FREE VIBRATIONS OF AN UNBOUNDED PERIODICALLY REINFORCED ELASTIC LAYER

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The object of analysis is an unbounded layer made of two isotropic, linear elastic materials and periodically laminated along the Ox^1 axis (cf. Fig.1). The layer is resting on the rigid base. It is assumed that the laminas are homogeneous and their number is very large. Hence we deal with a certain microstructured layer. The aim of this contribution is to propose a certain mass discretized model for the analysis of vibrations of the layer. It is shown that there exist two kinds of these vibrations which are independent of x^2 and x^3 coordinates.

1. OBJECT OF ANALYSIS

Let $Ox^1x^2x^3$ be the inertial coordinate system of Cartesian coordinates in the physical space. The elastic layer under considerations in its natural state occupies the region bounded by coordinate planes $x^1 = 0$ and $x^1 = L$. The layer is made of a very large number of thin laminas with a constant thickness $l/2$, where l/L is negligibly small when compared to 1. Hence l stands for the period of the structure and by means of

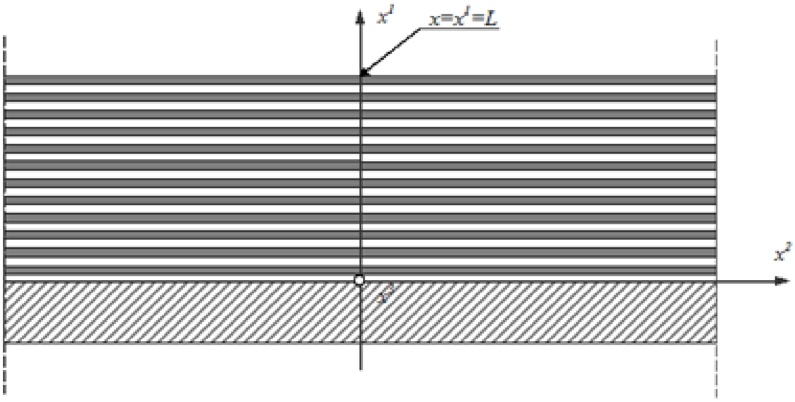

Fig. 1. Scheme of a part of the layer for $x^3 = 0$

$l/L \ll 1$ this structure can be referred to as microperiodic. The number of laminas is denoted by p, where $1/p \ll 1$. The scheme of the layer is shown in Figure 1.

The material properties of every lamina are determined by mass densities ρ_M, ρ_R, Young moduli E_M, E_R and Kirchhoff moduli G_M, G_R in the matrix and reinforcement materials, respectively. The object of consideration is the analysis of the free vibrations of the layer.

2. AIM OF THE CONTRIBUTION

The aim of contribution is to propose and apply a certain mass discretized model of the microstructured layer under consideration.

Free vibrations of the layer will be investigated under simplifying assumption that the piece wise constant mass distribution across the layer is approximated by masses concentrated exclusively on the interfaces between laminas. These masses are equal to $(\rho_M + \rho_R)l/2$ per unit area.

The main feature of the proposed model is that it describes the effect of the period length l on the values of free vibration frequencies. Obviously the proposed model has a physical sense if it describes wave lengths in the Ox^1- axis direction sufficiently large when compared to the microstructure period l.

3. INTRODUCTORY CONCEPTS

3.1 LAMINATED SPACE

The subsequent considerations will start with the concept of laminated space interpreted as 3D-space obtained by the formal extension of microstructure from interval $(0, L)$ to the whole Ox^1- axis. The scheme of the fragment of the space is shown in Figure 2.

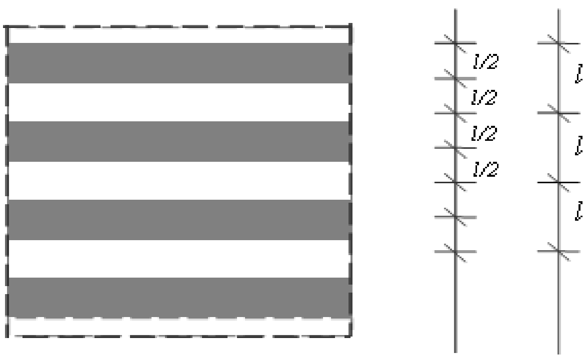

Fig. 2. A fragment of the space for $x^3 = 0$

In the vibration problem for the laminated space we shall independently deal with longitudinal or transversal waves along Ox^1 - axis.

