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Application Available Through the Web Page for Simulation of the SH Wave Scattering by an Elastic Rigidly Supported Inclusions

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Abstract. *Mathematical model of SH wave scattering by an elastic thin-walled rigidly supported inclusion is presented. The inclusion is replaced by the effective boundary condition that allow to simplify calculations of the scattered field significantly. Using the proposed model the problem is reduced to the system of hyper singular integral equations that is solved numerically. The numerical algorithm is implemented in the application that is available through the Web page and can be used as a tool in nondestructive testing of elastic materials with thin plane rigidly supported inclusions of arbitrary stiffness.*
Keywords: *numerical modelling, simulation tool, SH wave scattering, thin-walled inclusion.*

1. Introduction

The elastodynamic wave propagation in solids with thin interface elastic layers and thin coatings is one that has received much attention, particularly due to its importance in ultrasonic nondestructive evaluation of materials [1, 2, 3, 4], in acoustics [5], electromagnetics [6, 7, 8] and elastodynamics [9, 10, 11, 12, 13]. There are two problems. One problem may be posed: to describe the elastodynamic interaction of a thin-walled inhomogeneity with the host material by the boundary conditions, as the ratio of two characteristic lengths - the inclusion thickness and its diameter - tend to zero. The other problem is the inverse scattering problem of inclusion properties determination from the scattered data. Ultimately the interest, of course, lies in trying to determine the properties of such a thin-walled inclusion from the scattering data, but it seems clear that before that inverse problem can be studied with any hope of success, the direct scattering problem must be solved in an effective manner.

The goal of the present investigation is to obtain a solution of the direct scattering problem and implement it in application available through the Web page as a Java applet. The application can be useful in non-destructive evaluation of elastic materials with thin plane rigidly supported inclusions of arbitrary stiffness. In order to implement an algorithm the application uses Wolfram Mathematica. It consists of two modules. The first module responsible for the interaction with a user is a Java applet embedded in a Web page. The second application module is responsible for performing the calculations using Mathematica kernel. It communicates with the kernel through a J/Link interface.

2. Problem formulation

Let us consider a uniform medium in which there is a rigidly supported inclusion. The inhomogeneity occupies the region $S = \{|x_1| < a,\ |x_3| < h/2\},\ |x_2| < \infty$, where h is the thickness, (x_1, x_2, x_3) are the Cartesian coordinates and the quantity $\varepsilon = h/a$ is a small parameter. A plane, incident wave of the form

$$u^i(\mathbf{x}) = A_0 \exp\left[ik\left(l_1 x_1 + l_3 x_3\right)\right] \tag{1}$$

impinges on the inclusion, where $(l_1, l_3) = (\sin\theta_0, -\cos\theta_0)$ is the direction of incidence, A_0 is the amplitude, k is a wave number. Herein $u(\mathbf{x}) = u^i(\mathbf{x}) + u^s(\mathbf{x})$ is the displacement in the x_2 direction, $u^i(\mathbf{x})$ is the displacement in a homogeneous body, that characterises the applied load, $u^s(\mathbf{x})$ is the scattering field which satisfies the Sommerfeld radiation condition at infinity, from which it follows that:

$$u^s(\mathbf{x}) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}|+\mathrm{i}\pi/4}}{\sqrt{8\pi k|\mathbf{x}|}} \mathrm{f}(\omega; \mathbf{l}, \boldsymbol{\nu}),\ |\mathbf{x}| \to \infty,$$

where $f(\omega; \mathbf{l}, \boldsymbol{\nu})$ is the complex amplitude or far-field pattern of the scattering wave, $\boldsymbol{\nu} = \mathbf{x}/|\mathbf{x}| = (\sin\theta, \cos\theta)$ is the direction of observation, ω is the circular frequency.

The scattering problem of time harmonic SH waves is described by the Helmholtz wave equations

$$\Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0,\ \mathbf{x} \in R^2 \backslash S, \tag{2}$$

$$\Delta u^0(\mathbf{x}) + k_0^2 u^0(\mathbf{x}) = 0,\ \mathbf{x} \in S$$

where k_0 is the wave number for the inclusion, Δ is the Laplace operator, $u^0(\mathbf{x})$ is the displacement field in the inclusion.

