PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 33, Fasc. 2 (2013), pp. 287–299

CONSTRUCTION OF A COMPACT QUANTUM GROUP FOR TRANSPOSITION-COLORING FUNCTION

BY

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Abstract. We apply the Woronowicz construction of compact quantum group to the function which associates different parameters (colors) with transpositions generating the set of four-element permutation. We show that in the case when one of the parameters equals one, we get a non-trivial (non-commutative) compact quantum group which is a twisted product of $SU_{-1}(2)$ and the two-dimensional torus.

2000 AMS Mathematics Subject Classification: Primary: 81R50; Secondary: 20G42.

Key words and phrases: Quantum groups, twisted determinant condition, twisted product.

1. INTRODUCTION

Since the invention of the first example of non-trivial, non-commutative and non-cocommutative compact quantum group, the twisted $SU_q(2)$, several different constructions of these have been established (cf. [10], [7], [1]). In particular, in [10] the twisted special unitary group $SU_q(n)$ was obtained thanks to the result of categorial nature. According to it, the universal C*-algebra generated by abstract elements forming unitary matrix u (in the sense that $uu^* = 1 = u^*u$) and satisfying a modified (twisted) determinant condition admits the quantum group structure. The modification of the determinant condition depends on the choice of n^n constants $E_{i_1,i_2,\ldots,i_n}$, $i_k \in \{1, 2, \ldots, n\}$, being 'non-degenerate'. The non-degeneracy condition still leaves a lot of freedom to choose the constants – at least in theory. In practice, "unless the constants are chosen in a very special way, the quantum group is very small" (see [10], Remark following Theorem 1.4). These are the trivial cases when the underlying algebra is commutative and the quantum group reduces to the algebra of functions on a compact group. But what are the non-trivial quantum groups that can be obtained by this construction?

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* Supported by the Polish National Science Center Postdoctoral Fellowship No. 2012/04/S/ST1/00102.

** On leave from the Jagiellonian University.

Two concrete examples of objects successfully constructed with aid of the Woronowicz result are related to *functions on permutations*. By this notion we mean that the constants $E_{i_1,i_2,\dots,i_n}$ are non-zero and real whenever $(i_1, i_2, \dots, i_n)$ is a permuation of the set $\{1, 2, \dots, n\}$ and $E_{i_1,i_2,\dots,i_n} = 0$ if (some) indices repeat. In such a case the non-degeneracy condition is always satisfied. Moreover, it guarantees that no higher powers of one generator appear in the unitary or twisted determinant conditions (only the first order relations).

The first example is $SU_q(n)$, which comes from $f_1(\sigma) = (-q)^{\mathrm{inv}(\sigma)}$, the function counting the number of inversions in the permutation. The other example is the quantum group $U_q(2)$ (see [11]), where the function $f_2(\sigma) = (-q)^{3-\mathrm{c}(\sigma)}$ on S_3 was considered. Here, $\mathrm{c}(\sigma)$ denotes the number of cycles in the permutation σ. Note that whereas the function f_1 gives non-trivial objects for any $n \geqslant 2$, it was shown in [12] that when $n > 3$ the function f_2 leads to a trivial (commutative) quantum group. We shall show in this paper that a kind of generalization of Wysoczański's construction is possible for $n = 4$, but a different function should be considered.

More precisely, we want to consider the function on the group S_4 of permutations of four elements, which puts different parameters – 'colors' – on each transposition $\pi_i := (i, i + 1)$, $1 \leqslant i \leqslant 3$, and extends by zero onto $\mathbb{N}^4 \setminus S_4$. This gives *a priori* a three-parameter function. However, in most of the cases, the related quantum groups become trivial. There are two interesting cases: when one of the colors equals one or when two colors equal one.

In this note we shall focus on the first case and show that compact quantum group obtained via the Woronowicz construction is non-trivial. (The case of two colors equal to one will be treated elsewhere [3].) More precisely, the resulting object is a twisted product of $SU_{-1}(2)$ and the two-dimensional torus:

$$\mathbb{G} = SU_{-1}(2) \ltimes_\sigma T_2.$$

This quantum group inherits many properties from $SU_{-1}(2)$. In particular, it is of Kac type, and the representation theory of the related C*-algebra can be easily described using the results known for $SU_{-1}(2)$.

The paper is organized as follows. In Section 2 we give some necessary definitions and present a method to exhibit the relations which hold in the constructed quantum group. In Section 3 we describe the transposition-coloring function. Section 4 contains the computations of the relations among generators using the method described previously. The quantum group structure of the object we get is investigated in Section 5 and the representation theory of the related C*-algebra is shortly depicted in the last section.

2. GENERAL THEORY

We shall adopt the notation from [8]. Let φ be a non-constant real function on S_4 with $\varphi(\sigma) \neq 0$ for all $\sigma \in S_4$. We can assume that $\varphi\big((1, 2, 3, 4)\big) = 1$. Define the array

$$E_{i,j,k,l} = \begin{cases} \varphi(\sigma) & \text{if } (i,j,k,l) = \sigma \in S_4, \\ 0 & \text{if } (i,j,k,l) \notin S_4. \end{cases} \tag{2.1}$$

We say that the array E is *left nondegenerate* if the arrays $E_{1-}, \dots, E_{4-}$, where $E_{N-} = [E_{Nijk}]_{i,j,k=1}^4$, are linearly independent. Similarly, the array E is said to be *right nondegenerate* if $E_{-1}, \dots, E_{-4}$, where $E_{-n} = [E_{ijkN}]_{i,j,k=1}^4$, are linearly independent arrays.

