3.5 *If hypotheses H hold and $\lambda \in \mathcal{Y}$, then $[\lambda, +\infty) \subseteq \mathcal{Y}$.*

Proof Since by hypothesis $\lambda \in \mathcal{Y}$, we can find a solution $u_{\lambda} \in \text{int } C_+$ of $(P)_{\lambda}$ (see Proposition 3.3). Let $\mu > \lambda$ and consider the following truncation of the reaction in problem $(P)_{\mu}$:

$$h_{\mu}(z, \zeta) = \begin{cases} \mu f(z, u_{\lambda}(z)) & \text{if } \zeta \leq u_{\lambda}(z), \\ \mu f(z, \zeta) & \text{if } u_{\lambda}(z) < \zeta. \end{cases} \tag{3.6}$$

This is a Carathéodory function. Let

$$H_\mu(z, \zeta) = \int_0^\zeta h_\mu(z, s) ds$$

and consider the C^1 -functional $\psi_\mu : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$, defined by

$$\psi_\mu(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega H_\mu(z, u(z)) dz \quad \forall u \in W_0^{1,p}(\Omega).$$

As in the proof of Proposition 3.4, using hypotheses H(i) and (ii), we see that ψ_μ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\psi_\mu(u_\mu) = \inf_{u \in W_0^{1,p}(\Omega)} \psi_\mu(u),$$

so

$$\psi'_\mu(u_\mu) = 0$$

and thus

$$A_p(u_\mu) + A(u_\mu) = N_{h_\mu}(u_\mu). \tag{3.7}$$

On (3.7) we act with $(u_\lambda - u_\mu)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A_p(u_\mu), (u_\lambda - u_\mu)^+ \rangle + \langle A(u_\mu), (u_\lambda - u_\mu)^+ \rangle \\ &= \int_\Omega h_\mu(z, u_\mu)(u_\lambda - u_\mu)^+ dz \\ &= \int_\Omega \mu f(z, u_\lambda)(u_\lambda - u_\mu)^+ dz \\ &\geq \int_\Omega \lambda f(z, u_\lambda)(u_\lambda - u_\mu)^+ dz \\ &= \langle A_p(u_\lambda), (u_\lambda - u_\mu)^+ \rangle + \langle A(u_\lambda), (u_\lambda - u_\mu)^+ \rangle \end{aligned}$$

(see (3.6) and use the facts that $\mu > \lambda$ and $f \geq 0$), so

$$\int_{\{u_\lambda > u_\mu\}} (\|\nabla u_\lambda\|^{p-2} \nabla u_\lambda - \|\nabla u_\mu\|^{p-2} \nabla u_\mu, \nabla u_\lambda - \nabla u_\mu)_{\mathbb{R}^N} dz + \|\nabla(u_\lambda - u_\mu)^+\|_2^2 \leq 0,$$

thus

$$|\{u_\lambda > u_\mu\}|_N = 0$$

and hence $u_\lambda \leq u_\mu$.

Therefore, (3.7) becomes

$$A_p(u_\mu) + A(u_\mu) = \mu N_f(u_\mu),$$

so

$$u_\mu \in S(\mu) \subseteq \text{int } C_+,$$

hence $\mu \in \mathcal{Y}$. This proves that $[\lambda, +\infty) \subseteq \mathcal{Y}$. □

Proposition 3.6 *If hypotheses H hold, then for every $\lambda > \lambda_*$ problem $(P)_\lambda$ has at least two positive solutions*

$$u_0, \hat{u} \in \text{int } C_+, \quad u_0 \neq \hat{u}.$$

Proof Note that Proposition 3.5 implies that $(\lambda_*, +\infty) \subseteq \mathcal{Y}$. Let $\lambda_* < \vartheta < \lambda < \mu$. Then we can find $u_\vartheta \in S(\vartheta) \subseteq \text{int } C_+$ and $u_\mu \in S(\mu) \subseteq \text{int } C_+$. We have

$$-\Delta_p u_\vartheta - \Delta u_\vartheta = \vartheta f(z, u_\vartheta) \leq \lambda f(z, u_\vartheta) \quad \text{in } \Omega, \tag{3.8}$$

$$-\Delta_p u_\mu - \Delta u_\mu = \mu f(z, u_\mu) \geq \lambda f(z, u_\mu) \quad \text{in } \Omega \tag{3.9}$$

