Monatsh Math (2013) 170:161–178
DOI 10.1007/s00605-012-0468-8

Uniqueness of minimal Fourier-type extensions in L_1-spaces

Grzegorz Lewicki · Agnieszka Micek

Received: 5 December 2011 / Accepted: 20 December 2012 / Published online: 18 January 2013

Abstract We give a characterization of uniqueness of finite rank Fourier-type minimal extensions in L_1-norm. This generalizes the main result obtained by Lewicki (Proceedings of the Fifth International Conference on Function Spaces, Lecture Notes in Pure and Applied Mathematics, vol. 213, pp. 337–345, 1998) to the case of n-circular sets in $\mathbb{C}^n$.

Keywords Fourier projection · Minimal extension · Uniqueness of minimal extension

Mathematics Subject Classification (2010) 41A35 · 41A52 · 46E30

1 Introduction

We start with some notation which will be used in this paper. By $S_X(x, r)$ (S_X if $x = 0$ and $r = 1$) we denote a sphere in a Banach space X with the center x and the radius r, by $\mathrm{ext}(S_X)$ the set of extreme points of S_X and the symbol X^* stands for a dual

---

Communicated by K. Gröchenig.

G. Lewicki was supported by the State Committee for Scientific Research, Poland (Grant No. N N201 541 438). A. Micek was supported by the State Commitee for Scientific Research, Poland (Grant No. N N201 609 740).

G. Lewicki · A. Micek (✉)
Department of Mathematics, Jagiellonian University,
Łojasiewicza 6, Kraków, Poland
e-mail: Agnieszka.Micek@im.uj.edu.pl

G. Lewicki
e-mail: Grzegorz.Lewicki@im.uj.edu.pl

space of X. An element $x \in X$ is called a norming point for $f \in X^*$ if $x \in S_X$ and $f(x) = \|x\|$. For $z \in \mathbb{C}$, $\text{sgn} z = \overline{z}/|z|$ for $z \neq 0$ and 0 for $z = 0$.

Let Y be a linear subspace of X and let $\mathcal{L}(X, Y)$ ($\mathcal{L}(X)$ if $X = Y$) be the space of all linear, continuous operators from X into Y. Given $A \in \mathcal{L}(Y)$, an operator $P \in \mathcal{L}(X, Y)$ is called an extension of A (or a projection in the case of $A = \text{Id}_Y$) if $P|_Y = A$. The set of all extensions of A will be denoted by $\mathcal{P}_A(X, Y)$. An extension $P_0 \in \mathcal{P}_A(X, Y)$ is minimal if

$$\|P_0\| = \lambda_A(X, Y) = \inf\{\|P\| : P \in \mathcal{P}_A(X, Y)\}, \tag{1}$$

We write briefly $\mathcal{P}(X, Y)$ and $\lambda(Y, X)$ instead of $\mathcal{P}_{\text{Id}_Y}(X, Y)$ and $\lambda_{\text{Id}_Y}(Y, X)$ respectively. Basic results on minimal projections and extensions can be found in [1,7,9,12,13,15–18,20,22,24]. Set

$$\mathcal{L}_Y(X, Y) = \{L \in \mathcal{L}(X, Y) : L|_Y = 0\}. \tag{2}$$

Let π_n denote the space of all trigonometric polynomials of degree $\leqslant n$ and let $\mathcal{C}_0(2\pi)$ be the space of all continuous, real valued, 2π-periodic functions. The classical Fourier projection from $\mathcal{C}_0(2\pi)$ onto π_n is defined by a formula

$$(F_n f)(t) = (f * D_n)(t) = (1/2\pi) \int_0^{2\pi} f(s) D_n(t - s)\, ds, \tag{3}$$

where $D_n(t) = \sum_{j=-n}^{n} e^{ijt}$. It is well-known that F_n is the unique operator of minimal norm in the space $\mathcal{P}(\mathcal{C}_0(2\pi), \pi_n)$ [10,23]. Moreover, $\|F_n\| = \lambda(\pi_n, L_p[0, 2\pi])$ for $1 \leqslant p \leqslant \infty$, which follows from Rudin Theorem [8], however, in general, it is an open question if F_n is the unique minimal projection from $L_p[0, 2\pi]$ onto π_n for $p \neq 1, 2, +\infty$. Partial results concerning subject can be found in [26] and [27].

In this paper we study the problem of the unique minimality of the Fourier-type extensions in the space L_1. More precisely, let M be a set, Σ-σ-algebra of subsets of M, ν a positive measure on Σ such that (M, Σ, ν) is a complete measure space. By $L_1(M, \Sigma, \nu)$ denote a space of complex-valued, ν-measurable functions on M satisfying a condition

$$\|f\|_1 = \int_M |f(z)| d\nu(z) < +\infty.$$

To the end of this paper we assume that $(L_1(M, \Sigma, \nu))^* = L_\infty(M, \Sigma, \nu)$, which is satisfied, for example, if ν is σ-finite.

Definition 1 It is said that $w \in V \subset L_\infty(M, \Sigma, \nu)$, $w \neq 0$ is determined by its roots in the space V if and only if for any $g \in V$ the condition $g/w \in L_\infty(M, \Sigma, \nu)$ implies that $g = cw$ for some $c \in \mathbb{C}$.

Definition 2 A subspace $V \subset L_1(M, \Sigma, \nu)$ is called smooth if and only if each member of $V \setminus \{0\}$ is almost everywhere different from 0.

Take V a smooth, finite-dimensional subspace of $L_1(M, \Sigma, \nu)$ with a basis $\{v_1, \ldots, v_k\}$ and fix an operator $A \in \mathcal{L}(V)$. Observe that any $P \in \mathcal{P}_A(L_1(M, \Sigma, \nu), V)$ has a form

$$Pf = \sum_{j=1}^{k} \widehat{u_j}(f)v_j, \quad \widehat{u_j}(v_i) = a_{j,i}, i, j = 1, \ldots, k, \text{ where} \tag{4}$$

$[a_{i,j}]_{i,j=1}^{k}$ is a matrix of the operator A in the basis $\{v_1, \ldots, v_k\}$ and $\widehat{u} \in (L_1(M, \Sigma, \nu))^*$ denotes a functional associated with $u \in L_\infty(M, \Sigma, \nu)$ by

$$\widehat{u}(f) = \int_M f(z)u(z)\, d\nu(z). \tag{5}$$

Let $P \in \mathcal{P}_A(L_1(M, \Sigma, \nu), V)$ be given by (4) and $z, w \in M$. Define

$$\begin{aligned} x_z^P(w) &= \sum_{j=1}^{k} u_j(z)v_j(w), \\ V_j(z) &= \int_M v_j(w)\mathrm{sgn}(x_z^P(w))\, d\nu(w), \\ P_z &= \int_M |x_z^P(w)|\, d\nu(w). \end{aligned} \tag{6}$$