The *Woronowicz quantum group* related to the left and right nondegenerate array E is the pair $G = (\mathsf{A}, u)$, where A is the universal C^*-algebra generated by matrix coefficients $u = [u_{jk}]_{j,k=1}^4$ satisfying

(a) the *unitarity condition*:

$$\sum_{s=1}^4 u_{js} u_{ks}^* = \delta_{jk} 1 = \sum_{s=1}^4 u_{sj}^* u_{sk}, \tag{U}$$

(b) the *twisted determinant condition*:

$$\sum_{i,j,k,l}^4 E_{ijkl} u_{ai} u_{bj} u_{ck} u_{dl} = E_{abcd} \cdot 1. \tag{TD}$$

The existence of such a quantum group is ensured by Theorem 1.4 in [10] and the quantum group structure is imposed by the fact that u is the fundamental corepresentation, so the usual formulas on comultiplication, counit and coinverse hold. Namely,

$$\Delta(u_{jk}) = \sum_{s=1}^4 u_{js} \otimes u_{sk}, \quad \varepsilon(u_{jk}) = \delta_{jk}, \quad \kappa(u_{jk}) = u_{jk}^*.$$

The method to study the Woronowicz quantum group related to the array E (or to the function φ) is by the intertwining operator of the corepresentation u (cf. [8] and [11]).

For the standard basis $\{e_1, e_2, e_3, e_4\}$ in $\mathbb{C}^4$ let us consider

$$E : \mathbb{C} \mapsto \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$$

given by

$$\mathbb{C} \ni 1 \mapsto E(1) := \sum_{i,j,k,l} E_{ijkl} \cdot e_i \otimes e_j \otimes e_k \otimes e_l \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4.$$

Then the adjoint

$$E^* : \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \mapsto \mathbb{C}$$

is given by

$$E^*(e_a \otimes e_b \otimes e_c \otimes e_d) = E_{abcd} \cdot I.$$

It follows from the proof of Theorem 1.4 in [8] that the operator

$$P := (E^* \otimes I)(I \otimes E) : \mathbb{C}^4 \mapsto \mathbb{C}^4,$$

given by

$$e_a \overset{I\otimes E}{\longmapsto} \sum_{i,j,k,l} E_{ijkl} e_a \otimes e_i \otimes e_j \otimes e_k \otimes e_l \overset{E^*\otimes I}{\longmapsto} \sum_l \sum_{i,j,k} E_{aijk} E_{ijkl} \cdot e_l,$$

intertwines u, i.e. $(P \otimes I)u = u(P \otimes I)$. Since $E_{aijk} \neq 0 \neq E_{ijkl}$ only if (a, i, j, k), (i, j, k, l) are permutations, it follows that $a = l$, and thus P is the diagonal matrix:

$$P = \operatorname{diag}\{c_1, c_2, c_3, c_4\}, \quad c_a := \sum_{i,j,k} E_{aijk} E_{ijka}.$$

Moreover, different values on the diagonal induce different invariant subspaces, and thus a decomposition of the fundamental corepresentation u. More precisely, it follows from the equality

$$c_j u_{jk} = \big((P \otimes I)u\big)_{jk} = \big(u(P \otimes I)\big)_{jk} = c_k u_{jk}$$

that if $c_k \neq c_j$ then $u_{jk} = 0 = u_{kj}$. In particular, unless P is a multiple of identity matrix, u is reducible.

3. THE COLORING FUNCTION φ

Consider the function $\varphi : S_4 \mapsto \mathbb{R}$ which is defined as follows. Let $\pi_k := (k, k+1)$ for $k = 1, 2, 3$ be the transpositions generating S_4, and put $\varphi(\pi_k) := q_k$. We assume that $q_k \neq 0$.

We extend φ to S_4 multiplicatively. Namely, if $\sigma \in S_4$ has a minimal representation of the form $\sigma = \pi_{j_1} \ldots \pi_{j_s}$, then we look for different transpositions, and if

$$\{\pi_{j_1}, \ldots, \pi_{j_s}\} = \{\pi_{i_1}, \ldots, \pi_{i_t} : i_1 < \ldots < i_t\}$$

then

$$\varphi(\sigma) := q_{i_1} \ldots q_{i_t}.$$

For example,

$$\sigma = (1, 2, 4, 3) = \pi_3 \Rightarrow E_{1243} = q_3,$$
$$\sigma = (4, 1, 3, 2) = \pi_2\pi_3\pi_2\pi_1 \Rightarrow E_{4132} = \varphi(\sigma) = q_1 q_2 q_3.$$

The relations between the transpositions ensure that this extension is well defined.

