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SEVERAL WAYS OF CALCULATING THE GRADIENT, CURL AND DIVERGENCE UNDER ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

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ABSTRACT

Tianyu Yuan: Several Ways of Calculating the Gradient, Curl and Divergence under Orthogonal Curvilinear Coordinate Systems
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Calculating gradient, curl and divergence is very important in physics, especially in electrodynamics and fluid mechanics. To calculate the gradient, curl and divergence under orthogonal curvilinear coordinate systems, one must consider the Lame coefficients. Also, in many textbooks the calculation of gradient, curl and divergence under orthogonal coordinate systems are not well discussed.

In this thesis the concepts such as manifold, tensors, differential forms and Lame coefficients are defined and three different ways-differential form method, covariant derivative method, and Hodge star operator method-of calculating gradient, curl and divergence are discussed. The gradient, curl and divergence under three different orthogonal curvilinear coordinate systems are obtained.

Key words: Orthogonal Curvilinear Coordinate System, Gradient, Curl, Divergence, Lame Coefficient

CONTENTS

1. INTRODUCTION

Differential geometry is the application of the tools of differential calculus to the study of geometry. Gaspard Monge is considered the father of differential geometry[1], Elie Cartan is viewed as the father of exterior forms[2] and Shing-Shen Chern is known as the father of modern differential geometry[3]. The branches of differential geometry are Riemannian geometry, pseudo-Riemannian geometry, Finsler geometry, symplectic geometry, complex geometry, and so on. Differential geometry is widely used in physics, especially in field theories.

Calculating gradient, curl and divergence are very important in physics, especially in electrodynamics and fluid mechanics. In many textbooks the gradient, curl and divergence under orthogonal coordinate systems are obtained through coordinate transforms. But the gradient, curl and divergence under orthogonal coordinate systems are not easy to calculate and to remember. In this thesis the concepts such as manifold, tensors, differential forms and Lame coefficients are defined, and several differential-geometrical methods-differential form method, covariant derivative method, and Hodge star operator method-of calculating gradient, curl and divergence under orthogonal curvilinear coordinate systems are discussed.

As a result, we present the equations for the gradient, curl and divergence in Cartesian, cylindrical and spherical coordinates:

$$\mathrm{grad}f = \frac{\partial f}{\partial x}\boldsymbol{e}_x + \frac{\partial f}{\partial y}\boldsymbol{e}_y + \frac{\partial f}{\partial z}\boldsymbol{e}_z = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\boldsymbol{e}_\varphi + \frac{\partial f}{\partial z}\boldsymbol{e}_z = \frac{\partial f}{\partial R}\boldsymbol{e}_R + \frac{1}{R\cos\theta}\frac{\partial f}{\partial \varphi}\boldsymbol{e}_\varphi + \frac{1}{R}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta,$$

$$\mathrm{curl}\boldsymbol{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\boldsymbol{e}_x + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\boldsymbol{e}_y + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\boldsymbol{e}_z = \frac{1}{r}(\frac{\partial A_z}{\partial \varphi} - \frac{\partial (rA_\varphi)}{\partial z})\boldsymbol{e}_r + (\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r})\boldsymbol{e}_\varphi + \frac{1}{r}(\frac{\partial (rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi})\boldsymbol{e}_z = \frac{1}{R\cos\theta}(\frac{\partial A_\theta}{\partial \varphi} - \frac{\partial (A_\varphi \cos\theta)}{\partial \theta})\boldsymbol{e}_R + \frac{1}{R}(\frac{\partial A_R}{\partial \theta} - \frac{\partial (RA_\theta)}{R})\boldsymbol{e}_\varphi + \frac{1}{R}(\frac{\partial (RA_\varphi)}{\partial R} - \frac{1}{\cos\theta}\frac{\partial A_R}{\partial \varphi})\boldsymbol{e}_\theta,$$

$$\mathrm{div}\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}(\frac{\partial (rA_r)}{\partial r} + \frac{\partial A_\varphi}{\partial \varphi}) + \frac{\partial A_z}{\partial z} = \frac{1}{R^2\cos\theta}(\frac{\partial (R^2\cos\theta A_R)}{\partial R} + \frac{\partial (RA_\varphi)}{\partial \varphi} + \frac{\partial (R\cos\theta A_\theta)}{\partial \theta}).$$

2. PRELIMINARIES

In this chapter, the concepts of manifolds, tangent spaces, cotangent spaces, tensors, differential forms, exterior differentials, metric tensors, Riemannian manifolds, Lame coefficients, affine connections, covariant derivatives, musical isomorphisms and the Hodge star operator will be defined. These definitions will be used to calculate the gradient, curl and divergence under orthogonal curvilinear coordinates.

2.1 Manifold

Definition 2.1.1[4] A **topological manifold** M of dimension m is a Hausdorff space (that is, M is a topological space and for each pair p_1, p_2 of distinct points of M there exist neighborhoods V_1, V_2 of p_1, p_2 such that $V_1 \cap V_2 = \emptyset$) with following properties:

1) Each point $p \in M$ possesses a neighborhood V homeomorphic to an open subset U of $\mathbb{R}^m$;
2) M satisfies the **second countability axiom**, that is, M has a countable basis for its topology.

Assume the homeomorphism mentioned in Definition 1.1 is $\varphi_U: U \to \varphi_U(U)$, where $\varphi_U(U)$ is an open set of $\mathbb{R}^m$, then (U, φ_U) is called a **coordinate chart** of M. Since φ_U is a homomorphism, for any point $y \in U$, we can define the coordinates of $\varphi_U(y)$ as the coordinates of y, that is, to let

$$u^i = (\varphi_U(y))^i, \quad y \in U,\ i = 1, \cdots, m,$$

we call $u^i (1 \le i \le m)$ the **local coordinates** of the point $y \in U$.

Assume (U, φ_U) and (V, φ_V) are two coordinate charts of manifold M. We say that two coordinate charts (U, φ_U) and (V, φ_V) are C^r-**compatible**, if $U \cap V = \emptyset$ or if $U \cap V \neq \emptyset$ then $\varphi_V \circ {\varphi_U}^{-1}$ and $\varphi_U \circ {\varphi_V}^{-1}$ are C^r.

