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SYNCHRONIZATION OF TWO FORCED DOUBLE-WELL DUFFING OSCILLATORS WITH ATTACHED PENDULUMS

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We investigate the dynamics of two coupled Duffing oscillators with attached pendulums forced kinematically by a common signal. Our attention is focused on different kinds of synchronization which can appear in the considered system. Different types of coupling (spring, damper and spring and damper simultaneously) are taken into account. We show in a two-parameters space (amplitude and frequency of excitation) existence of complete and phase synchronization and asynchronous ranges.

Key word: Duffing pendulum system, bifurcational analysis, synchronization

1. Introduction

Investigation of systems consisting of pendulums is an issue which focused attention of many scientists. The synchronization between coupled systems with pendulums was discovered in the XVIIth century by the Dutch researcher Christian Huygens. He showed that a couple of mechanical clocks hanging from a common support synchronize (Huygens, 1665). There are a lot of practical applications of such systems (Blekhman, 1998). In recent years, many new types of synchronizations have been detected (Rulkov *et al.*, 1995; Rosenblum *et al.*, 1996; Boccaletti *et al.*, 2002; Stefanski *et al.*, 2007; Yanchuk *et al.*, 2008), which manifests a strong or weak interaction between coupled nonlinear oscillators. The strongest one is the complete synchronization (CS) (Pecora and Carroll, 1990, 1991). This idea is well-known and has been applied in many fields of science (Watts and Strogatz, 1998; Strogatz, 2001; Pikovsky *et al.*, 2001; Tass, 2003; Stefanski and Kapitaniak, 2003; Stefanski *et al.*, 2007; Balanov *et al.*, 2009). The CS has also been extended for slightly different systems in experimental investigations (Sekieta and Kapitaniak, 1996; Argyris *et al.*, 2005; Perlikowski *et al.*, 2008a). Another important type of synchronization is called the phase synchronization (PS) (Rosenblum *et al.*, 1996). The PS in nonlinear systems is defined as an appearance of the reaction between the phases of subsystems (or between the phase of the subsystems and the driving signal), while the amplitudes can still be chaotic and uncorrelated. This type of synchronization is typical in parametrically excited systems where different types of inner resonances can be observed (Miles, 1988; Clifford and Bishop, 1995, 1996).

In this work, we consider two Duffing-pendulum systems coupled by a spring and/or a damper. The dynamics of the investigated systems is complex even in the uncoupled case. Many published papers focus on behaviour of a pendulum attached to a forced linear oscillator. One of the first articles (Hatwal *et al.*, 1983a,b) gives approximate solutions by the method of harmonic balance in the primary parametric instability zone, which allows calculation of the separate regions of stable and unstable harmonic solutions. Further analysis allows understanding of the dynamics around primary and secondary resonances (Bajaj *et al.*, 1994; Cartmell and Lawson, 1994; Balthazar *et al.*, 2001; Song *et al.*, 2003; Warminski and Kecik, 2006). A good understanding of dynamics of systems with a linear base gives possibility to extend investigation to systems with a non-linear base. Non-linearity in the considered class of systems is usually introduced by changing the linear spring into a nonlinear one (Warminski *et al.*, 2001; Warminski and Kecik, 2009) or a magnetorheological damper (Kecik and Warminski, 2011).

In a few papers on this topic, one can find an analytical study of the dynamics of a Duffing-pendulum system around principal and secondary resonances (Warminski *et al.*, 2001; Warminski and Kecik, 2006, 2009; Vazquez-Gonzalez and Silva-Navarro, 2008; Macias-Cundapi *et al.*, 2008). Recently, we presented the complete bifurcation diagram containing oscillating and rotating solutions (Brzeski *et al.*, 2012). Our analysis also shows that in this class of two degree of freedom systems one can expect many coexisting solutions, hence their practical application should be preceded by careful investigation (Chudzik *et al.*, 2011; Maistrenko *et al.*, 1997).

In our previous papers (Perlikowski *et al.*, 2008b; Perlikowski, 2008) we describe the synchronization between two coupled single-well Duffing oscillators forced by the common signal. We show there detailed analysis of synchronization phenomena and compare different methods of synchronization detection.

In this paper, we consider two coupled (by the spring and/or damper) double-well Duffing systems each with the attached pendulum. We focus investigations on synchronization between them. Change from single-well to double-well Duffing oscillators cause an increase of complexity – even in the periodic case systems can oscillate around opposite equlibriums (Kapitaniak, 1985, 1988; Miles, 1989; Robinson, 1989; Yamasue and Hikihara, 2004; Czolczynski *et al.*, 2008).