On the interface of the medium for $\mathbf{x} \in \partial S$, the following coupling conditions hold:

$$u(\mathbf{x}) = u^0(\mathbf{x}),\ \frac{\partial u(\mathbf{x})}{\partial n^0} = \gamma_0 \frac{\partial u^0(\mathbf{x})}{\partial n^0},\ \mathbf{x} \in \partial S \setminus \left\{ x_3 = -\frac{h}{2} \right\}, \tag{3}$$

$$u(\mathbf{x}) = u^0(\mathbf{x}) = 0, x_3 = -h/2,\ \gamma_0 = \frac{\mu_0}{\mu}.$$

Herein, $\mathbf{n}^0$ denotes the outer direction normal to ∂S, a time factor $\exp(-i\omega t)$ is assumed and hereafter suppressed, μ and μ_0 are the Lamé elastic parameters of the medium and inclusion, respectively.

The field $u(x)$ satisfies in the domain $R^2 \backslash S$ the equation (2) and the following effective boundary conditions on the interval $|x_1| < a$ [4, 13]

$$u^+(x_1) = 2Zk^{-1} \frac{\partial u^+(x_1)}{\partial x_3},\ u^-(x_1) = 0, \tag{4}$$

$$Z = x\frac{\varepsilon}{2}\gamma_0^{-1},\ u^{\pm} = \lim_{\varepsilon \to 0} u(x_1, \pm\varepsilon),$$

which describes, asymptotically exactly up to the terms of $O(\varepsilon)$, a solution to the problem (2) and (3). Here Z is the normalised impedance of the material relative to the intrinsic impedance of the external medium.

From the Green theorem for $x \in R^2 \backslash S$ we obtain that the scattering field can be described in the form

$$u^s(\mathrm{x}) = \int_{-a}^{a} \left[kg(\mathrm{x}, \mathrm{y})\Phi_1(y_1) - \Phi_3(y_1) \frac{\partial g(\mathrm{x}, \mathrm{y})}{\partial y_3} \right]_{y_3=0} dy_1, \tag{5}$$

$$g(\mathrm{x}, \mathrm{y}) = -\frac{i}{4} H_0^{(1)}(k|\mathrm{x} - \mathrm{y}|),$$

$$k\Phi_1(x_1) = \left[\frac{\partial u^+(\mathrm{x})}{\partial x_3} - \frac{\partial u^-(\mathrm{x})}{\partial x_3}\right]_{x_3=0},$$

$$\Phi_3(x_1) = u^+(x_1) - u^-(x_1) = u^+(x_1).$$

Here $H_0^{(1)}$ is the Hankel function of the first kind.

As a middle surface of the scatterer is a plane, let us use the expansion of the fundamental solution of the Helmholtz equation (cylindrical wave) $g(\mathrm{x}, \mathrm{y})$ via plane waves. This will allow to deal with the symbols of the corresponding pseudo-differential operators only. As a result, from the equations (1), (2), (4) and (5), the following system of singular integral equations for $\Phi_\beta(x_1)$, $\beta = 1, 3$ is obtained

$$\Phi_1(x_1) + \frac{k}{Z}\int_{-a}^{a} \Phi_1(p) K_3(k|x_1 - p|)dp- \tag{6}$$

$$-k\int_{-a}^{a} \Phi_3(p) K_1(k|x_1 - p|)dp = q_1 \exp(ikl_1 x_1),$$

$$\Phi_3(x_1) + k\int_{-a}^{a} \Phi_1(p) K_3(k|x_1 - p|)dp = q_3 \exp(ikl_1 x_1),\ |x_1| < a,$$

$$K_1(|z|) = \frac{1}{2\pi}\int_\Gamma \gamma(\alpha) e^{\pm i\alpha z} d\alpha,$$

$$K_3(|z|) = \frac{1}{2\pi}\int_\Gamma \gamma^{-1}(\alpha) e^{\pm i\alpha z} d\alpha,$$

$$q_1 = A_0\left(2il_3 + \frac{2}{Z}\right),\ q_3 = 2A_0,\ \gamma = \sqrt{\alpha^2 - 1}.$$

Herein the contour Γ coincides with the real axis everywhere except for the branching points $\alpha = \pm 1$ and passes these points below in the right-hand half-plane of complex variable α and above in the left-hand one according to the limiting absorption principle and the point $\alpha = 0$ is situated below the contour Γ. The square root of the function $\gamma(\alpha)$ is defined by the condition $\operatorname{Im}\sqrt{\alpha^2 - 1} < 0$ for $|\alpha| < 1$.