A map $z \to P_z$ is called the Lebesgue function of the operator P. It is well known that

$$\|P\| = \operatorname*{ess\,sup}_{z\in M} P_z \text{ (see [14, Lem. 2]).} \tag{7}$$

Lemma 3 *Assume that $P_0 \in \mathcal{P}_A(L_1(M, \Sigma, \nu), V)$ and the Lebesgue function of the operator P_0 is constant on M (ν a.a.) Let $P_1, P_2 \in \mathcal{P}_A(L_1(M, \Sigma, \nu), V)$, $\|P_1\| = \|P_2\| = \|P_0\|$ and $P_0 = (P_1 + P_2)/2$. Then the Lebesgue functions of the operators $P_j : j = 1, 2$ are constant on M and for ν a.a. $z \in M$,*

$$sgn(x_z^{P_1}) = sgn(x_z^{P_2}) = sgn(x_z^{P_0}).$$

Proof Let

$$P_j(f) = \sum_{i=1}^{k} \widehat{u_{ji}}(f)v_i \,:\, j = 1, 2$$

for some $u_{ji} \in L_\infty(M, \Sigma, \nu) : j = 1, 2, i = 1, \dots, k$ [see (5)]. Then

$$P_0(f) = \frac{P_1(f) + P_2(f)}{2} = \sum_{i=1}^{k} \frac{1}{2}(\widehat{u_{1i}}(f) + \widehat{u_{2i}}(f))v_i.$$

For ν a.a. $z \in M$,

$$\begin{aligned} \|P_0\| = (P_0)_z &= \int_M \Big| \sum_{i=1}^{k} \frac{1}{2}[u_{1i}(z) + u_{2i}(z)]v_i(w) \Big| d\nu(w) \\ &\leqslant \frac{1}{2}\int_M \Big| \sum_{i=1}^{k} u_{1i}(z)v_i(w) \Big| d\nu(w) + \frac{1}{2}\int_M \Big| \sum_{i=1}^{k} u_{2i}(z)v_i(w) \Big| d\nu(w) \\ &= \frac{1}{2}(P_1)_z + \frac{1}{2}(P_2)_z \leqslant \frac{1}{2}(\|P_1\| + \|P_2\|) = \|P_0\|, \end{aligned}$$

so in the above inequalities we get equalities. In particular, for ν a.a. $z, w \in M$,

$$\Big| \sum_{i=1}^{k} [u_{1i}(z) + u_{2i}(z)]v_i(w) \Big| = \Big| \sum_{i=1}^{k} u_{1i}(z)v_i(w) \Big| + \Big| \sum_{i=1}^{k} u_{2i}(z)v_i(w) \Big|,$$

or equivalently

$$\operatorname{sgn}(x_z^{P_1}) = \operatorname{sgn}(x_z^{P_2}) = \operatorname{sgn}(x_z^{P_0})$$

and

$$(P_1)_z = (P_2)_z = \|P_0\|.$$

□

2 Main results

Now we introduce some notation which we will use in this section. We write briefly $re^{it} \in \mathbb{C}^n$ instead of $(r_1e^{it_1}, \dots, r_ne^{it_n}) \in \mathbb{C}^n$, and put $r = (r_1, \dots, r_n) \in [0, \infty)^n$, $t = (t_1, \dots, t_n) \in I$, $I = [0, 2\pi]^n$. The symbol $\mathbb{T}$ stands for the unit circle in a complex plane, i.e. $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Definition 4 A subset Z of $\mathbb{C}^n$ is called an n-circular set if for any $(z_1, \dots, z_n) \in Z$ and $(\delta_1, \dots, \delta_n) \in \mathbb{T}^n$, $(\delta_1 z_1, \dots, \delta_n z_n)$ belongs to Z.

A unit ball with p-norm for $p \geqslant 1$, $\mathbb{T}^n$ or $D_1 \times \cdots \times D_n$, where $D_j \subset \mathbb{C}$ for $j = 1, \dots, n$ denotes a classical geometric ring, are the examples of the n-circular sets.

Observe that any n-circular set Z can be written in a form

$$Z = \{re^{it} : r \in W \subset [0, \infty)^n,\ t \in I\}. \tag{8}$$

Let λ_W be a nonnegative measure on W such that $0 < \lambda_W(W) < \infty$. For example, for $Z = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and a Borel set $A \subset [0, 1]$, $\lambda_W(A) = \int_A r dr$ or for $Z = \bigcup_{j=1}^{p} \{z \in \mathbb{C} : |z| = r_j\}$, λ_W is a counting measure on $W = \{r_1, \ldots, r_p\}$. Let λ_I denote the normalized Lebesgue measure on I. Set

$$\mu = \lambda_W \times \lambda_I \text{ (the product measure of } \lambda_W \text{ and } \lambda_I \text{ on the set } W \times I). \tag{9}$$

Define a measure ν on Z associated with μ by

$$\nu(A) = \mu(\{(r, t) \in W \times I : re^{it} \in A\}). \tag{10}$$

Throughout the remainder of this paper the symbol $L_1(Z)$ stands for the space of all ν-measurable complex-valued functions on Z and such that

$$\|f\|_1 = \int_Z |f(z)|\, d\nu(z) = \iint_{W \times I} |f(re^{it})|\, d\mu(r, t) < \infty.$$

For $\beta \in \mathbb{N}^n$, $\alpha \in \mathbb{Z}^n$ define a function $e^{\beta,\alpha} \in L_1(Z)$ by

$$e^{\beta,\alpha}(re^{it}) = e^{\beta}(r) e^{\alpha}(e^{it}), \quad \text{where } e^{\gamma}(z) = \prod_{j=1}^{n} z^{\gamma_j} \text{ for } \gamma \in \mathbb{Z}^n, z \in \mathbb{C}^n \setminus \{0\}. \tag{11}$$

Fix for $j = 1, \ldots, k$ $a_j \in \mathbb{C}$, $\beta^j \in \mathbb{N}^n$ and $\alpha^j \in \mathbb{Z}^n$, $\alpha^i \neq \alpha^j$ for $i \neq j$. Set

$$V = \operatorname{span}\{e^{\beta^1,\alpha^1}, \ldots, e^{\beta^k,\alpha^k}\} \tag{12}$$

and

$$w = \sum_{j=1}^{k} a_j e^{\beta^j,\alpha^j}. \tag{13}$$

Define for $t \in I$ an operator $T_t : L_1(Z) \to L_1(Z)$ by

$$T_t(f)(ue^{is}) = f(ue^{i(s+t)}),\ s \in I. \tag{14}$$

Observe that T_t is an isometry and V is an invariant subspace of T_t. One can find that

$$T_t(e^{\beta,\alpha}) = e^{\beta,\alpha} \cdot e^{\alpha}(e^{it}). \tag{15}$$

Now we will search for a minimal extension of an operator $R_w = F_w|_V$, where $F_w \in \mathcal{L}(L_1(Z), V)$ is given by

$$(F_w f)(re^{it}) = (f * w)(re^{it}) = \iint_{W\times I} f(ue^{is})w(re^{i(t-s)})\, d\mu(u,s). \tag{16}$$

Remark 5 Let $n=1$, $Z=\mathbb{T}$, $V=\operatorname{span}\{e^{-k},\dots,e^k\}$ and $w=\sum_{j=-k}^{k} e^j$. Then F_w is a classical Fourier projection from $L_1(\mathbb{T})$ onto V.

