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MODELLING OF ANNULAR PLATES STABILITY WITH FUNCTIONALLY GRADED STRUCTURE INTERACTING WITH ELASTIC HETEROGENEOUS SUBSOIL

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This contribution deals with the modelling and analysis of stability problems for thin composite annular plates interacting with elastic heterogeneous subsoil. The object of analysis is an annular plate with a deterministic heterogeneous microstructure and the apparent properties smoothly varying along a radial direction. The aim of contribution is to formulate two macroscopic mathematical models describing stability of this plate. The considerations are based on a tolerance averaging technique. The general results are applied to the analysis of some special stability problems. The obtained results of critical forces with those obtained from finite element method are compared.
Keywords: functionally graded materials, stability, annular thin plates, heterogeneous subsoil

1. Introduction

The object of this contribution is a two-phased composite plate interacting with elastic micro-heterogeneous subsoil with two moduli (Fig. 1). The assumed model of elastic foundation is a generalization of the well-known Winkler model. The introduction of an additional modulus of horizontal deformability of the foundation makes it possible to describe the stability of the plate resting on a sufficiently fine net of elastic point supports such as piles or columns. The annular plates under consideration have a space varying microstructure and hence are described by partial differential equations with highly oscillating, non-continuous coefficients, which are not a good tool for application to engineering problems. Hence, various simplified models are proposed, replacing these plates by plates with effective properties described by smooth, slowly varying functions. The plates under consideration are made of an isotropic homogeneous matrix and isotropic homogeneous ribs which are situated along the radial direction.

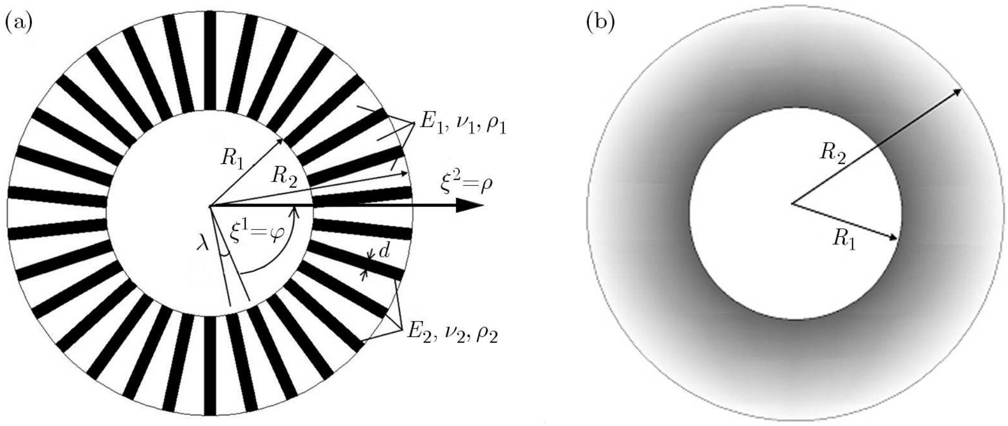

Fig. 1. Fragment of the midplane of a plate with longitudinally graded microstructure: (a) microscopic level, (b) macroscopic level

The plate and foundation have λ-periodic structure along the angular axis and slow gradation of effective properties in the radial direction. The period λ of inhomogeneity is assumed to be very small when compared to the characteristic length dimension of the plate along the angular axis. The apparent properties of the plate and foundation are constant in the angular direction and slowly graded in space in the radial direction. Hence we deal here with a special case of a functionally graded material (FGM) and functionally graded foundation properties.

Functionally graded materials are a class of composite materials where composition of each material constituent determines continuously varying effective properties of the composite. Many papers have been dedicated to analyse the behaviour of functionally graded (FGM) plates. The analysis of functionally graded plates subjected to in-plane compressive loading can be found in several papers. Javaheri and Eslani (2002) analyzed stability of rectangular FGM plates simply supported on all edges. It is assumed that Young's modulus varies along the thickness direction. In the paper of Tylikowski (2005) analysis of dynamic stability of FGM rectangular plates subjected to in-plane time dependent forces is presented. Material properties are graded in the thickness direction according to the volume fraction power law distribution. You *et al.* (2009) developed an analytical solution to determine deformations and stresses in circular disks made of an FGM subjected to internal and/or external pressure. The governing equations are derived from basic equations of axisymmetric, plane stress problem in elasticity. The mechanical properties of materials are functions of the radial coordinate. In the paper by Tung and Duc (2010) explicit expressions of postbuckling load-deflections curves by the Galerkin method are obtained. Material properties in simply supported rectangular plates are assumed to be graded along the thickness direction according the power law distribution of constituents. However, analyses of FGM plates resting on an elastic foundation are quite limited. In the paper by Benyoucef *et al.* (2010), the thick rectangular FGM plate with material properties graded in the thickness direction according to a simple power-law distribution in terms of volume fractions of constituents is analyzed. The plates are resting on a homogeneous elastic foundation. The foundation is modelled as a two-parameter Pasternak or one-parameter Winkler-type foundation. In the paper by Naderi and Saidi (2011), the exact solution of the buckling problem for FGM sector plates resting on a homogeneous elastic foundation with one modulus is presented. It is assumed that the modulus of elasticity E in the thickness direction varies according to a power law function.

The majority of the above mentioned papers deal with plates where it is assumed that the material properties vary along the plate thickness direction. In contrast to these papers, in the present contribution, we deal with effective properties of the plate material and foundation varying in the midplane of the plates.

The direct description of the plate under consideration leads to equations with highly oscillating and non-continuous coefficients. Hence, the aim of this contribution is to formulate averaged models described by equations with functional but smooth and slowly varying coefficients. Here we can mention these models which are based on the asymptotic homogenization technique for equations with non-uniformly oscillating coefficients, cf. Jikov *et al.* (1994). However, because the formulation of the averaged model by using the asymptotic homogenization is rather complicated from the computational point of view, these asymptotic methods are restricted to the first approximation. Hence, the averaged model obtained by using this method neglects the effect of the microstructure size on the overall response of the FGM-plate. The formulation of the averaged mathematical model for the analysis of stability of the plates under consideration will be based on the *tolerance averaging technique*. The general modelling procedures of this technique are given by Woźniak *et al.* (2008, 2010). One should also mention a few papers, where various special problems of microstructured media are presented; e.g. Matysiak (1995), Nagórko and Wągrowska (2002), Wierzbicki (1995). The applications of the tolerance averaging technique for the modelling of stability of various periodic composites were presented in a series papers, e.g. Baron (2003), Michalak (1998), Tomczyk (2005), Wierzbicki *et al.* (1997). The approach, based on the tolerance averaging technique, to formulate macroscopic mathematical models for functionally graded stratified media was proposed by Michalak *et al.* (2007), Ostrowski and Michalak (2011) for the heat conductions problem, and by Jędrysiak and Michalak (2011) for the stability of thin plates. In the paper by Michalak (2012), shells with functionally graded effective properties are analysed. Michalak and Wirowski (2012) analysed dynamic behaviour of thin annular FGM plates with gradation of the material properties along the specified direction.

Throughout the paper indices $i,j,k,\dots$ run over $1,2,3$, indices $\alpha,\beta,\gamma,\dots$ run over $1,2$. We also introduce non-tensorial indices $A,B,C,\dots$ which run over the sequence $1,\dots,N$. The summation convention holds for all aforementioned sub- and superscripts.