3. Numerical solution of the integral equations

In view of the edge condition we represent a solution of the integral equations (6) in the complete system of the Jacobi polynomials as

$$\Phi_\beta(x_1) = q_\beta(1 - p^2)^{\mu_\beta} \sum_{n=0}^{\infty} a_{n,\beta} P_n^{(\mu_\beta,\mu_\beta)}(p),\ p = x_1/a, \tag{7}$$

where $P_n^{(\mu_\beta,\mu_\beta)}$ are the Jacobi polynomials with the peculiarity exponents $\mu_1 = -3/4$, $\mu_3 = 1/4$ for $\gamma_0 = O(\varepsilon)$ and $\mu_1 = -3/2$, $\mu_3 = 1/2$ for $\gamma_0 = o(\varepsilon^{-1})$.

From the equations (6) and (7) we obtain an infinite system of linear algebraic equations for the unknown coefficients $a_{n,\beta}$:

$$\sum_{n=0}^{\infty}\left[a_{n,1}(A_{nm,1}+\frac{B_{nm,1}}{Zx})-a_{n,3}q_0^{-1}B_{nm,2}\right]=b_m, \tag{8}$$

$$\sum_{n=0}^{\infty}\left[a_{n,3}A_{nm,3}+a_{n,1}\frac{q_0}{x}B_{nm,1}\right]=b_m, \tag{9}$$

$$m=0,1,2,\ldots,\ q_0=\frac{q_1}{q_3},\ x=ka,$$

$$B_{mn,\beta}=(m+1)\sum_{j=0}^{\infty}i^{m-j}(j+1)B_{nj}^{\beta}I_{mj,\beta},$$

$$B_{nj}^{1}=\int_{-1}^{1}(1-p^2)^{\mu_1}P_n^{(\mu_1,\mu_1)}(p)U_j(p)dp,$$

$$B_{nj}^{3}=\int_{-1}^{1}(1-p^2)^{\mu_3}P_n^{(\mu_3,\mu_3)}(p)U_j(p)dp,$$

$$A_{nm,1}=\int_{-1}^{1}(1-p^2)^{\mu_3}P_n^{(\mu_1,\mu_1)}(p)U_m(p)dp,$$

$$A_{nm,3}=\int_{-1}^{1}(1-p^2)^{\mu_1}P_n^{(\mu_1,\mu_1)}(p)U_m(p)dp,$$

$$I_{mn,1}=\frac{1}{2}\int_{\Gamma}\frac{1}{\gamma(\alpha)}J_{m+1}(x\alpha)J_{n+1}(x\alpha)\alpha^{-2}d\alpha,$$

$$I_{mn,3}=\frac{1}{2}\int_{\Gamma}\gamma(\alpha)J_{m+1}(x\alpha)J_{n+1}(x\alpha)\alpha^{-2}d\alpha,$$

$$b_m=\frac{\pi}{xl_1}i^m(m+1)J_{m+1}(xl_1),\ \beta=1,3,$$

where $U_m(p)$ are the Chebyshev polynomials of the second kind and $J_{m+1}(\alpha)$ are the Bessel cylindrical functions.

The scattering amplitude

$$f(\omega; \mathbf{l}, \nu) = -\int_{-l}^{l} [\Phi_1(p) + ik\nu_3\Phi_3(p)]\, e^{-ik\nu_1 p} dp \tag{10}$$

taking into account the relations (7) is directly connected with the coefficients $a_{m,\beta}$.

It is obvious that in the numerical calculations the numbers m and n in the equations (7), (8) and (9) are limited and the quantities $B_{nm,\beta}$, $(\beta = 1, 3)$ and consequently $a_{m,\beta}$ can be calculated with a sufficient accuracy by the appropriate numerical procedures. Indeed, an accuracy of one percent is obtained if $m = n \approx 2x$.