The array $E = (E_{i,j,k,l})$ related to the function φ is then defined as described at the beginning of Section 2. Explicitly, it can be written in the following way:

$$\begin{array}{llll}
E_{1234} = 1, & E_{2134} = q_1, & E_{3124} = q_1q_2, & E_{4123} = q_1q_2q_3,\\
E_{1243} = q_3, & E_{2143} = q_1q_3, & E_{3142} = q_1q_2q_3, & E_{4132} = q_1q_2q_3,\\
E_{1324} = q_2, & E_{2314} = q_1q_2, & E_{3214} = q_1q_2, & E_{4213} = q_1q_2q_3,\\
E_{1342} = q_2q_3, & E_{2341} = q_1q_2q_3, & E_{3241} = q_1q_2q_3, & E_{4231} = q_1q_2q_3,\\
E_{1423} = q_2q_3, & E_{2413} = q_1q_2q_3, & E_{3412} = q_1q_2q_3, & E_{4312} = q_1q_2q_3,\\
E_{1432} = q_2q_3, & E_{2431} = q_1q_2q_3, & E_{3421} = q_1q_2q_3, & E_{4321} = q_1q_2q_3.
\end{array}$$

The diagonal entries of P are the following:

$$\begin{aligned}
c_1 &= q_1q_2q_3(1 + q_3 + q_2 + 3q_2q_3),\\
c_2 &= q_1q_2q_3(1 + q_3 + q_1q_2 + 3q_1q_2q_3),\\
c_3 &= q_1q_2q_3(1 + q_1 + q_2q_3 + 3q_1q_2q_3),\\
c_4 &= q_1q_2q_3(1 + q_1 + q_2 + 3q_1q_2).
\end{aligned}$$

In this paper we shall focus on the case

$$q_2 = 1 \quad \text{and} \quad q_1 \neq q_3,$$

which we tacitly assume in the sequel[1]. This implies that $c_2 = c_3$ and $c_1 \neq c_4$. Then it follows from the intertwining property of P that the matrix u is reducible and has the form

$$u = \begin{pmatrix} u_{11} & 0 & 0 & 0\\ 0 & u_{22} & u_{23} & 0\\ 0 & u_{32} & u_{33} & 0\\ 0 & 0 & 0 & u_{44}\end{pmatrix} := \begin{pmatrix} a & 0 & 0 & 0\\ 0 & x & y & 0\\ 0 & z & w & 0\\ 0 & 0 & 0 & b\end{pmatrix}.$$

In the sequel we denote by $\mathbb{G} = (\mathsf{A}, u)$ the quantum group which we get, by the Woronowicz theorem, from the coloring function φ, with $q_2 = 1$ and u as above.

4. RELATIONS IN THE ALGEBRA A

In this section we write down explicitly the relations in the algebra A of the Woronowicz quantum group $\mathbb{G} = (\mathsf{A}, u)$.

The unitarity condition (U) implies the following relations:

$$\begin{array}{llll}
\text{(A)} & aa^* = 1 = a^*a, & \text{(B)} & bb^* = 1 = b^*b,\\
\text{(C)} & xx^* + yy^* = 1, & \text{(D)} & y^*y + w^*w = 1,\\
\text{(E)} & zz^* + ww^* = 1, & \text{(F)} & x^*x + z^*z = 1,\\
\text{(G)} & xz^* + yw^* = 0, & \text{(H)} & x^*y + z^*w = 0.
\end{array}$$

[1]Note that if we assume $q_1 = 1$ and $q_3 \neq 1$ or $q_3 = 1$ and $q_2 \neq 1$ we get the same structure.

On the other hand, the twisted determinant condition (TD) applied to the permutations (written in square brackets) gives:

$$\begin{array}{llrllr}
(1) & [1234] & a(xw+yz)b=1, & (13) & [3124] & (wax+zay)b=1,\\
(2) & [1243] & a(xbw+ybz)=1, & (14) & [3142] & wabx+zaby=1,\\
(3) & [1324] & a(zy+wx)b=1, & (15) & [3214] & (wx+zy)ab=1,\\
(4) & [1342] & awbx+azby=1, & (16) & [3241] & (wx+yz)ba=1,\\
(5) & [1423] & ab(xw+yz)=1, & (17) & [3412] & wbax+zbay=1,\\
(6) & [1432] & ab(wx+zy)=1, & (18) & [3421] & (wbx+zby)a=1,\\
(7) & [2134] & (xaw+yaz)b=1, & (19) & [4123] & ba(xw+yz)=1,\\
(8) & [2143] & xabw+yabz=1, & (20) & [4132] & ba(wx+zy)=1,\\
(9) & [2314] & (xw+yz)ab=1, & (21) & [4213] & b(xaw+yaz)=1,\\
(10) & [2341] & (xw+yz)ba=1, & (22) & [4231] & b(xw+yz)a=1,\\
(11) & [2413] & xbaw+ybaz=1, & (23) & [4312] & b(wax+zay)=1,\\
(12) & [2431] & (xbw+ybz)a=1, & (24) & [4321] & b(wx+zy)a=1,
\end{array}$$

and

$$\begin{array}{llrllr}
(25) & [1224] & a(xy+yx)b=0, & (37) & [1242] & a(ybx+xby)=0,\\
(26) & [1422] & ab(xy+yx)=0, & (38) & [2124] & (xay+yax)b=0,\\
(27) & [2214] & (xy+yx)ab=0, & (39) & [2142] & xaby+yabx=0,\\
(28) & [2241] & (xy+yx)ba=0, & (40) & [2412] & xbay+ybax=0,\\
(29) & [4221] & b(xy+yx)a=0, & (41) & [2421] & (xby+ybx)a=0,\\
(30) & [4122] & ba(xy+yx)=0, & (42) & [4212] & b(xay+yax)=0,\\
\\
(31) & [1334] & a(zw+wz)b=0, & (43) & [1343] & a(zbw+wbz)=0,\\
(32) & [1433] & ab(zw+wz)=0, & (44) & [3134] & (zaw+waz)b=0,\\
(33) & [3314] & (zw+wz)ab=0, & (45) & [3143] & zabw+wabz=0,\\
(34) & [3341] & (zw+wz)ba=0, & (46) & [3413] & zbaw+wbaz=0,\\
(35) & [4133] & ba(zw+wz)=0, & (47) & [3431] & (zbw+wbz)a=0,\\
(36) & [4331] & b(zw+wz)a=0, & (48) & [4313] & b(zaw+waz)=0.
\end{array}$$

We see immediately that both parameters q_1 and q_3 disappear and the quantum group which we get by this construction will not be a parameter-deformation of any object.