Definition 2.1.2[5] Assume M is an m-dimensional topological manifold. If there is a family of coordinate charts $A = \{(U, \varphi_U), (V, \varphi_V), (W, \varphi_W), \cdots\}$ satisfies the following conditions, then we call A a C^r-**differential structure** on M:

1) $\{U, V, W, \cdots\}$ is an open cover of M;
2) any two coordinate charts that belong to A are C^r-compatible;
3) A is maximal, that is, for any coordinate chart of M, if it is C^r-compatible with every coordinate chart in A, then it belongs to A.

If there is a C^r-differential structure on M, then M is called a C^r-**differentiable manifold**. A C^∞-differentiable manifold is called a **smooth manifold**.

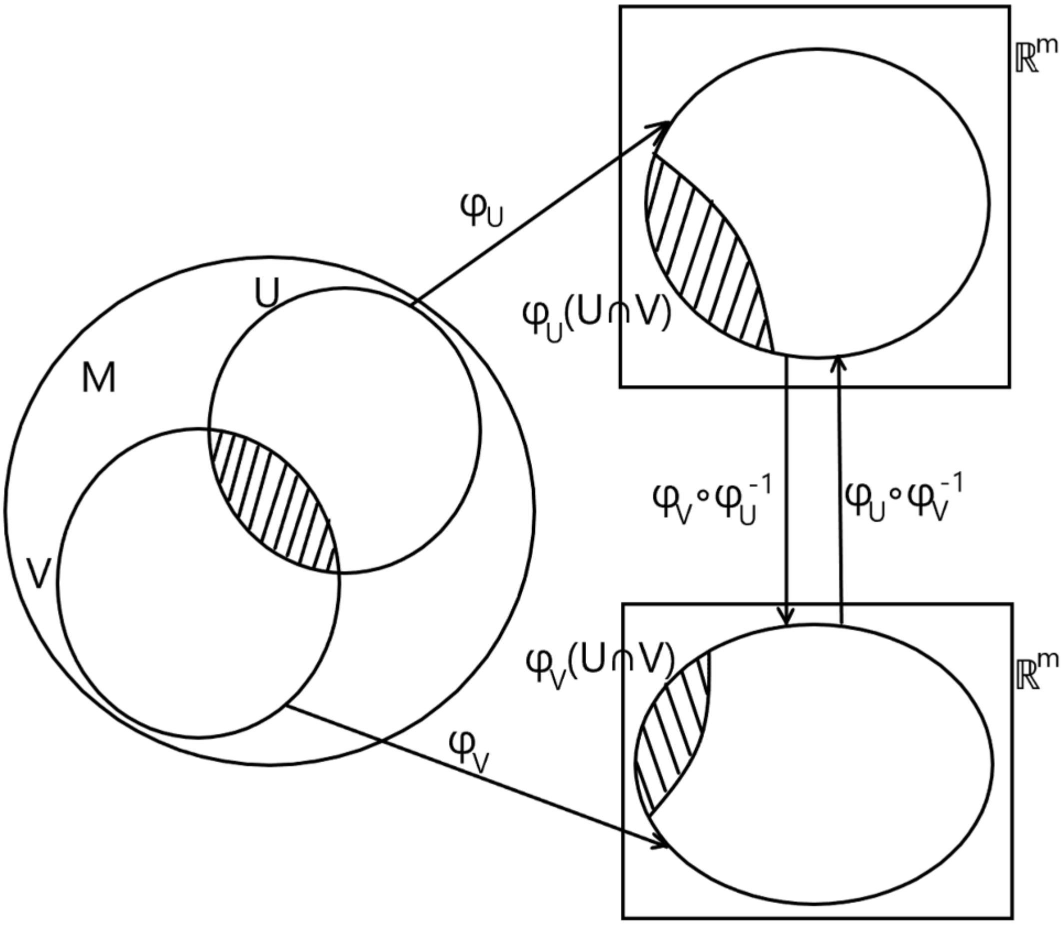

Figure 1. *Diagram for the definition of manifold*

2.2 Tangent space and cotangent space

Definition 2.2.1 The **directional derivative** of a function f of x in the direction of $\boldsymbol{v}$

$\mathrm{D}_{\boldsymbol{v}}(f) := \frac{d}{dt}[f(p + t\boldsymbol{v})]|_{t=0} = \sum_i \left[\frac{\partial f}{\partial x^i}\right](p)v^i$. We define a vector X_p at p, call it **tangent vector**, such that $X_p(f) := \mathrm{D}_X(f) = \sum_i \left[\frac{\partial f}{\partial x^i}\right](p)X^i$.

The **tangent space** at p is the space T_pM of all tangent vectors at p. The dual space of the tangent space of M at p T_p^*M is called the **cotangent space** of M at p.

2.3 Tensor

Let us define the concepts of tensor as in [4].

Let V be a n-dimensional vector space. A k-**tensor** on V is a real multilinear function defined on the product $V \times \cdots \times V$ of k copies of V. Denote the set of all k-tensors by $T^k(V^*)$.

Given a k-tensor T and an m-tensor S, we define their **tensor product** as the $(k+m)$-tensor $T \otimes S$ given by

$$T \otimes S(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+m}) = T(v_1, \cdots, v_k) \cdot S(v_{k+1}, \cdots, v_{k+m}).$$

Tensors in $T^k(V)$ are called **contravariant tensors** on V, while the elements of $T^k(V^*)$ are called **covariant tensors** on V. There are also **mixed** (k,m)-tensors on V, that is, multilinear functions defined on the product $V \times \cdots \times V \times V^* \times \cdots \times V^*$ of k copies of V and m copies of V^*. The space of all (k,m)-tensors on V is denoted by $T^{k,m}(V^*, V)$.

The **contraction** of the i^{th} $(i \leq k)$ upper index and j^{th} $(j \leq m)$ lower index of the tensor $T \in T^{k,m}(V^*, V)$ is defined as

$$C_j^i T := T(\cdot, \cdots, e^{\mu^*}, \cdot, \cdots; \cdot, \cdots, e_\mu, \cdot, \cdots,),$$

where e^{μ^*} takes the i^{th} upper index and e_μ takes the j^{th} lower index.

A tensor T is called **alternating** if

$$T(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -T(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

Let σ be a permutation. Set $\sigma(v_1, \cdots, v_k) = (v_{\sigma(1)}, \cdots, v_{\sigma(k)})$. Let us define an alternating k-tensor $\mathrm{Alt}(T)$, call it the **alternator** of T:

$$\mathrm{Alt}(T) := \frac{1}{k!} \sum_{\text{all possible } \sigma} (\mathrm{sgn}\sigma)(T \circ \sigma),$$

where

$$\mathrm{sgn}\sigma = \begin{cases} +1, & \sigma \text{ is even} \\ -1, & \sigma \text{ is odd} \end{cases}.$$

2.4 Differential forms

Let us define the concepts of differential forms as in [4].