Let $x = x_n \equiv n\frac{l}{2}$, $n = 0,\pm 1,\pm 2,\dots$ stand for a system of interfaces between adjacent laminas for the laminated space under consideration. Let H_R, H_M stand respectively either for E_R, E_M or G_R, G_M. Tripled $\{w(\cdot), H_R, H_M\}$ will be subsequently related either to longitudinal or to transversal waves. The corresponding displacement components, related to $Ox^1x^2x^3$ system will be denoted by $(w_1(\cdot), w_2(\cdot), w_3(\cdot))$. Consideration will be restricted to 1-D model of the laminated space by the assumption that $w = w(x,t)$, where $x \equiv x^1$, $x^1 \in R$, $t \in R$ and $w(\cdot) \in \{w_1(\cdot), w_2(\cdot), w_3(\cdot)\}$.

3.2 MASS DISCRETIZATION

The mass discretization will be formulated for the laminated space introduced above. The mass discretization assumption, introduced in this Section, will be extended on the whole laminated space. The displacement assigned to the *n*-th interface will be denoted by $w_n \equiv w(x_n, t)$, $t \in R$.

From the equilibrium equations for an arbitrary lamina we obtain:

$$w(x,t) = \begin{cases} 2\dfrac{w_{2n} - w_{2n-1}}{l}\left(x - x_{2n-1}\right) \; if \; x \in \left(x_{2n-1}, x_{2n}\right), \\ 2\dfrac{w_{2n+1} - w_{2n}}{l}\left(x - x_{2n}\right) \quad if \; x \in \left(x_{2n}, x_{2n+1}\right). \end{cases} \tag{1}$$

It has to be remembered that for laminated space, $n = 0,\pm 1,\pm 2,\dots$

Let us denote by $s_R = \left(s_R^{11}, s_R^{12}, s_R^{13}\right)$, $s_M = \left(s_M^{11}, s_M^{12}, s_M^{13}\right)$ stress components s^{11}, s^{12}, s^{13} in reinforcement and matrix material, respectively. Hence

$$\begin{aligned} & s_R = 2H_R\,\frac{w_{2n} - w_{2n-1}}{l} \;\; in \; \left(x_{2n-1}, \;\; x_{2n}\right), \\ & s_M = 2H_M\,\frac{w_{2n+1} - w_{2n}}{l} \;\; in \; \left(x_{2n}, \;\; x_{2n+1}\right), \\ & n = 0,\pm 1,\pm 2,\dots \end{aligned} \tag{2}$$

At the same time at every interface $x = x_n$, $n = 0,\pm 1,\pm 2,\dots$ the following dynamic equilibrium conditions hold:

$$4H_M\left(w_{2n+1}-w_{2n}\right)-H_R\left(w_{2n}-w_{2n-1}\right)=\frac{l}{2}\left(\rho_R+\rho_M\right)\ddot{w}_{2n},$$
$$4H_R\left(w_{2n+2}-w_{2n+1}\right)-H_M\left(w_{2n+1}-w_{2n}\right)=\frac{l}{2}\left(\rho_R+\rho_M\right)\ddot{w}_{2n+1}, \qquad (3)$$
$$n=0,\pm1,\pm2,\ldots$$

Equations (1)-(3) represent the mass discretized model equations for the elastic laminated space under consideration.

It has to be emphasized that the model equations (1)-(3) have a physical sense only if the mass discretization assumption is reliable. This situation takes place only if the wave lengths are large when compared to the period l.

3.3 DISPERSION RELATION

In order to describe the wave propagation along Ox^1 axis in a laminated space we have to introduce the concept of the wave length which will be denoted by λ. We also introduce the wave number $\kappa\equiv\frac{2\pi}{\lambda}$ and the dimensionless wave number $k\equiv\frac{2\pi l}{\lambda}=\kappa l$. The above terminology is based on that introduced in Brillouin [1] and many related papers. Following the line of approach given in Brillouin [1] we look for the solution of Equations (3) for $w_n(\cdot)$, $n=0,\pm1,\pm2,\ldots$ in the form:

$$w_{2n}=Aexpi\left(\omega t-nk\right),$$
$$w_{2n\pm1}=Bexpi\left[\omega t-\left(2n\pm1\right)\frac{k}{2}\right], \qquad (4)$$
$$n=0,\pm1,\pm2,\ldots$$

where A, B are arbitrary constants.

The non trivial solution to this system exists only if the following dispersion relation holds:

$$\left(\rho_R+\rho_M\right)l^2\omega_-^2=16\left(H_R+H_M\right)sin^2\frac{k}{4},$$
$$\left(\rho_R+\rho_M\right)l^2\omega_+^2=16\left(H_R+H_M\right)cos^2\frac{k}{4}, \qquad (5)$$

where ω_+ and ω_- are allowable free vibration frequencies.

