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RESEARCH OF PURCHASE OPTION IN CASE OF HEDGING WITH SET PROBABILITY

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The formulas defining option cost and also evolution in time of portfolio and capital for the European option of purchase in case of hedging with set probability (fractile hedging) at continuous time and diffusion model of the (B, S)-financial market have obtained. Some properties of solution are investigated.

1. Introduction

Financial tools used in markets become more various and generate refined enough streams of payments. The situation becomes complicated because fluctuation of interest rates and profitablenesses in markets is stochastic and mathematical models of these fluctuations are casual processes. Therefore such primary goal of participants of financial markets as price evaluation of financial tools can be solved only with attrstock of probability methods. At the same time, construction of mathematical model of the financial market and the analysis of processes which occur there, demand use of mathematical methods at high level. In connection to that, financial mathematics received great popularity. The main object of research of financial mathematics is various models of securities market [1, 2]. In the given work the research of nonclassical problem of the option theory is carried out – problems of hedging with probability of payment obligation performance, for the case of purchase option when unlike stationary options the payment obligation is carried out not with probability equal to one but with probability less than one, which corresponds more to realities of the financial market.

2. Problem statement

Let's consider the model of financial market as pairs of assets actives: nonrisk (bank account) B and risk (stocks) S, represented by their prices B_t and S_t, $t\in[0,T]$. In this case the (B, S) – market with continuous time is spoken about [1, 2]. Actives B and S we shall name *the basic actives* or *the basic securities*. Concerning bank account B it is supposed that $B=(B_t)_{t\geq 0}$ is the determined function submitting to the equation $dB_t=rB_tdt$, i. e. $B_t=B_0e^{rt}$, $B_0>0$, $r\geq0$, where r is the interest rate (bank percent). For the description of stock cost evolution $S=(S_t)_{t\geq0}$ we shall assume that consideration of the problem occurs on the Vinerovskiy stochastic basis $(\Omega,F,\mathbf{F}=(F_t)_{t\geq0},\mathrm{P})$ [1, 3].

Introduction in consideration of the Vinerovskiy process is caused by the role of casual component which defines «chaotic» structure in really observable fluctuations of stock prices. In this connection the model of «geometrical», or «economic», Brown movement $S=(S_t)_{t\geq0}$ was offered by P. Samuelson, according to which S is the casual process with

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \qquad (1)$$

where $W=(W_t)_{t\geq0}$ is the Vinerovskiy process, $\sigma>0$, $\mu\in R$. Using the formula Ито [3] from (1) we find that stochastic differential $dS_t=S_t(\mu dt+\sigma dW_t)$.

Let's imagine some investor having initial capital $X_0=x$ during the moment of time $t=0$, being on the bank account B and in stocks S according to portfolio $\pi_0=(\beta_0,\gamma_0)$, where β_0 is a part of the nonrisk active (the sum on the bank account), γ_0 is a part of the risk active (the sum invested in stocks). Thus, we shall find the initial capital $X_0=\beta_0B_0+\gamma_0S_0$. Similarly, let $\pi_t=(\beta_t,\gamma_t)$ is a pair describing condition of the investor's portfolio of securities during the moment of time $t>0$. Then the current capital $X_t=\beta_tB_t+\gamma_tS_t$.

Problem: To find the capital X_t, corresponding to it portfolio $\pi_t=(\beta_t,\gamma_t)$ and initial value $X_0=C_T$ of the capital as costs of the option secondary securities at which performance of the payment obligation is provided $X_T=f_T(S_T)$, where $f_T(S_T)$ is payment function with probability $P(A)=1-\varepsilon$, $0<\varepsilon<1$ [2, 4].

Let's notice that in a standard problem of option hedging performance of payment obligation is provided with probability equal to one [1], i. e. on each trajectory of the stochastic process S_t, modeling price evolution of the risk active.