Definition 2.4.1[4] A (k,m)-**tensor field** is a map that to each point $p \in M$ assigns a tensor $T \in T^{k,m}(T_p^*, T_p M)$.

Alternating tensor fields are very important objects called **forms**.

Definition 2.4.2[4] Let M be a smooth manifold. A **form of degree** k(or a k-form) on M is a field of alternating k-tensors defined on M, that is, a map ω that, to each point $p \in M$, assigns an element $\omega_p \in \Lambda^k(T_p^* M)$.

Definition 2.4.3[4] We now define the **wedge product** between alternating tensors: if $T \in \Lambda^k(V^*)$ and $S \in \Lambda^m(V^*)$, then $T \wedge S \in \Lambda^{k+m}(V^*)$ is given by

$$T \wedge S := \frac{(k+m)!}{k!m!} \mathrm{Alt}(T \otimes S).$$

The wedge product can also be applied to forms, and it is not difficult to prove some properties of wedge product:

Theorem 2.4.4[5] Assume $\xi, \xi_1, \xi_2 \in \Lambda^k(V^*)$, $\eta, \eta_1, \eta_2 \in \Lambda^l(V^*)$, $\zeta \in \Lambda^h(V^*)$, then

1) distributivity:
   i. $(\xi_1 + \xi_2) \wedge \eta = \xi_1 \wedge \eta + \xi_2 \wedge \eta,$
   ii. $\xi \wedge (\eta_1 + \eta_2) = \xi \wedge \eta_1 + \xi \wedge \eta_2;$
1) anti-commutativity: $\xi \wedge \eta = (-1)^{kl} \eta \wedge \xi;$
2) associativity: $(\xi \wedge \eta) \wedge \zeta = \xi \wedge (\eta \wedge \zeta).$

2.5 Exterior differential

Theorem 2.5.1[6] Let M be an n-dimensional smooth manifold, then there exists a unique mapping d, called the **exterior differential**, which maps k-forms to $(k+1)$-forms, satisfying:

1) for any ω_1, ω_2, $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2;$
2) if ω_1 is an r-form, then $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge \mathrm{d}\omega_2;$
3) if f is a differentiable function(0-form) on M, then $\mathrm{d}f$ is the differential of f;
4) if f is a differentiable function(0-form) on M, then $\mathrm{d}(\mathrm{d}f) = 0.$

The proof of this theorem can be found in books of differential geometry and is omitted.

2.6 Metric tensor

Definition 2.6.1[6] The metric tensor is the matrix $G = (g_{ij})$ with entries

$$g_{ij} := \langle e_i, e_j \rangle \ .$$

2.7 Riemannian manifold

Let us define the concept of a Riemannian manifold as in [6].

Let M be a smooth n-manifold and G is a symmetric 2-tensor on M. If $(U; u^i)$ is a local coordinate chart on M, then the tensor field G on U can be represented as

$$G = g_{ij} \mathrm{d}u^i \otimes \mathrm{d}u^j$$

where $g_{ij} = g_{ji}$ is a smooth function on U. Let $X = X^i \frac{\partial}{\partial u^i}$, $Y = Y^i \frac{\partial}{\partial u^i}$. Let

$$G(X, Y) = g_{ij} X^i Y^i$$

We call the tensor G nondegenerate at p, if there exists a vector $X \in T_p M$ such that

$$G(X,Y) = 0$$

for all $Y \in T_pM$, then $X = 0$. This is to say, G is nondegenerate at p if and only if the system of equations

$$g_{ij}(p)X^i = 0, 1 \le j \le m$$

Has only null solutions, that is, the determinant $\det(g_{ij}(p)) \neq 0$.

If for any $X \in T_p(M)$ there is $G(X,X) \ge 0$, then we say the tensor G is definite at p.

Definition 2.7.1 If on a m-dimension smooth manifold M there is a smooth everywhere nondegenerate symmetric 2-tensor field G, then we call M a **pseudo-Riemannian manifold**, while G is called the metric tensor of the pseudo-Riemannian manifold M.

If G is positive definite, then M is called a **Riemannian manifold**.

Example 1.6.2 It is clear that $M = \mathbb{R}^n$ with the metric $g = g_{ij}dx^i \otimes dx^i$ is a Riemannian manifold.

2.8 Lame coefficients

Let M be a three-dimensional oriented Riemannian manifold. Let x_1, x_2, x_3 be local coordinates. The square of line element $\mathrm{d}s^2 = {h_1}^2 \mathrm{d}{x_1}^2 + {h_2}^2 \mathrm{d}{x_2}^2 + {h_3}^2 \mathrm{d}{x_3}^2$, where h_1, h_2, h_3 are called **Lame coefficients**[9],

$$h_i := \left|\frac{\partial x}{\partial x_i}\right|.$$

For R^3, under Cartesian coordinates x, y, z, cylindrical coordinates r, φ, θ and spherical coordinates R, φ, θ, the square of line element

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2 = \mathrm{d}R^2 + R^2 sin^2\theta \mathrm{d}\varphi^2 + R^2\mathrm{d}\theta^2;$$

The Lame coefficients are $h_x = h_y = h_z = 1$; $h_r = 1$, $h_\theta = r$, $h_z = 1$; $h_R = 1$, $h_\varphi = Rsin\theta$, $h_\theta = R$.

Theorem 2.8.1[9] $g_{\alpha\beta} = {h_\alpha}^2 \delta_{\alpha\beta}$, $g^{\alpha\beta} = \frac{1}{{h_\alpha}^2}\delta^{\alpha\beta}$.

2.9 Connections and Covariant derivative

Definition 2.9.1[4] Let M be a smooth manifold. The set of all smooth vector fields on M is denoted by $\Xi(M)$. An **affine connection** on M is a map $\nabla: \Xi(M) \times \Xi(M) \to \Xi(M)$ such that

1) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$;

2) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$;

3) $\nabla_X(fY) = (X \cdot f)Y + f\nabla_X Y$

Ifor all $X, Y, Z \in \Xi(M)$ and $f, g \in C^\infty(M, \mathbb{R})$(we write $\nabla_X Y := \nabla(X, Y)$).

The vector field $\nabla_X Y$ is sometimes known as the **covariant derivative** of Y along X.