In Section 2, we present model of the considered system. In Section 3, we present an overview of possible synchronization scenarios. Section 4 is devoted to numerical investigation of synchronization. Finally, in Section 5 we conclude our work.

2. Model of the system

The model of the system taken under consideration is shown in Fig. 1. It consists of two mutually coupled subsystems with common driving. Each subsystem is composed of a double-well Duffing oscillator moving in the vertical direction and the pendulum suspended to the mass of the Duffing oscillator. Motion of each subsystem is described by two generalized coordinates: the vertical position of the Duffing oscillator is described by the coordinate y_i, and the angular displacement of the pendulum is given by the angle φ_i, where $i = 1, 2$. The systems are coupled thought the spring or/and damper.

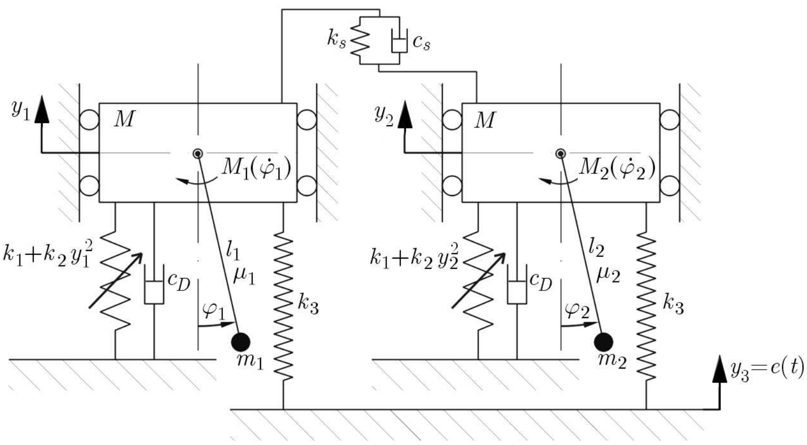

Fig. 1. Model of the system

2.1. Equations of motion

The evolution of the system can be described by the set of ODEs derived using Lagrange equations of the second kind. The kinetic energy T, potential energy V, and Rayleigh dissipation D are given by the following formulas

$$
\begin{aligned}
T &= \sum_{i=1}^{2}\Big[\frac{1}{2}(M+m_i+\mu_i)\dot{y}_i^2 + \Big(m_i+\frac{1}{2}\mu_i\Big)l_i\dot{y}_i\dot{\varphi}_i\sin\varphi_i + \frac{1}{2}\Big(m_i+\frac{1}{3}\mu_i\Big)l_i^2\dot{\varphi}_i^2\Big] \\
V &= \frac{1}{2}k_S\Big(y_1-y_2\Big)^2 + \sum_{i=1}^{2}\Big[\frac{1}{2}k_1y_i^2 + \frac{1}{4}k_2y_i^4 + \frac{1}{2}k_3\Big(y_i-y_3\Big)^2 + \Big(m_i+\frac{1}{2}\mu_i\Big)gl_i(1-\cos\varphi_i)\Big] \\
D &= \frac{1}{2}c_S(\dot{y}_1-\dot{y}_2)^2 + \sum_{i=1}^{2}\frac{1}{2}c_D y_i^2
\end{aligned}
\tag{2.1}
$$

where M is mass of the Duffing oscillators, m_i is mass at the end of each pendulum, μ_i and l_i correspond to mass and length of the rod of the i-th pendulum, k_1 and k_2 are linear and nonlinear parts of the spring stiffness and k_3 is stiffness of the forcing spring. The viscous damping coefficient of the Duffing oscillators is given by c_D. The coupling between subsystems is characterized by two parameters: stiffness k_S and viscous damping c_S.