Lemma 6 *The Lebesgue function of the operator F_w is constant and $\|F_w\| = \|w\|_1$.*

Proof By (13), (14) and (16),

$$\begin{aligned}(F_w f)(re^{it}) &= \iint_{W\times I} f(ue^{is})\left(\sum_{j=1}^{k} a_j e^{\beta^j,\alpha^j}\right)(re^{i(t-s)})\, d\mu(u,s)\\ &= \sum_{j=1}^{k} a_j \iint_{W\times I} f(ue^{is})e^{0,-\alpha_j}(ue^{is})\, d\mu(u,s) e^{\beta^j,\alpha^j}(re^{it}).\end{aligned}$$

Hence

$$F_w f = \sum_{j=1}^{k} a_j \widehat{e^{0,-\alpha_j}}(f) e^{\beta^j,\alpha^j}, \tag{17}$$

where for $v \in L_\infty(Z)$,

$$\hat{v}(f) = \iint_{W\times I} f(re^{it})v(re^{it})\, d\mu(r,t). \tag{18}$$

Combining it with (6) and (17) we get that for ν a.a. $ue^{is} \in Z$,

$$\begin{aligned}x^{F_w}_{ue^{is}}(re^{it}) &= \sum_{j=1}^{k} a_j e^{0,-\alpha_j}(ue^{is})e^{\beta^j,\alpha^j}(re^{it})\\ &= \sum_{j=1}^{k} a_j e^{\beta^j,\alpha^j}(re^{i(t-s)}) = w(re^{i(t-s)})\end{aligned} \tag{19}$$

and

$$\begin{aligned}(F_w)_{ue^{is}} &= \iint_{W\times I} |x^{F_w}_{ue^{is}}(re^{it})|\, d\mu(r,t) = \iint_{W\times I} |w(re^{i(t-s)})|\, d\mu(r,t)\\ &= \iint_{W\times I} |w(re^{it})|\, d\mu(r,t) = \|w\|_1.\end{aligned}$$

By (7),

$$\|F_w\| = \text{ess} \sup_{ue^{is} \in Z} (F_w)_{ue^{is}} = \|w\|_1.$$

□

Now we formulate three lemmas whose proofs go in the same manner as the proofs of Lemmas 1.3–1.5 in [19], so we omit them.

Lemma 7 *A subspace $V \subset L_1(Z)$ defined by (12) is smooth (see Definition 2).*

Lemma 8 *For N a finite subset of $\mathbb{Z}^n$ there exists a real function $f \in L_\infty(Z)$, $f \neq 0$ such that*

$$\iint\limits_{W \times I} f(re^{it}) e^{\beta,\alpha}(re^{it})\, d\mu(r,t) = 0 \ \text{for} \ \alpha \in N \ \text{and} \ \beta \in \mathbb{N}^n.$$

Lemma 9 *Assume that $g, w \in S_V$, $g/w \in L_\infty(Z)$, $sgn(g) = sgn(w)$ ν almost everywhere in Z. Then for any $\varepsilon \in \mathbb{R}$ such that $|\varepsilon| < (\|(g-w)/w\|_\infty)^{-1}$ we have*

$$sgn(w + \varepsilon(g - w)) = sgn(w) \ \nu \ a.e.$$

Now define

$$X = \text{span}\{e^{\beta,\alpha^j} : \beta \in \mathbb{N}^n, \quad j = 1, \ldots, k\} \text{ [see (11)]}. \tag{20}$$

Observe that (12) and (20) imply that $V \subset X$. Set

$$\mathcal{L}_X(L_1(Z), V) = \{L \in \mathcal{L}(L_1(Z), V) : L|_X = 0\} \tag{21}$$

and

$$\mathcal{P}_X(L_1(Z), V) = F_w + \mathcal{L}_X(L_1(Z), V) \text{ [see (2)]}. \tag{22}$$

We say that $P_0 \in \mathcal{P}_X(L_1(Z), V)$ is a minimal extension of the operator R_w in the set $\mathcal{P}_X(L_1(Z), V)$, if

$$\|P_0\| = \inf\{\|P\| : P \in \mathcal{P}_X(L_1(Z), V)\}.$$

It is easy to see that $\mathcal{P}_X(L_1(Z), V) \subset \mathcal{P}_{R_w}(L_1(Z), V)$. Denote

$$M_w^X \text{—the set of minimal extensions of } R_w \text{ in the space } \mathcal{P}_X(L_1(Z), V). \tag{23}$$

Theorem 10 *The operator F_w is the unique extension of R_w belonging to the space $\mathcal{P}_X(L_1(Z), V)$ and commutative with a group $\{T_t : t \in I\}$.*

Proof By the Stone–Weierstrass Theorem and a density of the continuous functions in the space $L_1(Z)$,

$$L_1(Z) = \overline{\mathrm{span}\{e^{\beta,\alpha} : \alpha \in \mathbb{Z}^n, \beta \in \mathbb{N}^n\}}. \tag{24}$$

Let $P = \sum_{j=1}^k \widehat{w_j}(\cdot)e^{\beta^j,\alpha^j}$ be a minimal extensionof R_w in the space $\mathcal{P}_X(L_1(Z), V)$ which commutes with the group of isometries $\{T_t : t \in I\}$. We show that $P(e^{\beta,\alpha}) = F_w(e^{\beta,\alpha})$ for $\beta \in \mathbb{N}^n, \alpha \in \mathbb{Z}^n$. If $\beta \in \mathbb{N}^n, \alpha \in \{\alpha^j : j = 1, \dots, k\}$ the above inequality follows from the fact that $P \in \mathcal{P}_X(L_1(Z), V)$. If $\beta \in \mathbb{N}^n, \alpha \notin \{\alpha^j : j = 1, \dots, k\}$ the condition

$$T_s \circ P(e^{\beta,\alpha}) = P \circ T_s(e^{\beta,\alpha}) \quad \text{for } s \in I$$

is equivalent to

$$\sum_{j=1}^k \widehat{w_j}(e^{\beta,\alpha})e^{\beta^j,\alpha^j} \cdot e^{\alpha^j}(e^{is}) = \sum_{j=1}^k \widehat{w_j}(e^{\beta,\alpha})e^{\beta^j,\alpha^j} \cdot e^{\alpha}(e^{is}), \quad s \in I.$$

