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Simulating the damped vibrations of a fractional oscillator with fuzzy initial conditions
(MAT111-15)

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Abstract: A Picard-like scheme using quadrature and differential quadrature rules, formerly introduced to solve integro-differential equations, is herein adapted to solve the problem of an oscillator with damping defined by the Riemann-Liouville fractional derivative and with fuzzy initial conditions. Considering fuzzy initial conditions has the meaning of a fuzzification of the problem via the Zadeh's extension principle. Following Zadeh, fuzziness is a way to take into account an uncertainty which cannot be identified as randomness. In the crisp domain, the proposed approach is able to approximate the reference analytical solutions with high accuracy and a relatively low computational cost. In the linear regime, the technique proposed becomes a non-recursive scheme, providing semi-analytical solutions by means of operational matrices and vectors of known quantities. In this sense, an example of application is given by the free damped vibrations of a linear oscillator in a medium with small viscosity, usually solved by using the method of multiple scales (in the crisp domain).

1. Introduction

In this manuscript, we investigate a class of fuzzy differential equations in presence of a Riemann-Liouville fractional derivative

$$L_t^{(2)}\tilde{u}(t) + \delta\, {}^{RL}D^{\beta}(\tilde{u}(t)) = f(\tilde{u}(t)) + \tilde{g}(t), \tag{1}$$

subject to the initial conditions

$$L_t^{(i)}\tilde{u}(0) = \tilde{a}_i, \qquad i = 0, 1 \tag{2}$$

where $L_t^{(i)}$ is the ith-order derivative operator with respect to t, ${}^{RL}D^{\beta}$ is the Riemann-Liouville fractional derivative of order β, with $0 < \beta < 1$. Besides, $f(\tilde{u}(t))$ is a functional form in $\tilde{u}$, $\tilde{g}(t)$ a given fuzzy-valued function and $\tilde{a}_i$ a fuzzy number, with $i = 0, 1$. Here, $\tilde{u}(t)$ represents the unknown fuzzy function for $t \in [0, 1]$. Obviously, assuming $[0, 1]$ as the problem domain is not a loss of generalization, because it is possible scaling the independent variable t.

It should be pointed out that (1) may be seen as the fuzzification via the Zadeh's extension principle of the same equation but without fuzzy variables and parameters. The introduction of 'fuzziness' in (1) means to take into account an uncertainty in the value of the equation variables which cannot be read as randomness [1]. Considering fuzziness in decision processes [2] and in regression analysis [3, 4] is very important. In the same way, it allows to model adequately realistic problems in science and engineering which traditionally involve ordinary differential equations [5, 6].

The problem corresponding to (1) in the crisp domain has been solved by means of several approaches [7], depending on the presence or not of nonlinearity and forcing term.

In the present work, we extend the approach proposed in [8] to solve (1). More precisely, we discuss a numerical scheme combining differential quadrature rules [9] (which provide high-order finite-difference approximations) and a Picard-like recursion into the fuzzy domain. In spite of its recursive nature, the proposed approach in the linear regime leads to a non-recursive approximate solution by means of operational matrices and vectors of known quantities. Notice that the scheme herein discussed is different from the one presented in [10].

As a first example application, we consider herein the free damped vibrations of a linear oscillator in a medium with small viscosity, which has been solved in the crisp domain by using the method of multiple scales [11].

Numerical simulations show that results obtained through our method are very accurate if compared in the crisp domain with the analytical solutions available in the literature.

2. Theoretical background

In this section, some basic notions are provided. Throughout, the set U will represent a nonempty and fixed (though arbitrary) closed interval of $\mathbb{R}$.