3. Finding of option price

Let's consider a problem of hedging with set probability $P(A)$ (fractile hedging) of standard option of the buyer (call-option) with function of payment $f_T=(S_T-K)^+=\max(0,S_T-K)$ [1]. Under the theorem 6.1 from [2] we have that the optimum strategy in a problem of fractile hedging coincides with perfect hedge of payment obligation $f=f_T\cdot I_A$, where I_A is a set (event) indicator A which has the view

$$A = \left\{\omega : \frac{dP}{dP^*} > \mathrm{const}\cdot f_T\right\}, \qquad (2)$$

P^* is martingal measure, i. e. the measure with respect to which the process $\hat{X}_t=X_t/B_t$ is martingal. Using that density process of the martingal measure P^* relatively to P is ([2], § 3.2)

$$\frac{dP^*_T}{dP_T} = \exp\left\{-\frac{\mu-r}{\sigma}W^*_T + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T\right\},$$

where $W^*_t = W_t + \left(\frac{\mu-r}{\sigma}\right)t$ is the Vinerovskiy process concerning the measure P^* we can write (2) as following:

$$A=\left\{\exp\left(\frac{\mu-r}{\sigma}W_T^*-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2T\right)>\text{const}\cdot(S_T-K)^+\right\}=$$

$$=\left\{\exp\left(\frac{\mu-r}{\sigma^2}\left(\ln S_0+\left(r-\frac{\sigma^2}{2}\right)T+\sigma W_T^*\right)\right)\times\right.$$

$$\left.\times\exp\left(-\frac{\mu-r}{\sigma^2}\left(\begin{matrix}\ln S_0+\left(r-\frac{\sigma^2}{2}\right)T-\\-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2T\end{matrix}\right)\right)>\text{const}\cdot(S_T-K)^+\right\}=$$

$$=\left\{S_T^{\frac{\mu-r}{\sigma^2}}\exp\left(-\frac{\mu-r}{\sigma^2}\left(\begin{matrix}\ln S_0+\\+\frac{\mu+r-\sigma^2}{2}\end{matrix}\right)T\right)>\text{const}\cdot(S_T-K)^+\right\}. \quad (3)$$

It is necessary to consider two cases separately: $(\mu-r)/\sigma^2\le1$, $(\mu-r)/\sigma^2>1$.

Remark 1: Further everywhere means $\Phi^{-1}(y)$ a function that is reverse to the Laplas' function

$$\Phi(y)=\int_{-\infty}^{y}\varphi(x)dx,\quad \varphi(x)=\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}.$$

Theorem 1. Let

$$y_0(T,S_0)=\left(\ln\frac{K}{S_0}-(r-\frac{\sigma^2}{2})T\right)\Big/\sigma\sqrt{T}. \quad (4)$$

Then at $(\mu-r)/\sigma^2\le1$ the option price is defined by the formula

$$C_T=S_0\left[\Phi\left(\frac{b_C}{\sqrt{T}}-\sigma\sqrt{T}\right)-\Phi(y_0(T,S_0)-\sigma\sqrt{T})\right]-$$

$$-Ke^{-rT}\left[\Phi\left(\frac{b_C}{\sqrt{T}}\right)-\Phi(y_0(T,S_0))\right], \quad (5)$$

where the constant b_C has the view

$$b_C=\sqrt{T}\Phi^{-1}(1-\varepsilon)+\frac{\mu-r}{\sigma}T. \quad (6)$$

Proof. To the case corresponds the following figure defining structure of hedging set A.

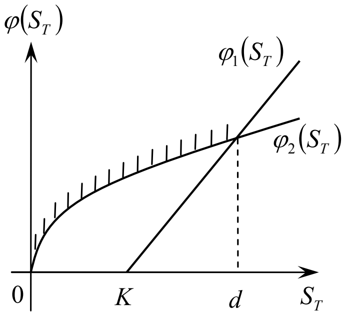

Fig. 1. *Structure of hedging set:* $(\mu-r)/\sigma^2\le1$; $\varphi_1(S_T)=\text{const}(S_T-K)^+$; $\varphi_2(S_T)=S_T^{(\mu-r)/\sigma^2}$

The shaded area is the area of inequality solution (3). Thus, in this case the set (event) A is represented in the form of

$$A=\{S_T<d\}=\{W_T^*<b\}=$$

$$=\left\{S_T<S_0\exp\left(\left(r-\frac{\sigma^2}{2}\right)T+b\sigma\right)\right\} \quad (7)$$

at restriction (E^* is averaging as far as P^*) [2]. From (7) we have that

$$P(A)=P\left\{S_T<S_0\exp\left(\left(r-\frac{\sigma^2}{2}\right)T+b\sigma\right)\right\}. \quad (8)$$