The relations (25)–(30) and (31)–(36), respectively, together with (A) and (B), give

$$\text{(K)}\quad xy+yx=0 \quad\text{and}\quad \text{(L)}\quad zw+wz=0.$$

So there are two anti-commuting pairs: (x,y) and (z,w). On the other hand, from (5), (6), and (19), together with (A) and (B), we see that a and b are both unitary and double commute:

$$\text{(M)}\qquad ab=ba,\quad ab^*=b^*a,$$

and that

$$\text{(N)} \qquad xw + yz = wx + zy = a^*b^*.$$

The next step is to show that both x and w commute with a and b and that

$$\text{(P)} \qquad x^* = abw, \quad w^* = abx.$$

This can be obtained from the following observations:

- Multiplying (12) by x^* on the left and using (F), (H), and (47) gives

$$\text{(P1)} \qquad \begin{aligned} x^* &= (x^*x)bwa + (x^*y)bza = (1 - z^*z)bwa - z^*wbza \\ &= bwa - z^*(zbwa + wbza) = bwa. \end{aligned}$$

- Multiplying (18) by w^* on the left and using (D), (H), and (41) gives

$$\text{(P2)} \qquad \begin{aligned} w^* &= (w^*w)bxa + (w^*z)bya = (1 - y^*y)bxa - y^*xbya \\ &= bxa - y^*(ybxa + xbya) = bxa. \end{aligned}$$

- Multiplying (D) by x^* on the left and using (K), (N), and (H) gives

$$\text{(P3)} \qquad \begin{aligned} x^* &= x^*y^*y + x^*w^*w = (-xy)^*y + (a^*b^* - zy)^*w \\ &= baw - y^*(x^*y + z^*w) = baw. \end{aligned}$$

- Comparing (P1) and (P3), we see that $aw = wa$. On the other hand, (P2) implies $x^* = awb$ and together with (P3) and (M) shows that $bw = wb$.

Similarly, we show that

$$\text{(R)} \qquad ay = ya, \quad by = yb, \quad az = za, \quad bz = zb, \quad y^* = abz, \quad z^* = aby.$$

From (E), (P), and (R) we deduce that

$$zz^* + x^*x = 1$$

which, together with (F), imply that z is normal. By the same reasoning, using (C), (F), and (P), we see that x is normal. Moreover, it follows from the relations $y^* = abz$ and $w^* = abx$ that y and w are normal. So we get one more family of the relations:

$$\text{(S)} \qquad xx^* = x^*x, \quad yy^* = y^*y, \quad zz^* = z^*z, \quad ww^* = w^*w.$$

Finally, (H), (P), and (R) imply that

$$\text{(T)} \qquad xy^* = y^*x \quad \text{and} \quad zw^* = w^*z.$$

We conclude that the quantum group is generated by four elements a, b, x, z, and the matrix u is of the form

$$u = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & x & a^*b^*z^* & 0 \\ 0 & z & a^*b^*x^* & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

where:

- a and b are unitaries and double commute,
- x and z are normal, double anti-commute and satisfy $xx^* + zz^* = 1$, i.e., the condition (A) holds true,
- x and z both commute with a and b.

Note that all the relations (1)–(48) follow from these ones when we put $z = a^*b^*y^*$ and $w = a^*b^*x^*$.

In the sequel we shall denote by A the universal C*-algebra generated by a, b, x, and z satisfying the above conditions.

5. QUANTUM GROUP STRUCTURE

In this section we study the quantum group structure of the Woronowicz quantum group $\mathbb{G} = (\mathsf{A}, u)$ related to transposition-coloring function with $q_2 = 1$. We show that, as a quantum group, $\mathbb{G}$ is the twisted product of $SU_{-1}(2)$ and T_2, the two-dimensional torus. Let us note that the construction of twisted product of quantum groups was introduced by Podleś and Woronowicz in [4] to describe the group structure of the noncompact quantum group $SL_q(2, \mathbb{C})$ as a twisted product of its two subgroups: the quantum group $SU_q(2)$ and the quantum Pontryagin dual of it. Later Wysoczański [11] applied this construction to describe $U_q(2)$ as a twisted product of $SU_q(2)$ and $U(1)$.