2. Preliminaries

The object of our considerations are annular functionally graded plates with microstructure given in Fig. 1 resting on a microheterogeneous foundation. Let us introduce the orthogonal curvilinear coordinate system $O\xi^1\xi^2\xi^3$ in the physical space occupied by a plate under consideration. Setting $\mathbf{x} \equiv (\xi^1,\xi^2)$ and $z = \xi^3$, it is assumed that the undeformed plate occupies the region $\Omega \equiv \{(\mathbf{x},z): \ -H/2 \leqslant z \leqslant H/2,\ \mathbf{x} \in \Pi\}$, where Π is the plate midplane and H is the plate thickness. We denote by $g_{\alpha\beta}$ a metric tensors and by $\epsilon_{\alpha\beta}$ a Ricci tensor. Here and in the sequel, a vertical line before the subscripts stands for the covariant derivative and $\partial_\alpha = \partial/\partial\xi^\alpha$, $\xi^1 = \varphi$, $\xi^2 = \rho$. The plate rests on the generalized Winkler foundation whose properties are characterized by vertical k_z and horizontal k_t foundation moduli. The foundation reaction according to Gomuliński (1967) has three components acting in the direction of the coordinates (z,ρ,φ)

$$R_z = k_z w \qquad R_\rho = k_t\frac{H}{2}\partial_\rho w \qquad R_\varphi = k_t\frac{H}{2}\frac{1}{\rho}\partial_\varphi w \tag{2.1}$$

The model equations for the stability of the considered plate will be obtained in the framework of the well-known second order non-linear theory for thin plates resting on the elastic foundation (Woźniak, 2001). Denoting the displacement field of the plate midsurface by $w(\mathbf{x},t)$, the external forces by $p(\mathbf{x},t)$ and by μ the mass density related to this midsurface, we obtain:
(i) strain-displacement and constitutive equations

$$\kappa_{\alpha\beta} = -w_{|\alpha\beta} \qquad m^{\alpha\beta} = -D^{\alpha\beta\gamma\mu}\kappa_{\gamma\mu} \tag{2.2}$$

where: $D^{\alpha\beta\gamma\mu} = 0.5D(g^{\alpha\mu}g^{\beta\gamma} + g^{\alpha\gamma}g^{\beta\mu} + \nu(\epsilon^{\alpha\gamma}\epsilon^{\beta\mu} + \epsilon^{\alpha\mu}\epsilon^{\beta\gamma}))$, $D = Eh^3/12(1-\nu^2)$.
(ii) the strain energy averaged over the plate thickness

$$E(\xi^\lambda) = \frac{1}{2}D^{\alpha\beta\gamma\delta}w_{|\alpha\beta}w_{|\gamma\delta} + \frac{1}{2}n^{\alpha\beta}w_{|\alpha}w_{|\beta} + \frac{1}{2}k_z(w)^2 + \frac{1}{2}\frac{h^2}{4}k_t\delta^{\alpha\beta}\partial_\alpha w\partial_\beta w \tag{2.3}$$

(iii) kinetic energy

$$K(\xi^\alpha) = \frac{1}{2}\mu\dot{w}\dot{w} \tag{2.4}$$

The governing equations of the plate under consideration can be described by the well-known principle of stationary action. We introduce the action functional defined by

$$\mathcal{A}(w(\cdot)) = \int_\Pi\int_{t_0}^{t_1}[\mathcal{L}(\mathbf{y},w_{|\alpha\beta}(\mathbf{y},t),w_{|\alpha}(\mathbf{y},t),\dot{w}(\mathbf{y},t),w(\mathbf{y},t)) + pw]\,dt\,d\mathbf{y} \tag{2.5}$$

with the Lagrangian defined by

$$
\begin{aligned}
&\mathcal{L}(\cdot, w_{|\alpha\beta}, \dot{w}, \partial_\alpha w, w) = K(\cdot, \dot{w}) - E(\cdot, w_{|\alpha\beta}, \partial_\alpha w, w) \\
&\quad = \frac{1}{2}\Big(\mu\dot{w}\dot{w} - n^{\alpha\beta} w_{|\alpha} w_{|\beta} - D^{\alpha\beta\gamma\delta} w_{|\alpha\beta} w_{|\gamma\delta} - k_z w w - \frac{h^2}{4} k_t \delta^{\alpha\beta} w_{|\alpha} w_{|\beta}\Big)
\end{aligned}
\tag{2.6}
$$

where $n^{\alpha\beta}$ are in-plane forces, and the Kronecker-deltas $\delta^{\alpha\beta}$ will be treated as a tensor; $\delta^{11} = 1/\rho^2$, $\delta^{22} = 1$.

The principle stationary action applied to the functional $\mathcal{A}$ with the Lagrangian $\mathcal{L}$, defined by Eq. (2.6), leads to the Euler-Lagrange equation

$$
\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{w}} - \frac{\partial \mathcal{L}}{\partial w} + \Big(\frac{\partial \mathcal{L}}{\partial w_{|\alpha}}\Big)_{|\alpha} - \Big(\frac{\partial \mathcal{L}}{\partial w_{|\alpha\beta}}\Big)_{|\alpha\beta} = p
\tag{2.7}
$$

and the equilibrium equations

$$
(D^{\alpha\beta\gamma\delta} w_{|\gamma\delta})_{|\alpha\beta} - (n^{\alpha\beta}_{|\beta})_{|\alpha} - \frac{h^2}{4}(k_t \delta^{\alpha\beta} w_{|\beta})_{|\alpha} + k_z w + \mu\ddot{w} = p
\tag{2.8}
$$

This direct description leads to plate equations with discontinuous and highly oscillating coefficients. These equations are too complicated to be used in the engineering analysis and will be used as the starting point in the tolerance modelling procedure.

3. Averaged models

Let us introduce the polar coordinates system $O\xi^1\xi^2$, $0 \leqslant \xi^1 \leqslant \varphi$, $R_1 \leqslant \xi^2 \leqslant R_2$ so that the undeformed midplane of the annular plate occupies the region $\Pi \equiv [0,\varphi] \times [R_1, R_2]$. Let λ, $0 < \lambda \ll \varphi$, be the known microstructure parameter. Denote Π_Δ as a subset of Π of points with coordinates determined by conditions $(\xi^1, \xi^2) \in (\lambda/2, \varphi - \lambda/2) \times (R_1, R_2)$. An arbitrary cell with a center at the point with coordinates $(\xi^1\xi^2)$ in Π_Δ will be determined by $\Delta(\xi^1,\xi^2) = (\xi^1 - \lambda/2, \xi^1 + \lambda/2) \times \{\xi^2\}$. At the same time, the thickness h of the plate under consideration is supposed to be constant and small compared to the microstructure parameter λ.

In order to derive averaged model equations, we applied the tolerance averaging approach. This technique based on the concept of tolerance and indiscrenibility relations. The general modelling procedures of this technique and basic concepts of this technique, as a tolerance parameter, a tolerance periodic function, a slowly varying function, a highly oscillating function are given by Woźniak *et al.* (2008, 2010).

We mention here only the averaging operator. For an arbitrary integrable function $f(\cdot)$, the averaging operator over the cell $\Delta(\cdot)$ is defined by

$$
\langle f\rangle(\xi^1,\xi^2) = \frac{1}{\lambda}\int\limits_{\xi^1-\lambda/2}^{\xi^1+\lambda/2} f(\eta,\xi^2)\, d\eta
\tag{3.1}
$$

for every $\xi^1 \in [\lambda/2, \varphi - \lambda/2]$, $\xi^2 \in [R_1, R_2]$.

3.1. Tolerance model

The tolerance averaging technique will be applied to equations (2.1)-(2.7) in order to derive averaged model equations. The first assumption in this technique is micro-macro decomposition of the displacement field

$$w(\xi^1,\xi^2,t)=w^0(\xi^1,\xi^2,t)+h^A(\xi^1)V_A(\xi^1,\xi^2,t) \qquad A=1,\dots,N \tag{3.2}$$

for $\xi^\alpha \in \Pi$ and $t\in(t_0,t_1)$.

The modelling assumption states that the functions $w^0(\cdot,\xi^2,t)\in SV_\delta^2(\Omega,\Delta)$, $V_A(\cdot,\xi^2,t)\in SV_\delta^2(\Omega,\Delta)$ are slowly varying functions together with all partial derivatives. The functions $w^0(\cdot,\xi^2,t)$, $V_A(\cdot,\xi^2,t)$ are the basic unknowns of the modelling problem. The functions $h^A(\cdot)$ are known, dependent on the microstructure length parameter λ, fluctuation shape functions.