Hence

$$\sum_{j=1}^k \widehat{w_j}(e^{\beta,\alpha})e^{\beta^j,\alpha^j} \cdot (e^{\alpha^j}(e^{is}) - e^{\alpha}(e^{is})) = 0, \quad s \in I.$$

By linear independence of the functions $\{e^{\beta^j,\alpha^j}\}_{j=1}^k$ we get that $\widehat{w_j}(e^{\beta,\alpha}) = 0$ for $j = 1, \dots, k$. Hence $P(e^{\beta,\alpha}) = 0 = F_w(e^{\beta,\alpha})$ for $\alpha \notin \{\alpha^1, \dots, \alpha^k\}$, $\beta \in \mathbb{N}^n$, which by (24) shows that $P = F_w$. □

Theorem 11 *F_w is a minimal extensionof R_w in the set $\mathcal{P}_X(L_1(Z), V)$ and for any $P \in \mathcal{P}_X(L_1(Z), V)$,*

$$F_w = \int_I T_s^{-1} P T_s \, d\lambda_I(s).$$

Proof Let $P \in M_w^X$ [see (23)]. Define

$$Q_P = \int_I T_s^{-1} P T_s \, d\lambda_I(s).$$

By (15) we obtain that $Q_P \in \mathcal{P}_X(L_1(Z), V)$. Properties of the Lebesgue measure imply that Q_P is an operator commutative with a group of isometries $\{T_t : t \in I\}$. By Theorem 10, $Q_P = F_w$. Since $\{T_t : t \in I\}$ are the isometries, $\|F_w\| = \|Q_P\| \leqslant \|P\|$, which completes the proof of minimality of an operator F_w in the space $\mathcal{P}_X(L_1(Z), V)$. □

A convenient tool for studying minimality of Fourier-type extensions in the space $\mathcal{P}_A(L_1(M, \Sigma, \nu), V)$ is the following:

Theorem 12 ([5, Cor. 1]) *Let V be a smooth, k-dimensional subspace of $L_1(M, \Sigma, \nu)$ with a basis $\{v_1, \ldots, v_k\}$. Then P is a minimal extension of the operator A if and only if two conditions are satisfied:*

(i) *the Lebesgue function of the operator P is constant on M;*
(ii) *there exist a matrix $B = [B_{ij}]_{i,j=1}^k$ and a positive function Φ such that for $v = (v_1, \ldots, v_k)$,*

$$\Phi(z)(V_1(z), \ldots, V_k(z)) = Bv(z) = \left[\sum_{j=1}^{k} B_{ij} v_j(z)\right]_{i=1}^{k}. \tag{25}$$

Theorem 13 *Assume that $\#\{a_j \neq 0 : j = 1, \ldots, k\} \geqslant 2$ [see (13)]. Let Z be an n-circular setsuch that $\{e^{\beta^j}|_W\}_{j=1}^k$ are linearly independent functions. Then F_w is not a minimal extensionof an operator R_w.*

Proof Assume on the contrary that $F_w \in \mathcal{P}_{R_w}(L_1(Z), V)$ is a minimal extensionof an operator R_w. By Theorem 10, F_w commutes with a group $G = \{T_t : t \in I\}$. Taking ν_1 a Haar measure on G and $\int_I {T_s}^{-1} B T_s d\nu_1(s)$ instead of a matrix B we can assume that the matrix B from Theorem 12 is commutative with G. It is easy to check that such a matrix is diagonal. Set $B = diag(B_1, \ldots, B_k)$. We calculate [see (6) and (19)],

$$\begin{aligned} V_j(re^{it}) &= \iint\limits_{W\times I} e^{\beta^j,\alpha^j}(ue^{is})\mathrm{sgn}(x_{re^{it}}^{F_w}(ue^{is}))\, d\mu(u,s) \\ &= \iint\limits_{W\times I} e^{\beta^j,\alpha^j}(ue^{is})\mathrm{sgn}(w(ue^{i(s-t)}))\, d\mu(u,s) \\ &= \iint\limits_{W\times I} e^{\beta^j,\alpha^j}(ue^{i(s+t)})\mathrm{sgn}(w(ue^{is}))\, d\mu(u,s) \\ &= e^{\alpha^j}(e^{it}) \iint\limits_{W\times I} e^{\beta^j,\alpha^j}(ue^{is})\mathrm{sgn}(w(ue^{is}))\, d\mu(u,s) = C_j e^{\alpha^j}(e^{it}), \end{aligned}$$

where $C_j = \iint\limits_{W\times I} e^{\beta^j,\alpha^j}(ue^{is})\mathrm{sgn}(w(ue^{is}))d\mu(u,s)$. Now the condition (ii) of Theorem 12 [see (25)] is equivalent to

$$\Phi(re^{it})(C_1 e^{\alpha^1}(e^{it}), \ldots, C_k e^{\alpha^k}(e^{it})) = (B_1 e^{\beta^1,\alpha^1}(re^{it}), \ldots, B_k e^{\beta^k,\alpha^k}(re^{it})). \tag{26}$$

By [5, Lem. 4], dim span$\{V_1, \ldots, V_k\} \geqslant 2$. Hence there exist $j_1, j_2 \in \{1, \ldots, k\}$ such that $C_{j_1} \neq 0$ i $C_{j_2} \neq 0$ and

$$\Phi(re^{it}) = \frac{B_{j_1}}{C_{j_1}} e^{\beta^{j_1}}(r) = \frac{B_{j_2}}{C_{j_2}} e^{\beta^{j_2}}(r) \quad \text{for } r \in W,$$

which leads to a contradiction with a linear independence of the functions $e^{\beta^{j_1}}|_W$ and $e^{\beta^{j_2}}|_W$. By Theorem 12, F_w is not a minimal extensionof R_w. $\square$

Remark 14 [19] In the case $Z = \mathbb{T}^n$, the operator F_w is a minimal extensionof R_w in the whole space $\mathcal{P}_{R_w}(L_1(\mathbb{T}^n), V)$ [it is sufficient to take $\Phi \equiv 1$ and $B_j = C_j$ for $j = 1, \ldots, k$ in the equality (26)].

Theorem 15 *An operator F_w is an extreme point of the set $S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V)$ if and only if $w/\|w\|_1$ is a unique norming point $g \in S_V$ for a functional*

$$L_1(Z) \ni h \mapsto \int\limits_Z h(z) sgn(w)(z)\, d\nu(z) = \iint\limits_{W\times I} h(re^{it}) sgn(w)(re^{it})\, d\mu(r,t)$$

such that $g/w \in L_\infty(Z)$.