Definition 1. *A* **fuzzy number** $\tilde{u}$ *is defined by a membership function* $\mu_u(x) : U \to [0, 1]$*, and it satisfies the following properties:*

- $\tilde{u}$ *is* **normal***, that is* $\sup_{x \in U} \mu_u(x) = 1$*,*
- $\tilde{u}$ *is* **convex** *on* U*, meaning that* $\mu_u(\alpha x + (1 - \alpha)y) \geq \min(\mu_u(x), \mu_u(y))$ *for each* $x, y \in U$ *and each* $\alpha \in [0, 1]$*,*
- $\tilde{u}$ *is upper semi-continuous and*
- $[\tilde{u}]_0 = cl(\{x \in U : \mu_u(x) > 0\})$ *is compact, where cl denotes closure in the standard topology of* U*.*

Definition 2. *An* α**-cut** *of the fuzzy number* $\tilde{u}$ *is the crisp set defined by*

$$[\tilde{u}]_\alpha = \{x \in U : \mu_u(x) \geq \alpha\}, \qquad \alpha > 0. \tag{3}$$

Notice that for $\alpha = 0$, the α-cut of a fuzzy number $\tilde{u}$ reduces to $[\tilde{u}]_0$ in Definition 1.

Definition 3. *The* **parametric form** *of the fuzzy number* $\tilde{u}$ *is a pair* $[\underline{u}(\alpha), \overline{u}(\alpha)]$ *satisfying the following properties for each* $\alpha \in [0,1]$*:*

1. $\underline{u}(\alpha)$ *is a bounded, left-continuous, monotonic increasing function over* $[0,1]$*,*
2. $\overline{u}(\alpha)$ *is a bounded, left-continuous, monotonic decreasing function over* $[0,1]$ *and*
3. $\underline{u}(\alpha) \leq \overline{u}(\alpha)$*.*

The notation $[\tilde{u}]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$ *is employed if such form exists.*

In particular, a fuzzy *triangular* number $\tilde{u}$ is generally identified by an ordered triplet of numbers (d_C, d_L, d_R), where d_C is the center, d_L and d_R are the left and the right spreads, respectively. Using an α-cut operation, any triangular fuzzy number may be written as

$$[\tilde{u}]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)] = [d_C + (\alpha - 1)d_L, d_C + (1 - \alpha)d_R], \tag{4}$$

for each $\alpha \in [0,1]$.

In what follows, $\tilde{f}(x)$ denotes a continuous and Lebesgue-integrable fuzzy-valued function on the bounded interval $[a,b]$ and $\Gamma(.)$ represents the Gamma function.

The concept of strongly generalized H-differentiability [12], was extended in [13] to the context of fractional derivatives. In the latter work, the following definition was considered.

Definition 4. *Let* $0 < \beta < 1$*. The fuzzy-valued function* $\tilde{f}(x)$ *is a* **Riemann-Liouville fuzzy fractional differentiable function** *of order* β *at* $x_0 \in (a,b)$ *if either*

$${}^{RL}D^\beta \tilde{f}(x_0) = \lim_{h \to 0^+} \frac{\tilde{\phi}(x_0 + h) \ominus \tilde{\phi}(x_0)}{h} = \lim_{h \to 0^+} \frac{\tilde{\phi}(x_0) \ominus \tilde{\phi}(x_0 - h)}{h} \tag{5}$$

or

$${}^{RL}D^\beta \tilde{f}(x_0) = \lim_{h \to 0^+} \frac{\tilde{\phi}(x_0) \ominus \tilde{\phi}(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{\tilde{\phi}(x_0 - h) \ominus \tilde{\phi}(x_0)}{-h}, \tag{6}$$

where ${}^{RL}D^\beta \tilde{f}(x_0)$ *denotes the Riemann- Liouville fuzzy fractional derivative of* $\tilde{f}$ *at* x_0*, and*

$$\tilde{\phi}(x) = \frac{1}{\Gamma(1-\beta)} \int_0^x \tilde{f}(s)(x-s)^{-\beta} ds. \tag{7}$$

Following [13], the α-cut representation of the Riemann- Liouville fuzzy fractional derivative is

$$[{}^{RL}D^{\beta}\tilde{f}(x_0)]_{\alpha} = \begin{cases} [{}^{RL}D^{\beta}\underline{f}(x_0,\alpha), {}^{RL}D^{\beta}\overline{f}(x_0,\alpha)], & \text{for (5)}, \\ \\ [{}^{RL}D^{\beta}\overline{f}(x_0,\alpha), {}^{RL}D^{\beta}\underline{f}(x_0,\alpha)], & \text{for (6)}, \end{cases} \tag{8}$$

for each $0 \leq \alpha \leq 1$.

