A NEW APPROACH TO DYNAMIC BUCKLING LOAD ESTIMATION FOR PLATE STRUCTURES

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In the paper the authors propose to estimate the dynamic buckling load relatively to static buckling load for imperfect plates, therefore the dynamic load factor DLF^* is introduced as a ratio of pulse load amplitude and the static buckling load for imperfect plates. The calculations have been made to check how the way of dynamic load factor determination influences the critical amplitude of pulse load leading to dynamic buckling. The results for composite plate loaded by compressive pulse load of different durations for different imperfection amplitude are presented in figures and discussed.

1. INTRODUCTION

The problem of dynamic buckling of thin walled structures such as shells and plates subjected to in-plane pulse loading has been widely investigated starting from sixties of previous century (see e.g. works [2], [11], [13]). These pulse loads may be of various durations and shapes (rectangular, sinusoidal, triangular, trapezoidal, etc.) being approximations of real load courses. Depending on the so-called "pulse intensity" different phenomena may occur – impact for pulses of high amplitudes and durations in range of microseconds or quasi-static behaviour if amplitude is low and duration is twice of period of fundamental natural vibrations. For pulses of intermediate intensity (amplitudes in range of static buckling load and durations close to one half or one of period of fundamental natural vibrations) the phenomenon of dynamic buckling occurs. It is known that at pulse loads of short duration (in range of milliseconds) the dynamic structure carrying capacity is larger than static one. However it should be remembered that for plate structure, in contrary to the static behaviour, the bifurcation dynamic load does not exist. The phenomenon of dynamic buckling takes place only for initially imperfect structures. Initial imperfections' magnitude in connection with pulse shape and its duration are crucial parameters in dynamic buckling load estimation [6], [7].

Dynamic buckling load is usually determined on the basis of dynamic buckling criterion that oneself seems to be problematic. Commonly used Budiansky-Roth-Hutchinson criterion [1] was formulated for structures having limit point or unstable postbifurcation path. Its application to plate structure behaviour, with stable postbuckling path, is based rather on accepted practice. Therefore in subject literature one can find a lot of stability criteria - most of them are based on state of displacements or state of stresses. The most popular are Volmir criterion [13] and Budiansky-Hutchinson [3] criterion.

Some degree of uncertainty of all mentioned criteria brought the researches of new dynamic stability criteria basing on Jacobian matrix eigenvalues analysis (see Kubiak [8]) or applying phase portraits criterion (see Teter [12]).

In most papers the results of calculations are presented as the relations between maximal dimensionless dynamic defection of a structure and ratio of pulse amplitude versus static buckling load determined for perfect structure (i.e. bifurcation load). In literature, following Budiansky and Hutchinson, the quotient of pulse amplitude and static bucking load is termed as Dynamic Load Factor (*DLF*). The critical value of *DLF* is determined on the assumed dynamic buckling criterion. It can be easily seen that for very small imperfections the dynamic buckling load determined on the basis of Budiansky-Hutchinson criterion is greater than static one and in some range of loads the dynamic deflections are smaller than static ones. For larger values of imperfections the character of curves changes (it becomes similar to static course) and dynamic buckling load is less than static one for a perfect plate.

Therefore the question arises - should the buckling load of imperfect structure be determined relatively to the static critical load of a perfect structure as it is commonly assumed? Perhaps the information of dynamic load carrying capacity would be more evident with regard to static buckling load determined for imperfect structure.

In this paper the authors propose to estimate the dynamic buckling load relatively to static buckling load for imperfect plates, therefore the dynamic load factor *DLF*[*] is introduced as a ratio of pulse load amplitude and the static buckling load for imperfect plates. The calculations have been made to check how the way of *DLF* determination influences the critical amplitude of pulse load leading to dynamic buckling.

2. METHOD OF SOLUTION

A non-linear stability problem has been solved by means of the Koiter's asymptotic theory [5]. The displacement field $\overline{U}$, and sectional force field $\overline{N}$ have been expanded into the power series with respect to the parameter ξ, - the buckling linear eigenvector amplitude (normalised with the equality condition between the maximum deflection and the thickness of the first plate).

$$\overline{U} = \lambda\overline{U}^{(0)} + \xi_i\overline{U}^{(i)} + \xi_i\xi_j\overline{U}^{(ij)} + ...$$
$$\overline{N} = \lambda\overline{N}^{(0)} + \xi_i\overline{N}^{(i)} + \xi_i\xi_j\overline{N}^{(ij)} + ... \tag{1}$$