Let $\widetilde{h}^A(\cdot)$, $\partial_1\widetilde{h}^A(\cdot)$ stand for the periodic approximation of $h^A(\cdot)$, $\partial_1 h^A(\cdot)$ in the cell Δ, respectively. Due to the fact that $w(\cdot,\xi^2,t)$ are tolerance periodic functions, it can be observed that the periodic approximation of $w_h(\cdot,\xi^2,t)$ and $\partial_\alpha w_h(\cdot,\xi^2,t)$ in $\Delta(\cdot)$ have the form

$$\begin{aligned}
&w_h(y,\xi^2,t)=w^0(\xi^\alpha,t)+h^A(y)V_A(\xi^\alpha,t)\\
&\partial_\alpha w_h(y,\xi^2,t)=\partial_\alpha w^0(\xi^\alpha,t)+\partial_1 h^A(y)V_A(\xi^\alpha,t)+h^A(y)\partial_2 V_A(\xi^\alpha,t)\\
&\dot{w}_h(y,\xi^2,t)=\dot{w}^0(\xi^\alpha,t)+h^A(y)\dot{V}_A(\xi^\alpha,t)
\end{aligned} \tag{3.3}$$

for every $\xi^\alpha\in\Pi$, almost every $y\in\Delta(\xi^\alpha)$ and every $t\in(t_0,t_1)$.

The tolerance model equations will be obtained by the averaging of the functional $\mathcal{A}$, Eq. (2.5). Substituting decomposition (3.2) of the displacement field into the Lagrangian $\mathcal{L}(\xi^\alpha,w,w_{|\alpha\beta},w_{|\alpha},\dot{w})$, described by equation (2.6), and using the tolerance averaging technique, we obtain

$$\mathcal{A}_h(w^0,V_A)=\int\limits_{t_0}^{t_1}\int\limits_{\Pi}[\langle\mathcal{L}\rangle+\langle p(\cdot)\rangle w^0(\cdot)+\langle p(\cdot)h^A(\cdot)\rangle V_A(\cdot)]\,d\xi^\alpha\,dt \tag{3.4}$$

where averaged Lagrangian (2.6) has the form

$$\begin{aligned}
\langle\mathcal{L}\rangle&=\frac{1}{2}\langle\mu\rangle\dot{w}^0\dot{w}^0+\langle\mu h^A\rangle\dot{w}^0\dot{V}_A+\frac{1}{2}\langle\mu h^A\varphi^B\rangle\dot{V}_A\dot{V}_B+\langle p\rangle w^0+\langle ph^A\rangle V_A\\
&-\frac{1}{2}\langle D^{\alpha\beta\gamma\mu}\rangle w^0_{|\alpha\beta}w^0_{|\gamma\mu}-\langle D^{11\gamma\mu}h^A_{|11}\rangle w^0_{|\gamma\mu}V_A-\langle D^{22\gamma\mu}h^A\rangle V_{A|22}w^0_{|\gamma\mu}\\
&-2\langle D^{12\gamma\mu}h^A_{|1}\rangle w^0_{|\gamma\mu}V_{A|2}-\langle D^{1122}h^A_{|11}h^B\rangle V_AV_{B|22}-\frac{1}{2}\langle D^{1111}h^A_{|11}h^B_{|11}\rangle V_AV_B\\
&-2\langle D^{1212}h^A_{|1}h^B_{|1}\rangle V_{A|2}V_{B|2}-\frac{1}{2}\langle D^{2222}h^Ah^B\rangle V_{A|22}V_{B|22}-\frac{1}{2}\langle k_z\rangle w^0w^0\\
&-\langle k_zh^A\rangle w^0V_A-\frac{1}{2}\langle k_zh^Ah^B\rangle V_BV_A-\frac{H^2}{8}\langle k_tg^Ag^B\rangle\delta^{22}V_{A|2}V_{B|2}\\
&-\frac{H^2}{4}\langle k_th^A_{|1}\rangle\delta^{1\beta}w^0_{|\beta}V_A-\frac{H^2}{4}\langle k_th^A\rangle\delta^{2\beta}w^0_{|\beta}V_{A|2}+\frac{H^2}{4}\langle k_th^A_{|1}h^B_{|1}\rangle\delta^{11}V_AV_B\\
&-\frac{H^2}{8}\langle k_t\rangle\delta^{\alpha\beta}w^0_{|\alpha}w^0_{|\beta}-\frac{1}{2}\langle n^{\alpha\beta}\rangle w^0_{|\alpha}w^0_{|\beta}-\langle n^{1\beta}h^A_{|1}\rangle w^0_{|\beta}V_A-\langle n^{2\beta}h^A\rangle V_{A|2}w^0_{|\beta}\\
&-\frac{1}{2}\langle n^{11}h^A_{|1}h^B_{|1}\rangle V_AV_B-\langle n^{12}h^A_{|1}h^B_{|1}\rangle V_AV_{B|2}-\frac{1}{2}\langle n^{22}h^Ah^B\rangle V_{A|2}V_{B|2}
\end{aligned} \tag{3.5}$$

Applying the principle of stationary action to the averaged functional $\mathcal{A}_h$, the Euler-Lagrange equations take the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\frac{\partial\langle\mathcal{L}\rangle}{\partial\dot{w}^0}-\Big(\frac{\partial\langle\mathcal{L}\rangle}{\partial w^0_{|\alpha\beta}}\Big)_{|\alpha\beta}+\Big(\frac{\partial\langle\mathcal{L}\rangle}{\partial w^0_{|\alpha}}\Big)_{|\alpha}-\frac{\partial\langle\mathcal{L}\rangle}{\partial w^0}=\langle p\rangle\\
&\frac{\partial}{\partial t}\frac{\partial\langle\mathcal{L}\rangle}{\partial\dot{V}_A}-\Big(\frac{\partial\langle\mathcal{L}\rangle}{\partial V_{A|22}}\Big)_{|22}+\Big(\frac{\partial\langle\mathcal{L}\rangle}{\partial V_{A|2}}\Big)_{|2}-\frac{\partial\langle\mathcal{L}\rangle}{\partial V_A}=\langle ph^A\rangle
\end{aligned} \tag{3.6}$$

Using averaged Lagrangian (3.5), we obtain the following system of equations describing stability of the plate resting on the microheterogeneous foundation

$$
\begin{aligned}
&(\langle D^{\alpha\beta\gamma\mu}\rangle w^0_{|\gamma\mu})_{|\alpha\beta} + (\langle D^{\alpha\beta 11}h^A_{|11}\rangle V_A)_{|\alpha\beta} + (\langle D^{\alpha\beta 22}h^A\rangle V_{A|22})_{|\alpha\beta} \\
&\quad + \langle k_z\rangle w^0 + \langle k_z h^A\rangle V_A - \frac{H^2}{4}(\langle k_t\rangle \delta^{\alpha\beta} w^0_{|\beta})_{|\alpha} - \frac{H^2}{4}(\langle k_t h^A_{|1}\rangle \delta^{1\beta} V_A)_{|\beta} \\
&\quad - \frac{H^2}{4}(\langle k_t h^A\rangle \delta^{2\beta} V_{A|2})_{|\beta} - (N^{\alpha\beta} w^0_{|\beta})_{|\alpha} + \langle \mu\rangle \ddot{w}^0 = \langle p\rangle \\
&\langle D^{11\gamma\mu} h^A_{|11}\rangle w^0_{|\gamma\mu} + \langle D^{1122} h^A_{|11} h^B\rangle V_{B|22} + \langle D^{1111} h^A_{|11} h^B_{|11}\rangle V_B \\
&\quad + (\langle D^{22\gamma\mu} h^A\rangle w^0_{|\gamma\mu})_{|22} + (\langle D^{1122} h^A h^B_{|11}\rangle V_B)_{|22} + (\langle D^{2222} h^A h^B\rangle V_{B|22})_{|22} \\
&\quad - 2(\langle D^{12\gamma\mu} h h^A_{|1}\rangle w^0_{|\gamma\mu})_{|2} - 4(\langle D^{1212} h^A_{|1} h^B_{|1}\rangle V_{B|2})_{|2} + \langle k_z h^A h^B\rangle V_B \\
&\quad + \langle k_z h^A\rangle w^0 - \frac{H^2}{4}(\langle k_t h^A\rangle \delta^{2\beta} w^0_{|\beta})_{|2} + \frac{H^2}{4}\langle k_t h^A_{|1}\rangle \delta^{11} w^0_{|\beta} \\
&\quad - \frac{H^2}{4}(\langle k_t h^A h^B\rangle \delta^{22} V_{B|2})_{|2} + \frac{H^2}{4}\langle k_t h^A_{|1} h^B_{|1}\rangle \delta^{11} V_B \\
&\quad - (N^{22}\langle h^A h^B\rangle V_{B|2})_{|2} + N^{11}\langle h^A_{|1} h^B_{|1}\rangle V_B + \langle \mu h^A h^B\rangle \ddot{V}_B = \langle p h^A\rangle
\end{aligned}
\tag{3.7}
$$