With regard to (1), we assume that sufficient derivatives of the solution exist. There are different notions of fuzzy derivatives [14], but we refer to H-differentiability [15], which has the advantage that it always exists and provides fuzzy numbers as the solution of fuzzy differential equations.

In order to obtain numerical solutions of (1), we will use the α-cut approach, which allows to replace the computation of a function of a fuzzy number by a sequence of interval computations on successive α-cuts. This approach follows the idea that, in many practical applications, it is sufficient to know the solution of the problem in a finite set of points in order to approximate the solution over the entire domain.

Let $[\tilde{u}(t)]_{\alpha}$ be the α-cut of a function $\tilde{u}(t)$. In light of [16], we may rewrite $[\tilde{u}(t)]_{\alpha} = [\underline{u}(t,\alpha), \overline{u}(t,\alpha)]$. Assuming that the derivatives exist, we obtain (see [15])

$$\begin{cases} L_t^{(2)}\underline{u}(t) + \delta^{RL}D^{\beta}\underline{u}(t,\alpha) &= \underline{f}(\underline{u}(t,\alpha), \overline{u}(t,\alpha), \alpha) + \underline{g}(t,\alpha), \\ L_t^{(2)}\overline{u}(t) + \delta^{RL}D^{\beta}\overline{u}(t,\alpha) &= \overline{f}(\underline{u}(t,\alpha), \overline{u}(t,\alpha), \alpha) + \overline{g}(t,\alpha), \\ L_t^{(i)}\underline{u}(0,\alpha) &= \underline{a}_i(\alpha), \qquad i = 0,1 \\ L_t^{(i)}\overline{u}(0,\alpha) &= \overline{a}_i(\alpha), \qquad i = 0,1 \end{cases} \tag{9}$$

Alternatively, we can write (9) as

$$\begin{cases} L_t^{(2)}\mathbf{U}(t,\alpha) + \delta^{RL}D^{\beta}\mathbf{U}(t,\alpha) &= \mathbf{F}(\mathbf{U}(t,\alpha),\alpha) + \mathbf{G}(t,\alpha), \\ L_t^{(i)}\mathbf{U}(0,\alpha) &= \mathbf{a}_i(\alpha), \quad i = 0,1 \end{cases} \tag{10}$$

where

$$\mathbf{U}(t,\alpha)^T = (\underline{u}(t,\alpha), \overline{u}(t,\alpha)), \tag{11}$$

$$\mathbf{F}(\mathbf{U}(t,\alpha),\alpha)^T = \big(\underline{f}(\underline{u}(t,\alpha), \overline{u}(t,\alpha), \alpha), \overline{f}(\underline{u}(t,\alpha), \overline{u}(t,\alpha), \alpha)\big), \tag{12}$$

$$\mathbf{G}(t,\alpha)^T = \big(\underline{g}(t,\alpha), \overline{g}(t,\alpha)\big), \tag{13}$$

$$\mathbf{a}_i(\alpha)^T = (\underline{a}_i(\alpha), \overline{a}_i(\alpha)). \tag{14}$$