Notation[11] $\nabla_{e_j}\boldsymbol{v} := v^s{}_{;j}\boldsymbol{e}_s$;

$v^i{}_{,j} := \partial_j v^i$.

Definition 2.9.2[9] The **Christoffel symbol of the second kind** $\Gamma^\alpha_{\beta\gamma} := \frac{1}{2} g^{\alpha\lambda}(g_{\lambda\gamma,\beta} + g_{\beta\lambda,\gamma} - g_{\gamma\beta,\lambda})$.

Theorem 2.9.3[9] $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\frac{1}{h^2_\alpha}(\frac{\partial(h^2_\gamma)}{\partial x^\beta}\delta_{\alpha\gamma} + \frac{\partial(h^2_\beta)}{\partial x^\gamma}\delta_{\alpha\beta} - \frac{\partial(h^2_\gamma)}{\partial x^\alpha}\delta_{\gamma\beta})$.

Theorem 2.9.4[9] $v^i{}_{;j} = v^i{}_{,j} + v^k\Gamma^i{}_{kj}$.

2.10 Hodge star operator and musical isomorphisms

Let us define the concepts of Hodge star operator and musical isomorphisms as in [8].
Musical isomorphisms $\flat$ and $\sharp$

Definition 2.10.1 Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors, α be a differential form and g be the metric tensor. Then

$$v^\flat(\boldsymbol{u}) = g(\boldsymbol{v}, \boldsymbol{u}),$$
$$g(\alpha^\sharp, \boldsymbol{v}) = \alpha(v).$$

Hodge star operator $\star$

Definition 2.10.2 The **Hodge star operator** $\star$ is a mapping that maps k-vectors to $(n-k)$-vectors, for $0 \le k \le n$:

$$\alpha \wedge (\star\, \beta) = \langle \alpha, \beta \rangle\ \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n.$$

Theorem 2.10.3 $(\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3)^\sharp = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial x_3}\boldsymbol{e}_3$.

Theorem 2.10.4 $(A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3)^\flat = A_1h_1\mathrm{d}x_1 + A_2h_2\mathrm{d}x_2 + A_3h_3\mathrm{d}x_3$.

Theorem 2.10.5 $\star\,(\mathrm{d}x_1 \wedge \mathrm{d}x_2) = \mathrm{d}x_3$, $\star\,(\mathrm{d}x_2 \wedge \mathrm{d}x_3) = \mathrm{d}x_1$, $\star\,(\mathrm{d}x_3 \wedge \mathrm{d}x_1) = \mathrm{d}x_2$

3. SEVERAL WAYS OF CALCULATING GRADIENT, CURL AND DIVERGENCE UNDER ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

In this chapter, three different methods of calculating the gradient, curl and divergence under orthogonal curvilinear coordinate systems will be presented.

3.1 Calculating gradient, curl and divergence under orthogonal curvilinear coordinate systems using differential forms

Let us first calculate the gradient, curl and divergence under orthogonal curvilinear coordinates using differential forms. We follow the procedure presented in [7].

In an oriented three-dimensional Euclidean space, every vector $\boldsymbol{A}$ corresponds to a 1-form $\omega^1_{\boldsymbol{A}}$ and a 2-form $\omega^2_{\boldsymbol{A}}$. Let $\boldsymbol{A}$, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be vectors.

Define as follows:

$\omega^1_{\boldsymbol{A}}(\boldsymbol{\xi}) := (\boldsymbol{A}, \boldsymbol{\xi})$, where $(\boldsymbol{A}, \boldsymbol{\xi})$ is the inner product of $\boldsymbol{A}$ and $\boldsymbol{\xi}$;

$\omega^2_{\boldsymbol{A}}(\boldsymbol{\xi}, \boldsymbol{\eta}) := (\boldsymbol{A}, \boldsymbol{\xi}, \boldsymbol{\eta})$, where $(\boldsymbol{A}, \boldsymbol{\xi}, \boldsymbol{\eta})$ is the triple product of $\boldsymbol{A}$, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

Suppose that in the coordinates (x_1, x_2, x_3) the vector field A has the form $A = A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3$, with smooth component functions A_1, A_2 and A_3.

Then we have

$$\omega^1_{\boldsymbol{A}}(\boldsymbol{e_i}) = (\boldsymbol{A}, \boldsymbol{e_i}) = A_i.$$

Also, the 1-form $\omega^1_{\boldsymbol{A}}$ decomposes over the basis $\mathrm{d}x_i$, meaning that

$$\omega^1_{\boldsymbol{A}} = a_1\mathrm{d}x_1 + a_2\mathrm{d}x_2 + a_3\mathrm{d}x_3$$

for unique component functions A_1, A_2 and A_3(see [4] p.195). We can represent these component functions using the Lame coefficients, since from the equation

$$\mathrm{d}s^2 = {h_1}^2{\mathrm{d}x_1}^2 + {h_2}^2{\mathrm{d}x_2}^2 + {h_3}^2{\mathrm{d}x_3}^2$$

we get that

$$\mathrm{d}x_1(\boldsymbol{e}_i) = \frac{1}{h_i},$$

and using that we obtain

$$\omega^1_{\boldsymbol{A}}(\boldsymbol{e_i}) = (a_1\mathrm{d}x_1 + a_2\mathrm{d}x_2 + a_3\mathrm{d}x_3)(\boldsymbol{e}_i) = a_i\mathrm{d}x_i(\boldsymbol{e}_i) = \frac{a_i}{h_i}.$$

Combining these two results we get

$$a_i = A_i h_i,$$

and thus we can write

$$\omega_{\boldsymbol{A}}^1 = A_1 h_1 \mathrm{d}x_1 + A_2 h_2 \mathrm{d}x_2 + A_3 h_3 \mathrm{d}x_3.$$

In the same way, we have

$$\omega_{\boldsymbol{A}}^2(\boldsymbol{e}_j, \boldsymbol{e}_k) = (\boldsymbol{A}, \boldsymbol{e}_j, \boldsymbol{e}_k) = A_i (ijk = 123{,}231{,}312).$$