The generalized forces are given by the following formula

$$
Q = \sum_{i=1}^{2} M_i(\dot{\varphi}_i)
\tag{2.2}
$$

where $M_i(\dot{\varphi}_i) = c_i\dot{\varphi}_i$ is damping momentum of the i-th pendulum with the damping coefficient c_i. The dampers of the pendulums are located in pivot points (not shown in Fig. 1). Using formulas (2.1) and (2.2), one can derive four coupled second order ODEs

$$
\begin{aligned}
&(M+m_1+\mu_1)\ddot{y}_1 + c_D\dot{y}_1 + (-k_1+k_3)y_1 + k_2y_1^3 + \Big(m_1+\frac{1}{2}\mu_1\Big)l_1(\ddot{\varphi}_1\sin\varphi_1 + \dot{\varphi}_1^2\cos\varphi_1) \\
&\qquad + c_S(\dot{y}_1-\dot{y}_2) + k_S(y_1-y_2) = k_3y_3 \\
&(M+m_2+\mu_2)\ddot{y}_2 + c_D\dot{y}_2 + (-k_1+k_3)y_2 + k_2y_2^3 + \Big(m_2+\frac{1}{2}\mu_2\Big)l_2(\ddot{\varphi}_2\sin\varphi_2 + \dot{\varphi}_2^2\cos\varphi_2) \\
&\qquad + c_S(\dot{y}_2-\dot{y}_1) + k_S(y_2-y_1) = k_3y_3 \\
&\Big(m_1+\frac{1}{3}\mu_1\Big)l_1^2\ddot{\varphi}_1 + c_1\dot{\varphi}_1 + \Big(m_1+\frac{1}{2}\mu_1\Big)l_1(\ddot{y}_1+g)\sin\varphi_1 = 0 \\
&\Big(m_2+\frac{1}{3}\mu_2\Big)l_2^2\ddot{\varphi}_2 + c_2\dot{\varphi}_2 + \Big(m_2+\frac{1}{2}\mu_2\Big)l_2(\ddot{y}_2+g)\sin\varphi_2 = 0
\end{aligned}
\tag{2.3}
$$

In numerical calculations, we assumed that the Duffing oscillators and pendulums are identical. We used the following values of their parameters: $M = 2.0\,\mathrm{kg}$, $k_1 = 900\,\mathrm{N/m^3}$, $k_2 = 200000\,\mathrm{N/m}$, $c_D = 3.9\,\mathrm{Ns/m}$. We have distinguished parameters of the pendulums suspended to each subsystem and assumed the following values of pendulums parameters: $m_1 = 0.1\,\mathrm{kg}$, $\mu_1 = 0.1\,\mathrm{kg}$, $l_1 = 0.05\,\mathrm{m}$, $c_1 = 5.94\cdot10^{-5}\,\mathrm{Nms}$, $m_2 = 0.1\,\mathrm{kg}$, $\mu_2 = 0.1\,\mathrm{kg}$, $l_2 = 0.05\,\mathrm{m}$, $c_2 = 5.94\cdot10^{-5}\,\mathrm{Nms}$. The Duffing oscillators are kinematically forced thought linear springs $k_3 = 200\,\mathrm{N/m}$. Motion of the base is described by a harmonic function: $y_3 = e(t) = A\cos(\omega t)$, where $A = 0.04\,\mathrm{m}$. We neglect the static deflation of Duffing systems.

2.2. Dimensionless equations of motion:

Introducing dimensionless time $\tau = tp_1$, where $p_1 = \sqrt{[g(m_1+\mu_1/2)]/[l_1(m_1+\mu_1/3)]}$ is the natural frequency of first pendulum we reach dimensionless equations

$$
\begin{aligned}
&(1+m_{1B}+\mu_{1B})\ddot{x}_1+c_B\dot{x}_1+(-k_{1B}+k_{3B})x_1+k_{2B}x_1^3+\Big(m_{1B}+\frac{1}{2}\mu_{1B}\Big)\\
&\quad\cdot l_{1B}(\ddot{\varphi}_1\sin\varphi_1+\dot{\varphi}_1^2\cos\varphi_1)+c_{SB}(\dot{x}_1-\dot{x}_2)+k_{SB}(x_1-x_2)=k_{3B}A_B\cos(\mu\tau)\\
&(1+m_{2B}+\mu_{2B})\ddot{x}_2+c_B\dot{x}_2+(-k_{1B}+k_{3B})x_2+k_{2B}x_2^3+\Big(m_{2B}+\frac{1}{2}\mu_{2B}\Big)\\
&\quad\cdot l_{2B}(\ddot{\varphi}_2\sin\varphi_2+\dot{\varphi}_2^2\cos\varphi_2)+c_{SB}(\dot{x}_2-\dot{x}_1)+k_{SB}(x_2-x_1)=k_{3B}A_B\cos(\mu\tau)\\
&\Big(m_{1B}+\frac{1}{3}\mu_{1B}\Big)l_{1B}^2\ddot{\varphi}_1+c_{1B}\dot{\varphi}_1+\Big(m_{1B}+\frac{1}{2}\mu_{1B}\Big)l_{1B}\ddot{x}_1\sin\varphi_1\\
&\quad+\Big(m_{1B}+\frac{1}{3}\mu_{1B}\Big)l_{1B}^2\sin\varphi_1=0\\
&\Big(m_{2B}+\frac{1}{3}\mu_{2B}\Big)l_{2B}^2\ddot{\varphi}_2+c_{2B}\dot{\varphi}_2+\Big(m_{2B}+\frac{1}{2}\mu_{2B}\Big)l_{2B}\ddot{x}_2\sin\varphi_2\\
&\quad+\Big(m_{2B}+\frac{1}{2}\mu_{2B}\Big)ol_{1B}l_{2B}\sin\varphi_2=0
\end{aligned}
\tag{2.4}
$$