3. Methodology

3.1. Integral and differential quadrature

Integral and differential quadrature are very similar in the underlying principle. The term integral in addiction to quadrature was introduced by C. Shu in order to highlight the difference between the two approaches [17]. Integral quadrature consists in replacing the integral of a function $u(t)$ over an interval, e.g. $[0,1]$, by a weighted sum of the functional values $u_1, u_2, \ldots, u_N$ at the discrete points $0 = t_1 < t_2 < \ldots < t_N = 1$, that is

$$\int_0^1 u(t)dt = \sum_{i=1}^{N} C_i u_i. \tag{15}$$

The differential quadrature (DQ) rules provide the approximation of the rth-order derivative of the function $u(t)$ at a point $t = t_j$ by a weighted sum [9] as follows

$$\left[\frac{d^r u}{dt^r}\right]_{t=t_j} = \sum_{i=1}^{N} A_{ji}^{(r)} u_i, \qquad i,j = 1,2,\ldots,N \tag{16}$$

where the constants $A_{ji}^{(r)}$ are the weight coefficients computed in N grid points. By letting $u(t) = \sum_{i=1}^{N} l_i(t)u_i$, where $l_i(t)$ is the Lagrange polynomial at the point t_i, for each $i = 1,2,\ldots,N$, one gets

$$C_i(t) = \int_0^x l_i(t)dt, \qquad A_i^{(r)}(t) = \frac{d^r l_i}{dt^r}(t). \tag{17}$$

Obviously, it is $A_{ji}^{(r)} = A_i^{(r)}(t_j)$ for every $i,j = 1,2,\ldots,N$.

By using Lagrange's polynomials as test functions, then there is no restriction on the choice of the grid points, which may be equally spaced or not. Usually grid coordinates are obtained through the Gauss-Tchebychev-Lobatto (GCL) distribution. As a result of the fuzzification of the integral and differential quadrature rules, one has respectively:

$$\left[\int_0^1 \tilde{u}(t)dt\right]_\alpha = \left[\int_0^1 \underline{u}(t)dt, \int_0^1 \overline{u}(t)dt\right]_\alpha = \left[\sum_{i=1}^{N} C_i \underline{u}_i, \sum_{i=1}^{N} C_i \overline{u}_i\right], \tag{18}$$

$$\left[\frac{d^r \tilde{u}}{dt^r}\right]_\alpha = \left[\frac{d^r \underline{u}}{dt^r}, \frac{d^r \overline{u}}{dt^r}\right]_\alpha = \left[\sum_{i=1}^{N} A_i^{(r)}(t)\underline{u}_i(\alpha), \sum_{i=1}^{N} A_i^{(r)}(t)\overline{u}_i(\alpha)\right]. \tag{19}$$

3.2. The Picard-like approach

In this section we extend the approach discussed in [8] to solve (10). Let us consider then the inverse operator

$$L_t^{-1}(\cdot) = \int_0^t \int_0^t (\cdot)dtdt. \tag{20}$$

In light of Theorems 5–8 in [18] , we may apply (20) on both sides of (10). So by recalling the definition of the Riemann–Liouville fractional derivative, we obtain

$$\mathbf{U}(t,\alpha) = \mathbf{A}(\alpha)\mathbf{w} + L_t^{-1}\Big(\mathbf{F}(\mathbf{U}(t,\alpha),\alpha) + \mathbf{G}(t,\alpha) - \delta^{RL}D^{\beta}\mathbf{U}(t,\alpha)\Big), \tag{21}$$

where

$$\mathbf{w}^T = (1,t)\,, \tag{22}$$

$$\mathbf{A}(\alpha) = \begin{pmatrix} \underline{a}_0(\alpha) & \underline{a}_1(\alpha) \\ \overline{a}_0(\alpha) & \overline{a}_1(\alpha) \end{pmatrix}. \tag{23}$$

Here, for $i = 0, 1$, we convey that $\underline{a}_i(\alpha)$ and $\overline{a}_i(\alpha)$ will represent the ith derivative with respect to t of $\underline{u}$ and $\overline{u}$ respectively, at the point $(0,\alpha)$.