It was assumed that the dimensionless amplitude of the initial deflections (imperfections) corresponding to the considered buckling mode (for *s*-th buckling mode) is:

$$\overline{U} = \xi_s^*\overline{U}^{(i)}. \tag{2}$$

By substituting expansions (1) into equations of equilibrium with neglected inertia terms (static buckling problem) and boundary conditions, the boundary problem of the zero (superscript [(0)] in Equations (1) and further), first (superscript [(i)]) and second (superscript [(ij)]) order has been obtained [8], [9]. The zero approximation describes the prebuckling state, whereas the first order approximation allows for determination of critical loads and the buckling modes corresponding to them, taking into account minimisation with respect to the number of half-waves *m* in the lengthwise direction. The second order approximation is reduced to a linear system of differential heterogeneous

equations, which right-hand sides depend on the force field and the first order displacements only.

Having found the solutions to the first and second order of the boundary problem, the coefficients a_{ijs}, b_{ijks} have been determined [8]:

$$a_{ijs} = \frac{\boldsymbol{\sigma}^{(i)} * L_{11}(\mathbf{U}^{(j)},\mathbf{U}^{(s)}) + 0.5\boldsymbol{\sigma}^{(s)} * L_{11}(\mathbf{U}^{(i)},\mathbf{U}^{(j)})}{-\lambda_s \boldsymbol{\sigma}^{(0)} * L_2(\mathbf{U}^{(s)})},$$

(3)

$$b_{ijks} = \frac{2\boldsymbol{\sigma}^{(i)} * L_{11}(\mathbf{U}^{(jk)},\mathbf{U}^{(s)}) + \boldsymbol{\sigma}^{(ij)} * L_{11}(\mathbf{U}^{(k)},\mathbf{U}^{(s)})}{-\lambda_s \boldsymbol{\sigma}^{(0)} * L_2(\mathbf{U}^{(s)})},$$

where: λ_s - is the critical load corresponding to the s-th mode, L_{11} is the bilinear operator, L_2 is the quadratic operator and $\boldsymbol{\sigma}^{(i)}$, $\boldsymbol{\sigma}^{(ij)}$ are the stress field tensors in the first and second order.

The postbuckling static equilibrium paths for coupled buckling can be described by the equation:

$$\left(1 - \frac{\lambda}{\lambda_s}\right)\xi_s + a_{ijs}\xi_i\xi_j + b_{ijks}\xi_i\xi_j\xi_k = \frac{\lambda}{\lambda_s}\xi_s^*; \quad (s = 1,\ldots,N),$$

(4)

which for the uncoupled problem have the form:

$$\left(1 - \frac{\lambda}{\lambda_{cr}}\right)\xi + a_{111}\xi^2 + b_{111}\xi^3 = \xi^* \frac{\lambda}{\lambda_{cr}}$$

(5)

where λ_{cr} is the critical load value.

When the equilibrium path is symmetrical ($a_{111} = 0$) the eq. (5) have the form:

$$\left(1 - \frac{\lambda}{\lambda_{cr}}\right)\xi + b_{111}\xi^3 = \xi^* \frac{\lambda}{\lambda_{cr}}$$

(6)

In a special case, i.e. for the so-called ideal structure without initial imperfections ($\xi^*=0$) the postbuckling equilibrium path is defined by the equation:

$$\frac{\lambda}{\lambda_{cr}} = 1 + b_{111}\xi^2$$

(7)

In the dynamic analysis (while finding the frequency of natural vibrations [9]), the independent non-dimensional displacement ξ and the load factor λ become a function dependent on time, and dynamic terms were added to equations describing postbuckling equilibrium path. Neglecting the forces associated with the inertia terms of prebuckling state and the second-order approximations, and taking into account the orthogonality conditions for the displacement field in the first $\overline{U}^{(i)}$ and second-order approximation $\overline{U}^{(ij)}$, the Lagrange equations can be written as:

$$\frac{1}{\omega_s^2}\ddot{\xi}_s + \left(1 - \frac{\lambda}{\lambda_s}\right)\xi_s + a_{ijs}\xi_i\xi_j + b_{ijks}\xi_i\xi_j\xi_k = \xi_s^* \frac{\lambda}{\lambda_s}; \quad (s=1,2,\ldots,N)$$

(8)

where ω_s is a natural frequency with mode corresponding to buckling mode; a_{ijs} and b_{ijks} are the coefficients (Eq. 4) describing the postbuckling behaviour of the structure (independent of time); however the parameters of load λ and the displacement ξ are the functions of time t.