As with respect to (1)

$$S_T=S_0\exp\left\{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_T\right\}, \quad (9)$$

then the use of (9) in (8) gives that

$$P(A)=P\left\{\begin{matrix}S_0\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_T\right)<\\<LS_0\exp\left(\left(r-\frac{\sigma^2}{2}\right)T+b\sigma\right)\end{matrix}\right\}=$$

$$=P\left\{\begin{matrix}\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_T\right)<\\<\exp\left(\left(r-\frac{\sigma^2}{2}\right)T+b\sigma\right)\end{matrix}\right\}=$$

$$=P\left\{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma W_T<\left(r-\frac{\sigma^2}{2}\right)T+b\sigma\right\}=$$

$$=P\{\sigma W_T<b\sigma-(\mu-r)T\}=$$

$$=P\left\{W_T<b-\left(\frac{\mu-r}{\sigma}\right)T\right\}. \quad (10)$$

With respect that the Vinerovskiy process W_t has a normal distribution with zero average and dispersion t, from (10) we get

$$P(A)=\Phi\left(\left(b-\frac{\mu-r}{\sigma}T\right)\Big/\sqrt{T}\right),$$

where $P(A)=1-\varepsilon$ is probability of successful hedging, $0<\varepsilon<1$. Hence, for constant finding $b=b_c$ we have a condition

$$\left(b-\frac{\mu-r}{\sigma}T\right)\Big/\sqrt{T}=\Phi^{-1}(1-\varepsilon),$$

which proves (6). According to *the Theorem* 1 from [1] we have that price of purchase option is defined by the formula

$$C_T=e^{-rT}F_T(S_0), \quad (11)$$

where $F_T(S_0)=E^*\{I_A\cdot f_T(S_T)|S_0\}$. Then with respect to the function for purchase option we shall receive

$$F_T(S_0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b_C/\sqrt{T}}f_T\left(S_0\exp\left\{\begin{matrix}\sigma\sqrt{T}y+\\+(r-\frac{\sigma^2}{2})T\end{matrix}\right\}\right)e^{-y^2/2}dy=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{y_0(T,S_0)}^{b_C/\sqrt{T}}\left(S_0\exp\left\{\begin{matrix}\sigma\sqrt{T}y+\\+(r-\frac{\sigma^2}{2})T\end{matrix}\right\}-K\right)e^{-y^2/2}dy, \quad (12)$$

where $y_0(T,S_0)$ is the solution of the equation

$$S_0\exp\{\sigma\sqrt{T}y+(r-\frac{\sigma^2}{2})T\}-K=0,$$

which is defined by the formula (4). Having substituted $F_T(S_0)$ of kind (12) in (11) we shall receive in view of (6) that

$$C_T = e^{-rT} F_T(S_0) =$$

$$= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{y_0(T,S_0)}^{b_C/\sqrt{T}} \left(S_0 \exp\left\{ \begin{matrix} \sigma\sqrt{T}y + \\ +(r - \frac{\sigma^2}{2})T \end{matrix} \right\} - K \right) e^{-y^2/2} dy =$$

$$= \frac{S_0}{\sqrt{2\pi}} \int_{y_0(T,S_0)}^{b_C/\sqrt{T}} e^{-\frac{1}{2}[y-\sigma\sqrt{T}]^2} dy - \frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{y_0(T,S_0)}^{b_C/\sqrt{T}} e^{-y^2/2} dy =$$

$$= S_0 \left[\Phi\left(\frac{b_C}{\sqrt{T}} - \sigma\sqrt{T} \right) - \Phi(y_0(T,S_0) - \sigma\sqrt{T}) \right] -$$

$$-Ke^{-rT} \left[\Phi\left(\frac{b_C}{\sqrt{T}} \right) - \Phi(y_0(T,S_0)) \right],$$

That is we came to (5). The theorem is proved.

Theorem 2. Let $y_0(T,S_0)$ be defined by the formula (4). Then at $(\mu-r)/\sigma^2>1$ the option price is defined by the formula

$$C_T = S_0 \left[\begin{matrix} \Phi\left(\frac{b_1}{\sqrt{T}} - \sigma\sqrt{T} \right) - \Phi(y_0(T,S_0) - \sigma\sqrt{T}) + \\ +\Phi\left(-\frac{b_2}{\sqrt{T}} + \sigma\sqrt{T} \right) \end{matrix} \right] -$$

$$-Ke^{-rT} \left[\Phi\left(\frac{b_1}{\sqrt{T}} \right) - \Phi(y_0(T,S_0)) + \Phi\left(-\frac{b_2}{\sqrt{T}} \right) \right], \quad (13)$$

where constants b_1 and b_2 are such, that they satisfy to the condition

$$P(A) = 1 - \varepsilon = \Phi\left(\left(b_1 - \frac{\mu - r}{\sigma} T \right) \Big/ \sqrt{T} \right) +$$

$$+\Phi\left(\left(-b_2 + \frac{\mu - r}{\sigma} T \right) \Big/ \sqrt{T} \right). \quad (14)$$

Proof. To the case corresponds $(\mu-r)/\sigma^2>1$ the following analogue, fig. 1.