For $\mathbb{G}$ related to transposition-coloring function with $q_2 = 1$, the fundamental representation u is reducible and decomposes into a direct sum of two one-dimesional representations a and b and the two-dimensional representation w:

$$u = a \oplus b \oplus w, \qquad w = \begin{pmatrix} x & a^*b^*z^* \\ z & a^*b^*x^* \end{pmatrix}.$$

The *-bialgebra structure on $\mathcal{A} := {}^*\text{-alg}\{a, b, x, z\}$ is given by the comultiplication

$$\Delta(a) = a \otimes a, \qquad \Delta(x) = x \otimes x + a^*b^*z^* \otimes z,$$
$$\Delta(b) = b \otimes b, \qquad \Delta(z) = z \otimes x + a^*b^*x^* \otimes z,$$

the counit

$$\varepsilon(a) = \varepsilon(b) = \varepsilon(x) = 1, \quad \varepsilon(z) = 0,$$

and the coinverse

$$\kappa(a) = a^*, \quad \kappa(b) = b^*, \quad \kappa(x) = x^*, \quad \kappa(z) = abz.$$

REMARK 5.1. The modular properties of the quantum group $\mathbb{G}$ can be revealed by studying $\bar{u}$, the conjugate corepresentation of u, i.e. the matrix with entries $(\bar{u})_{ij} = u^*_{ij}$. Using the relations between generators described in the previous section, one can easily check that $\bar{u}\bar{u}^* = I = \bar{u}^*\bar{u}$. This means that $\mathbb{G}$ is a subgroup of the free unitary quantum group $A_u(4)$ defined by Wang [7] and, in particular, is of Kac type. Thus

$$\kappa(\alpha^*) = \kappa(\alpha)^* \quad \text{for } \alpha \in \{a, b, x, z\}.$$

For a more general reasoning see also [6].

We see that a and b are group-like elements which are unitaries in the center of $\mathcal{A}$. They generate a C*-algebra isomorpic to the algebra $C(T_2)$ of continuous functions on the two-dimensional (commutative) torus. On the other hand, x and z generate the C*-algebra isomorphic to $SU_{-1}(2)$:

$$xx^* + zz^* = 1 = x^*x + z^*z, \quad zz^* = z^*z, \quad xz + zx = 0, \quad xz^* + z^*x = 0.$$

This leads to the observation that the quantum group A, as a C*-algebra, is the spatial tensor product:

$$\mathsf{A} = C\big(SU_{-1}(2)\big) \otimes C(T_2).$$

We shall show that as a quantum group A is the twisted product of $SU_{-1}(2)$ and the two-dimensional commutative torus T_2. Let $\mathbb{G}_1 = (\mathsf{A}_1, w_1)$ be the quantum group $SU_{-1}(2)$ with the fundamental corepresentation

$$w_1 = \begin{pmatrix} x & z^* \\ z & x^* \end{pmatrix}.$$

The comultiplication is then given by

$$\Delta(x) = x \otimes x + z^* \otimes z, \quad \Delta(z) = z \otimes x + x^* \otimes z,$$

the counit is $\varepsilon_1(x) = 1$ and $\varepsilon_1(z) = 0$, and the coinverse is $\kappa_1(x) = x^*$, $\kappa_1(z) = z$.

Let also $\mathbb{G}_2 = (\mathsf{A}_2, w_2)$ be the commutative group with the fundamental corepresentation

$$w_2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

The comultiplication is $\Delta(a) = a \otimes a$, $\Delta(b) = b \otimes b$, the counit $\varepsilon_2(a) = \varepsilon_2(b) = 1$, and the coinverse $\kappa_2(a) = a^*$, $\kappa_2(b) = b^*$.

PROPOSITION 5.1. *Let us define the morphisms*

$$p_1 = \mathrm{id}_1 \otimes \varepsilon_2 : \mathsf{A} = \mathsf{A}_1 \otimes \mathsf{A}_2 \to \mathsf{A}_1, \quad p_2 = \varepsilon_1 \otimes \mathrm{id}_2 : \mathsf{A} = \mathsf{A}_1 \otimes \mathsf{A}_2 \to \mathsf{A}_2.$$

Then, up to the isomorpism $\xi \otimes \alpha \cong \xi\alpha$ for $\xi \in \mathsf{A}_1$ and $\alpha \in \mathsf{A}_2$, the following relations hold:

$$\Delta_j \circ p_j = (p_j \otimes p_j) \circ \Delta, \quad \varepsilon_j \circ p_j = \varepsilon, \quad \kappa_j \circ p_j = p_j \circ \kappa, \quad j = 1, 2. \tag{5.1}$$

P r o o f. Indeed, due to linear, (anti-)multiplicative and involutive properties of Δ's, ε's, and κ's, it is enough to check the relations (5.1) for the generators of the form $\xi \otimes \alpha$, where $\xi \in \{1, x, z\}$ and $\alpha \in \{1, a, b\}$. Moreover, since b behaves under Δ, ε, and κ exactly as a, we can restrict the computation to $1 \otimes a$, $x \otimes a$, and $z \otimes a$.