We have assumed that the forces $n^{\alpha\beta}$ can be represented by a decomposition

$$
n^{\alpha\beta}(\xi^\gamma) = N^{\alpha\beta}(\xi^\gamma) + \widetilde{n}^{\alpha\beta}(\xi^\gamma) \tag{3.8}
$$

where $N^{\alpha\beta} = \langle n^{\alpha\beta}\rangle$ is a slowly varying function and $\widetilde{n}^{\alpha\beta}(\cdot)$ is the fluctuating part of the forces $n^{\alpha\beta}(\cdot)$, such that $\langle \widetilde{n}^{\alpha\beta}\rangle = 0$. In Eq. (3.5), we have assumed that the fluctuating part $\widetilde{n}^{\alpha\beta}(\cdot)$ of the forces $n^{\alpha\beta}(\cdot)$ is very small compared to their averaged part $N^{\alpha\beta}(\cdot)$, and hence $\langle n^{22} h^A h^B\rangle \cong N^{22}\langle h^A h^B\rangle$.

3.2. Asymptotic model

For the asymptotic modelling procedure we recall only the concept of highly oscillating function. We shall not deal with the notion of the tolerance periodic function as well as slowly-varying function. For every parameter $\varepsilon = 1/n$, $n = 1, 2, \ldots$ we define the scaled cell $\varDelta_\varepsilon \equiv (-\varepsilon l/2, \varepsilon l/2)$ and by $\varDelta_\varepsilon(x) = x + \varDelta_\varepsilon$ the scaled cell with a centre at $\xi^\alpha \in \overline{\varPi}$.

The mass density $\mu(\cdot)$, moduli of the foundation $k_z(\cdot)$, $k_t(\cdot)$ and tensor of elasticity $D^{\alpha\beta\gamma\delta}(\cdot)$ are assumed to be highly oscillating discontinuous functions for almost every $\xi^\alpha \in \overline{\varPi}$. If $\mu(\cdot), k_z(\cdot), k_t(\cdot), D^{\alpha\beta\gamma\delta}(\cdot) \in HO^0_\delta(\varPi, \varDelta)$ then for every $\xi^\alpha \in \overline{\varPi}$ there exist functions $\mu(y, \xi^2)$, $k_z(y, \xi^2)$, $k_t(y, \xi^2)$, $D^{\alpha\beta\gamma\delta}(y, x^2)$ which are periodic approximations of the functions $\mu(\cdot)$, $k_z(\cdot)$, $k_t(\cdot)$, $D^{\alpha\beta\gamma\delta}(\cdot)$, respectively.

The asymptotic modelling procedure begins with decomposition of the displacement as a family of fields

$$
w_\varepsilon(y, \xi^2, t) = w^0(y, \xi^2, t) + \varepsilon^2 \widetilde{h}^A\Big(\frac{y}{\varepsilon}, \xi^2\Big) V_A(y, \xi^2, t) \qquad y \in \varDelta_\varepsilon(\xi^\alpha) \tag{3.9}
$$

where $\widetilde{h}^A(y, \xi^2)$ is a periodic approximation of highly oscillating functions $h^A(\cdot)$. From formula (3.3) we obtain

$$
\begin{aligned}
&\partial_\alpha w_\varepsilon(y, \xi^2, t) = \partial_\alpha w^0(y, \xi^2, t) + \varepsilon \partial_1 \widetilde{h}^A\Big(\frac{y}{\varepsilon}, \xi^2\Big) V_A(y, \xi^2, t) + \varepsilon^2 \widetilde{h}^A\Big(\frac{y}{\varepsilon}, \xi^2\Big) \partial_2 V_A(y, \xi^2, t) \\
&\partial_{\alpha\beta} w_\varepsilon(y, \xi^2, t) = \partial_{\alpha\beta} w^0(y, \xi^2, t) + \partial_{11} \widetilde{h}^A\Big(\frac{y}{\varepsilon}, \xi^2\Big) V_A(y, \xi^2, t) \\
&\quad + 2\varepsilon \partial_1 \widetilde{h}^A\Big(\frac{y}{\varepsilon}, \xi^2\Big) \partial_2 V_A(y, \xi^2, t) + \varepsilon^2 \widetilde{h}^A\Big(\frac{y}{\varepsilon}, \xi^2\Big) \partial_{22} V_A(y, \xi^2, t) \\
&\dot{w}_\varepsilon(y, \xi^2, t) = \dot{w}^0(y, \xi^2, t) + \varepsilon^2 \widetilde{h}^A\Big(\frac{y}{\varepsilon}, \xi^2\Big) \dot{V}_A(y, \xi^2, t)
\end{aligned}
\tag{3.10}
$$

Under limit passage $\varepsilon \to 0$ for $y \in \Delta_\varepsilon(\xi^\alpha)$, $\xi^\alpha \in \overline{\Pi}$ we rewrite formulae (3.9) and (3.10) in the form

$$\begin{aligned} &w_\varepsilon(y,\xi^2,t) = w^0(y,\xi^2,t) + O(\varepsilon) \\ &\partial_\alpha w_\varepsilon(y,\xi^2,t) = \partial_\alpha w^0(y,\xi^2,t) + O(\varepsilon) \\ &\partial_{\alpha\beta} w_\varepsilon(y,\xi^2,t) = \partial_{\alpha\beta} w^0(y,\xi^2,t) + \partial_{11}\widetilde{h}^A\Big(\frac{y}{\varepsilon},\xi^2\Big)V_A(y,\xi^2,t) + O(\varepsilon) \\ &\dot{w}_\varepsilon(y,\xi^2,t) = \dot{w}^0(y,\xi^2,t) + O(\varepsilon) \end{aligned} \tag{3.11}$$

For a periodic approximation of the Lagrangian $\mathcal{L}$, we have

$$\begin{aligned} &\widetilde{\mathcal{L}}_\varepsilon\Big(\frac{y}{\varepsilon},\ \xi^2,\ w^0\Big(\frac{y}{\varepsilon},\xi^2,t\Big) + O(\varepsilon),\ \partial_\alpha w^0\Big(\frac{y}{\varepsilon},\xi^2,t\Big) + O(\varepsilon),\ \dot{w}^0\Big(\frac{y}{\varepsilon},\xi^2,t\Big) + O(\varepsilon), \\ &\qquad \partial_{\alpha\beta} w^0\Big(\frac{y}{\varepsilon},\xi^2,t\Big) + \partial_{11}\widetilde{h}^A\Big(\frac{y}{\varepsilon},\xi^2\Big)V_A\Big(\frac{y}{\varepsilon},\xi^2,t\Big) + O(\varepsilon)\Big) \end{aligned} \tag{3.12}$$