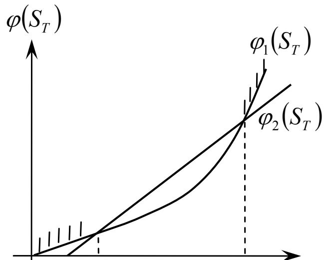

Fig. 2. *Structure of set hedging:* $(\mu-r)/\sigma^2>1$; $\varphi_1(S_T)=\text{const}(S_T-K)^+$; $\varphi_2(S_T)=S_T^{(\mu-r)/\sigma^2}$

In this case the set A has the following structure: $A=\{S_T<d_1\}\cup\{S_T<d_2\}=\{W_T^*<b_1\}\cup\{W_T^*>b_2\}$. As sets $\{S_T<d_1\}$ also $\{S_T<d_2\}$ do not cross, then

$$P(A) = P\{S_T < d_1\} + P\{S_T > d_2\} =$$

$$= P\{S_T < S_0 \exp((r - \frac{\sigma^2}{2})T + b_1\sigma)\} +$$

$$+P\{S_T > S_0 \exp((r - \frac{\sigma^2}{2})T + b_2\sigma)\}. \quad (15)$$

With respect to (9) from (15) follows

$$P(A) = P\{W_T < b_1 - \frac{\mu - r}{\sigma}T\} + P\{W_T > b_2 - \frac{\mu - r}{\sigma}T\}. \quad (16)$$

As W_t has normal distribution with zero average and dispersion t, with respect to the property $\Phi(x)+\Phi(-x)=1$, from (16) follows (14). Thus, we found that constants b_1 and b_2 should satisfy to the condition (14), but they are not found in an obvious kind. Similarly (12)

$$F_T(S_0) =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b_1/\sqrt{T}} f_T(S_0 \exp\{\sigma\sqrt{T}y + \frac{r-\sigma^2}{2}\}) \cdot e^{-\frac{y^2}{2}} dy +$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{b_1/\sqrt{T}}^{\infty} f_T(S_0 \exp\{\sigma\sqrt{T}y + \frac{r-\sigma^2}{2}\}) \cdot e^{-\frac{y^2}{2}} dy. \quad (17)$$

Further transformations on development of the formula (13) after substitution (17) in (11) are similar to transformations on development of the formula (5) and consequently are not presented. The theorem is proved.

4. Finding of capital and portfolio

Let's designate through $\pi^*=(\beta^*,\gamma^*)$ the minimal hedge (optimum portfolio), where β^* is a part of the nonrisk active (the sum on the bank account), and γ^* is a part of the risk active (the sum invested in stocks). Negative value β^* or γ^* means taking corresponding active on credit.

Theorem 3. For the case $(\mu-r)/\sigma^2\le1$ the capital X_t and portfolio $\pi_t^*=(\beta_t^*,\gamma_t^*)$ are defined by the formulas

$$X_t = S_t \left[\begin{matrix} \Phi\left(\frac{b_C}{\sqrt{T-t}} - \sigma\sqrt{T-t} \right) - \\ -\Phi(y_0(T-t,S_t) - \sigma\sqrt{T-t}) \end{matrix} \right] -$$

$$-Ke^{-r(T-t)} \left[\Phi\left(\frac{b_C}{\sqrt{T-t}} \right) - \Phi(y_0(T-t,S_t)) \right], \quad (18)$$

$$\gamma_t^* = \Phi\left(\frac{b_C}{\sqrt{T-t}} - \sigma\sqrt{T-t} \right) -$$

$$-\Phi(y_0(T-t,S_t) - \sigma\sqrt{T-t}), \quad (19)$$

$$\beta_t^* = -\frac{K}{B_T} \left[\Phi\left(\frac{b_C}{\sqrt{T-t}} \right) - \Phi(y_0(T-t,S_t)) \right], \quad (20)$$

where $y_0(T-t,S_t)$ and b_C are defined by the formulas (4) and (6) with replacements $T\to(T-t)$ and $S_0\to S_t$.