After denotation $a^2\equiv\frac{16}{l^2\left(\rho_R+\upsilon_M\right)}\left(H_R+H_M\right)$ the diagram of dispersion relation is shown in Figure 3.

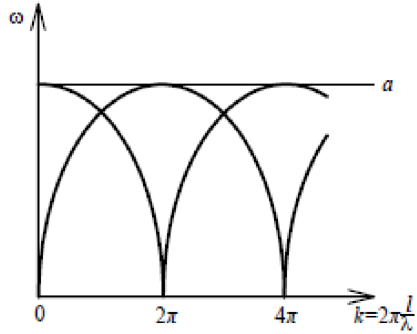

Fig.3. Diagram of dispersion relation

From the physical reliability off the mass discretization assumption it follows that formula (5) and the corresponding diagram have the physical meaning only for small values of parameter k with respect to 1.

3.4 BOUNDARY CONDITIONS

Now assume that the constants *A*, *B* in (4) are imaginary. In this case:

$$\begin{aligned} &w_{2n} = -A\sin\omega t\sin kn, \\ &w_{2n\pm1} = -B\sin\omega t\sin\left(k\frac{2n\pm1}{2}\right), \\ &n = 0,\pm1,\pm2,\dots \end{aligned} \tag{6}$$

Subsequently we shall restrict considerations to the layer made of p laminas; we recall that $1/p \ll 1$. If the number p of laminas is even then the formula $(6)_1$ takes place for $n=0,1,\dots,\frac{p}{2}$. If the number p of laminas is odd then the formula $(6)_2$ takes place for $n=0,1,\dots,\frac{p-1}{2}$.

Since the layer rests on the rigid base, for $x^1 = 0$ we obtain $w(0,t) = w_0 = 0$. At the same time the upper bound $x^1 = L = pl$ is free of tractions $w_p - w_{p-1} = 0$. Hence the value of s is equal to zero in the interval $(x_{p-1},\ x_p)$.
Condition on the upper boundary implies that:

1. If a number p of laminas is even then $-A\sin k\frac{p}{2} + B\sin k\frac{p-1}{2} = 0$,
2. If a number p of lamina is odd then $-B\sin k\frac{p}{2} + A\sin k\frac{p-1}{2} = 0$.

The boundary condition on the plane $x^1 = 0$ is satisfied identically.

The above boundary conditions are represented by the interrelation between real constants (vibration amplitudes) A and B and take place for the 1-D model of the laminated layer under consideration.

4. MODELLING RESULTS

The model of free vibrations is obtained directly from the dispersion relation. For free vibration frequencies we obtain the following formulas:

$$\omega_-^2 = \frac{16(H_R + H_M)}{l^2(\rho_R + \rho_M)} \sin^2 \frac{k}{4}, \quad \omega_+^2 = \frac{16(H_R + H_M)}{l^2(\rho_R + \rho_M)} \cos^2 \frac{k}{4}. \tag{7}$$

We recall that the above formulas have a physical sense only if the dimensionless wave number k is not large when compared to 1.
For $k \ll 1$, the formula (7) can be rewritten in the asymptotic form:

$$\omega_-^2 = \frac{(H_R + H_M)}{(\rho_R + \rho_M)} \kappa^2 + o(\kappa^2), \quad \omega_+^2 = \frac{16(H_R + H_M)}{l^2(\rho_R + \rho_M)} + O(\kappa^2). \tag{8}$$

The first terms on the right hand side of the following formulas represent the asymptotic approximation of higher and lower free vibration frequencies.

For a homogeneous layer we obtain $\rho = \rho_R = \rho_M$, $H = H_R = H_M$ and formula (8) takes the form:

$$\omega_-^2 = \frac{16H}{\rho} \kappa^2 + o(\kappa^2), \quad \omega_+^2 = \frac{16H}{\rho l^2} + O(\kappa^2). \tag{9}$$

Formulas (7)-(9) represent the final results of the 1-D modelling procedure proposed in this contribution.

5. CONCLUDING REMARKS

The characteristic feature of the free vibration analysis are two independent simple formulas for the lower frequency ω_- and higher frequency ω_+.

It can be seen that the obtained general results depend on the microstructure parameter l. The asymptotic formulas for the lower free vibrations ω_- for sufficiently large wave length λ (eg. for $\lambda > L$) are dispersion less e.a. they are independent on l, while the upper frequencies are constant and very large.

REFERENCES

[1] Brillouin L., Wave propagations in periodic structures, 1946, Mc Graw-Hill Books Company, Inc. Second Edition.