The relation $\varepsilon_j \circ p_j = \varepsilon$ $(j = 1, 2)$ is straightforward since $\varepsilon_j \circ p_j(\xi \otimes \alpha) = \varepsilon_1(\xi)\varepsilon_2(\alpha) = \varepsilon(\xi)\varepsilon(\alpha) = \varepsilon(\xi \otimes \alpha)$ for any $j = 1, 2$. The other two relations need to be checked directly on the generators. For example, for $1 \otimes a$ we have

$$\begin{aligned}
\Delta_1 \circ p_1(1 \otimes a) &= \Delta_1(1)\varepsilon_2(a) = 1 \otimes 1 = p_1(1 \otimes a) \otimes p_1(1 \otimes a)\\
&= (p_1 \otimes p_1)\big((1 \otimes a) \otimes (1 \otimes a)\big) \cong (p_1 \otimes p_1)(a \otimes a)\\
&= (p_1 \otimes p_1)\Delta(a) \cong (p_1 \otimes p_1)\Delta(1 \otimes a),\\
\kappa_1 \circ p_1(1 \otimes a) &= \kappa_1(1)\varepsilon_2(a) = 1 = p_1(1 \otimes a^*) \cong p_1(a^*)\\
&= p_1 \circ \kappa(a) \cong p_1 \circ \kappa(1 \otimes a),
\end{aligned}$$

and similarly

$$\begin{aligned}
\Delta_2 \circ p_2(1 \otimes a) &= \Delta_2(a) = a \otimes a = p_2(1 \otimes a) \otimes p_2(1 \otimes a) \cong (p_2 \otimes p_2)(a \otimes a)\\
&= (p_2 \otimes p_2)\Delta(a) \cong (p_2 \otimes p_2)\Delta(1 \otimes a),\\
\kappa_2 \circ p_2(1 \otimes a) &= \kappa_2(a) = a^* = p_2(1 \otimes a^*) \cong p_2(a^*) = p_2 \circ \kappa(a)\\
&\cong p_2 \circ \kappa(1 \otimes a).
\end{aligned}$$

On the other hand, for the generator $x \otimes 1$ we have

$$\begin{aligned}
\Delta_1 \circ p_1(x \otimes 1) &= \Delta_1(x)\varepsilon_2(1) = x \otimes x + z^* \otimes z\\
&= (p_1 \otimes p_1)\big((x \otimes 1) \otimes (x \otimes 1) + (z^* \otimes a^*b^*) \otimes (z \otimes 1)\big)\\
&\cong (p_1 \otimes p_1)(x \otimes x + a^*b^*z^* \otimes z) = (p_1 \otimes p_1)\Delta(x)\\
&\cong (p_1 \otimes p_1)\Delta(x \otimes 1).
\end{aligned}$$

For the generator $x \otimes a$ we have

$$\begin{aligned}
(p_1 \otimes p_1) \circ \Delta(x \otimes a) &\cong (p_1 \otimes p_1) \circ \Delta(xa) = (p_1 \otimes p_1)(xa \otimes xa + b^*z^* \otimes za)\\
&\cong (p_1 \otimes p_1)(x \otimes a \otimes x \otimes a + z^* \otimes b^* \otimes z \otimes a)\\
&= x \otimes x + z^* \otimes z = \Delta_1(x) = \Delta_1 \circ p_1(x \otimes a),\\
(p_2 \otimes p_2) \circ \Delta(x \otimes a) &\cong (p_2 \otimes p_2)(x \otimes a \otimes x \otimes a + z^* \otimes b^* \otimes z \otimes a)\\
&= a \otimes a = \Delta_2(a) = \Delta_2 \circ p_2(x \otimes a),\\
p_1 \circ \kappa(x \otimes a) &= p_1\big(\kappa(xa)\big) = p_1(a^*x^*) \cong p_1(x^* \otimes a^*) = x^*\\
&= \kappa_1(x) = \kappa_1 \circ p_1(x \otimes a),\\
p_2 \circ \kappa(x \otimes a) &\cong p_2(x^* \otimes a^*) = a^* = \kappa_2(a) = \kappa_2 \circ p_2(x \otimes a).
\end{aligned}$$

A similar reasoning shows that the relations (5.1) hold for $z \otimes a$. ∎

It follows from the relations (5.1) that G_1 and G_2 are subgroups of G in the sense of Podleś and Woronowicz [4]. Moreover, we can define the *-algebra homomorphism $\sigma : \mathsf{A}_1 \otimes \mathsf{A}_2 \to \mathsf{A}_2 \otimes \mathsf{A}_1$ by the values on generators:

$$\sigma(1 \otimes \alpha) = \alpha \otimes 1, \quad \sigma(x \otimes \alpha) = \alpha \otimes x, \quad \sigma(z \otimes \alpha) = a^* b^* \alpha \otimes z, \tag{5.2}$$

where $\alpha \in \{1, a, b, a^*, b^*\}$. In particular, $\sigma(z \otimes a) = b^* \otimes z$ and $\sigma(z \otimes b) = a^* \otimes z$.

PROPOSITION 5.2. *The mapping* $\sigma : \mathsf{A}_1 \otimes \mathsf{A}_2 \to \mathsf{A}_2 \otimes \mathsf{A}_1$ *is a* **-isomorphism satisfying*

$$\Delta = (\mathrm{id}_1 \otimes \sigma \otimes \mathrm{id}_2)(\Delta_1 \otimes \Delta_2), \quad \varepsilon = \varepsilon_1 \otimes \varepsilon_2, \quad \kappa = s(\kappa_2 \otimes \kappa_1)\sigma,$$

where $s : \mathsf{A}_1 \otimes \mathsf{A}_2 \ni \xi \otimes \alpha \mapsto \alpha \otimes \xi \in \mathsf{A}_2 \otimes \mathsf{A}_1$ *is the flip automorphism.*

P r o o f. In the proof α will always denote an element from $\{1, a, b\}$. Then α is in the center of A and, moreover, $\Delta(\alpha) = \Delta_2(\alpha) = \alpha \otimes \alpha$, $\varepsilon(\alpha) = 1$, and $\kappa(\alpha) = \alpha^*$. The relation $\varepsilon = \varepsilon_1 \otimes \varepsilon_2$ is evident since $\varepsilon|_{\mathsf{A}_j} = \varepsilon_j$, $j = 1, 2$, on the generators.