Proof. Under *the Theorem* 1 from [1] we have that

$$X_t = e^{-r(T-t)} F_{T-t}(S_t), \quad (21)$$

where $F_{T-t}(S_t)=E^*\{I_A;f_T(S_T)|S_t\}$. Then, having substituted expression (12) for function $F_T(S_0)$ with replacements $T\to(T-t)$ and $S_0\to S_t$ in (21), we get the formula defining evolution of the current capital

$$X_t = e^{-r(T-t)} F_{T-t}(S_t) =$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{y_0(T-t,S_t)}^{b_C/\sqrt{T-t}} \left(S_t \exp\left\{ \begin{matrix} \sigma\sqrt{T-t}y + \\ +(r - \frac{\sigma^2}{2})(T-t) \end{matrix} \right\} - K \right) e^{-y^2/2} dy =$$

$$= \frac{S_t}{\sqrt{2\pi}} \int_{y_0(T-t,S_t)-\sigma\sqrt{T}}^{\frac{b_C}{\sqrt{T-t}}-\sigma\sqrt{T}} e^{-\frac{y^2}{2}} dy - \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} \int_{y_0(T-t,S_t)}^{b_C/\sqrt{T-t}} e^{-y^2/2} dy =$$

$$= S_t \left[\Phi\left(\frac{b_C}{\sqrt{T-t}} - \sigma\sqrt{T-t} \right) - \Phi(y_0(T-t,S_t) - \sigma\sqrt{T-t}) \right] -$$

$$-Ke^{-r(T-t)} \left[\Phi\left(\frac{b_C}{\sqrt{T-t}} \right) - \Phi(y_0(T-t,S_t)) \right],$$

which proves the formula (18).

Under the formulas (4.22), (4.23) from [1] we have:

$$\gamma_t^* = e^{-r(T-t)} \frac{\partial F_{T-t}(s)}{\partial s}(S_t), \qquad (22)$$

$$\beta_t^* = \frac{1}{B_T}\left[F_{T-t}(S_t) - S_t \frac{\partial F_{T-t}(s)}{\partial s}(S_t) \right] = \frac{X_t - \gamma_t^* S_t}{B_t}. \qquad (23)$$

With respect to (18), (21)

$$e^{-r(T-t)} F_{T-t}(s) =$$

$$= s\left[\Phi\left(\frac{b_C}{\sqrt{T-t}} - \sigma\sqrt{T-t} \right) - \Phi(y_0(T-t,s) - \sigma\sqrt{T-t}) \right] -$$

$$-Ke^{-r(T-t)} \left[\Phi\left(\frac{b_C}{\sqrt{T-t}} \right) - \Phi(y_0(T-t,s)) \right].$$

Then from (22) follows

$$\gamma_t^* = \frac{\partial}{\partial s}(e^{-r(T-t)} F_{T-t}(s))\Big|_{S_t} =$$

$$= \begin{bmatrix} \Phi\left(\frac{b_C}{\sqrt{T-t}} - \sigma\sqrt{T-t} \right) - \\ -\Phi\left(y_0(T-t,S_t) - \sigma\sqrt{T-t} \right) \end{bmatrix} -$$

$$-s \frac{\partial}{\partial s} \Phi(y_0(T-t,s) - \sigma\sqrt{T-t})\Big|_{S_t} +$$

$$+Ke^{-r(T-t)} \frac{\partial}{\partial s} \Phi(y_0(T-t,s))\Big|_{S_t}. \qquad (24)$$

With respect to the function $y_0(T-t,s)$ typewe get

$$\frac{\partial}{\partial s}\Phi(y_0(T-t,s)) = \frac{\partial}{\partial s}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y_0(T-t,s)} e^{\frac{-x^2}{2}} dx \right) =$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-(y_0(T-t,s))^2}{2}} \frac{\partial}{\partial s} y_0(T-t,s) =$$

$$= -\frac{1}{s\sigma\sqrt{2\pi(T-t)}} e^{\frac{-(y_0(T-t,s))^2}{2}}. \qquad (25)$$

Then

$$\frac{\partial}{\partial s}\Phi(y_0(T-t,s) - \sigma\sqrt{T-t}) =$$

$$= -\frac{1}{s\sigma\sqrt{2\pi(T-t)}} e^{\frac{-(y_0(T-t,s)-\sigma\sqrt{T-t})^2}{2}}. \qquad (26)$$