where: $y_{st}=\frac{M+m_1+\mu_1}{|k_1+k_3|}$, $o=\frac{m_{1B}+\mu_{1B}/3}{m_{1B}+\mu_{1B}/2}$, $A_B=\frac{A}{y_{st}}$, $\mu=\frac{\omega}{p_1}$, $c_B=\frac{c_D}{Mp_1}$, $c_{SB}=\frac{c_S}{Mp_1}$, $k_{SB}=\frac{k_S}{Mp_1^2}$, $k_{1B}=\frac{k_1}{Mp_1^2}$, $k_{2B}=\frac{k_2y_{st}^2}{Mp_1^2}$, $k_{3B}=\frac{k_3}{Mp_1^2}$, and $l_{iB}=\frac{l_i}{y_{st}}$, $m_{iB}=\frac{m_i}{M}$, $\mu_i=\frac{\mu_i}{M}$, $c_{iB}=\frac{c_i}{Mp_1y_{st}^2}$, $x_i=\frac{y_i}{y_{st}}$, $\dot{x}_i=\frac{\dot{y}_i}{y_{st}p_1}$, $\dot{\varphi}_i=\frac{\dot{\varphi}_i}{p_1}$, $\ddot{x}_i=\frac{y_i}{y_{st}p_1^2}$, $\ddot{\varphi}_i=\frac{\ddot{\varphi}_i}{p_1^2}$ for $i=1,2$.

The dimensionless parameters have the following values: $o=0.8889$, $A_B=1.297$, $l_{1B}=1.622$, $m_{1B}=0.05$, $\mu_1=0.05$, $l_{2B}=1.622$, $m_{2B}=0.05$, $\mu_2=0.05$, $c_B=0.1321$, $c_{1B}=0.002104$, $c_{2B}=0.002104$, $k_{1B}=2.039$, $k_{2B}=0.4307$, $k_{3B}=0.4531$. The dimensionless parameters of coupling: k_{SB}, c_{SB} and the excitation frequency μ are control parameters. Pendulums have a linear resonance at $\mu=1.0$, while for the Duffing oscillators the linear resonance occurs at $\mu=1.2$. Such diversification means that during the increasing frequency of excitation the resonance of pendulums occurs faster than the resonance of the Duffing oscillators, and we do not observe an overlapping of those two resonances.

3. Types of synchronization

Generally, between coupled oscillators, one can observe different types of synchronization. The strongest relation between coupled oscillators is the CS (Pecora and Carroll, 1990, 1991) which takes place when the identical subsystems exhibit motion with the same amplitude and frequency. In the analyzed system, the sufficient condition for this type of synchronization is

$$
\lim_{t\to\infty}\|\mathbf{z}(t)-\mathbf{w}(t)\|=0 \tag{3.1}
$$

where $\mathbf{z}(t)$ and $\mathbf{w}(t)$ are state vectors of the two coupled systems.

When Duffing oscillators are in the CS state, the pendulums are also completely synchronized. In our system, there is no transfer of vertical forces between Duffing oscillators and pendulums, hence there is no qualitative difference between in-phase and anti-phase synchronization. In-phase or anti-phase motion depends only on initial conditions. Therefore, both in-phase and anti-phase synchronization of the pendulums is described as complete synchronization.