Proof Assume that there exists $g \in S_V$, $g \neq w$, $g/w \in L_\infty(Z)$ such that g is a norming point for $\widehat{\mathrm{sgn}(w)}$ [see (18)], i.e. $\mathrm{sgn}(w) = \mathrm{sgn}(g)$ ν a.e. Define $h = g - w \in V$. By Lemma 8, there exists a real function $f \in L_\infty(Z)$ which is orthogonal to $e^{\beta,\alpha}$ for $\beta \in \mathbb{N}^n$, $\alpha \in \tilde{V} - \tilde{V}$, where $\tilde{V} = \{\alpha : e^{\beta,\alpha} \in V\}$. We can assume that $\|f\|_\infty < (\|h/w\|_\infty)^{-1}$. By Lemma 9,

$$\mathrm{sgn}(w)(re^{it}) = \mathrm{sgn}\left(w \pm f(ue^{is}) \cdot h\right)(re^{it}) \quad \text{for } \nu \text{ a.a. } re^{it}, ue^{is} \in Z. \tag{27}$$

Set $Q_1 = F_w + L$ and $Q_2 = F_w - L$, where

$$(Lk)(re^{it}) = \iint\limits_{W\times I} f(ue^{is}) k(ue^{is}) h(re^{i(t-s)})\, d\mu(u,s). \tag{28}$$

For any $k \in L_1(Z)$ a function $Lk \in V$ because $h = \sum_{j=1}^k B_j e^{\beta_j, \alpha^j}$ for some $B_j \in \mathbb{C}$, $j = 1 \ldots, k$ and

$$\begin{aligned} Lk(re^{it}) &= \iint\limits_{W\times I} f(ue^{is}) k(ue^{is}) \left(\sum_{j=1}^k B_j e^{\beta^j, \alpha^j}(re^{i(t-s)}) \right) d\mu(u,s) \\ &= \sum_{j=1}^k B_j \widehat{f e^{0,-\alpha^j}}(k) e^{\beta^j, \alpha^j}(re^{it}). \end{aligned}$$

Observe that $L \neq 0$. We calculate,

$$x^L_{ue^{is}}(re^{it}) = \sum_{j=1}^{k} B_j f(ue^{is}) e^{0,-\alpha^j}(ue^{is}) e^{\beta^j,\alpha^j}(re^{it}) \\ = \sum_{j=1}^{k} B_j e^{\beta^j,\alpha^j}(re^{i(t-s)}) f(ue^{is}) = h(re^{i(t-s)}) f(ue^{is}). \tag{29}$$

By the properties of f,

$$L(e^{\beta,\alpha}) = \sum_{j=1}^{k} B_j \widehat{f e^{0,-\alpha^j}}(e^{\beta,\alpha}) \\ = \sum_{j=1}^{k} B_j \iint\limits_{W\times I} f(ue^{is}) e^{\beta,\alpha-\alpha^j}(ue^{is})\, d\mu(u,s) = 0 \quad \text{for } \alpha \in \tilde{V}.$$

Hence Q_1 and Q_2 are the minimal extensions of R_w and $Q_j \neq F_w : j = 1, 2$ (since $L \neq 0$). By (19) and (27)–(29) for ν a.a. $ue^{is} \in Z$ and $j = 1, 2$,

$$(Q_j)_{ue^{is}} = \iint\limits_{W\times I} |x^{F_w \pm L}_{ue^{is}}(re^{it})|\, d\mu(r,t) = \iint\limits_{W\times I} |(x^{F_w}_{ue^{is}} \pm x^L_{ue^{is}})(re^{it})|\, d\mu(r,t) \\ = \iint\limits_{W\times I} |w(re^{i(t-s)}) \pm h(re^{i(t-s)}) f(ue^{is})|\, d\mu(r,t) \\ = \iint\limits_{W\times I} \operatorname{sgn}[(w \pm h)(re^{i(t-s)}) f(ue^{is})][(w \pm h)(re^{i(t-s)}) f(ue^{is})]\, d\mu(r,t) \\ = \iint\limits_{W\times I} \operatorname{sgn}(w)(re^{i(t-s)})(w(re^{i(t-s)}) \pm h(re^{i(t-s)}) f(ue^{is}))\, d\mu(r,t) \\ = \iint\limits_{W\times I} |w(re^{i(t-s)})|\, d\mu(r,t) \pm f(ue^{is}) \iint\limits_{W\times I} (\operatorname{sgn}(w)\cdot(g-w))(re^{i(t-s)})\, d\mu(r,t) \\ = \|w\|_1 \pm f(ue^{is})(\|g\|_1 - \|w\|_1) = \|w\|_1 = \|F_w\|.$$

Applying (7) we obtain that $\|Q_1\| = \|Q_2\| = \|F_w\|$. Since $F_w = (Q_1 + Q_2)/2$, F_w is not an extreme point of the set $\mathcal{P}_{R_w}(L_1(Z), V) \cap S(0, \|w\|_1)$.

Now assume that F_w is not an extreme point of the set $\mathcal{P}_{R_w}(L_1(Z), V) \cap S(0, \|w\|_1)$. Hence there exist $P_1, P_2 \in S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V)$ such that $P_j \neq F_w : j = 1, 2$ and $F_w = (P_1 + P_2)/2$. Define $L = (P_1 - P_2)/2$. Then $P_1 = F_w + L$ and $P_2 = F_w - L$. By Lemma 6, the Lebesgue function of the operator F_w is constant.

We have $\|F_w + L\| = \|F_w - L\| = \|w\|_1$ and by Lemma 3, for ν a.a. $ue^{is} \in Z$,

$$(F_w + L)_{ue^{is}} = (F_w - L)_{ue^{is}} = \|w\|_1,$$
$$\operatorname{sgn}(x^{F_w+L}_{ue^{is}}) = \operatorname{sgn}(x^{F_w-L}_{ue^{is}}) = \operatorname{sgn}(x^{F_w}_{ue^{is}}). \tag{30}$$

Let us fix $ue^{is} \in Z$ satisfying (30) and such that

$$x^L_{ue^{is}} \neq 0 \text{ and } x^{F_w}_{ue^{is}}(re^{i(t+s)}) = w(re^{it}) \text{ [see (19)]}. \tag{31}$$