From (25), (26) with respect to the function $y_0(T-t,s)$ type follows that

$$s\frac{\partial}{\partial s}\Phi(y_0(T-t,s) - \sigma\sqrt{T-t}) -$$

$$-Ke^{-r(T-t)} \frac{\partial}{\partial s}\Phi(y_0(T-t,s)) =$$

$$= -\frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{(y_0(T-t,s)-\sigma\sqrt{T-t})^2}{2}} +$$

$$+\frac{Ke^{-r(T-t)}}{s\sigma\sqrt{2\pi(T-t)}} e^{-\frac{(y_0(T-t,s)-\sigma\sqrt{T-t})^2}{2}} =$$

$$= -\frac{1}{\sigma\sqrt{2\pi(T-t)}} \times$$

$$\times e^{-\frac{(y_0(T-t,s))^2}{2}} \left[\frac{Ke^{-r(T-t)}}{s} - \frac{Ke^{-r(T-t)}}{s} \right] = 0. \qquad (27)$$

Substitution of (27) in (24) leads to (19). From (18), (19), (23) follows

$$\beta_t^* = \frac{S_t}{B_t}[\Phi(\frac{b_C}{\sqrt{T-t}} - \sigma\sqrt{T-t}) -$$

$$-\Phi(y_0(T-t,S_t) - \sigma\sqrt{T-t})] -$$

$$-\frac{Ke^{-r(T-t)}}{B_t}[\Phi(\frac{b_C}{\sqrt{T-t}}) - \Phi(y_0(T-t,S_t))] -$$

$$-\frac{S_t}{B_t}[\Phi(\frac{b_C}{\sqrt{T-t}} - \sigma\sqrt{T-t}) -$$

$$-\Phi(y_0(T-t,S_t) - \sigma\sqrt{T-t})] =$$

$$= -\frac{Ke^{-r(T-t)}}{B_t}[\Phi(\frac{b_C}{\sqrt{T-t}}) - \Phi(y_0(T-t,S_t))] =$$

$$= -\frac{K}{B_T}[\Phi(\frac{b_C}{\sqrt{T-t}}) - \Phi(y_0(T-t,S_t))],$$

,

that is we came to (20). The theorem is proved.

Theorem 4. For the case $(\mu-r)/\sigma^2>1$ the capital X_t and portfolio $\pi_t^*=(\beta_t^*,\gamma_t^*)$ are defined by the formulas

$$X_t = S_t \begin{bmatrix} \Phi\left(\frac{b_1}{\sqrt{T-t}} - \sigma\sqrt{T-t} \right) - \\ -\Phi(y_0(T-t,S_t) - \sigma\sqrt{T-t}) + \\ +\Phi\left(-\frac{b_2}{\sqrt{T-t}} + \sigma\sqrt{T-t} \right) \end{bmatrix} -$$

$$-Ke^{-r(T-t)} \begin{bmatrix} \Phi\left(\frac{b_1}{\sqrt{T-t}} \right) - \Phi(y_0(T-t,S_t)) + \\ +\Phi\left(-\frac{b_2}{\sqrt{T-t}} \right) \end{bmatrix}, \qquad (28)$$

$$\gamma_t^* = \Phi\left(\frac{b_1}{\sqrt{T-t}} - \sigma\sqrt{T-t} \right) -$$

$$-\Phi(y_0(T-t,S_t) - \sigma\sqrt{T-t}) +$$

$$+\Phi\left(-\frac{b_2}{\sqrt{T-t}} + \sigma\sqrt{T-t} \right), \qquad (29)$$

$$\beta_t^* = -\frac{K}{B_T}\left[\begin{array}{l}\Phi\left(\frac{b_1}{\sqrt{T-t}}\right) - \Phi(y_0(T-t,S_t)) + \\ +\Phi\left(-\frac{b_2}{\sqrt{T-t}}\right)\end{array}\right]. \quad (30)$$

where $y_0(T-t,S_t)$, b_1 and b_2 are found under the formulas (4), (14) with replacements $T\to(T-t)$ and $S_0\to S_t$.

The proof of the given theorem is conducted similarly to the proof *of the Theorem* 3 with use of the formula (17) instead of (12).