Using successive approximations, the solution $\mathbf{U}(t,\alpha)$ can be expressed as

$$\mathbf{U}(t,\alpha) = \sum_{k=0}^{\infty}\mathbf{U}_k(t,\alpha), \tag{24}$$

where $\mathbf{U}_k^T(t,\alpha) = (\underline{u}_k(t,\alpha), \overline{u}_k(t,\alpha))$ has to be determined recursively using the formulas

$$\mathbf{U}_0(t,\alpha) = \mathbf{A}(\alpha)\mathbf{w} + L_t^{-1}(\mathbf{G}(t,\alpha)), \tag{25}$$

$$\mathbf{U}_{k+1}(t,\alpha) = L_t^{-1}\left(-\delta^{RL}D^{\beta}\mathbf{U}_k(t,\alpha) + \mathbf{F}(\mathbf{U}_k(t,\alpha),\alpha)\right). \tag{26}$$

Consider fixed (though arbitrary) partitions $t_1 < t_2 < \ldots < t_{N-1} < t_N$ of the interval $[0,1]$.

Additionally, if we let $\underline{u}_{k,j}(\alpha) = \underline{u}_k(t_j,\alpha)$ and $\overline{u}_{k,j}(\alpha) = \overline{u}_k(t_j,\alpha)$, then

$$\mathbf{U}_k^T(\alpha) = \left(\underline{u}_{k,1}(\alpha),\ldots,\underline{u}_{k,N}(\alpha),\overline{u}_{k,1}(\alpha),\ldots,\overline{u}_{k,N}(\alpha)\right). \tag{27}$$

Using numerical integration and the differential quadrature rules, we obtain that

$$\mathbf{U}_0(t,\alpha) = \mathbf{A}(\alpha)\mathbf{w} + \mathbf{C}(t)\mathbf{Q}(\alpha), \tag{28}$$

$$\mathbf{U}_{k+1}(t,\alpha) = \mathbf{C}(t)\left[-\gamma\mathbf{B}\mathbf{U}_k(\alpha) + \mathbf{F}(\mathbf{U}_k(\alpha),\alpha)\right], \tag{29}$$

where $\gamma = \delta/\Gamma(1-\beta)$ and

$$\mathbf{Q}^T(\alpha) = \left(\underline{g}(t_1,\alpha),\ldots,\underline{g}(t_N,\alpha),\overline{g}(t_1,\alpha),\ldots,\overline{g}(t_N,\alpha)\right), \tag{30}$$

$$\mathbf{C}(t) = \begin{pmatrix} \mathbf{C}_0(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_0(t) \end{pmatrix}, \tag{31}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_0 \end{pmatrix} \tag{32}$$

with $\mathbf{C}_0(t) = (C_1(t), \ldots, C_N(t))$ and $\mathbf{B}_0 = \mathbf{A}^{(1)}\mathbf{C}^{(\beta)}$. Here, $\mathbf{A}^{(1)}$ denotes the matrix of the DQ weighting coefficients and $\mathbf{C}^{(\beta)}$ the matrix of which entries are, for each $i = 1, \ldots, N$,

$$C_i^{\beta}(t) = \int_0^t l_i(s)(t-s)^{-\beta} ds, \tag{33}$$

with $l_i(s)$ being the Lagrange polynomial at the point x_i.

For the linear case, i.e. $f(\tilde{u}(t)) = -\omega_0^2 \tilde{u}(t)$, one gets

$$\mathbf{U}_{k+1}(t, \alpha) = \mathbf{H}(t)\mathbf{U}_k(\alpha) \tag{34}$$

where

$$\mathbf{H}(t) = \begin{pmatrix} \mathbf{H}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_0 \end{pmatrix} \tag{35}$$

with $\mathbf{H}_0(t) = -\mathbf{C}_0(t)\left(\gamma \mathbf{B}_0 + \omega_0^2 \mathbf{I}\right)$, $\mathbf{I}$ being the identity matrix of size N.