Also, the 2-form $\omega_{\boldsymbol{A}}^2$ decomposes over the basis $\mathrm{d}x_j \wedge \mathrm{d}x_k$, meaning that

$$\omega_{\boldsymbol{A}}^2 = \alpha_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \alpha_2 \mathrm{d}x_3 \wedge \mathrm{d}x_1 + \alpha_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2$$

for unique component functions A_1, A_2 and A_3. We can represent these component functions using the Lame coefficients, since from the equation

$$\mathrm{d}s^2 = {h_1}^2 {\mathrm{d}x_1}^2 + {h_2}^2 {\mathrm{d}x_2}^2 + {h_3}^2 {\mathrm{d}x_3}^2$$

we get that

$$\mathrm{d}x_j \wedge \mathrm{d}x_k(\boldsymbol{e}_j, \boldsymbol{e}_k) = \frac{1}{h_j h_k}.$$

Combining these two results we get

$$\alpha_i = A_i h_j h_k,$$

and thus we can write

$$\omega_{\boldsymbol{A}}^2 = A_1 h_2 h_3 \mathrm{d}x_2 \wedge \mathrm{d}x_3 + A_2 h_3 h_1 \mathrm{d}x_3 \wedge \mathrm{d}x_1 + A_3 h_1 h_2 \mathrm{d}x_1 \wedge \mathrm{d}x_2,$$

Theorem 3.1.1 The exterior differentiation of the 0-form, 1-form and 2-form corresponds to the gradient, curl, and divergence:

$\mathrm{d}f = \omega_{\mathrm{grad}f}^1$, $\mathrm{d}\omega_{\boldsymbol{A}}^1 = \omega_{\mathrm{curl}\boldsymbol{A}}^2$, $\mathrm{d}\omega_{\boldsymbol{A}}^2 = (\mathrm{div}\boldsymbol{A})\omega^3$($\omega^3$ is the volume element of M).

Proof: The equations are independent of the chosen coordinate system, so to prove this, it is enough to prove that it holds under Cartisian coordinates:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z,$$

$$\omega_{\mathrm{grad}f}^1 = \frac{\partial f}{\partial x} h_x \mathrm{d}x + \frac{\partial f}{\partial y} h_y \mathrm{d}y + \frac{\partial f}{\partial z} h_z \mathrm{d}z = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z,$$

so

$$\mathrm{d}f = \omega_{\mathrm{grad}f}^1;$$

because

$$\omega_{\boldsymbol{A}}^1 = A_1 h_1 \mathrm{d}x_1 + A_2 h_2 \mathrm{d}x_2 + A_3 h_3 \mathrm{d}x_3,$$

by Theorem 2.4.4 we have

$$
\begin{aligned}
\mathrm{d}\omega_{\boldsymbol{A}}^{1} &= \mathrm{d}\left(A_x h_x \mathrm{d}x + A_y h_y \mathrm{d}y + A_z h_z \mathrm{d}z\right) = \mathrm{d}\left(A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z\right) \\
&= \mathrm{d}A_x \wedge \mathrm{d}x + \mathrm{d}A_y \wedge \mathrm{d}y + \mathrm{d}A_z \wedge \mathrm{d}z \\
&= \left(\frac{\partial A_x}{\partial x}\mathrm{d}x + \frac{\partial A_x}{\partial y}\mathrm{d}y + \frac{\partial A_x}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x + \left(\frac{\partial A_y}{\partial x}\mathrm{d}x + \frac{\partial A_y}{\partial y}\mathrm{d}y + \frac{\partial A_y}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \\
&+ \left(\frac{\partial A_z}{\partial x}\mathrm{d}x + \frac{\partial A_z}{\partial y}\mathrm{d}y + \frac{\partial A_z}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \\
&= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y,
\end{aligned}
$$

also,

$$
\begin{aligned}
\omega_{\mathrm{curl}\boldsymbol{A}}^{2} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) h_y h_z \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) h_z h_x \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) h_x h_y \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y,
\end{aligned}
$$

then we have

$$\mathrm{d}\omega_{\boldsymbol{A}}^{1} = \omega_{\mathrm{curl}\boldsymbol{A}}^{2},$$

because

$$\omega_{\boldsymbol{A}}^{2} = A_1 h_2 h_3 \mathrm{d}x_2 \wedge \mathrm{d}x_3 + A_2 h_3 h_1 \mathrm{d}x_3 \wedge \mathrm{d}x_1 + A_3 h_1 h_2 \mathrm{d}x_1 \wedge \mathrm{d}x_2,$$

by Theorem 2.4.4 we have

$$
\begin{aligned}
\mathrm{d}\omega_{\boldsymbol{A}}^{2} &= \mathrm{d}(A_1 h_2 h_3 \mathrm{d}y \wedge \mathrm{d}z + A_2 h_3 h_1 \mathrm{d}z \wedge \mathrm{d}x + A_3 h_1 h_2 \mathrm{d}x \wedge \mathrm{d}y) \\
&= \mathrm{d}\left(A_x \mathrm{d}y \wedge \mathrm{d}z + A_y \mathrm{d}z \wedge \mathrm{d}x + A_z \mathrm{d}x \wedge \mathrm{d}y\right) \\
&= \mathrm{d}A_x \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}A_y \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}A_z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial A_x}{\partial x}\mathrm{d}x + \frac{\partial A_x}{\partial y}\mathrm{d}y + \frac{\partial A_x}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial A_y}{\partial x}\mathrm{d}x + \frac{\partial A_y}{\partial y}\mathrm{d}y + \frac{\partial A_y}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}z \\
&\wedge \mathrm{d}x + \left(\frac{\partial A_z}{\partial x}\mathrm{d}x + \frac{\partial A_z}{\partial y}\mathrm{d}y + \frac{\partial A_z}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,
\end{aligned}
$$

also,

$$(\mathrm{div}\boldsymbol{A})\omega^3 = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) h_x h_y h_z \mathrm{d}x_x \wedge \mathrm{d}x_y \wedge \mathrm{d}x_z = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

then we have

$$\mathrm{d}\omega_{\boldsymbol{A}}^{2} = (\mathrm{div}\boldsymbol{A})\omega^3.$$

Qed.