4. Description of the application

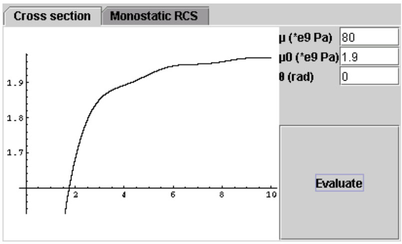
In order to implement the algorithm, Java language and Wolfram Mathematica were used. They were chosen so that the application could be available for a wide range of operating systems and computer hardware. Programs developed in Java can be run in any operating system that provides Java Virtual Machine (JVM), which in fact means any modern operating system. Mathematica package, which is a leading software for advanced mathematical computations, is also available for a wide variety of systems.

The application consists of two modules. The first one is a Java applet embedded on a Web page that a user can request from a Web server using a Web browser. The graphical user interface of the applet was created by means of the standard Java visual components library Swing. The applet obtains parameters from the user and sends them to the second module that resides on the Web server computer from which a Web page with an applet has been downloaded. That module performs computations in a number of steps. First, it starts the Mathematica kernel through a standard interface J/Link. Then it sends to the kernel a request to load definitions of functions used in the algorithm implementation as well as numerical system coefficient tables. Subsequently it sends the parameters fed by the user and a request to perform calculations and obtains their results from the kernel through J/Link. Finally, it closes the kernel and sends the output to the applet. The final step is the visualisation of the output by the applet.

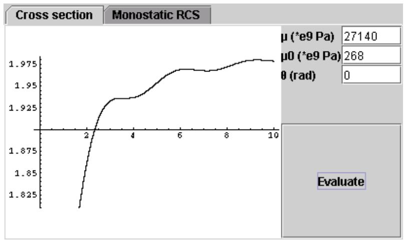
The application allows to obtain a total scattering cross-section for a wave scattered from inclusion located in an elastic material. The following parameters: angle of observation and shear modulus of the inclusion as well as of the material are provided by the user.

In Fig. 1 the dimensionless total cross-section $\sigma^0 = \sigma(0)/2a$, where $\sigma(\theta_0) = \frac{1}{A_0 k}\mathrm{Im} f(\omega; \mathbf{l}, \mathbf{l})$, for the incident angles $\theta_0 = 0$ is plotted as a function of x.

In Fig. 2 the monostatic radar cross section (RCS) is plotted as a function of the incident angle ($20 \log \left[A_0^{-1} |f(\omega, \mathbf{l}, -\mathbf{l})|\right]$) for $x = 15$.

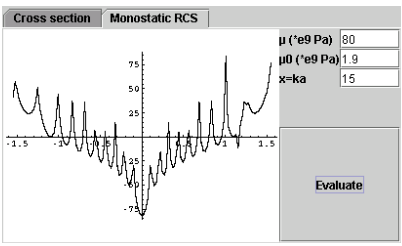


a)

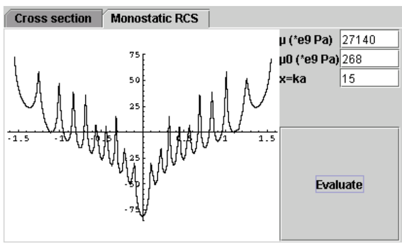


b)

Figure 1. The normalised total cross-section versus dimensionless wave number $x = ka$ for normal incidence ($\theta_0 = 0$) of SH wave for parameters a) $\mu_0 = 1.9 \cdot 10^9$Pa, $\mu = 80 \cdot 10^9$Pa, $\varepsilon = 0.05$, b) $\mu_0 = 268 \cdot 10^9$Pa, $\mu = 27140 \cdot 10^9$Pa, $\varepsilon = 0.05$.



a)



b)

Figure 2. Monostatic RCS for $x = 15$ for the same parameters as in Fig. 1, $\varepsilon = 0.05$.

5. Conclusion

In this paper the application that can be used as a tool in nondestructive testing of elastic materials with thin plane inclusions is presented. The application implements the algorithm that uses mathematical model of SH wave scattering by an elastic thin-walled rigidly supported inclusion. Java programming language and Wolfram Mathematica were used to elaborate the application. Application graphical user interface is implemented as Java applet embedded on a Web page. Such a choice causes that the application is available for a wide range of operating systems and computer hardware.

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