Without loss of generality we can assume that $\|w\|_1 = 1$. Set

$$h = T_s(x^L_{ue^{is}}) \text{ [see (14)]}, \quad g = w + h. \tag{32}$$

Since $h \neq 0$, $g \neq w$. Observe that by (30)–(32) for ν a.a. $re^{it} \in Z$,

$$\begin{aligned} g(re^{it}) &= (w + h)(re^{it}) = w(re^{it}) + T_s(x^L_{ue^{is}})(re^{it}) \\ &= x^{F_w}_{ue^{is}}(re^{i(t+s)}) + x^L_{ue^{is}}(re^{i(t+s)}) = x^{F_w+L}_{ue^{is}}(re^{i(t+s)}) \end{aligned} \tag{33}$$

and

$$\operatorname{sgn}(g)(re^{it}) = \operatorname{sgn}(x^{F_w+L}_{ue^{is}})(re^{i(t+s)}) = \operatorname{sgn}(x^{F_w}_{ue^{is}})(re^{i(t+s)}) = \operatorname{sgn}(w)(re^{it}),$$
$$\operatorname{sgn}(w - h)(re^{it}) = \operatorname{sgn}(x^{F_w-L}_{ue^{is}})(re^{i(t+s)}) = \operatorname{sgn}(x^{F_w}_{ue^{is}})(re^{i(t+s)}) = \operatorname{sgn}(w)(re^{it}).$$

Hence

$$\operatorname{sgn}(g) = \operatorname{sgn}(w) = \operatorname{sgn}(w - h) \ \nu \text{ a.e.} \tag{34}$$

By (30) and (33),

$$\begin{aligned} \|g\|_1 &= \iint\limits_{W \times I} |g(re^{it})| d\mu(r, t) = \iint\limits_{W \times I} |x^{F_w+L}_{ue^{is}}(re^{i(t+s)})| \, d\mu(r, t) \\ &= \iint\limits_{W \times I} |x^{F_w+L}_{ue^{is}}(re^{it})| \, d\mu(r, t) = (F_w + L)_{ue^{is}} = \|w\|_1 = 1. \end{aligned}$$

Now we show that $\frac{g}{w} \in L_\infty(Z)$. Assume on the contrary that $\frac{g}{w} \notin L_\infty(Z)$. Then for any $k \in \mathbb{N}$ there exists $A_k \subset Z : \nu(A_k) > 0$ and $|g(z)/w(z)| > k$ for $z \in A_k$. Let $z \in A_k$ and $a_z = \operatorname{sgn}(w)(z)$. By (34),

$$a_z(w(z) + h(z)) > k a_z w(z)$$

and

$$a_z(w(z) - h(z)) < (2 - k)|w(z)| \leqslant 0 \quad \text{for } k > 2.$$

The above inequality implies that

$$\operatorname{sgn}(w-h)(z) \neq \operatorname{sgn}(w)(z) \quad \text{for } z \in A_k, k > 2;$$

a contradiction with (34). □

Lemma 16 *F_w is the unique minimal extensionof R_w in the space $\mathcal{P}_X(L_1(Z), V)$ if and only ifF_w is an extreme point of the set M_w^X [see (23)].*

The proof of Lemma 16 goes in the same manner as the proof of [19, Lem. 1.2]*, so we omit it.*

Now we can formulate a main result of this paper, a theorem, which characterize the uniqueness of minimal Fourier-type extensions in the set $\mathcal{P}_X(L_1(Z), V)$.

Theorem 17 *The operator F_w is the unique minimal extensionof $R_w = F_w|_V$ in the set $\mathcal{P}_X(L_1(Z), V)$ if and only if$w/\|w\|_1$ is a unique norming point $g \in V$ for a functional*

$$L_1(Z) \ni h \mapsto \iint\limits_{W \times I} h(re^{it}) sgn(w)(re^{it})\, d\mu(r,t)$$

such that $g/w \in L_\infty(Z)$.

Proof Assume that there exists $g \in S_V, g \neq w, g/w \in L_\infty(Z)$ such that g is a norming point for $\widehat{\operatorname{sgn}(w)}$ [see (18)], i.e. $\operatorname{sgn}(w) = \operatorname{sgn}(g)\ \nu$ a.e. Notice that the operator L constructed in Theorem 15 [see (28)] is actually an element of the space $\mathcal{L}_X(L_1(Z), V)$ and extensions $Q_1 = F_w + L$ and $Q_2 = F_w - L$ belong to the set $\mathcal{P}_X(L_1(Z), V)$. By the first part of the proof of Theorem 15, F_w is not an extreme point of the set M_w^X [see (23)].

Now assume that $w/\|w\|_1$ is a unique norming point $g \in V$ for a functional

$$L_1(Z) \ni h \mapsto \iint\limits_{W \times I} h(re^{it})\operatorname{sgn}(w)(re^{it})\, d\mu(r,t)$$

such that $g/w \in L_\infty(Z)$. By Theorem 15, F_w is not an extreme point of the set $S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V)$. Since $M_w^X \subset S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V)$, we get that

$$\operatorname{ext}\left[S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V)\right] \subset \operatorname{ext} M_w^X$$

and F_w is not an extreme point of the set M_w^X. By Lemma 16 the proof is complete. □

Directly from Theorem 17, reasoning as in the proof of [19, Cor. 1.9], we get the following result:

Corollary 18 *Let $w \in V$, $w \neq 0$ is determined by its roots in V (Definition 1). Then the operator F_w is the unique minimal extensionof R_w in the set $\mathcal{P}_X(L_1(Z), V)$*

The next theorem shows how large the set of minimal extensions can be. We present it without proof. The reader interested in the method of proof is referred to [19, Th. 1.7].

Theorem 19 *Let*

$$S_w = span\{P - F_w : P \in M_w^X\}.$$

If the operator F_w is not a unique minimal extensionof the operator R_w in the set $\mathcal{P}_X(L_1(Z), V)$, then $\dim(S_w) = \infty$.

Theorem 20 (Daugavet [11]) *Let K be a compact set without isolated points. If $L : \mathcal{C}_{\mathbb{K}}(K) \to \mathcal{C}_{\mathbb{K}}(K)$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is a compact operator, then*

$$\|Id + L\| = 1 + \|L\|.$$

Denote

$$(\mathcal{P}_X(L_1(Z), V))^* = \{P^* : P \in \mathcal{P}_X(L_1(Z), V)\}.$$

We say that an operator ${P_0}^* \in (\mathcal{P}_X(L_1(Z), V))^*$ is an element of best approximation to $A : L_\infty(Z) \to L_\infty(Z)$ in the set $(\mathcal{P}_X(L_1(Z), V))^*$ if

$$\|A - {P_0}^*\| = \inf\{\|A - P^*\| : P^* \in (\mathcal{P}_X(L_1(Z), V))^*\}.$$

In the same manner as in [20, Th.1.9] we can prove:

Theorem 21 *Let Z be a compact n-circular set. An identity operator $Id : L_\infty(Z) \to L_\infty(Z)$ has the unique element of best approximation in $(\mathcal{P}_X(L_1(Z), V))^*$ if and only ifw$/\|w\|_1$ is a unique norming point $g \in V$ for a functional*

$$L_1(Z) \ni h \mapsto \iint\limits_{W \times I} h(re^{it}) sgn(w)(re^{it})\, d\mu(r, t)$$

such that $g/w \in L_\infty(Z)$.