If $\varepsilon \to 0$ then $\widetilde{\mathcal{L}}_\varepsilon$ by means of property of the mean value, see Jikov *et al.* (1994), weakly tends to

$$\begin{aligned} &\mathcal{L}_0\Big(\xi^\alpha, w^0(\xi^\alpha,t), \partial_\alpha w^0(\xi^\alpha,t), \dot{w}^0(\xi^\alpha,t), \partial_{\alpha\beta} w^0(\xi^\alpha,t), V_A(\xi^\alpha,t)\Big) \\ &\quad = \frac{1}{|\Delta|}\int\limits_{\Delta(x)} \widetilde{\mathcal{L}}\Big(y,\xi^\alpha, w^0(\xi^\alpha,t), \partial_\alpha w^0(\xi^\alpha,t), \dot{w}^0(\xi^\alpha,t), \partial_{\alpha\beta} w^0(\xi^\alpha,t) \\ &\quad + \partial_{11}\widetilde{h}^A(y,\xi^2)V_A(\xi^\alpha,t)\Big)\ dy \end{aligned} \tag{3.13}$$

The asymptotic action functional has the form

$$\mathcal{A}_\varepsilon^0(w^0,V_A) = \int\limits_{t_0}^{t_1}\int\limits_{\Pi} \mathcal{L}_0\Big(\xi^\alpha, w^0(\cdot), \partial_\alpha w^0(\cdot), w^0_{|\alpha\beta}(\cdot), V_A(\cdot), \dot{w}^0(\cdot)\Big)\ d\xi^\alpha\ dt \tag{3.14}$$

where the Lagrangian is given by

$$\begin{aligned} &\mathcal{L}_0(\xi^\alpha, w^0, \partial_\alpha w^0, w^0_{|\alpha\beta}, V_A, \dot{w}^0) = \frac{1}{2}\langle D^{\alpha\beta\gamma\mu}\rangle w^0_{|\alpha\beta} w^0_{|\gamma\mu} + \langle D^{11\gamma\mu} h^A_{|11}\rangle V_A w^0_{|\gamma\mu} \\ &\quad + \frac{1}{2}\langle D^{1111} h^A_{|11} h^B_{|11}\rangle V_A V_B + \frac{1}{2}\langle k_z\rangle w^0 w^0 + \frac{H^2}{8}\langle k_t\rangle \delta^{\alpha\beta}\partial_\alpha w^0 \partial_\beta w^0 \\ &\quad + \frac{1}{2}\langle n^{\alpha\beta}\rangle \partial_\alpha w^0 \partial_\beta w^0 - \frac{1}{2}\langle \mu\rangle \dot{w}^0 \dot{w}^0 - \langle p\rangle w^0 \end{aligned} \tag{3.15}$$

Applying the principle of stationary action, we derive the Euler-Lagrangian equations

$$\begin{aligned} &\frac{\partial}{\partial t}\frac{\partial \mathcal{L}_0}{\partial \dot{w}^0} - \Big(\frac{\partial \mathcal{L}_0}{\partial w^0_{|\alpha\beta}}\Big)_{|\alpha\beta} + \partial_\alpha\Big(\frac{\partial \mathcal{L}_0}{\partial w^0_{|\alpha}}\Big) - \frac{\partial \mathcal{L}_0}{\partial w^0} = \langle p\rangle \\ &\frac{\partial \mathcal{L}_0}{\partial V_A} = 0 \qquad\qquad A = 1,\dots,N \end{aligned} \tag{3.16}$$

Substituting formulae (3.15) into equations (3.16), we obtain the following system of equations describing the stability of the plate under consideration

$$\begin{aligned} &(\langle D^{\alpha\beta\gamma\mu}\rangle w^0_{|\gamma\mu})_{|\alpha\beta} + (\langle D^{11\alpha\beta} h^A_{|11}\rangle V_A)_{|\alpha\beta} + \langle k_z\rangle w^0 - \frac{H^2}{4}\partial_\alpha(\langle k_t\rangle \delta^{\alpha\beta}\partial_\beta w^0) \\ &\quad - (N^{\alpha\beta} w^0_{|\beta})_{|\alpha} + \langle \mu\rangle \ddot{w}^0 = \langle p\rangle \\ &\langle D^{11\alpha\beta} h^A_{|11}\rangle w^0_{|\alpha\beta} + \langle D^{1111} h^A_{|11} h^B_{|11}\rangle V_B = 0 \end{aligned} \tag{3.17}$$

Eliminating V_A from second equation (3.17)

$$V_A = -\frac{\langle D^{11\gamma\mu} h^B_{|11}\rangle}{\langle D^{1111} h^A_{|11} h^B_{|11}\rangle} w^0_{|\gamma\mu} \tag{3.18}$$

and denoting the effective elastic moduli

$$D^{\alpha\beta\gamma\mu}_{eff} = \langle D^{\alpha\beta\gamma\mu}\rangle - \frac{\langle D^{11\gamma\mu} h^B_{|11}\rangle}{\langle D^{1111} h^A_{|11} h^B_{|11}\rangle}\langle D^{11\alpha\beta} h^A_{|11}\rangle \tag{3.19}$$

we arrive at the following asymptotic model equation for the averaged displacement of the plate midplane $w^0(\xi^\alpha, t)$

$$(\langle D^{\alpha\beta\gamma\mu}_{eff}\rangle w^0_{|\gamma\mu})_{|\alpha\beta} + \langle k_z\rangle w^0 - \frac{H^2}{4}\partial_\alpha(\langle k_t\rangle \delta^{\alpha\beta}\partial_\beta w^0) - (N^{\alpha\beta} w^0_{|\beta})_{|\alpha} + \langle \mu\rangle \ddot{w}^0 = \langle p\rangle \tag{3.20}$$

Equations (3.18)-(3.20) represent the asymptotic model of the stability of the thin plate interacting with microheterogeneous subsoil.

The coefficients of model equations (3.7), (3.20) are smooth functions of the radial coordinate $\rho \in (R_0, R_1)$ in contrast to equations in direct description with the discontinuous and highly oscillating coefficients.

4. Applications

In order to illustrate the model equations (3.7) and (3.20), we shall investigate a simple problem of the linear polar-symmetrical stability of the annular plate clamped on its boundary (Fig. 2). The considered composite plate is interacting with heterogeneous elastic subsoil.

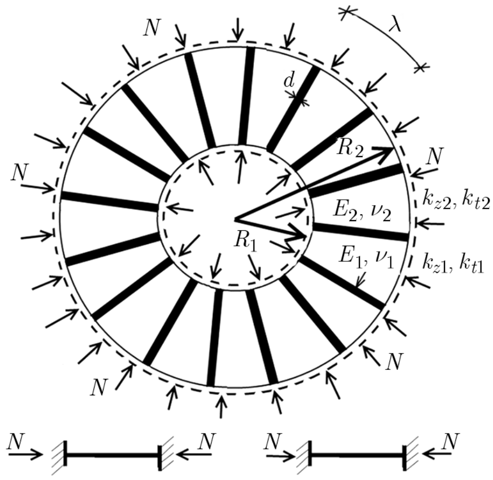

Fig. 2. The annular plate with a longitudinally graded structure

The important point of the tolerance modeling approach is the determination of the fluctuation shape functions (FSF). Our analysis we restrict to the case when we have only one fluctuation shape function, hence $A, B = 1$ and $V_A(\xi^\alpha, t) = V(\xi^\alpha, t)$. The calculation of the fluctuation shape functions is usually very difficult. Hence we apply an approximate form of the fluctuation shape function analogous to dynamic analysis. For one-dimensional cell under consideration $\Delta(\xi^1, \xi^2)$, as the fluctuation, shape function we assume

$$h(\cdot) = \lambda^2 \cos\frac{2\pi\xi^1}{\lambda} \tag{4.1}$$

The comparison of results for the exact and this approximate form of the fluctuation shape function, we can find in papers by Jędrysiak (2001), Jędrysiak and Michalak (2005). For differences between the value of Young's modulus $0.25 \leqslant E_b/E_m \leqslant 4$ and for the ratio $0.25 \leqslant d/(\lambda\rho) \leqslant 0.75$ (d – the widths of the ribs, $\lambda\rho$ – microstructure length parameter) the error for the approximate form of FSF is smaller than 10%.