For the uncoupled buckling, i.e. the single-mode buckling (where index $s = N = 1$), the equations of motion may be written in the form:

$$\frac{1}{\omega_1^2}\ddot{\xi}_1 + \left(1 - \frac{\lambda}{\lambda_1}\right)\xi_1 + a_{111}\xi_1^2 + b_{111}\xi_1^3 = \xi_1^* \frac{\lambda}{\lambda_1}; \tag{9}$$

It is assumed that in the initial moment of time $t = 0$ the non-dimensional displacement ξ, as well as the velocity of displacement are equal to zero, i.e.:

$$\xi(t=0) = 0 \quad \text{and} \quad \dot{\xi}(t=0) = 0. \tag{10}$$

3. DYNAMIC BUCKLING CRITERIA

The results of calculations presented in the paper allow discussing the effect of application of the following criteria:

- the simplest criterion, proposed by Volmir [13] - the dynamic critical load corresponds to the amplitude of pulse force (of constant duration) at which the maximal plate deflection is equal to some constant value k (k=one half or one plate thickness).
- Budiansky&Hutchinson [3] stability criterion that states: dynamic stability loss occurs when the maximal plate deflection grows rapidly with the small variation of the load amplitude.

4. SUBJECT OF CONSIDERATION

A square simply supported composite plate with length to thickness ratio equals 100 is considered. The plate subjects to unidirectional compression in form of a rectangular pulse load P of finite duration. The unloaded edges of plate remain straight and parallel during loading. The analysed plate is made of epoxy-glass composite with fibre volume fraction f equals 0.5. The material properties for epoxy resin and glass fibre are given in Table 1.

Table 1. Assumed material properties

material type:	E [GPa]	ν	ρ [kg/m^3]
epoxy resin	3.5	0.33	1249
glass fibre	71	0.22	2450

For numerical calculation the orthotropic model was assumed. The necessary values of Young's modules and Poison's ratios were calculated using equations based on mixture theory [4], which are as follows:

$$E_x = E_m(1 - f) + E_f f,$$

$$E_y = E_m \frac{E_m(1 - \sqrt{f}) + E_f \sqrt{f}}{E_m[1 - \sqrt{f}(1 - \sqrt{f})] + E_f \sqrt{f}(1 - \sqrt{f})},$$

$$\nu_{yx} = \nu_m(1 - \sqrt{f}) + \nu_f \sqrt{f},$$

$$G = G_m \frac{G_m \sqrt{f}(1 - \sqrt{f}) + G_f[1 - \sqrt{f}(1 - \sqrt{f})]}{G_m \sqrt{f} + G_f(1 - \sqrt{f})}.$$

(11)

where: E_m and E_f are the Young's modules of elasticity for matrix and fibre, respectively, G_m and G_f are the shear modules for matrix (subscript m) and fibre (subscript f), ν_m and ν_f are the Poisson's ratios for matrix and fibre and $f = V_f /(V_m + V_f)$ is the fibre volume fraction.

5. RESULTS OF CALCULATIONS

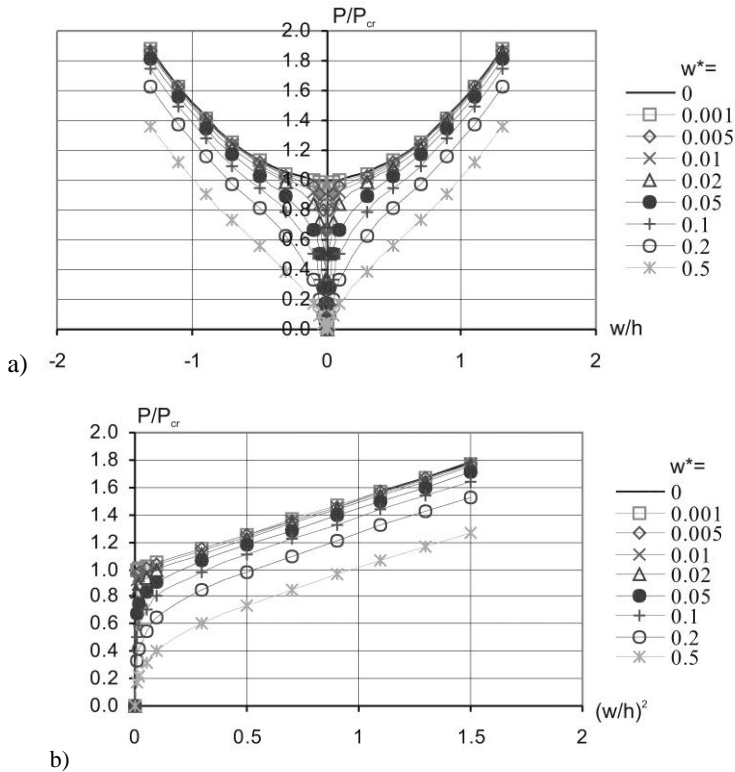### 5.1. STATIC BUCKLING LOAD AND POSTBUCKLING BEHAVIOUR