(recall that $f \geq 0$ and $\vartheta < \lambda < \mu$). As in the proof of Proposition 3.5, we can show that $u_\vartheta \leq u_\mu$. We introduce the following truncation of the reaction in problem $(P)_\lambda$:

$$g_\lambda(z, \zeta) = \begin{cases} \lambda f(z, u_\vartheta(z)) & \text{if } \zeta < u_\vartheta(z), \\ \lambda f(z, \zeta) & \text{if } u_\vartheta(z) \leq \zeta \leq u_\mu(z), \\ \lambda f(z, u_\mu(z)) & \text{if } u_\mu(z) < \zeta. \end{cases} \tag{3.10}$$

This is a Carathéodory function. We set

$$G_\lambda(z, \zeta) = \int_0^\zeta g_\lambda(z, s) ds$$

and consider the C^1 -functional $\widehat{\psi}_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \int_\Omega G_\lambda(z, u) dz \quad \forall u \in W_0^{1,p}(\Omega).$$

It is clear from (3.10) that $\widehat{\psi}_\lambda$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\psi}_\lambda(u_0) = \inf_{u \in W_0^{1,p}(\Omega)} \widehat{\psi}_\lambda(u),$$

so

$$\widehat{\psi}'_\lambda(u_0) = 0$$

and thus

$$A_p(u_0) + A(u_0) = N_{g_\lambda}(u_0). \tag{3.11}$$

Acting on (3.11) with $(u_\vartheta - u_0)^+ \in W_0^{1,p}(\Omega)$ and next with $(u_0 - u_\mu)^+ \in W_0^{1,p}(\Omega)$ (similarly as in the proof of Proposition 3.5), we get

$$u_\vartheta \leq u_0 \leq u_\mu.$$

Hence, we have

$$u_0 \in [u_\vartheta, u_\mu],$$

where $[u_\vartheta, u_\mu] = \{u \in W_0^{1,p}(\Omega) : u_0(z) \leq u(z) \leq u_\mu(z) \text{ for almost all } z \in \Omega\}$.

Then (3.11) becomes

$$A_p(u_0) + A(u_0) = \lambda N_f(u_0)$$

(see (3.10)), so

$$u_0 \in S(\lambda) \subseteq \text{int } C_+.$$

Let

$$a(y) = \|y\|^{p-2}y + y \quad \forall y \in \mathbb{R}^N.$$

Then $a \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ (recall that $p > 2$) and

$$\nabla a(y) = \|y\|^{p-2} \left(I + (p-2) \frac{y \otimes y}{\|y\|^2} \right) + I \quad \forall y \in \mathbb{R}^N,$$

so

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \|\xi\|^2 \quad \forall y, \xi \in \mathbb{R}^N.$$

Note that

$$\text{div } a(\nabla u) = \Delta_p u + \Delta u \quad \forall u \in W_0^{1,p}(\Omega).$$

So, we can apply the tangency principle of Pucci and Serrin [19, p.35] and infer that

$$u_\vartheta(z) < u_0(z) \quad \forall z \in \Omega. \tag{3.12}$$

Let $\varrho = \|u_0\|_\infty$ and let $\xi_\varrho > 0$ be as postulated by hypothesis H(iv). Then

$$\begin{aligned} & -\Delta_p u_\vartheta(z) - \Delta u_\vartheta(z) + \vartheta \xi_\varrho u_\vartheta(z)^{p-1} \\ & = \vartheta f(z, u_\vartheta(z)) + \vartheta \xi_\varrho u_\vartheta(z)^{p-1} \\ & \leq \vartheta f(z, u_0(z)) + \vartheta \xi_\varrho u_0(z)^{p-1} \\ & \leq \lambda f(z, u_0(z)) + \vartheta \xi_\varrho u_0(z)^{p-1} \\ & = -\Delta_p u_0(z) - \Delta u_0(z) + \vartheta \xi_\varrho u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega \end{aligned}$$

(see hypothesis H(iv) and use the facts that $\lambda > \vartheta$ and $f \geq 0$), so

$$u_0 - u_\vartheta \in \text{int } C_+ \tag{3.13}$$

(see (3.12) and Proposition 2.3).