3 Applications

Now we show some applications of Theorem 17.

Theorem 22 [25, Th.14.3.3] *Let Ω be a bounded domain in $\mathbb{C}^n$, $n > 1$. If f and g are holomorphic functions on Ω, continuous in $\bar{\Omega}$ and such that*

$$|f(z)| \leqslant |g(z)| \quad for \quad z \in \partial\Omega,$$

then

$$|f(z)| \leqslant |g(z)| \quad for \quad z \in \Omega.$$

Directly from Theorem 17 and Theorem 22 we get the following two examples:

Example 23 (Uniqueness) Let D be a bounded n-circular domain in $\mathbb{C}^n$, $n \geqslant 2$ and let $Z = \partial D$. Assume that V is the space of algebraic polynomials of n complex variables of degree $\leqslant k$ and fix $w \in V$. We prove that the operator F_w is the unique minimal extensionof R_w in the set $\mathcal{P}_X(L_1(Z), V)$. Indeed, by Theorem 17 it is sufficient to show that if $g \in V$ satisfies the conditions $\operatorname{sgn}(g)(z) = \operatorname{sgn}(w)(z)$ for $z \in Z$, $\|g|_Z\|_1 = 1$ and $g/w \in L_\infty(Z)$, then $g|_Z = w|_Z/\|w|_Z\|_1$. Take g as we mentioned above. Define $F_1 = g + iw$, $F_2 = g - iw$. Observe that F_1 and F_2 are holomorphic functions on D and continuous in $\bar{D}$. By assumptions, $g(z)\overline{w(z)} \in \mathbb{R}]$ for $z \in Z$ and

$$|F_1(z)| = |F_2(z)| = \sqrt{|g(z)|^2 + |w(z)|^2}, z \in Z.$$

Applying twice Theorem 22, we get that $|F_1(z)| = |F_2(z)|$ for $z \in D$. Put $G = D \setminus \{F_2 = 0\}$ and $h(z) = F_1(z)/F_2(z)$ for $z \in G$. Note that h is holomorphic in the domain G and $|h| = 1$ on G. Since nonconstant holomorphic functions are open mappings, $h|_G = c$ for some $c \in \mathbb{C}$. A condition

$$\|g|_Z\|_1 = \left\| \frac{w|_Z}{\|w|_Z\|_1} \right\|_1 = 1$$

implies that $g|_Z = w|_Z/\|w|_Z\|_1$.

Reasoning as in Example 23 we get:

Example 24 (Uniqueness) Let Z be an n-circular domain, $n \geqslant 2$. Assume that V is a space of algebraic polynomials of n complex variables of degree $\leqslant k$ and fix $w \in V$. Then the operator F_w is the unique minimal extensionof R_w in the set $\mathcal{P}_X(L_1(Z), V)$.

Now we give an example of n-circular set$Z$, a smooth space V and $w \in V$, for which the operator F_w is not a unique minimal extensionof R_w in the set $\mathcal{P}_X(L_1(Z), V)$.

Example 25 (Nonuniqueness) Let V be a space of algebraic polynomials of n complex variables of degree $\leqslant k$. Set $Z = \mathbb{T}^n$. For $s \in \mathbb{T}$ define

$$h(s) = s^2 + l(1+b)s + b,$$
$$k(s) = s^2 + m(1+b)s + b,$$

where $l, m \in (0, \infty) \setminus \{1\}$, $m \neq l$, $|b| = 1$, $b \neq -1$ are such that polynomials h and k have all roots outside of $\mathbb{T}$ and

$$ml(1 + Reb) + (m + l)Re((1 + \overline{b})s) \geqslant 0 \text{ for } s \in \mathbb{T}.$$

Observe that by our assumptions,

$$\begin{aligned} h(s)\overline{k(s)} &= (s^2 + b + l(1+b)s)(\overline{s}^2 + \overline{b} + m(1+\overline{b})\overline{s}) \\ &= |s^2 + b|^2 + m(1+\overline{b})s + m(1+b)\overline{s} + l(1+b)\overline{s} + l(1+\overline{b})s + ml|1+b|^2 \\ &= |s^2 + b|^2 + 2ml(1 + Reb) + 2(m+l)Re[(1+\overline{b})s] \geqslant 0. \end{aligned}$$

Consider any polynomial l of n variables of degree $k-2$. Put

$$w(s,t)=h(s)l(s,t),\quad g(s,t)=k(s)l(s,t)\quad \text{for } s\in\mathbb{T}, t\in\mathbb{T}^{n-1}.$$

Then $g,w\in V$, $g/w\in L_\infty(Z)$ and $w(s,t)\overline{g(s,t)}=h(s)\overline{k(s)}|l(s,t)|^2\geqslant 0$ for $(s,t)\in\mathbb{T}^n$. Hence $\mathrm{sgn}(w)=\mathrm{sgn}(g)$ and by Theorem 17, F_w is not a minimal extensionof R_w in the set M_w^X. In this case the assertion of Theorem 19 is fulfilled.

Other examples in which there is more than one minimal extensionof R_w can be found in [19].

Notice that methods of proofs used in this paper can be applied not only in the case of n-circular sets, but also to $Z=[0,2\pi]^n\times\mathbb{T}^n$. More precisely, let $L_1([0,2\pi]^n\times\mathbb{T}^n)$ denote the space of complex-valued functions, Lebesgue measurable on $[0,2\pi]^n\times\mathbb{T}^n$ and such that

$$\|f\|_1=(1/2\pi)^{2n}\iint\limits_{I\times I}|f(u,e^{is})|\,duds=\iint\limits_{I\times I}|f(u,e^{is})|\,d\mu(u,s)<\infty,$$

where

$$\mu=\lambda_I\times\lambda_I,\ \lambda_I\text{—a normalized Lebesgue measure on } I=[0,2\pi]^n. \tag{35}$$

For $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n$ and $r=(r_1,\ldots,r_n)\in[0,2\pi]^n$ put

$$\begin{aligned}G^\alpha&=\{f: f(r)=f_1(r_1)\cdot\ldots\cdot f_n(r_n),\quad f_j=\cos(\alpha_j\cdot)\text{ or}\\ f_j&=\sin(\alpha_j\cdot): j=1,\ldots,n\}\end{aligned} \tag{36}$$