Another type of synchronization appears when the difference between the position of Duffing oscillators after every period remains the same. This can be observed in three cases. The first one corresponds to the situation when masses oscillate around opposite equilibria. The second case appears when one pendulum is locked with an excitation frequency with a different ratio than the other one (mostly it is $2:1$ and $1:1$). The third case can be observed when one pendulum is in the stable equilibrium position and the other one is oscillating. All specified cases fulfil the following condition

$$\forall t \qquad |[\mathbf{z}(t) - \mathbf{w}(t)] - [\mathbf{z}(t+T) - \mathbf{w}(t+T)]| = 0 \tag{3.2}$$

where T is the period of motion, and can be classified as the PS (Pikovsky *et al.*, 2001).

Nevertheless, when Duffing oscillate periodically around the same or opposite equlibria one can observe the CS between pendulums when the following condition is fulfilled

$$\lim_{t\to\infty} \Big\| |\mathbf{\Phi}(t)| - |\mathbf{\Theta}(t)| \Big\| = 0 \tag{3.3}$$

where $\mathbf{\Phi}(t)$ and $\mathbf{\Theta}(t)$ are state vectors of the pendulums.

Phase synchronization between pendulums occurs when one pendulum is locked with an excitation with a different ratio than the second one. Hence, pendulums oscillate with different amplitude. In our system, the sufficient condition for the PS of the pendulums is

$$\forall t \qquad |[\mathbf{\Phi}(t) - \mathbf{\Theta}(t)] - [\mathbf{\Phi}(t+T) - \mathbf{\Theta}(t+T)]| = 0 \tag{3.4}$$

4. Results of numerical simulations

In our numerical simulations, we investigate synchronization between the two considered subsystems. We take into account three different types of coupling: (1) only by a damper, (2) only by a spring and (3) combinated by the spring and damper. To obtain more general overview on synchronization properties, for each case of coupling, we calculate three diagrams in a two-dimensional space: couping coefficient (damping coefficient and/or spring stiffness) and frequency of the external excitation. The first one shows synchronization between Duffing oscillators with fixed pendulums (the mass of each oscillator is equal to $(M + m_i)$). The second and third panels show synchronization between the Duffing oscillators and pendulums, respectively (pendulums oscillate freely). Such comparison enables us to present the influence of the attached pendulums on the synchronization. To distinguish different types of synchronization, we use symbols presented in Table 1.

Table 1. List of symbols used to distinguish different states of the system

Symbols for Duffing oscillators			
For periodic orbits (PO)		For chaotic attractors	
$\star$	complete synchronization (CS)	$\bullet$	complete synchronization (CS)
$\blacktriangledown$	phase synchronization (PS)	$\circ$	no synchronization
$\times$	no synchronization		

Symbols for pendulums			
no symb.	both pendulums are in their equilibrium position (stable one)		
$+$	one pendulum is equilibrium position (stable one) and the another one is oscillating		
For periodic orbits (PO)		For chaotic attractors	
$\star$	complete synchronization (CS) – phase or antiphase	$\bullet$	complete synchronization (CS)
$\blacktriangledown$	phase synchronization (PS)	$\circ$	no synchronization
$\times$	no synchronization		

We always start integration for the lowest value of frequency excitation $(\mu = 0.6)$ with the same initial conditions for all values of coupling: Duffing oscillators are in an unstable steady state ($x_{1,2} = \dot{x}_{1,2} = 0.0$) and the pendulums are slightly perturbed from the hanging-down position ($\varphi_{1,2} = 0.0001$, $\dot{\varphi}_{1,2} = 0.0$).

In Fig. 2, we show case (1) where the coupling is realized only by the damper. The damping coefficient $c_{SB} \in (0, 0.35)$ and the excitation frequency $\mu \in (0.6, 2.4)$. This range of parameters includes linear resonances of the Duffing system and pendulum and a $2:1$ resonance of the pendulum. In Fig. 2a, we consider the system with fixed pendulums. When Duffing oscillators are in the periodic regime, the CS occurs in a large range of c_s, and in a few cases we observe the PS (which coexists with the CS). For chaotic motion of Duffing oscillators, the value of the coupling coefficient is crucial and strongly influences the synchronization properties. In the chaotic regime the CS is possible only for larger values of c_s. Worth to notice is the fact that for a higher frequency of excitation μ, velocities of the Duffing oscillators are larger. This implies that forces generated in the coupling damper are also higher. This explains why the minimum value of c_s for which the CS occurs decreases with an increasing excitation frequency μ.