Consequence. Let $\tilde{C}_T, \tilde{X}_t, \tilde{\gamma}_t, \tilde{\beta}_t^*$ designate corresponding values at $\varepsilon=0$, when the perfect hedging is reached. Then

$$\tilde{C}_T = S_0\Phi(\sigma\sqrt{T} - y_0(T,S_0)) - Ke^{-rT}\Phi(-y_0(T,S_0)), \quad (31)$$

$$\tilde{X}_t = S_t\Phi(\sigma\sqrt{T-t} - y_0(T-t,S_t)) - \\ -Ke^{-r(T-t)}\Phi(-y_0(T-t,S_t)), \quad (32)$$

$$\gamma_t^* = \Phi(\sigma\sqrt{T-t} - y_0(T-t,S_t)), \quad (33)$$

$$\beta_t^* = -\frac{K}{B_T}\Phi(-y_0(T-t,S_t)). \quad (34)$$

Proof. Let's consider the case $(\mu-r)/\sigma^2\leq 1$. As $\Phi^{-1}(1)=\infty$, then from (6) follows that $b_C=\infty$ at $\varepsilon=0$. As $\Phi(\infty)=1$, then according to (5)

$$\tilde{C}_T = S_0[1-\Phi(y_0(T,S_0)-\sigma\sqrt{T})] - \\ -Ke^{-rT}[1-\Phi(y_0(T,S_0))]. \quad (35)$$

As $\Phi(x)+\Phi(-x)=1$, then (31) follows from (35). Formulas (32)–(34) follow from (18)–(20) similarly.

Let's consider now the case $(\mu-r)/\sigma^2>1$. Then at $\varepsilon=0$ with respect to the property of the function $\Phi(x)$ from (14) follows that

$$\Phi\left(\left(b_1 - \frac{\mu-r}{\sigma}T\right)\Big/\sqrt{T}\right) = \Phi\left(\left(b_2 - \frac{\mu-r}{\sigma}T\right)\Big/\sqrt{T}\right),$$

i. e. $b_1=b_2=b$.

Hence, according to (13),

$$\tilde{C}_T = S_0\left[\begin{array}{l}\Phi\left(\frac{b}{\sqrt{T}} - \sigma\sqrt{T}\right) - \Phi(y_0(T,S_0)-\sigma\sqrt{T}) + \\ +\Phi\left(-\frac{b}{\sqrt{T}} + \sigma\sqrt{T}\right)\end{array}\right] - \\ -Ke^{-rT}\left[\Phi\left(\frac{b}{\sqrt{T}}\right) - \Phi(y_0(T,S_0)) + \Phi\left(-\frac{b}{\sqrt{T}}\right)\right]. \quad (36)$$

Then the formula (31), with respect to the property $\Phi(x)+\Phi(-x)=1$, follows from (36) in obvious manner, and in similar manner formulas (32)–(34) follow from (28)–(30). Consequence is proved.

Remark 2: In case of, when fractile hedging switches over to perfect hedging (with probability equal to one), results *of Theorems* 1–4 switch over to results *of the Theorem* 2 from [1] as formulas (31)–(34) represent formulas (4.26), (4.29)–(4.31) from [1].

5. Coefficients of sensitivity

Dependences of the option price on parameters μ and ε, absent in the solution of the perfect hedging problem, [1] are of interest. These dependences are defined by the values $C_T^\mu=dC_T/d\mu$ and $C_T^\varepsilon=dC_T/d\varepsilon$.

Theorem 5. For the case $(\mu-r)/\sigma^2\leq 1$ coefficients of sensitivity C_T^μ and C_T^ε are defined by the formulas

$$C_T^\mu = \frac{\sqrt{T}}{\sigma}\left[S_0\varphi\left(\frac{b_C}{\sqrt{T}} - \sigma\sqrt{T}\right) - Ke^{-rT}\varphi\left(\frac{b_C}{\sqrt{T}}\right)\right], \quad (37)$$

$$C_T^\varepsilon = \sqrt{2\pi}\cdot e^{\frac{(\Phi^{-1}(1-\varepsilon))^2}{2}}\left[\begin{array}{l}Ke^{-rT}\varphi\left(\frac{b_C}{\sqrt{T}}\right) - \\ -S_0\varphi\left(\frac{b_C}{\sqrt{T}} - \sigma\sqrt{T}\right)\end{array}\right]. \quad (38)$$

Proof. According to (4) and (6)

$$\frac{dy_0(T,S_0)}{d\mu} = \frac{d(y_0(T,S_0)-\sigma\sqrt{T})}{d\mu} = 0,$$

$$\frac{d\left(\frac{b_C}{\sqrt{T}}\right)}{d\mu} = \frac{d\left(\frac{b_C}{\sqrt{T}} - \sigma\sqrt{T}\right)}{d\mu} = \frac{\sqrt{T}}{\sigma}.$$