Fig.1. Dimensionless load P/P_{cr} vs. dimensionless deflection (a) or square of dimensionless deflection (b)

For ideal plate structure the critical load can be determined from eigenvalue analysis but for structures with imperfection the buckling load may be determined basing on pre- and post-buckling behaviour. The authors decided to employ two well-known methods for identification of the critical load that are usually applied to the results of experimental investigations. The inflection point method (P-w) which is very similar to "top of the knee" method and alternative (P-w^2) method was used [9].

The postbuckling equilibrium paths for plates with initial imperfections corresponding to the buckling mode and different amplitudes $w^*=w_0/h$ (where: w_0 - amplitude of imperfection, h - thickness of the plate) are presented in Fig. 1. Using mentioned above methods the buckling load P_{cr}^*/P_{cr} for a plate with geometrical imperfections has been found and presented in Table 2 (where P_{cr}^* - buckling compressive force for imperfect plate and P_{cr} - bifurcational load).

Table 2. P_{cr}^*/P_{cr} for different amplitude of initial imperfection

determination method: initial imperfection amplitude w*	**P-w** P_{cr}^*/P_{cr}	**P-w^2** P_{cr}^*/P_{cr}
0.001	0.999	0.999
0.005	0.998	0.994
0.01	0.995	0.986
0.02	0.990	0.968
0.05	0.968	0.925
0.1	0.925	0.863
0.2	0.834	0.751
0.5	0.589	0.494

It follows from the Table 2 that lower values of P_{cr}^* were obtained using P-w^2 method and the differences between the results of both the methods are growing with the increase of imperfection amplitude value.

In further investigations the values of P_{cr}^* found on the basis of the inflection point method (*P-w* method) were taken into account.

5.2. ANALYSIS OF DYNAMIC RESPONSE OF A PLATE

In most publications dealing with dynamic buckling problem the amplitude of initial imperfection has been assumed as $w^* = 0.01$ for which the buckling load decrease is very small ca 1%.

The dynamic load factor *DLF* is defined as the ratio of an amplitude of pulse load to the critical static buckling load for ideal structures. Presented below calculations were made in aim to check how the way of *DLF* estimation influences the critical amplitude of pulse load leading to dynamic buckling. The authors propose to introduce a dynamic load factor $DLF^*=P/P_{cr}^*$ - a pulse load amplitude divided by the static buckling load for imperfect structures.

The calculations were performed for two values of pulse duration $T_0=T$ and $T_0=0.5T$ (where T - period of natural fundament flexural vibration of a plate, for assumed material properties and geometry $T=0.59$ ms).

In Tables 3 and 4 the critical dynamic load factors DLF_{cr} (determined in the conventional way) and DLF^*_{cr} (determined from $DLF^*(w/h)$ relations, where the amplitude of pulse loading is divided by buckling load for imperfect structures) are presented. The critical value of dynamic load factor DLF^*_{cr} was calculated as the amplitude of pulse load divided by static buckling load for structure with initial imperfection using inflection point method (see Table 2, P-w column).

Table 3. DLF_{cr} and DLF_{cr}^* for different amplitude of initial imperfection, $T_0=T$

Assumed criterion:	Volmir criterion $(w/t)_{cr}=1$	Budiansky-Hutchinson criterion	Volmir criterion $(w/t)_{cr}=1$	Budiansky-Hutchinson criterion
initial imperfection amplitude w^*	DLF_{cr}	DLF_{cr}	DLF^*_{cr}	DLF^*_{cr}
0.01	1.49	1.4 – 1.6	*1.49*	*1.4 – 1.6*
0.02	1.31	1.2 – 1.3	*1.32*	*1.2 – 1.3*
0.05	1.17	0.8 – 0.9	*1.21*	*0.9 – 1.1*
0.1	1.07	0.8 – 0.9	*1.15*	*0.9 – 1.0*
0.2	1.13	0.7 – 0.8	*1.10*	*0.84 – 0.96*
0.5	0.63	0.4 – 0.5	*1.08*	*0.7 – 0.85*