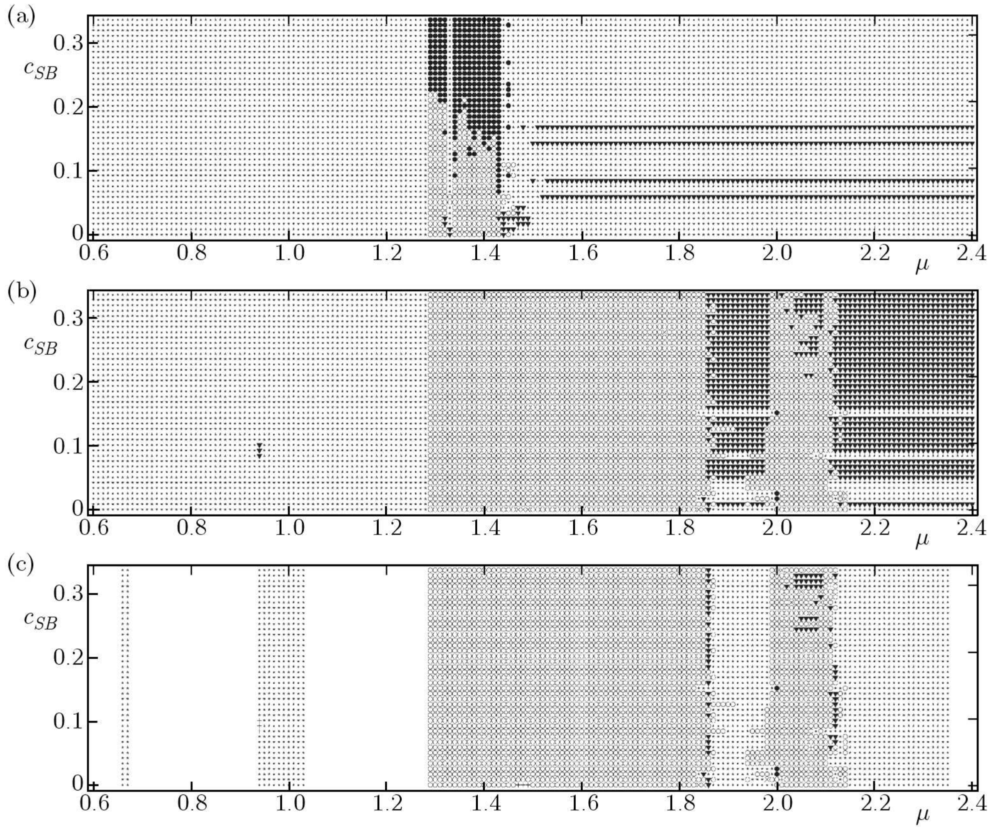

Fig. 2. Synchronization of Duffing oscillators and pendulums, subsystems coupled by the linear damper, (a) synchronization between Duffing oscillators with fixed pendulums, (b) synchronization between Duffing oscillators and (c) synchronization between pendulums

Figures 2b and 2c correspond to the system with unfixed pendulums. Comparing Fig. 2a and Fig. 2b (synchronization between Duffing oscillators) one can notice that for systems with oscillating pendulums, the area where the Duffing oscillators have chaotic motion is much wider and synchronization in that range does not occur for all considered values of c_s. Therefore, for the system with pendulums, the coupling strength is not so crucial. It is also important that the area where the PS occurs is larger than in the case with fixed pendulums. By analyzing Fig. 2c (synchronization of pendulums) one can notice that motion of the pendulums is observed only near parametric resonances ($1:1$, $2:1$) and in areas where the Duffing oscillators have chaotic behaviour. Outside the mentioned ranges, the pendulums reach hanging-down positions. Synchronization between the pendulums occur only for periodic motion while for chaotic solutions they stay unsynchronized. For this type of coupling, mostly the CS occurs, and regions where the PS can be observed are small. One can see that for $\mu \in (1.85, 1.97)$ and for $\mu > 2.15$ the Duffing systems synchronize their phases while the pendulums are in the CS state. This situation is observed because the Duffing components oscillate around opposite wells.