Because

$$\mathrm{d}f = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\mathrm{d}x_2 + \frac{\partial f}{\partial x_3}\mathrm{d}x_3,$$

by

$$\mathrm{d}f = \omega^1_{\mathrm{grad}f}$$

we have

$$\mathrm{grad}f = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial x_3}\boldsymbol{e}_3; \tag{1}$$

Because

$$\mathrm{d}\omega^1_{\boldsymbol{A}} = \left(\frac{\partial(A_3h_3)}{\partial x_2} - \frac{\partial(A_2h_2)}{\partial x_3}\right)\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \left(\frac{\partial(A_2h_2)}{\partial x_1} - \frac{\partial(A_1h_1)}{\partial x_2}\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \left(\frac{\partial(A_1h_1)}{\partial x_3} - \frac{\partial(A_3h_3)}{\partial x_1}\right)\mathrm{d}x_3 \wedge \mathrm{d}x_1,$$

by

$$\mathrm{d}\omega^1_{\boldsymbol{A}} = \omega^2_{curl\boldsymbol{A}}$$

we have

$$\mathrm{curl}\boldsymbol{A} = \frac{1}{h_2h_3}\left(\frac{\partial(A_3h_3)}{\partial x_2} - \frac{\partial(A_2h_2)}{\partial x_3}\right)\boldsymbol{e_1} + \frac{1}{h_1h_3}\left(\frac{\partial(A_1h_1)}{\partial x_3} - \frac{\partial(A_3h_3)}{\partial x_1}\right)\boldsymbol{e_2} + \frac{1}{h_1h_2}\left(\frac{\partial(A_2h_2)}{\partial x_1} - \frac{\partial(A_1h_1)}{\partial x_2}\right)\boldsymbol{e_3} =$$

$$\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e_1} & h_2\boldsymbol{e_2} & h_3\boldsymbol{e_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}; \tag{2}$$

Because

$$\mathrm{d}\omega^2_{\boldsymbol{A}} = \left(\frac{\partial}{\partial x_1}(A_1h_2h_3) + \frac{\partial}{\partial x_2}(A_2h_1h_3) + \frac{\partial}{\partial x_3}(A_3h_1h_2)\right)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3,$$

by

$$\mathrm{d}\omega^2_{\boldsymbol{A}} = (\mathrm{div}\boldsymbol{A})\omega^3 = (\mathrm{div}\boldsymbol{A})h_1h_2h_3\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$$

we have

$$(\mathrm{div}\boldsymbol{A}) = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x_1}(A_1h_2h_3) + \frac{\partial}{\partial x_2}(A_2h_1h_3) + \frac{\partial}{\partial x_3}(A_3h_1h_2)\right). \tag{3}$$

3.2 Calculating gradient, curl, divergence under orthogonal curvilinear coordinate systems using covariant derivatives

Let us then calculate the gradient, curl and divergence under orthogonal curvilinear coordinates using covariant derivatives as in [9].

By definition, use the differential operator to act on the scalar field we get the gradient, the contraction of the covariant derivative of the vector field with $\delta^{\alpha\beta\gamma} = \frac{1}{h_1h_2h_3}\varepsilon_{\alpha\beta\gamma}$ ($\varepsilon_{ijk} = +1$ if $ijk =$ 123,231,312; $\varepsilon_{ijk} = -1$ if $ijk =$ 321,132,213; $\varepsilon_{ijk} = 0$ if $i = j$ or $j = k$ or $k = i$) is the curl, and the contraction of the differential operator with the vector field is the divergence.

The results of this method are the same as first method:

for the gradient

$$\nabla f := \nabla_\alpha f \boldsymbol{e}_\alpha = f_{;\alpha} \boldsymbol{e}_\alpha,$$

by Theorem 2.9.4 and Theorem 2.9.3 we have

$$\nabla f = f_{,\alpha} \boldsymbol{e}_\alpha = \frac{1}{h_i}\frac{\partial f}{\partial x_i} \boldsymbol{e}_i;$$

for the curl

$$\nabla \times \boldsymbol{A} := \delta^{\alpha\beta\gamma} A_{\beta;\alpha} \boldsymbol{e}_\gamma,$$

by Theorem 2.9.4 and Theorem 2.9.3 we have

$$\begin{aligned}\nabla \times \boldsymbol{A} = \delta^{\alpha\beta\gamma} \partial_\alpha A_\beta \boldsymbol{e}_\gamma &= \frac{1}{h_1 h_2 h_3} \varepsilon_{\alpha\beta\gamma} \partial_\alpha A_\beta \boldsymbol{e}_\gamma \\ &= \frac{1}{h_2 h_3}\left(\frac{\partial(A_3 h_3)}{\partial x_2} - \frac{\partial(A_2 h_2)}{\partial x_3}\right)\boldsymbol{e_1} + \frac{1}{h_1 h_3}\left(\frac{\partial(A_1 h_1)}{\partial x_3} - \frac{\partial(A_3 h_3)}{\partial x_1}\right)\boldsymbol{e_2} \\ &+ \frac{1}{h_1 h_2}\left(\frac{\partial(A_2 h_2)}{\partial x_1} - \frac{\partial(A_1 h_1)}{\partial x_2}\right)\boldsymbol{e_3} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e_1} & h_2\boldsymbol{e_2} & h_3\boldsymbol{e_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix};\end{aligned}$$

for the divergence

$$\nabla \cdot \boldsymbol{A} := \nabla_\alpha A^\alpha,$$

by Theorem 2.9.4 and Theorem 2.9.3 we have

$$\begin{aligned}\nabla \cdot \boldsymbol{A} = A^\mu_{,\mu} + \Gamma^\mu_{\mu\alpha} A^\alpha &= \frac{1}{h_\mu}\frac{\partial A^\mu}{\partial x^\alpha} + \frac{1}{2}\frac{1}{h_\mu^2}\left(\frac{\partial(h_\alpha^2)}{\partial x^\mu}\delta_{\mu\alpha} + \frac{\partial(h_\mu^2)}{\partial x^\alpha}\delta_{\mu\mu} - \frac{\partial(h_\alpha^2)}{\partial x^\mu}\delta_{\alpha\mu}\right)A^\alpha \\ &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x_1}(A_1 h_2 h_3) + \frac{\partial}{\partial x_2}(A_2 h_1 h_3) + \frac{\partial}{\partial x_3}(A_3 h_1 h_2)\right).\end{aligned}$$

3.3 Calculating gradient, curl and divergence under orthogonal curvilinear coordinate systems using Hodge star operator

Finally, let us calculate the gradient, curl and divergence under orthogonal curvilinear coordinates using Hodge star operator as in [10].