Table 4. DLF_{cr} and DLF_{cr}^* for different amplitude of initial imperfection, $T_0=0.5T$

Assumed criterion:	Volmir criterion $(w/t)_{cr}=1$	Budiansky-Hutchinson criterion	*Volmir criterion $(w/t)_{cr}=1$*	*Budiansky-Hutchinson criterion*
initial imperfection amplitude w^*	DLF_{cr}	DLF_{cr}	DLF^*_{cr}	DLF^*_{cr}
0.01	3.07	4.4 – 4.6	*3.08*	*4.4 – 4.6*
0.02	2.47	3.4 – 3.6	*2.49*	*3.4 – 3.6*
0.05	1.89	2.0 – 2.8	*1.86*	*2.5 – 2.9*
0.1	1.40	1.6 – 1.8	*1.51*	*1.7 – 1.9*
0.2	1.01	1.0 – 1.2	*1.21*	*1.2 – 1.4*
0.5	0.62	0.4 – 0.6	*1.01*	*0.7 – 1.0*

The courses of $DLF(w/h)$ and $DLF^*(w/h)$ for three values of imperfection amplitudes and two pulse durations ($T_0=T$ and $T_0=0.5T$) are presented in Figs 2-4. In these figures the static postbuckling curves P/P_{cr} (for flat plate) and P/P_{cr}^* (for imperfect plate) are also drawn.

It can be noticed that for relatively small imperfection amplitude $w^*=0.01$ (Fig. 2) the curves $DLF(w/h)$ and $DLF^*(w/h)$ cover up for given pulse duration T_0. Also the static postbuckling curves overlap (excluding the initial range of deflections). In this case the character of dynamic responses strongly depends on the assumed pulse duration - for shorter pulse the deflections are small and the dynamic buckling load is at least three times greater (see Table 3 and 4).

When the amplitude of imperfections grows up to the value $w^*=0.1$ (Fig. 3) the differences between DLF and DLF^* curves are clearly visible for both pulse durations and static postbuckling curves P/P_{cr} (for flat plate) and P/P_{cr}^* differ as well. The character

of dynamic responses for pulse durations $T_0=T$ and $T_0=0.5T$ is similar but the dynamic buckling load for shorter pulse is twice the one for $T_0=T$.
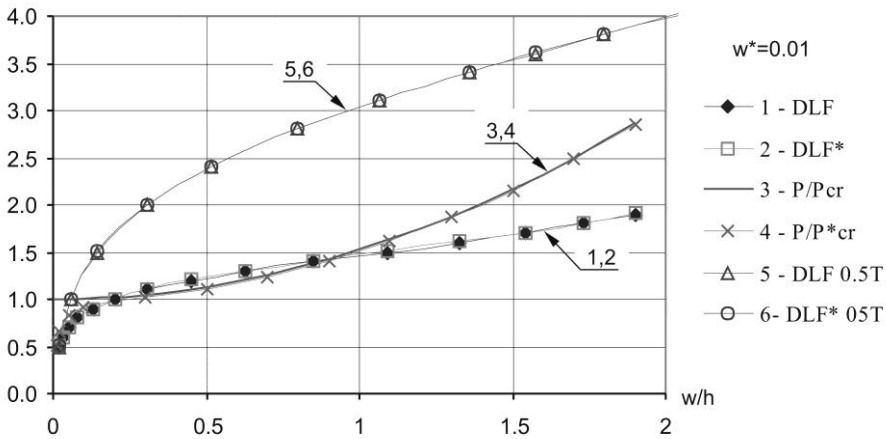