In a similar fashion, we show that

$$u_\mu - u_0 \in \text{int } C_+. \tag{3.14}$$

From (3.13) and (3.14), it follows that

$$u_0 \in \text{int}_{C_0^1(\overline{\Omega})} [u_\vartheta, u_\mu]. \tag{3.15}$$

From (3.10), we see that

$$\varphi_\lambda|_{[u_\vartheta, u_\mu]} = \widehat{\psi}_\lambda|_{[u_\vartheta, u_\mu]} + \xi_\lambda^*$$

for some $\xi_\lambda^* \in \mathbb{R}$.

So, (3.15) implies that u_0 is a local $C_0^1(\overline{\Omega})$ -minimizer of φ_λ . Invoking Proposition 2.3, we have that

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda. \tag{3.16}$$

Hypotheses H(i), (ii) and (iii) imply that for given $\varepsilon > 0$ and $r > p$, we can find $c_2 = c_2(\varepsilon, r) > 0$ such that

$$F(z, \zeta) \leq \frac{\varepsilon}{p} \zeta^p + c_2 \zeta^r \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geq 0. \tag{3.17}$$

Then for all $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \varphi_\lambda(u) &= \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \lambda \int_\Omega F(z, u) dz \\ &\geq \frac{1}{p} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \frac{\lambda \varepsilon}{p} \|u^+\|_p^p - \lambda c_2 \|u^+\|_r^r \\ &\geq \frac{1}{p} \left(1 - \frac{\lambda \varepsilon}{\widehat{\lambda}_1(p)}\right) \|\nabla u\|_p^p + \frac{1}{2} \|\nabla u\|_2^2 - \lambda c_3 \|u\|^r \\ &\geq \frac{1}{p} \left(1 - \frac{\lambda \varepsilon}{\widehat{\lambda}_1(p)}\right) \|\nabla u\|_p^p - \lambda c_3 \|u\|^r \end{aligned} \tag{3.18}$$

for some $c_3 > 0$ (see (3.17) and (2.3)).

Choose $\varepsilon \in (0, \frac{\widehat{\lambda}_1(p)}{\lambda})$. Then, from (3.18) and since $r > p$, we infer that u_0 is a local minimizer of φ_λ . Without any loss of generality, we may assume that $\varphi_\lambda(0) = 0 \leq \varphi_\lambda(u_0)$ (the analysis is similar if the opposite inequality holds). By virtue of (3.16), as in Gasiński and Papageorgiou [20] (see the proof of Theorem 2.12), we can find $0 < \varrho < \|u_0\|$ such that

$$\varphi_\lambda(0) = 0 \leq \varphi_\lambda(u_0) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \varrho\} = \eta_\varrho^\lambda. \tag{3.19}$$

Recall that φ_λ is coercive, hence it satisfies the Palais-Smale condition. This fact and (3.19) permit the use of the mountain pass theorem (see Theorem 2.1). So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\eta_\phi^\lambda \leq \varphi_\lambda(\hat{u}) \tag{3.20}$$

and

$$\varphi'_\lambda(\hat{u}) = 0. \tag{3.21}$$

From (3.20) and (3.19), we have that $\hat{u} \neq 0$, $\hat{u} = u_0$. From (3.21), it follows that $\hat{u} \in S(\lambda) \subseteq \text{int } C_+$. □

Next, we examine what happens at the critical parameter λ_* .