Theorem 3.3.1 The gradient, curl and divergence can be calculated using the musical isomorphism and the Hodge star operator as

$$\mathrm{grad} f = (\mathrm{d}f)^\sharp;$$

$$\mathrm{curl}\boldsymbol{A} = (\star\, \mathrm{d}(\boldsymbol{A}^\flat))^\sharp;$$

$$\mathrm{div}\boldsymbol{A} = \star\, \mathrm{d}(\star\, \boldsymbol{A}^\flat).$$

Proof: The equations are independent of the chosen coordinate system, so to prove this, it is enough to show that it hold under the Cartisian coordinates:

by Theorem 2.10.3, Theorem 2.10.4, and Theorem 2.10.5, we have

$$(\mathrm{d}f)^\sharp = (\frac{\partial f}{\partial \mathrm{x}}\mathrm{d}x + \frac{\partial f}{\partial \mathrm{y}}\mathrm{d}y + \frac{\partial f}{\partial \mathrm{z}}\mathrm{dz})^\sharp = \frac{\partial f}{\partial x}\boldsymbol{e}_x + \frac{\partial f}{\partial y}\boldsymbol{e}_y + \frac{\partial f}{\partial z}\boldsymbol{e}_z = \mathrm{grad}f;$$

$$\begin{aligned}
(\star\, \mathrm{d}(\boldsymbol{A}^\flat))^\sharp &= (\star\, \mathrm{d}((A_x\boldsymbol{e}_x + A_y\boldsymbol{e}_y + A_z\boldsymbol{e}_\mathrm{z})^\flat))^\sharp = (\star\, \mathrm{d}(A_x\mathrm{dx} + A_y\mathrm{dy} + A_z\mathrm{dz}))^\sharp \\
&= (\star\, (\mathrm{d}A_x \wedge \mathrm{d}x + \mathrm{d}A_y \wedge \mathrm{d}y + \mathrm{d}A_z \wedge \mathrm{dz}))^\sharp \\
&= (\star\, ((\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\mathrm{dy}\wedge\mathrm{dz} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\mathrm{dz}\wedge\mathrm{dx} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\mathrm{d}x\wedge\mathrm{d}y))^\sharp \\
&= ((\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\mathrm{dx} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\mathrm{d}y + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\mathrm{dz})^\sharp \\
&= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{e}_z = \mathrm{curl}\boldsymbol{A};
\end{aligned}$$

$$\begin{aligned}
\star\, \mathrm{d}(\star\, \boldsymbol{A}^\flat) &= \star\, \mathrm{d}\left(\star\, (A_x\boldsymbol{e}_x + A_y\boldsymbol{e}_y + A_z\boldsymbol{e}_z)^\flat\right) = \star\, \mathrm{d}\left(\star\, (A_x\mathrm{dx} + A_y\mathrm{dy} + A_z\mathrm{dz})\right) \\
&= \star\, \mathrm{d}(A_x\mathrm{dy}\wedge\mathrm{dz} + A_y\mathrm{dz}\wedge\mathrm{dx} + A_z\mathrm{d}x\wedge\mathrm{dy}) \\
&= \star\, (\mathrm{d}A_x \wedge \mathrm{d}y \wedge \mathrm{dz} + \mathrm{d}A_y \wedge \mathrm{dz}\wedge \mathrm{d}x + \mathrm{d}A_z \wedge \mathrm{dx}\wedge\mathrm{dy}) \\
&= \\
&\star\left(\left(\frac{\partial A_x}{\partial x}\mathrm{dx} + \frac{\partial A_x}{\partial y}\mathrm{dy} + \frac{\partial A_x}{\partial z}\mathrm{dz}\right)\wedge\mathrm{dy}\wedge\mathrm{dz} + \left(\frac{\partial A_y}{\partial x}\mathrm{dx} + \frac{\partial A_y}{\partial y}\mathrm{dy} + \frac{\partial A_y}{\partial z}\mathrm{dz}\right)\wedge\mathrm{dz}\right. \\
&\left.\wedge\,\mathrm{d}x + \left(\frac{\partial A_z}{\partial x}\mathrm{d}x + \frac{\partial A_z}{\partial y}\mathrm{d}y + \frac{\partial A_z}{\partial z}\mathrm{dz}\right)\wedge\mathrm{dx}\wedge\mathrm{dy}\right) \\
&= \star\left(\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{dz}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \mathrm{div}\boldsymbol{A}.
\end{aligned}$$

Qed.

The results of this method are the same as first method:

by Theorem 2.10.3, Theorem 2.10.4, and Theorem 2.10.5, we have

$$\mathrm{grad}f = (\mathrm{d}f)^\sharp = (\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3)^\sharp = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial x_3}\boldsymbol{e}_3;$$

$$\begin{aligned}
\mathrm{curl}\boldsymbol{A} &= (\star\,\mathrm{d}(\boldsymbol{A}^\flat))^\sharp = (\star\,\mathrm{d}((A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3)^\flat))^\sharp = (\star\,\mathrm{d}(A_1h_1\mathrm{d}x_1 + A_2h_2\mathrm{d}x_2 + A_3h_3\mathrm{d}x_3))^\sharp \\
&= (\star\,(\mathrm{d}(A_1h_1)\wedge\mathrm{d}x_1 + \mathrm{d}(A_2h_2)\wedge\mathrm{d}x_2 + \mathrm{d}(A_3h_3)\wedge\mathrm{d}x_3))^\sharp \\
&= (\star\,((\frac{\partial(A_3h_3)}{\partial x_2} - \frac{\partial(A_2h_2)}{\partial x_3})\mathrm{d}x_2\wedge\mathrm{d}x_3 + (\frac{\partial(A_1h_1)}{\partial x_3} - \frac{\partial(A_3h_3)}{\partial x_1})\mathrm{d}x_3\wedge\mathrm{d}x_1 + (\frac{\partial(A_2h_2)}{\partial x_1} \\
&- \frac{\partial(A_1h_1)}{\partial x_2})\mathrm{d}x_1\wedge\mathrm{d}x_2))^\sharp \\
&= ((\frac{\partial(A_3h_3)}{\partial x_2} - \frac{\partial(A_2h_2)}{\partial x_3})\mathrm{d}x_1 + (\frac{\partial(A_1h_1)}{\partial x_3} - \frac{\partial(A_3h_3)}{\partial x_1})\mathrm{d}x_2 + (\frac{\partial(A_2h_2)}{\partial x_1} \\
&- \frac{\partial(A_1h_1)}{\partial x_2})\mathrm{d}x_3)^\sharp \\
&= \frac{1}{h_2h_3}(\frac{\partial(A_3h_3)}{\partial x_2} - \frac{\partial(A_2h_2)}{\partial x_3})\boldsymbol{e_1} + \frac{1}{h_1h_3}(\frac{\partial(A_1h_1)}{\partial x_3} - \frac{\partial(A_3h_3)}{\partial x_1})\boldsymbol{e_2} + \frac{1}{h_1h_2}(\frac{\partial(A_2h_2)}{\partial x_1} \\
&- \frac{\partial(A_1h_1)}{\partial x_2})\boldsymbol{e_3} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e_1} & h_2\boldsymbol{e_2} & h_3\boldsymbol{e_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix};
\end{aligned}$$