Then with respect to the formula (5) and designations for the function $\varphi(x)$ (see *Remark 1*)

$$C_T^\mu = \frac{dC_T}{d\mu} = S_0\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{b_C}{\sqrt{T}}-\sigma\sqrt{T}\right)^2}\cdot\frac{\sqrt{T}}{\sigma}\right) - \\ -Ke^{-rT}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{b_C}{\sqrt{T}}\right)^2}\cdot\frac{\sqrt{T}}{\sigma}\right) = \\ = \sqrt{\frac{T}{2\pi\sigma^2}}\left[S_0e^{-\frac{1}{2}\left(\frac{b_C}{\sqrt{T}}-y\sqrt{T}\right)^2} - Ke^{-rT}e^{-\frac{1}{2}\left(\frac{b_C}{\sqrt{T}}\right)^2}\right] = \\ = \frac{\sqrt{T}}{\sigma}\left[S_0\varphi\left(\frac{b_C}{\sqrt{T}} - \sigma\sqrt{T}\right) - Ke^{-rT}\varphi\left(\frac{b_C}{\sqrt{T}}\right)\right],$$

i. e. we came to the formula (37). Using the rule of differentiation of inverse functions

$$\frac{d(\phi^{-1}(x))}{dx} = \left[\frac{d\phi(\phi^{-1}(x))}{dx}\right]^{-1},$$

we have that

$$\frac{d(\Phi^{-1}(1-\varepsilon))}{d\varepsilon} = \left[\frac{d\Phi(\Phi^{-1}(1-\varepsilon))}{d\varepsilon}\right]^{-1} = -\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}(\Phi^{-1}(1-\varepsilon))^2}.$$

Then from the formula (6) follows that

$$db_C/d\varepsilon = -\sqrt{2\pi T}e^{\frac{1}{2}(\Phi^{-1}(1-\varepsilon))^2}.$$

Thus

$$C_T^\varepsilon = \frac{dC_T}{d\varepsilon} = S_0\left[-e^{-\frac{1}{2}\left(\frac{b_C}{\sqrt{T}}-\sigma\sqrt{T}\right)^2}e^{\frac{(\Phi^{-1}(1-\varepsilon))^2}{2}}\right] -$$

$$-Ke^{-rT}\left[-e^{-\frac{1}{2}\left(\frac{b_C}{\sqrt{T}}\right)^2} e^{\frac{(\Phi^{-1}(1-\varepsilon))^2}{2}}\right]=$$

$$=\sqrt{2\pi}e^{\frac{(\Phi^{-1}(1-\varepsilon))^2}{2}}\left[\begin{array}{l}\frac{1}{\sqrt{2\pi}}Ke^{-rT}e^{-\frac{1}{2}\left(\frac{b_C}{\sqrt{T}}\right)^2}-\\ -\frac{1}{\sqrt{2\pi}}S_0e^{-\frac{1}{2}\left(\frac{b_C}{\sqrt{T}}-\sigma\sqrt{T}\right)^2}\end{array}\right]=$$

$$=\sqrt{2\pi}e^{\frac{(\Phi^{-1}(1-\varepsilon))^2}{2}}\left[\begin{array}{l}Ke^{-rT}\varphi\left(\frac{b_C}{\sqrt{T}}\right)-\\ -S_0\varphi\left(\frac{b_C}{\sqrt{T}}-\sigma\sqrt{T}\right)\end{array}\right],$$

i. e. we came to the formula (38). The theorem is proved.

6. Conclusion

1. In case of the perfect hedging, with respect to (31)–(34), the decision possesses essential lack, because it does not depend on the parameter of variability μ describing the tendency in price change of the risk active: if μ=0, then the price on the average fluctuates near the initial value S_0, i. e. the process S_t behaves as martingal; if μ>0, then the price on the average increases, i. e. the process S_t behaves as submartingal; if μ<0, then the price on the average decreases, i. e. the process S_t behaves as supermartingal. The obtained solution of the hedging problem with the probability removes this lack, as it contains the dependence on μ through b_C, b_1 and b_2.
2. Numeral researches have shown, that $C_T^{\mu}>0$ and $C_T^{\varepsilon}<0$, i. e. with increase of the parameter μ the price of purchase option will increase, and with increase of the parameter ε it will decrease. Really, with increase of μ there is an average increase in prices of the risk active S_t, which reduces risk for the buyer of purchase option, and it is necessary to pay more for smaller risk. With increase of ε the probability of successful hedging $P(A)=1-\varepsilon$ decreases, which leads to risk increase, and it is necessary to pay less for the increased risk.

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