Let $\mathbf{D}$ be the matrix

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_0 \end{pmatrix} \tag{36}$$

where $\mathbf{D}_0^T = [\mathbf{H}_0(t_1), \ldots, \mathbf{H}_0(t_N)]^T$. Since

$$\mathbf{U}_k(\alpha) = \mathbf{D}\mathbf{U}_{k-1}(\alpha) = \mathbf{D}^k \mathbf{U}_0(\alpha), \tag{37}$$

the truncation of (24) after p terms becomes

$$\mathbf{U}^{[p]}(t, \alpha) = \mathbf{U}_0(t, \alpha) + \mathbf{H}(t) \sum_{k=0}^{p-1} \mathbf{D}^k \mathbf{U}_0(\alpha), \tag{38}$$

where $\mathbf{U}_0^T(\alpha) = \left(\underline{u}_{0,1}(\alpha), \ldots, \underline{u}_{0,N}(\alpha), \overline{u}_{0,1}(\alpha), \ldots, \overline{u}_{0,N}(\alpha)\right)$.

For the remainder of this work, we will represent the spectral radius of the matrix $\mathbf{D_0}$ by $\rho(\mathbf{D_0})$.

Lemma 5. *Suppose that* $\rho(\mathbf{D}_0) \leq 1$. *Then the solution* $\mathbf{U}(t, \alpha)$ *in* (24) *is given by*

$$\mathbf{U}(t, \alpha) = \mathbf{U}_0(t, \alpha) + \mathbf{H}(t)(\mathbf{I} - \mathbf{D})^{-1}\mathbf{U}_0(\alpha). \tag{39}$$

Proof. The proof follows as that of Theorem 1 in [8], by considering that in (38) there is a geometric series of matrices and that the spectral radius of $\mathbf{D}$ is equal to that of $\mathbf{D}_0$. □

It is the case to point out that an error bound can be derived similarly to the ones presented in [8], [10].

4. Simulations

As an example application, we discuss herein the free damped vibrations of a linear oscillator in a medium with small viscosity. The analytical solution of this problem in the crisp domain has been obtained in [11] by means of the method of multiple scales. In reason of the Zadeh's extension principle and in particular the α-cut approach, we compared our solution on successive α-cuts with the corresponding analytical solution [11] in the crisp domain.

We assumed null initial velocity and a triangular fuzzy number as initial displacement, that is

$$[\tilde{a}_0]_\alpha = [\alpha - 1, 1 - \alpha] \tag{40}$$

Besides, we fixed $\delta = 10^{-4}$ and $\omega_0 = 1$. In our simulations, we assumed $N = 9$ GCL points. We used (39) in view that the condition $\rho(\mathbf{D}_0) \leq 1$ holds. Figure 1 shows the approximate solution for $\beta = 0.75$.

With regard to the same value of β, the graphs of the approximate and analytical solutions obtained for some values of α are depicted in figure 2. The graphs are referred to the functions $\underline{u}$ (below) and $\overline{u}$ (above).

We obtained similar accurate results also for other values of β, such as $\beta = 0.5$ and $\beta = 0.995$. In our simulations, we observed that the maximum error, intended as the distance between the approximate and the analytical solution in accordance with the α-cut approach, is of the order of 10^{-5}.

5. Conclusions

In this work, a Picard-like numerical scheme which was formerly employed to solve integro–differential equations, has been extended to handle fuzzy initial value problems involving Riemann–Liouville fuzzy fractional derivatives. In the linear regime, the technique proposed here is a non-recursive scheme. As an example application, the problem of free damped vibrations of a linear oscillator in a medium with small viscosity has been considered. Numerical results show the effectiveness of the proposed approach.