4.1. Tolerance model

Let us introduce the polar coordinate system $O\xi^1\xi^2$, where $\varphi = \xi^1$ is the angular coordinate and $\rho = \xi^2$ – radial coordinate. Setting $w^0 = w^0(\rho)$, $V = V(\rho)$, we obtain from equations (3.7) the following system of equations describing the stability of annular plates interacting with heterogeneous subsoil

$$
\begin{aligned}
&\partial_{22}(\langle\widetilde{D}^{2222}\rangle\partial_{22}w^0) + \frac{1}{\rho}\partial_{22}(\langle\widetilde{D}^{2211}\rangle\partial_2 w^0) + \frac{2}{\rho}\partial_2(\langle\widetilde{D}^{2222}\rangle\partial_{22}w^0) + \frac{1}{\rho^3}\langle\widetilde{D}^{1111}\rangle\partial_2 w^0\\
&\quad - \frac{1}{\rho^2}\partial_2(\langle\widetilde{D}^{1111}\rangle\partial_2 w^0) - \frac{1}{\rho}\partial_2(\langle\widetilde{D}^{2211}\rangle\partial_{22}w^0) + \frac{2}{\rho^4}\langle\widetilde{D}^{2211}h_{|11}\rangle V - \frac{2}{\rho^3}\partial_2(\langle\widetilde{D}^{2211}h_{|11}\rangle V)\\
&\quad + \frac{1}{\rho^2}\partial_{22}(\langle\widetilde{D}^{2211}h_{|11}\rangle V) + \frac{2}{\rho^4}\langle\widetilde{D}^{1111}h_{|11}\rangle V - \frac{1}{\rho^3}\partial_2(\langle\widetilde{D}^{1111}h_{|11}\rangle V) + \partial_{22}(\underline{\langle\widetilde{D}^{2222}h\rangle}\partial_{22}V)\\
&\quad + \frac{2}{\rho}\partial_2(\underline{\langle\widetilde{D}^{2222}h\rangle}\partial_{22}V) - \frac{1}{\rho}\partial_2(\underline{\langle\widetilde{D}^{1122}h\rangle}\partial_{22}V) + \langle k_z\rangle w^0 + \underline{\langle k_z h\rangle}V - \frac{H^2}{4}\langle k_t\rangle\frac{1}{\rho}\partial_2 w^0\\
&\quad - \frac{H^2}{4}\partial_2(\langle k_t\rangle\partial_2 w^0) - \frac{H^2}{4}\partial_2(\underline{\langle k_t h\rangle}\partial_2 V) - \frac{1}{\rho}N_\varphi\partial_2 w^0 - N_\rho\partial_{22}u = 0 \qquad (4.2)\\
&\frac{1}{\rho^3}\langle\widetilde{D}^{1111}h_{|11}\rangle\partial_2 w^0 + \frac{1}{\rho^2}\langle\widetilde{D}^{1122}h_{|11}\rangle\partial_{22}w^0 + \frac{1}{\rho^4}\langle\widetilde{D}^{1111}h_{|11}h_{|11}\rangle V + \frac{1}{\rho^2}\underline{\langle\widetilde{D}^{1111}h_{|11}h\rangle}\partial_{22}V\\
&\quad + \partial_{22}\Big(\frac{1}{\rho}\underline{\langle\widetilde{D}^{2211}h\rangle}\partial_2 w^0\Big) + \partial_{22}(\underline{\langle\widetilde{D}^{2222}h\rangle}\partial_{22}w^0) + \partial_{22}\Big(\frac{1}{\rho^2}\underline{\langle\widetilde{D}^{2211}h_{|11}h\rangle}V\Big)\\
&\quad + \partial_{22}(\underline{\langle\widetilde{D}^{2222}hh\rangle}\partial_{22}V) - 4\partial_2\Big(\frac{1}{\rho^2}\underline{\langle\widetilde{D}^{1212}h_{|1}h_{|1}\rangle}\partial_2 V\Big) + \underline{\langle k_z hh\rangle}V - \frac{H^2}{4}\partial_2(\underline{\langle k_t h\rangle}\partial_2 w^0)\\
&\quad - \frac{H^2}{4}\partial_2(\underline{\langle k_t hh\rangle}\partial_2 V) + \frac{H^2}{4}\underline{\langle k_t h_{|1}h_{|1}\rangle}\frac{1}{\rho^2}V - N_\rho\underline{\langle hh\rangle}\partial_{22}V + \frac{1}{\rho^2}N_\varphi\underline{\langle h_{|1}h_{|1}\rangle}V = 0
\end{aligned}
$$

where we have denoted $\widetilde{D}^{2222} = D^{2222}$, $\widetilde{D}^{1122} = \rho^2 D^{1122}$, $\widetilde{D}^{1111} = \rho^4 D^{1111}$, $N_\varphi = \rho^2 N^{11}$, $N_\rho = N^{22}$. Equations (4.2) represent a system of two partial differential equations for the averaged deflection $w^0(\cdot)$ and the fluctuation amplitude $V(\cdot)$. The boundary conditions for the clamped plate are given by

$$
\begin{aligned}
&w^0(\rho = R_1) = w^0(\rho = R_2) = 0 \qquad && \partial_2 w^0(\rho = R_1) = \partial_2 w^0(\rho = R_2) = 0\\
&V(\rho = R_1) = V(\rho = R_2) = 0 \qquad && \partial_2 V(\rho = R_1) = \partial_2 V(\rho = R_2) = 0
\end{aligned}
\qquad (4.3)
$$

Since $h(\cdot) \in O(\lambda^2)$, the underlined moduli depend on the microstructure length parameter λ. Hence, the tolerance model equations describe the microstructure length-scale effect on the stability of the plate under consideration.

4.2. Asymptotic model

For analysis of the asymptotic model we use equations (3.20). Denoting $D_r(\rho) = D_{eff}^{2222}$, $D_\varphi(\rho) = \rho^4 D_{eff}^{1111}$, $D_{r\varphi}(\rho) = \rho^2 D_{eff}^{1122}$, $K_z = \langle k_z\rangle$, $K_t = \langle k_t\rangle$ we obtain from equation (3.20) a single equation describing stability for the asymptotic model of the plate under consideration

$$\begin{aligned}&\partial_{22}(D_r\partial_{22}w^0)+\frac{2}{\rho}\partial_2\Big(\Big(D_r-\frac{1}{2}D_{r\varphi}\Big)\partial_{22}w^0\Big)+\frac{1}{\rho}\partial_{22}(D_{r\varphi}\partial_2 w^0)-\frac{1}{\rho^2}\partial_2(D_\varphi\partial_2 w^0)\\&\quad+\frac{1}{\rho^3}D_\varphi\partial_2 w^0+K_z w^0-\frac{H^2}{4}K_z\frac{1}{\rho}\partial_2 w^0-\frac{H^2}{4}\partial_2(K_t\partial_2 w^0)-N_\rho\partial_{22}w^0-\frac{1}{\rho}N_\varphi\partial_2 w^0=0\end{aligned}\tag{4.4}$$

The above equation represents the single partial differential equation for the averaged deflection $w^0(\cdot)$ and has the form similar to the equation for buckling of the annular plate with cylindrical orthotropy.