Figure 3 corresponds to the system coupled only by the spring ($c_{SB} = 0$) – case (2). The stiffness k_{SB} varies from 0 to 1.6, and similarly to Fig. 2, we check synchronization of the Duffing oscillators and pendulums for different values of the excitation frequency μ. Similarly to the situation shown in Fig. 2a, one can notice that when motion of the Duffing oscillators is periodic, synchronization always occurs (independent on the coupling k_{SB} value). Therefore, the strength of the coupling changes only the type of synchronization (from the PS to the CS and vice-versa) modifying the region of periodic motion (systems oscillate around the same or opposite wells). When the Duffing subsystems oscillate chaotically, the synchronization occurs for $k_{SB} > 0.72$ regardless of the value of μ.

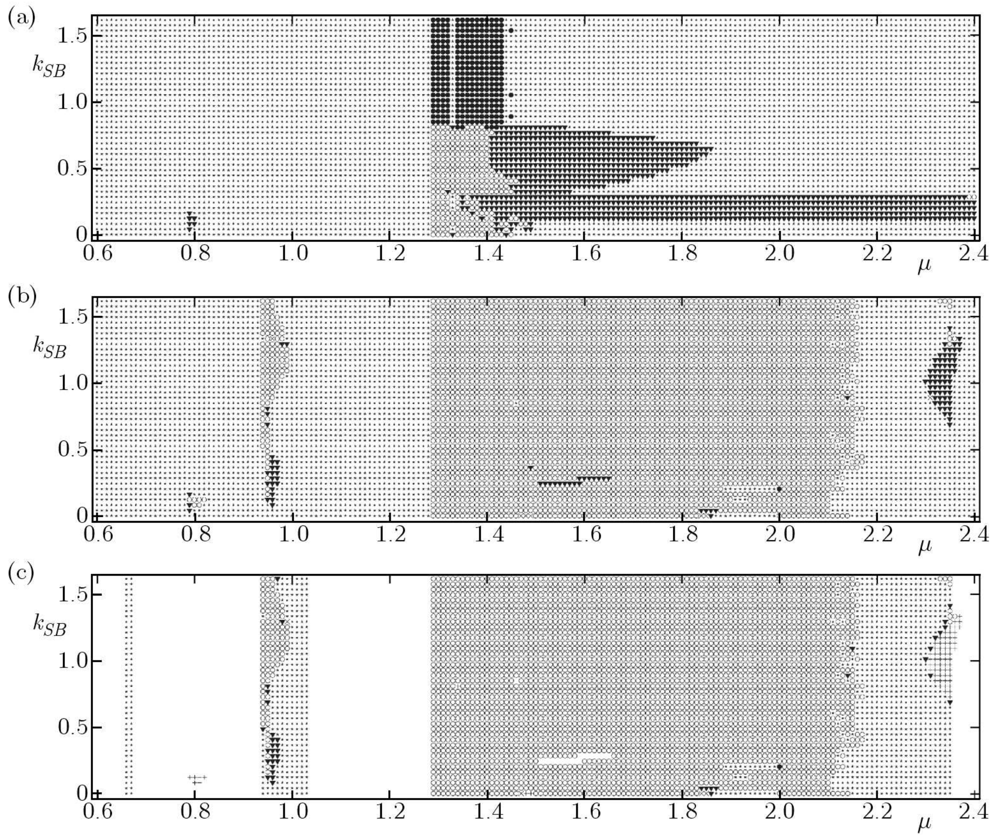

Fig. 3. Synchronization of Duffing oscillators and pendulums, subsystems are coupled by the linear spring, (a) synchronization between Duffing oscillators with fixed pendulums, (b) synchronization between Duffing oscillators and (c) synchronization between pendulums

Analyzing system with pendulums (Figs. 3b,c), one can notice that contrary to the system coupled by the damper, we do not observe periodic windows in the chaotic range of the excitation frequency μ. Chaotic attractors that exist for $\mu < 1.29$ (around the principal resonance of pendulum) cause asynchronous motion of the systems. The regions where the PS of Duffing oscillators occur are larger in the case of fixed pendulums and this is a contrary situation to the phenomenon observed in the system coupled with the damper. In Fig. 3c one can notice ranges where only one pendulum is oscillating and the second one is in a stable equilibrium position (marked as plus). This phenomenon is observed only in this coupling scheme, but theoretically is possible for all types of coupling.