For $1 \otimes \alpha$ we have

$$\begin{aligned}
&(\mathrm{id}_1 \otimes \sigma \otimes \mathrm{id}_2)(\Delta_1 \otimes \Delta_2)(1 \otimes \alpha) = (\mathrm{id}_1 \otimes \sigma \otimes \mathrm{id}_2)\big((1 \otimes 1) \otimes (\alpha \otimes \alpha)\big)\\
&= 1 \otimes \sigma(1 \otimes \alpha) \otimes \alpha = (1 \otimes \alpha) \otimes (1 \otimes \alpha) \cong \alpha \otimes \alpha = \Delta(\alpha) \cong \Delta(1 \otimes \alpha),\\
&s(\kappa_2 \otimes \kappa_1)\sigma(1 \otimes \alpha) = s(\kappa_2 \otimes \kappa_1)(\alpha \otimes 1) = s(\alpha^* \otimes 1) = 1 \otimes \alpha^* \cong \alpha^*\\
&= \kappa(\alpha) \cong \kappa(1 \otimes \alpha).
\end{aligned}$$

For $x \otimes \alpha$ we expand the two sides of the relation for Δ obtaining

$$\begin{aligned}
(\mathrm{id}_1 \otimes \sigma \otimes \mathrm{id}_2)(\Delta_1 \otimes \Delta_2)(x \otimes \alpha) &= (\mathrm{id}_1 \otimes \sigma \otimes \mathrm{id}_2)(x \otimes x + z^* \otimes z) \otimes (\alpha \otimes \alpha)\\
&= (\mathrm{id}_1 \otimes \sigma \otimes \mathrm{id}_2)(x \otimes x \otimes \alpha \otimes \alpha + z^* \otimes z \otimes \alpha \otimes \alpha)\\
&= x \otimes \alpha \otimes x \otimes \alpha + z^* \otimes a^* b^* \alpha \otimes z \otimes \alpha,\\
\Delta(x \otimes \alpha) \cong \Delta(x\alpha) &= (x \otimes x + a^* b^* z^* \otimes z)(\alpha \otimes \alpha)\\
&= x\alpha \otimes x\alpha + a^* b^* z^* \alpha \otimes z\alpha\\
&\cong (x \otimes \alpha) \otimes (x \otimes \alpha) + (z^* \otimes a^* b^* \alpha) \otimes (z \otimes \alpha),
\end{aligned}$$

and immediately see that they are equal. Similarly,

$$\begin{aligned}
s(\kappa_2 \otimes \kappa_1)\sigma(x \otimes \alpha) &= s(\kappa_2 \otimes \kappa_1)(\alpha \otimes x) = s(\alpha^* \otimes x^*) = x^* \otimes \alpha^* \cong x^* \alpha^*\\
&= \kappa(\alpha x) = \kappa(x\alpha) \cong \kappa(x \otimes \alpha).
\end{aligned}$$

Finally, for $z\otimes\alpha$ we have

$$\begin{aligned}(\mathrm{id}_1\otimes\sigma\otimes\mathrm{id}_2)(\Delta_1\otimes\Delta_2)(z\otimes\alpha) &= (\mathrm{id}_1\otimes\sigma\otimes\mathrm{id}_2)(z\otimes x+x^*\otimes z)\otimes(\alpha\otimes\alpha)\\ &= (\mathrm{id}_1\otimes\sigma\otimes\mathrm{id}_2)(z\otimes x\otimes\alpha\otimes\alpha+x^*\otimes z\otimes\alpha\otimes\alpha)\\ &= z\otimes\alpha\otimes x\otimes\alpha+x^*\otimes a^*b^*\alpha\otimes z\otimes\alpha,\\ \Delta(z\otimes\alpha)\cong\Delta(z\alpha) &= (z\otimes x+a^*b^*x^*\otimes z)(\alpha\otimes\alpha)\\ &= z\alpha\otimes x\alpha+a^*b^*x^*\alpha\otimes z\alpha\\ &\cong (z\otimes\alpha)\otimes(x\otimes\alpha)+(x^*\otimes a^*b^*\alpha)\otimes(z\otimes\alpha),\end{aligned}$$

and

$$\begin{aligned}s(\kappa_2\otimes\kappa_1)\sigma(z\otimes\alpha) &= s(\kappa_2\otimes\kappa_1)(a^*b^*\alpha\otimes z)\\ &= s(ab\alpha^*\otimes z)=z\otimes ab\alpha^*\cong(zab)\alpha^*\\ &= \kappa(z)\kappa(\alpha)=\kappa(z\alpha)\cong\kappa(z\otimes\alpha).\end{aligned}$$

To complete the proof we observe that, by unitarity of a and b, the inverse of σ is

$$\sigma^{-1}(\alpha\otimes 1)=1\otimes\alpha,\quad \sigma^{-1}(\alpha\otimes x)=x\otimes\alpha,\quad \sigma^{-1}(\alpha\otimes z)=abz\otimes\alpha,$$

so σ is an isomorphism. ∎

COROLLARY 5.1. *The quantum group* $\mathbb{G}=(\mathsf{A},u)$ *is a twisted product of its quantum subgroups*:

$$\mathbb{G}=SU_{-1}(2)\ltimes_\sigma T_2,$$

where $\sigma:\mathsf{A}_1\otimes\mathsf{A}_2\to\mathsf{A}_2\otimes\mathsf{A}_1$ *is given by* (5.2).