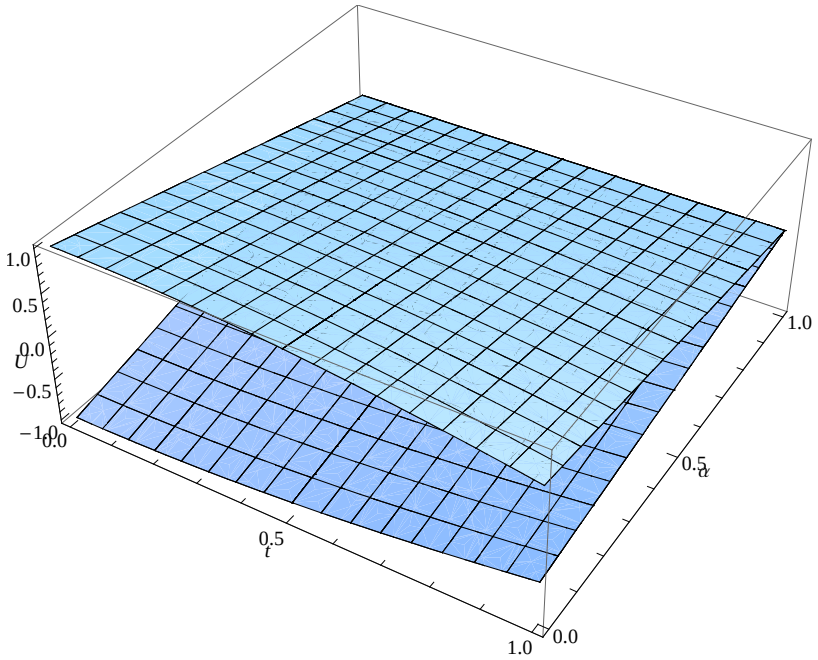

Figure 1. Approximate solution for $\beta = 0.75$. The graphs correspond to the functions $\underline{u}$ (below) and $\overline{u}$ (above).

Acknowledgments

Dr. Tomasiello wishes to thank Prof. Awrejcewicz for his kind support encouraging this work.

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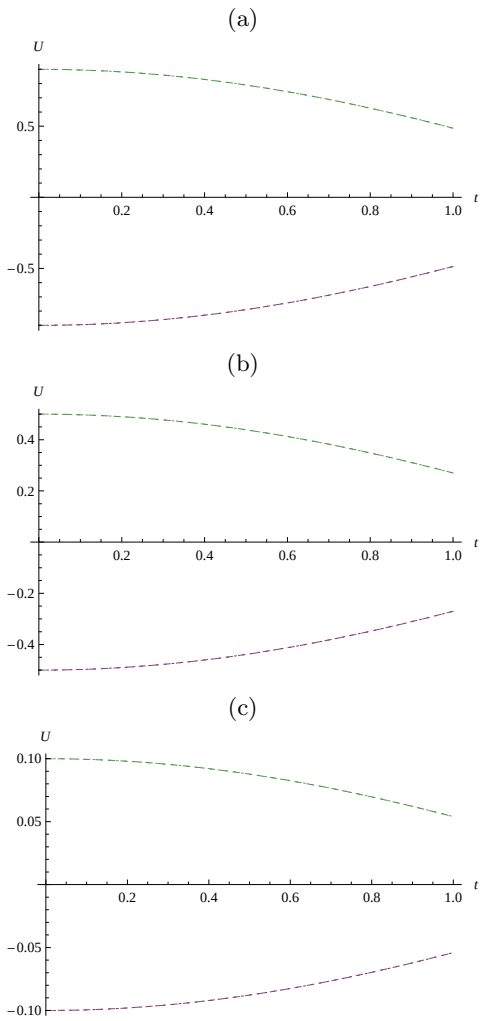

Figure 2. Graphs of the numerical (dotted line) and the analytical solutions (dashed line) for $\beta = 0.75$ and (a) $\alpha = 0.1$, (b) $\alpha = 0.5$, (c) $\alpha = 0.9$.

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