Let

$$G^\alpha=\{G_j^\alpha: j=1,\ldots,2^n\}.$$

Let $\alpha\in\mathbb{Z}^n$, $\beta\in\mathbb{Z}^n$ and $j\in\{1,\ldots,2^n\}$. Define a function

$$G_j^{\alpha,\beta}: L_1([0,2\pi]^n\times\mathbb{T}^n)\ni(r,e^{it})\mapsto G_j^\alpha(r)e^\beta(e^{it})\in\mathbb{C}, \tag{37}$$

where e^β is given by the formula (11). Let

$$\begin{aligned}V&=\mathrm{span}\{G_j^{\alpha^p,\beta^p}: p=1,\ldots,k,\ j=1,\ldots,2^n\},\\ w&=\sum_{\substack{p=1\ldots,k\\ j=1,\ldots,2^n}} b_{p,j}G_j^{\alpha^p,\beta^p},\end{aligned} \tag{38}$$

for some $\alpha^p \in \mathbb{Z}^n$, $\beta^p \in \mathbb{Z}^n$ such that $(\alpha^p, \beta^p) \neq (\alpha^m, \beta^m)$ for $p \neq m$ and $b_{p,j} \in \mathbb{C}$. Set

$$(F_w f)(r, e^{it}) = (f * w)(r, e^{it}) = \iint_{I \times I} f(u, e^{is}) w(r - u, e^{i(t-s)})\, d\mu(u, s). \tag{39}$$

Applying a group of isometries $\tilde{G} = \{T_{u,s}(f)(r, e^{it}) = f(r+u, e^{i(s+t)})\}_{u,s \in [0,2\pi]^n}$ instead of $G = \{T_t\}_{t \in [0,2\pi]^n}$ [see (14)] and reasoning in the same manner as in the case of n-circular sets, we can obtain the following:

Theorem 26 *The operator F_w is the unique minimal extensionof R_w in the set $\mathcal{P}_{R_w}(L_1([0, 2\pi]^n \times \mathbb{T}^n), V)$ if and only if $w/\|w\|_1$ is a unique norming point $g \in V$ for a functional*

$$L_1([0, 2\pi]^n \times \mathbb{T}^n) \ni h \mapsto \iint_{I \times I} h(r, e^{it}) sgn(w)(r, e^{it})\, d\mu(r, t)$$

such that $g/w \in L_\infty([0, 2\pi]^n \times \mathbb{T}^n)$.

Here we have the uniqueness in the whole space $\mathcal{P}_{R_w}(L_1([0, 2\pi]^n \times \mathbb{T}^n), V)$ because the group $\tilde{G}$ is so big that F_w is the unique operator in $\mathcal{P}_{R_w}(L_1([0, 2\pi]^n \times \mathbb{T}^n), V)$ commuting with $\tilde{G}$. As a consequence, F_w has a minimal norm in the space $\mathcal{P}_{R_w}(L_1(Z), V)$, which is not true in the case of an arbitrary n-circular set.

References

1. Baronti, M., Papini, P.L.: Norm-one projections onto subspaces of ℓ_p. Ann. Mat. Pura Appl **152**, 53–61 (1988)
2. Chalmers, B.L., Lewicki, G.: Maximal complex symmetric spaces. J. Funct. Anal. **200**, 1–22 (2003)
3. Chalmers, B.L., Lewicki, G.: A proof of the Grünbaum conjecture. Stud. Math. **200**(2), 103–129 (2010)
4. Chalmers, B.L., Lewicki, G.: Three-dimensional subspace of $l_\infty^{(5)}$ with maximal projection constant. J. Funct. Anal. **257**, 553–592 (2009)
5. Chalmers, B.L., Metcalf, F.T.: The determination of minimal projections and extensions in L_1. Trans. Am. Soc. **329**(1), 289–305 (1992)
6. Chalmers, B.L., Metcalf, F.T.: A characterization and equations for minimal projections and extensions. J. Oper. Theory **32**, 31–46 (1994)
7. Cheney, E.W., Morris, P.D.: On the existence and characterization of minimal projections. J. Reine Angew. Math. **270**, 61–76 (1974)
8. Cheney, E.W., Light, W.A.: Approximation theory in tensor product spaces. Lecture Notes in Mathematics, vol. 1169. Approximation Theory in Tensor Product Spaces (1985)
9. Cheney, E.W., Price, K.H.: Minimal projections. In: Talbot, A. (ed.) Approximation Theory, pp. 261–289. Academic Press, New York (1970)
10. Cheney, E.W., Hobby, C.R., Morris, P.D., Schurer, F., Wulbert, D.E.: On the minimal property of the Fourier projection. Trans. Am. Math. Soc. **143**, 249–258 (1969)

11. Daugavet, I.K.: A property of compact operators in the space C. Uspekhi Mat. Nauk **18**, 157–158 (1963) (in Russian)
12. Deutsch, F., Li, W.: Strong uniqueness, projections in L1. J. Approx. Theory **66**, 198–224 (1991)
13. Franchetti, C.: Projections onto hyperplanes in Banach spaces. J. Approx. Theory **38**, 319–333 (1983)
14. Franchetti, C., Cheney, E.W.: Minimal projections in L1-spaces. Duke Math. J. **43**, 501–510 (1976)
15. Fisher, S.D., Morris, P.D., Wulbert, D.E.: Unique minimality of Fourier projections. Trans. Am. Math. Soc. **265**, 235–246 (1981)
16. König, H., Tomczak-Jaegermann, N.: Norms of minimal projections. J. Funct. Anal. **119**, 253–280 (1994)
17. Lambert, P.V.: Minimum norm property of the Fourier projection in spaces of continuous functions. Bull. Soc. Math. Belg. **21**, 359–369 (1969)
18. Lambert, P.V.: Minimum norm property of the Fourier projection in L_1 -spaces. Bull. Soc. Math. Belg. **21**, 370–391 (1969)
19. Lewicki, G.: On the unique minimality of the Fourier-type extensions in L_1-space. In: Proceedings Fifth Internat. Conf. On Function Spaces, Poznañ 1998. Lecture Notes in Pure and Applied Mathematics, vol. 213, pp. 337–345 (1998)
20. Lewicki, G., Marino, G., Pietramala, P.: Fourier-type minimal extensions in real L1-space. Rocky Mt. J. Math. **30**, 1025–1037 (2000)
21. Lewicki, G., Skrzypek, L.: On the properties of Chalmers–Metcalf operator. In: Proceedings of Banach Spaces and their Applications in Analysis, Oxford (2006)
22. Light, W.A.: Minimal projections in tensor-product spaces. Math. Z. **191**, 633–643 (1986)
23. Loziñski, S.M.: On a class of linear operators. Dokl. Akad. Nauk SSSR **61**, 193–196 (1948)
24. Odyniec, W., Lewicki, G.: Mimimal projections in Banach spaces. Lecture Notes in Mathematics, vol. 1449. Springer, Berlin (1990)
25. Rudin, W.: Function Theory in the Unit Ball of $\mathbb{C}^n$. Springer, New York (1980)
26. Shekhtman, B., Skrzypek, L.: On the uniqueness of the Fourier projection in Lp spaces. J. Concr. Appl. Math. (2012, to appear)
27. Shekhtman, B., Skrzypek, L.: Norming points and unique minimality of orthogonal projections. Abstr. Appl. Anal. (2006)