4.3. Numerical results for the asymptotic model

In order to derive the critical value of forces for buckling of the plate under consideration we shall use the asymptotic model equation. We look for the solution to equation (4.4), where the problem of the forces N_ρ and N_φ will be solve similarly to plate with cylindrical orthotropy (cf. Mossakowski, 1960)

$$N_\rho=-N\Big(\frac{\rho}{R_2}\Big)^{k-1}\qquad N_\varphi=-Nk\Big(\frac{\rho}{R_2}\Big)^{k-1}\qquad k=\sqrt{\frac{D_\varphi}{D_r}}\tag{4.5}$$

Substituting (4.5) into equation (4.4) we obtain differential operator in the form

$$\begin{aligned}L(w^0)&=\partial_{22}(D_r\partial_{22}w^0)+\frac{2}{\rho}\partial_2\Big(\Big(D_r-\frac{1}{2}D_{r\varphi}\Big)\partial_{22}w^0\Big)+\frac{1}{\rho}\partial_{22}(D_{r\varphi}\partial_2 w^0)\\&\quad-\frac{1}{\rho^2}\partial_2(D_\varphi\partial_2 w^0)+\frac{1}{\rho^3}D_\varphi\partial_2 w^0+K_z w^0-\frac{h^2}{4}K_t\frac{1}{\rho}\partial_2 w^0-\frac{h^2}{4}\partial_2(K_t\partial_2 w^0)\\&\quad+N\Big(\frac{\rho}{R_2}\Big)^{k-1}\partial_{22}w^0+N\frac{1}{\rho}k\Big(\frac{\rho}{R_2}\Big)^{k-1}\partial_2 w^0=0\end{aligned}\tag{4.6}$$

Operator (4.4) has smoothly varying functional coefficients along the radial direction. Hence, in most cases, solutions to specific problems for the plates under consideration have to be obtained using approximate methods. In order to obtain the approximate solution to equation (4.4) for the annular clamped plate interacting with heterogeneous subsoil, the Galerkin method will be used. The smallest value of critical forces can be obtained from the following equation

$$\int\limits_{R1}^{R_2} L(f(\rho))f(\rho)\,d\rho=0\tag{4.7}$$

As the function $f(\rho)$, we assume the first shape function of stability for the isotropic annular clamped plate with the radius $R_1=1.0\,\mathrm{m}$ and $R_2=3.0\,\mathrm{m}$ resting on the elastic homogeneous foundation

$$\begin{aligned}f(r)&=w_1\Big(J_0\Big(1.1875\frac{\rho}{R_1}\Big)+23.9767Y_0\Big(1.1875\frac{\rho}{R1}\Big)+16.0964J_0\Big(3.8405\frac{\rho}{R_1}\Big)\\&\quad+11.1116Y_0\Big(3.8405\frac{\rho}{R1}\Big)\Big)\end{aligned}\tag{4.8}$$

where $J_0(\cdot)$, $Y_0(\cdot)$ are Bessel's functions of the first and second kind, respectively.

4.3.1. Comparison of the test tasks with results from the finite element method

In order to verify the correctness of the derived equations, we analysed the obtained results for a test task. We shall investigate the simple problem of polar-symmetrical stability of an annular clamped plate. We compare the value of critical forces from the asymptotic model with the results from the finite element method (Abaqus program). The following material and geometrical parameters of the plate were assumed: matrix: $E_m = E_1 = 210$, 150 and 69 GPa, $\nu_1 = 0.3$, ribs: $E_r = E_2 = 210$ GPa, $\nu_1 = 0.3$, number of periodic cells $N = 60$, thickness of the plate $H = 0.05$ m, internal radius $R_1 = 1$ m, external radius $R_2 = 3$ m, width of ribs $d = 0.75\lambda R_1 = 0.75(2\pi/60)R_1$ and the foundation parameters: vertical modulus of elasticity of foundation below the matrix $k_{zm} = 25\,\mathrm{MN/m^3}$, vertical modulus of elasticity of foundation below the ribs $k_{zr} = 50\,\mathrm{MN/m^3}$, horizontal modulus of elasticity of foundation below the matrix $k_{tm} = 0\,\mathrm{MN/m^3}$, horizontal modulus of tangent elasticity of the foundation below the ribs $k_{tr} = 0\,\mathrm{MN/m^3}$.

The value of critical forces was calculated in two ways: by making use of asymptotic model equations (AS) and through the finite element method (FEM, Abaqus program). These results are summarized in Table 1

Table 1. The comparison of results for the test task calculated by two independent methods

Matrix	Ribs	Matrix	Ribs	AM (asym. model)	FEM (Abaqus)	Ratio
Modulus E_m [GPa]	Modulus E_r [GPa]	Elast. found. k_{zm} [MN/m^3]	Elast. found. k_{zr} [MN/m^3]	N_{cr} [kN/m]	N_{cr} [kN/m]	$\frac{N_{cr\,\mathrm{FEM}}}{N_{cr\,\mathrm{AS}}}$
210	210	0	0	24303	23843	0.98
150	210	0	0	20321	19533	0.96
69	210	0	0	14239	13181	0.93
210	210	50	50	39212	38756	0.99
150	210	50	50	34990	34054	0.97
69	210	50	50	28090	26615	0.95
210	210	25	50	34720	34275	0.99
150	210	25	50	30510	29712	0.97
69	210	25	50	23790	22690	0.95

The above table shows that the results obtained from equations for the tolerance averaging technique coincide with the results from the well-known finite element method.

4.3.2. Influence of material properties of the plate and foundation on the critical forces

The aim of this Subsection is to investigate the influence of material properties of the plate and foundation on the value of the critical forces. The material and geometrical parameter of the plate we assume identical as in the above example.

In Fig. 3a, there is shown a diagram of the value of critical forces N_{kr} [MN] versus $k = k_{z2}/k_{z1}$, where k_{z2} is the vertical modulus of the foundation under the matrix and k_{z1} under the ribs. The diagram is derived for the ratio $k_{t1}/k_{z1} = 0.5$, $k_{t2}/k_{z2} = 0.5$, horizontal and vertical moduli of elasticity of the foundation and $k_{z1} = 500.0\,\mathrm{MN/m^3}$.

In Fig. 3b, there is shown the influence of the ratio $p = k_t/k_z$, horizontal k_t and vertical k_z modulus of the elastic foundation. The diagrams in Fig. 3b are derived for the ratio $k_{z2}/k_{z1} = 0.1$, $k_{t2}/k_{t1} = 0.1$ and the vertical modulus $k_{z1} = 5000.0\,\mathrm{MN/m^3}$. Diagram $N_{kr1}(p)$ shows the smallest value of critical forces for the plate thickness $H = 0.05$ m and $N_{kr2}(p)$ for the plate thickness $H = 0.20$ m.

4.4. Numerical results for the tolerance model

We look for an approximate solution to equations (4.2) similarly to the asymptotic model using the Galerkin method.

For the tolerance model, we obtain two values of critical forces, for macro and micro buckling.

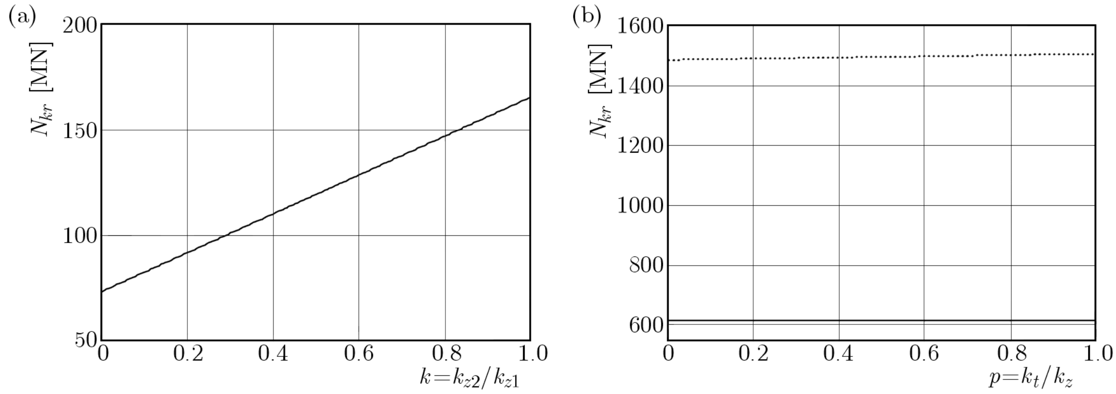

Fig. 3. Diagrams of the value of critical forces N_{kr} [MN] versus: (a) $k = k_{z2}/k_{z1}$, (b) $p = k_t/k_z$

4.4.1. Influence of the number of cells on the critical forces

The aim of this Subsection is to investigate the influence of the microstructure length parameter $\lambda = 2\pi/\alpha$ on the value of critical forces. In Fig. 4, diagrams of the value of critical forces versus numbers of the cells α are shown. The diagrams are derived for the annular clamped plate with geometry: $H = 0.05\,\mathrm{m}$, $R_1 = 1\,\mathrm{m}$, $R_2 = 3\,\mathrm{m}$. The material parameters of the matrix are: $E_m = E_1 = 69\,\mathrm{GPa}$, $\nu_1 = 0.3$ and of the ribs: $E_r = E_2 = 210\,\mathrm{GPa}$, $\nu_2 = 0.3$. The width of the ribs is $d = 0.75\lambda R_1$, subsoil moduli: $k_{z2}/k_{z1} = k_{t2}/k_{t1} = 0.2$, $k_{t1}/k_{z1} = k_{t2}/k_{z2} = 0.2$, $k_{z1} = 50.0\,\mathrm{MN/m^3}$.