$$\begin{aligned}
(\mathrm{div}\boldsymbol{A}) &=\star\,\mathrm{d}(\star\,\boldsymbol{A}^\flat) =\star\,\mathrm{d}(\star\,(A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3)^\flat) =\star\,\mathrm{d}(\star\,(A_1\mathrm{d}x_1 + A_2\mathrm{d}x_2 + A_3\mathrm{d}x_3)) \\
&=\star\,(\mathrm{d}(A_1h_2h_3)\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3 + \mathrm{d}(A_2h_1h_3)\wedge\mathrm{d}x_3\wedge\mathrm{d}x_1 + \mathrm{d}(A_3h_1h_2)\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2) \\
&=\star\,((\frac{\partial(A_1h_2h_3)}{\partial x_1}\mathrm{d}x_1 + \frac{\partial(A_1h_2h_3)}{\partial x_2}\mathrm{d}x_2 + \frac{\partial(A_1h_2h_3)}{\partial x_3}\mathrm{d}x_3)\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3 \\
&+ (\frac{\partial(A_2h_1h_3)}{\partial x_1}\mathrm{d}x_1 + \frac{\partial(A_2h_1h_3)}{\partial x_2}\mathrm{d}x_2 + \frac{\partial(A_2h_1h_3)}{\partial x_3}\mathrm{d}x_3)\wedge\mathrm{d}x_3\wedge\mathrm{d}x_1 \\
&+ (\frac{\partial(A_3h_1h_2)}{\partial x_1}\mathrm{d}x_1 + \frac{\partial(A_3h_1h_2)}{\partial x_2}\mathrm{d}x_2 + \frac{\partial(A_3h_1h_2)}{\partial z}\mathrm{d}x_3)\wedge\mathrm{d}x_1\wedge\mathrm{d}x_2) \\
&=\star\,((\frac{\partial(A_1h_2h_3)}{\partial x_1} + \frac{\partial(A_2h_1h_3)}{\partial x_2} + \frac{\partial(A_3h_1h_2)}{\partial x_3})\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3) \\
&= \frac{1}{h_1h_2h_3}(\frac{\partial}{\partial x_1}(A_1h_2h_3) + \frac{\partial}{\partial x_2}(A_2h_1h_3) + \frac{\partial}{\partial x_3}(A_3h_1h_2)).
\end{aligned}$$

4. THE GRADIENT, CURL AND DIVERGENCE UNDER CARTESIAN COORDINATE SYSTEM, CYLINDRICAL COORDINATE SYSTEM AND SPHERICAL COORDINATE SYSTEM

In this chapter, the gradient, curl and divergence under Cartesian coordinate system, cylindrical coordinate system and spherical coordinate system will be computed.

From (1), (2) and (3), we can calculate that

$$\mathrm{grad}f=\frac{\partial f}{\partial x}\boldsymbol{e}_x+\frac{\partial f}{\partial y}\boldsymbol{e}_y+\frac{\partial f}{\partial z}\boldsymbol{e}_z=\frac{\partial f}{\partial r}\boldsymbol{e}_r+\frac{1}{r}\frac{\partial f}{\partial \varphi}\boldsymbol{e}_\varphi+\frac{\partial f}{\partial z}\boldsymbol{e}_z=\frac{\partial f}{\partial R}\boldsymbol{e}_R+\frac{1}{R\cos\theta}\frac{\partial f}{\partial \varphi}\boldsymbol{e}_\varphi+\frac{1}{R}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta,$$

$$\mathrm{curl}\boldsymbol{A}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\boldsymbol{e}_x+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\boldsymbol{e}_y+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\boldsymbol{e}_z=\frac{1}{r}\left(\frac{\partial A_z}{\partial \varphi}-\frac{\partial(rA_\varphi)}{\partial z}\right)\boldsymbol{e}_r+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\boldsymbol{e}_\varphi+$$
$$\frac{1}{r}\left(\frac{\partial(rA_\varphi)}{\partial r}-\frac{\partial A_r}{\partial \varphi}\right)\boldsymbol{e}_z=\frac{1}{R\cos\theta}\left(\frac{\partial A_\theta}{\partial \varphi}-\frac{\partial(A_\varphi\cos\theta)}{\partial \theta}\right)\boldsymbol{e}_R+\frac{1}{R}\left(\frac{\partial A_R}{\partial \theta}-\frac{\partial(RA_\theta)}{R}\right)\boldsymbol{e}_\varphi+\frac{1}{R}\left(\frac{\partial(RA_\varphi)}{\partial R}-\frac{1}{\cos\theta}\frac{\partial A_R}{\partial \varphi}\right)\boldsymbol{e}_\theta,$$

$$\mathrm{div}\boldsymbol{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=\frac{1}{r}\left(\frac{\partial(rA_r)}{\partial r}+\frac{\partial A_\varphi}{\partial \varphi}\right)+\frac{\partial A_z}{\partial z}=\frac{1}{R^2\cos\theta}\left(\frac{\partial(R^2\cos\theta A_R)}{\partial R}+\frac{\partial(RA_\varphi)}{\partial \varphi}+\frac{\partial(R\cos\theta A_\theta)}{\partial \theta}\right).$$

5. SUMMARY

The aim of this thesis is to discuss three differential-geometrical methods to calculate the gradient, curl and divergence under orthogonal curvilinear coordinates. The first method is the differential form method, the second method is the covariant derivative method and the third one is the Hodge star operator method. The gradient, curl and divergence under orthogonal curvilinear coordinates can be calculated also without using differential geometry just by using coordinate transform.

As a result we have

$$\mathrm{grad}f = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial x_3}\boldsymbol{e}_3,$$

$$\mathrm{curl}\boldsymbol{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e_1} & h_2\boldsymbol{e_2} & h_3\boldsymbol{e_3} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix},$$

$$(\mathrm{div}\boldsymbol{A}) = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x_1}(A_1 h_2 h_3) + \frac{\partial}{\partial x_2}(A_2 h_1 h_3) + \frac{\partial}{\partial x_3}(A_3 h_1 h_2)\right).$$

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