Proposition 3.7 *If hypotheses H hold, then $\lambda_* \in \mathcal{Y}$.*

Proof Let $\{\lambda_n\}_{n \geq 1} \subseteq \mathcal{Y}$ be a sequence such that

$$\lambda_* < \lambda_n \quad \forall n \geq 1$$

and

$$\lambda_n \searrow \lambda_* \quad \text{as } n \rightarrow +\infty.$$

For every $n \geq 1$, we can find $u_n \in \text{int } C_+$, such that

$$A_p(u_n) + A(u_n) = \lambda_n N_f(u_n). \tag{3.22}$$

We claim that the sequence $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. Arguing indirectly, suppose that the sequence $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is unbounded. By passing to a suitable subsequence if necessary, we may assume that $\|u_n\| \rightarrow +\infty$. Let

$$y_n = \frac{u_n}{\|u_n\|} \quad \forall n \geq 1.$$

Then $\|y_n\| = 1$ and $y_n \in \text{int } C_+$ for all $n \geq 1$. From (3.22), we have

$$A_p(y_n) + \frac{1}{\|u_n\|^{p-2}} A(y_n) = \frac{\lambda_n N_f(u_n)}{\|u_n\|^{p-1}} \quad \forall n \geq 1. \tag{3.23}$$

Recall that

$$f(z, \zeta) \leq c_1 \zeta^{p-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geq 0$$

(see (3.1)), so the sequence $\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\}_{n \geq 1} \subseteq L^{p'}(\Omega)$ is bounded. This fact and hypothesis H(ii) imply that at least for a subsequence, we have

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \rightharpoonup 0 \quad \text{weakly in } L^{p'}(\Omega) \tag{3.24}$$

(see Gasinski and Papageorgiou [20]). Also, passing to a subsequence if necessary, we may assume that

$$y_n \rightharpoonup y \text{ weakly in } W_0^{1,p}(\Omega), \tag{3.25}$$

$$y_n \rightarrow y \text{ in } L^p(\Omega). \tag{3.26}$$

On (3.23) we act with $y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow +\infty$ and use (3.24) and (3.26). Then

$$\lim_{n \rightarrow +\infty} \left(\langle A_p(y_n), y_n - y \rangle + \frac{1}{\|u_n\|^{p-2}} \langle A(y_n), y_n - y \rangle \right) = 0,$$

so

$$\lim_{n \rightarrow +\infty} \langle A_p(y_n), y_n - y \rangle \leq 0.$$

Using Proposition 2.4, we have that

$$y_n \rightarrow y \text{ in } W_0^{1,p}(\Omega)$$

and so

$$\|y\| = 1. \tag{3.27}$$

Passing to the limit as $n \rightarrow +\infty$ in (3.23) and using (3.24), (3.27) and the fact that $p > 2$, we obtain

$$A_p(y) = 0,$$

so $y = 0$, which contradicts (3.27).

This proves that the sequence $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, passing to a subsequence if necessary, we may assume that

$$u_n \rightharpoonup u^* \text{ weakly in } W_0^{1,p}(\Omega), \tag{3.28}$$

$$u_n \rightarrow u^* \text{ in } L^p(\Omega). \tag{3.29}$$

On (3.22) we act with $u_n - u^* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow +\infty$ and use (3.28) and (3.29). Then

$$\lim_{n \rightarrow +\infty} \left(\langle A_p(u_n), u_n - u^* \rangle + \langle A(u_n), u_n - u^* \rangle \right) = 0,$$

so

$$\limsup_{n \rightarrow +\infty} \langle A_p(u_n), u_n - u^* \rangle \leq 0$$

(since A is monotone) and thus

$$u_n \rightarrow u_* \quad \text{in } W_0^{1,p}(\Omega) \tag{3.30}$$

(see Proposition 2.4).

Therefore, if in (3.22) we pass to the limit as $n \rightarrow +\infty$ and use (3.30), then

$$A_p(u_*) + A(u_*) = \lambda_* N_f(u_*)$$

and so $u_* \in C_+$ is a solution of problem $(P)_{\lambda_*}$.