Results of simulations of the system with both types of coupling ($c_{SB} \neq 0$, $k_{SB} \neq 0$) are presented in Fig. 4 (case (3)). We decide to hold the same ranges of coupling parameters as in the previous cases, and the coupling is given by the proportional sum of c_{SB} and k_{SB}. The values of coupling by the damper and spring is shown in the right and left axis of Fig. 4, respectively. Systems that are coupled by both a spring and damper exhibit dynamics that is characteristic for both types of previous cases and can be regarded as their generalization. Nevertheless, there are also some uniqueness. In Fig. 4a, the CS in the chaotic state occurs for lower values of coupling than in separate coupling cases. The range of the PS is smaller than in the case when the systems are coupled only by the spring and larger than in the case when the coupling is realized only by the damper. In Fig. 4b, one can observe a noticeable increase in the area of periodic window, which also occurs in Fig. 3b (for which both pendulums are in the stable equilibrium position). The area of the PS between the pendulums can be observed in a much wider range than in Fig. 2c and Fig. 3c, and it is typical for sufficiently large values of c_{SB} and k_{SB}.

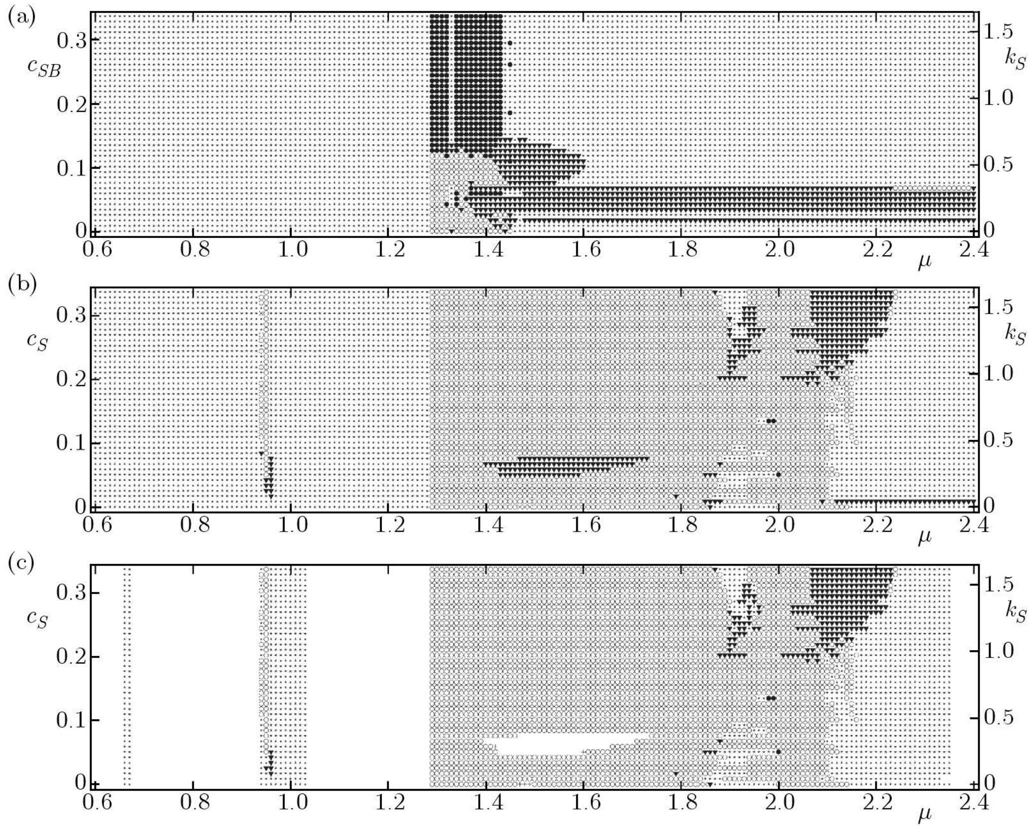

Fig. 4. Synchronization of Duffing oscillators and pendulums subsystems coupled by the linear spring and damper, (a) synchronization between Duffing oscillators with fixed pendulums, (b) synchronization between Duffing oscillators and (c) synchronization between pendulums

5. Conclusion

In this paper, we analyse dynamical behaviour of two coupled Duffing oscillators with attached pendulums. We show how different types of couplings affect synchronization between the subsystems. From the practical point of view, periodic motion of the coupled system is the best, and the coupling which causes periodization is prominent. One can see that for different types of couplings, the dynamics is generally similar because in most cases motion of pendulums is observed around parametric resonances, where a locking phenomenon occurs. For the coupling realized by a spring and damper, chaotic ranges are smallest especially for the $2:1$ resonance.

The coexistence of different types of synchronization is visible by the rapid changes in behaviour of the subsystems.

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