REMARK 5.2. The quantum group $\mathbb{G}$ will inherit the main properties (like C*-algebraic properties, representation theory, subgroups, etc.) from $SU_{-1}(2)$, studied in [13], [5], and recently in [2]. As an example, in the next section we discuss briefly the representation theory of the related C*-algebra.

6. *-REPRESENTATIONS OF A

Let us remind (cf. [5], Corollary 5.23, and [2], Proposition 1.5, see also [13]) that there are three types of irreducible *-representations of $A=C\big(SU_{-1}(2)\big)$ (we use the notation where x and z are generators of $SU_{-1}(2)$):

1. $\pi_{(\xi,0)}:C\big(SU_{-1}(2)\big)\to\mathbb{C}$, $\pi_{(\xi,0)}(x)=\xi$, $\pi_{(\xi,0)}(z)=0$, $|\xi|=1$;
2. $\pi_{(0,\zeta)}:C\big(SU_{-1}(2)\big)\to\mathbb{C}$, $\pi_{(0,\zeta)}(x)=0$, $\pi_{(0,\zeta)}(z)=\zeta$, $|\zeta|=1$;
3. $\pi_{(\xi,\zeta)}:C\big(SU_{-1}(2)\big)\to M_2(\mathbb{C})$,

$$\pi_{(\xi,\zeta)}(x)=\begin{pmatrix}\xi&0\\0&-\xi\end{pmatrix},\quad \pi_{(\xi,\zeta)}(z)=\begin{pmatrix}0&\zeta\\ \zeta&0\end{pmatrix}.$$

In particular, the irreducible *-representations are one- or two-dimensional only.

Moreover, the complete family of inequivalent irreducible *-representations of $A = C\big(SU_{-1}(2)\big)$ is indexed by a quotient of

$$SU(2) \approx \{(\xi,\zeta) \in \mathbb{C}^2 : |\xi|^2 + |\zeta|^2 = 1\}$$

by the equivalence relation: if $\xi\zeta \neq 0$ then $(\xi,\zeta) \sim (\xi',\zeta')$ iff $\xi = \pm\xi'$ and $\zeta = \pm\zeta'$.

By the reasoning as in the proof of Proposition 1.5 in [2] one can show the following.

PROPOSITION 6.1. *The irreducible *-representations of $\mathbb{G}$ are one- or two-dimensional and are determined by four constants λ, μ, ξ, ζ such that $|\lambda| = 1$, $|\mu| = 1$, and $|\xi|^2 + |\zeta|^2 = 1$. They are of the form*

$$\pi(a) = \lambda I, \quad \pi(b) = \mu I, \quad \pi(x) = \pi_{(\xi,\zeta)}(x), \quad \pi(z) = \pi_{(\xi,\zeta)}(z). \tag{6.1}$$

Two *-representations $\pi_{\lambda,\mu,\xi,\zeta}$ and $\pi_{\lambda',\mu',\xi',\zeta'}$ are equivalent iff $\lambda = \lambda'$, $\mu = \mu'$, $\xi = \pm\xi'$, and $\zeta = \pm\zeta'$.

REFERENCES

[1] T. Banica and R. Speicher, *Liberation of orthogonal Lie groups*, Adv. Math. 222 (4) (2009), pp. 1461–1501.

[2] S. Barlak, *K-theory of $SUq(2)$ for $q = -1$*, available on arXiv:1302.0305v5 [math.OA].

[3] A. Kula, *Compact quantum group for transposition-coloring function II*, in preparation.

[4] P. Podleś and S. L. Woronowicz, *Quantum deformation of Lorentz group*, Comm. Math. Phys. 130 (1990), pp. 381–431.

[5] M. Skeide, *The Lévy–Khintchine formula for the quantum group $SU_q(2)$*, PhD thesis, Heidelberg 1994.

[6] P. M. Sołtan, *Quantum Bohr compactification*, Illinois J. Math. 49 (4) (2005), pp. 1245–1270.

[7] Shuzhou Wang, *Free products of compact quantum groups*, Comm. Math. Phys. 167 (3) (1995), pp. 671–692.

[8] S. L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. 111 (1987), pp. 613–665.

[9] S. L. Woronowicz, *Twisted $SU(2)$ group. An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. 23 (1) (1987), pp. 117–181.

[10] S. L. Woronowicz, *Tannaka–Kreĭn duality for compact matrix pseudogroups. Twisted $SU(N)$ groups*, Invent. Math. 93 (1) (1988), pp. 35–76.

[11] J. Wysoczański, *Twisted product structure and representation theory of the quantum group $U_q(2)$*, Rep. Math. Phys. 54 (3) (2004), pp. 327–347.

[12] J. Wysoczański, *On the Woronowicz's twisted product construction of quantum groups, with comments on related cubic Hecke algebra*, RIMS Kôkyûroku 1658 (2009), pp. 124–135.

[13] S. Zakrzewski, *Matrix pseudogroups associated with anti-commutative plane*, Lett. Math. Phys. 21 (1991), pp. 309–321.

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Received on 13.5.2013;
revised version on 28.6.2013