In Fig. 4a, the diagram of the value of critical forces for macro buckling of the plate versus number of the cells α is shown. The value of critical forces for $\alpha > 25$-30 is independent of the number of the cells and conforms with the results from the asymptotic model. Let us note that the number of the microstructure cells should be bigger than 30 to provide the correct solution for the tolerance and asymptotic models.

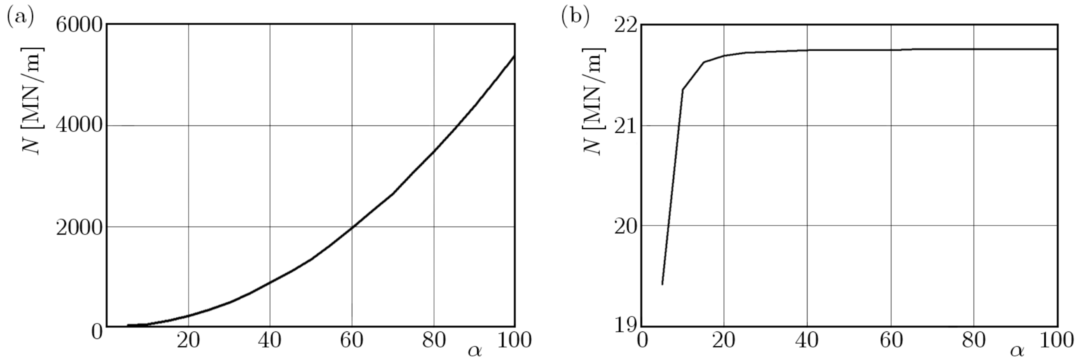

Fig. 4. The value of critical forces N [MN/m] for: (a) macro buckling, (b) micro buckling versus the number of the cells α

In Fig. 4b, the diagram of the value of critical forces for micro buckling of the plate versus number of the cells α is shown. As one should expect, the value of critical forces for micro buckling grows with the increasing number of the microstructure cells.

5. Conclusions

- The composite plate interacting with elastic heterogeneous subsoil having a functionally graded structure is described by model equations involving only smooth coefficients in contrast to the coefficients in equations for direct description, which are non-continuous and highly oscillating.

- Since the proposed model equations have smooth and slowly varying functional coefficients, hence in most cases, solutions to specific problems of stability of the functionally graded plate under consideration have to be obtained using well known numerical methods.
- The contribution contains two model equations – tolerance model equations (3.7) with coefficients depending on the microstructure length λ and simplified asymptotic model equations (3.20).
- Solutions to the boundary value problems formulated in the framework of the proposed models have the physical sense only if they are slowly varying in the distinguished directions. The number of the microstructure cells should be bigger than 30. This requirement determines the range of physical applicability of the proposed model.
- The horizontal foundation modulus has negligibly small influence on the critical force for polar-symmetrical buckling of the plates under consideration.
- Analysing the obtained results, we can observe that the differences between the value of macro buckling critical forces for the tolerance model and the asymptotic model are negligibly small.

References

1. Baron E., 2003, On dynamic stability of an uniperiodic medium thickness plate band, *Journal of Theoretical and Applied Mechanics*, **41**, 305-321
2. Benyoucef S., Mecgab I., Tounsi A., Fekrar A., Atmane H., Bedia A.A., 2010, Bending of thick functionally graded plates resting on Winkler-Pasternak elastic foundation, *Mechanics of Composite Materials*, **46**, 425-434
3. Gomulinski A ., 1967, Determination of eigenvalues for circular plates resting on elastic foundation with two moduli, *Archives of Civil Engineering*, **2**, 183-203
4. Javaheri R., Eslami MR., 2002, Bukling of functionally graded plates under in-plane compressive loading, *Zeitschrift für Angewandte Mathematik und Mechanik*, **82**, 277-283
5. Jędrysiak J., 2001, A contribution to the modeling of dynamic problems for periodic plates, *Engineering Transactions*, **49**, 65-87
6. Jedrysiak J., Michalak B., 2005, Some remarks on dynamic results for averaged and exact models of thin periodic plates, *Journal of Theoretical and Applied Mechanics*, **43**, 405-425
7. Jędrysiak J., Michalak B., 2011, On the modelling of stability problems for thin plates with functionally graded structure, *Thin-Walled Structures*, **49**, 627-635
8. Jikov V.V., Kozlov C.M., Oleinik O.A., 1994, *Homogenization of Differential Operators and Integral Functionals*, Berlin-Heidelberg, Springer Verlag
9. Matysiak S.J., 1995, On the microlocal parameter method in modeling of periodically layered thermoelastic composites, *Journal of Theoretical and Applied Mechanics*, **33**, 481-487
10. Michalak B., 1998, Stability of elastic slightly wrinkled plates, *Acta Mechanica*, **130**, 111-119
11. Michalak B., 2012, Dynamic modeling thin skeletonal shallow shells as 2D structures with nonuniform microstructures, *Archive of Applied Mechanics*, **82**, 949-961
12. Michalak B., Wirowski A., 2012, Dynamic modelling of thin plate made of certain functionally graded materials, *Meccanica*, **47**, 1487-1498
13. Michalak B., Woźniak C., Woźniak M., 2007, Modelling and analysis of certain functionally graded heat conductor, *Archive of Applied Mechanics*, **77**, 823-834
14. Mossakowski J., 1960, Buckling of circular plates with cylindrical orthotropy, *Archiwum Mechaniki Stosowanej*, **12**, 583-595

15. Naderi A., Saidi A.R., 2011, Exact solution for stability analysis of moderately thick functionally graded sector plates on elastic foundation, *Composite Structures*, **93**, 629-638

16. Nagórko W., Wągrowska M., 2002, A contribution to modeling of composite solids, *Journal of Theoretical and Applied Mechanics*, **40**, 149-158

17. Ostrowski P., Michalak B., 2011, Non-stationary heat transfer in hollow cylinder with functionally graded material properties, *Journal of Theoretical and Applied Mechanics*, **49**, 385-399

18. Tomczyk B., 2005, On stability of thin periodically densely stiffened cylindrical shells, *Journal of Theoretical and Applied Mechanics*, **43**, 427-455

19. Tylikowski A., 2005 Dynamic stability of functionally graded plate under in-plane compression, *Mathematical Problems in Engineering*, **4**, 411-424

20. Tung H., Duc N., 2010, Nonlinear analysis of stability for functionally graded plates under mechanical and thermal loads, *Composite Structures*, **92**, 1184-1191

21. Wierzbicki E., 1995, Nonlinear macro-micro dynamics of laminated structures, *Journal of Theoretical and Applied Mechanics*, **33**, 1-17

22. Wierzbicki E., Woźniak C., Woźniak M., 1997, Stability of micro-periodic materials under finite deformations, *Archives of Mechanics*, **49**, 143-158

23. Woźniak C. (Edit.), 2001, *Mechanics of Elastic Plates and Shells* (in Polish), PWN, Warszawa

24. Woźniak C., Michalak B., Jędrysiak J., (Edit.), 2008, *Thermomechanics of Microheterogeneous Solids and Structures*, Wydawnictwo Politechniki Łódzkiej, Łódź

25. Woźniak C. et al., 2010, *Mathematical Modelling and Analysis in Continuum Mechanics of Microstructured Media*, Silesian Technical University Press, Gliwice

26. You L.H., Wang J.X., Tang B.P., 2009, Deformations and stresses in annular disks made of functionally graded materials subjected to internal and/or external pressure, *Meccanica*, **44**, 283-292

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