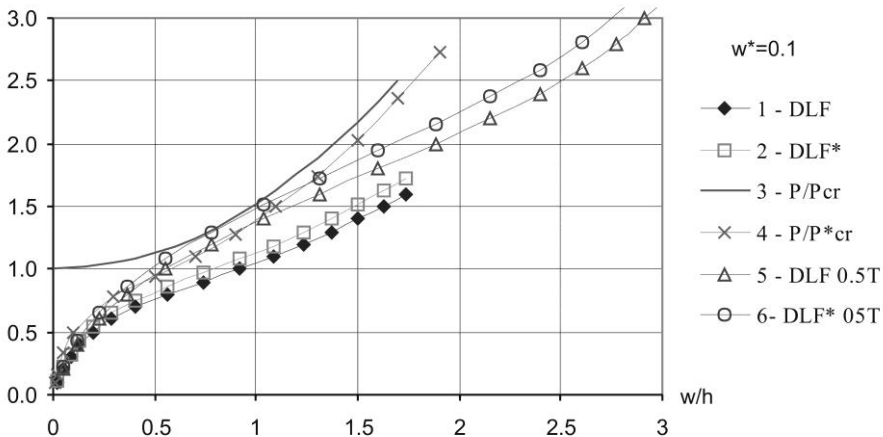Fig. 2. Static and dynamic dimensionless responses versus dimensionless deflection for w*=0.01.

Fig. 3. Static and dynamic dimensionless responses versus dimensionless deflection for w*=0.1.

For relatively large value of *w** (the imperfection amplitude equals one half of plate thickness) - Fig. 4, the results show that the dynamic responses of a plate do not depend on pulse load duration (the relations *DLF(w/h)* for $T_0=T$ and for $T_0=0.5T$ cover up and also *DLF*(w/h)* for both assumed pulses overlap) and moreover the courses of *DLF*(w/h)* are almost identical as the static postbuckling curve P/P_{cr}^*. It should be underlined that the differences between the courses of *DLF* (estimated as the ratio of

pulse load amplitude to static bifurcational load) and *DLF** (estimated as the ratio of pulse load amplitude to static buckling load of imperfect plate) are clearly visible.
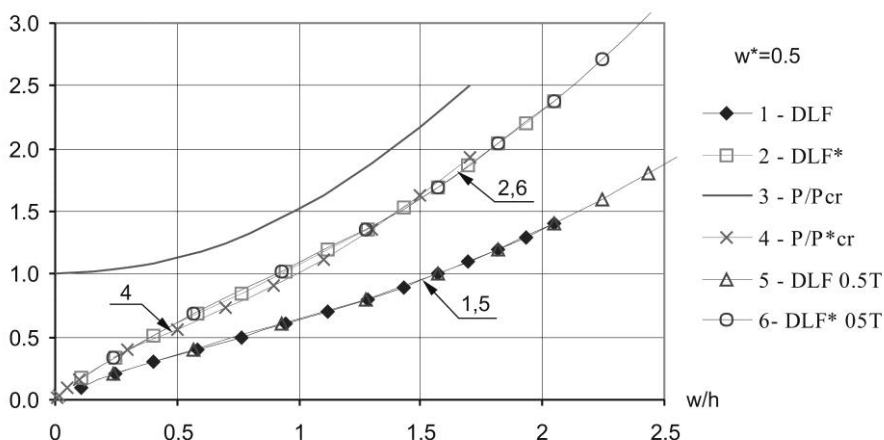

Fig. 4. Static and dynamic dimensionless responses versus dimensionless deflection for w*=0.5.

6. CONCLUSIONS

On the basis of the results analysis of the conducted calculations following conclusions can be drawn:

- For small values of imperfection amplitude (in range of hundredth parts of plate thickness) the differences between the dynamic response known as *DLF (w/h)* (ratio of pulse load amplitude to static bifurcational load versus dimensionless deflection) and proposed relation *DLF* (w/h)* (ratio of pulse load amplitude to static buckling load for imperfect plate versus dimensionless deflection) are less than 1% and the curves are practically identical. In this case the pulse load duration time strongly affects the dynamic buckling load value and the character of dynamic response of considered plate.
- For rather large values of imperfection amplitude (in range of one half and greater) the influence of pulse duration on courses of *DLF(w/h)* and *DLF*(w/h)* has shown to be negligible but the dynamic responses *DLF(w/h)* and *DLF*(w/h)* differ significantly. It should be strongly underlined that the proposed relations *DLF*(w/h)* calculated for two values of pulse load are almost identical as static postbuckling curve P/P_{cr}^* what allows us to conclude that for large imperfection amplitude values the static and dynamic behaviour of a plate is practically the same (for considered pulse durations). This fact can be only observed if proposed relation *DLF*(w/h)* is applied in calculations.

The authors are aware that the analysis conducted in this work is restricted to a single plate and for pulses of rectangular shape; therefore further investigations should be performed to confirm the above conclusions.

ACKNOWLEDGMENTS

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