We need to show that $u_* \neq 0$. From (3.22), we have

$$\begin{cases} -\Delta_p u_n(z) - \Delta u_n(z) = \lambda_n f(z, u_n(z)) & \text{in } \Omega \\ u_n|_{\partial\Omega} = 0 \end{cases} \quad \forall n \geq 1.$$

From Ladyzhenskaya and Ural'tseva [17, p.286], we know that we can find $M_1 > 0$ such that

$$\|u_n\|_\infty \leq M_1 \quad \forall n \geq 1.$$

Then applying Theorem 1 of Lieberman [18], we can find $\beta \in (0, 1)$ and $M_2 > 0$ such that

$$u_n \in C_0^{1,\beta}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\beta}(\overline{\Omega})} \leq M_2 \quad \forall n \geq 1.$$

Recall that $C_0^{1,\beta}(\overline{\Omega})$ is embedded compactly in $C_0^1(\overline{\Omega})$. So, by virtue of (3.28), we have

$$u_n \rightarrow u_* \quad \text{in } C_0^1(\overline{\Omega}).$$

Suppose that $u_* = 0$. Then

$$u_n \rightarrow 0 \quad \text{in } C_0^1(\overline{\Omega}). \tag{3.31}$$

Hypothesis H(iii) implies that for a given $\varepsilon > 0$, we can find $\delta \in (0, \varepsilon]$ such that

$$f(z, \xi) \leq \varepsilon \xi^{p-1} \quad \text{for almost all } z \in \Omega, \text{ all } \xi \in [0, \delta]. \tag{3.32}$$

From (3.31), it follows that we can find $n_0 \geq 1$ such that

$$u_n(z) \in [0, \delta] \quad \forall z \in \overline{\Omega}, \text{ all } n \geq n_0. \tag{3.33}$$

Therefore, for almost all $z \in \Omega$ and all $n \geq n_0$, we have

$$-\Delta_p u_n(z) - \Delta u_n(z) = \lambda_n f(z, u_n(z)) \leq \lambda_n \varepsilon u_n(z)^{p-1}$$

(see (3.32) and (3.33)), so

$$\|\nabla u_n\|_p^p \leq \lambda_n \varepsilon \|u_n\|_p^p \leq \frac{\lambda_n}{\lambda_1(p)} \varepsilon \|\nabla u_n\|_p^p \quad \forall n \geq n_0$$

(see (2.3)), thus

$$\frac{\widehat{\lambda}_1(p)}{\varepsilon} \leq \lambda_n \quad \forall n \geq n_0$$

and so

$$\frac{\widehat{\lambda}_1(p)}{\varepsilon} \leq \lambda_*$$

Let $\varepsilon \searrow 0$ to get a contradiction. This proves that $u_* \neq 0$ and so $u_* \in S(\lambda_*) \subseteq \text{int } C_+$, hence $\lambda_* \in \mathcal{Y}$. \square

The bifurcation-type theorem summarizes the situation for problem $(P)_\lambda$.

Theorem 3.8 *If hypotheses H hold, then there exists $\lambda_* > 0$ such that*

(a) *for every $\lambda > \lambda_*$ problem $(P)_\lambda$ has at least two positive solutions:*

$$u_0, \widehat{u} \in \text{int } C_+;$$

(b) *for $\lambda = \lambda_*$ problem $(P)_\lambda$ has at least one positive solution $u_* \in \text{int } C_+$;*

(c) *for $\lambda \in (0, \lambda_*)$ problem $(P)_\lambda$ has no positive solution.*

Remark 3.9 As the referee pointed out, it is an interesting problem to produce an example in which, at the bifurcation point $\lambda^* > 0$, the equation has exactly one solution. We believe that the recent paper of Gasiński and Papageorgiou [21] on the existence and uniqueness of positive solutions will be helpful. Concerning the existence of nodal solutions for $\lambda \in (0, \lambda^*)$, we mention the recent paper of Gasiński and Papageorgiou [22], which studies the $(p, 2)$ -equations and produces nodal solutions for them.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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