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Abstract

Let $\mathcal{L}$ be a local system on the complement $X^\star$ of a normal crossing divisor (NCD) Y in a smooth analytic variety X and let $j : X^\star = X - Y \to X$ denotes the open embedding. The purpose of this paper is to describe a weight filtration W on the direct image $\mathbf{j}_\star\mathcal{L}$ and in case a morphism $f : X \to D$ to a complex disc is given with $Y = f^{-1}(0)$, the weight filtration on the complex of nearby cocycles $\Psi_f(\mathcal{L})$ on Y. A comparison theorem shows that the filtration coincides with the weight defined by the logarithm of the monodromy and provides the link with various results on the subject.

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1§. Introduction

We consider a local system $\mathcal{L}$ on the complement $X^\star$ of a normal crossing divisor (NCD) Y in a smooth analytic variety X and let $j : X^\star = X - Y \to X$ denotes the open embedding. The purpose of this paper is to describe a weight filtration W on the direct image $\mathbf{j}_\star\mathcal{L}$ (denoted also $\mathbf{R}j_\star\mathcal{L}$) in the category of perverse sheaves and in case a morphism $f : X \to D$ to a complex disc is given with $Y = f^{-1}(0)$, the weight filtration on the complex of nearby cocycles $\Psi_f(\mathcal{L})$ on Y. This subject started with Deligne's paper [8] when $\mathcal{L} = \ C$ (Steenbrink treated the complex of nearby cycles in [32]), then developed extensively again after the discovery of the theory of intersection cohomology [17] perverse sheaves [2] and the purity theorem [2], [5], [24]. The theory of differential modules introduced by Kashiwara proved to be fundamental in the understanding of the problem as it appeared in the work of Malgrange [28], Kashiwara [24]-[26] and later M. Saito [29],[30]. It is interesting to treat the problem by the original logarithmic methods as proposed in the note [14], in order to obtain topological interpretation of these results in the direction of [27](for another direction [15]). Let me explain now the problems encountered in the construction of a mixed Hodge structure (MHS) for X proper on the cohomology $H^*(X - Y, \mathcal{L})$.

Hypothesis 1. Let $\mathcal{L}$ be a local system defined over $\mathbb{Q}$, on the complement of the normal crossing divisor (NCD) Y in a smooth analytic variety X, $(\mathcal{L}_X, \nabla)$ the canonical extension of $\mathcal{L}^{\mathbb{C}} = \mathcal{L} \otimes \mathbb{C}$ [6] with a meromorphic connection ∇ having a regular singularity along Y in X and the associated DeRham logarithmic complex $\Omega^*_X(LogY) \otimes \mathcal{L}_X$ defined by ∇ (in the text we write L and $\mathcal{L}$ for the rational as well complex vector spaces).

In order to construct a weight filtration W by subcomplexes of $\Omega^*_X(LogY) \otimes \mathcal{L}_X$ we need a precise description of the correspondence with the local system $\mathcal{L}$. For all subset M of I, let $Y_M = \cap_{i\in M} Y_i$ and $Y^*_M = Y_M - \cup_{i\notin M} Y_i$, $j_M: Y^*_M \to Y_M$ the locally closed embeddings, then $Y_M - Y^*_M$ is a NCD in Y_M and the open subsets Y^*_M of Y_M form with X^* a natural stratification of X (we suppose the NCD $Y = \cup_{i\in I} Y_i$ equal to the union of irreducible and smooth components Y_i for i in I). All extensions of $\mathcal{L}$ that we will introduce will be constructible with respect to this stratification and even perverse. In fact we will need a combinatorial model of $\Omega^*_X(LogY) \otimes \mathcal{L}_X$ for the description of the weight.

If we consider a point $y \in Y^*_M$ and a variation of Hodge structures on $\mathcal{L}$ of weight m, locally defined by a nilpotent orbit L and a set of nilpotent endomorphisms $N_i, i \in M$, the nilpotent orbit theorem [4], [24] states that it degenerates along Y_M^* into a variation of MHS with weight filtration $W^M = W(\Sigma_{i\in M}N_i)$ shifted by m, however this result doesn't lead directly to a structure of mixed Hodge complex since what happens at the intersection of Y_M and Y_K for two subsets M and K of I couldn't be explained until the discovery of perverse sheaves. The MHS we are looking for cannot be obtained from Hodge complexes defined by smooth and proper varieties, so it was only after the purity theorem [2], [5], [24] and the work on perverse sheaves [2], that the weight filtration $\mathcal{W}(\mathcal{N})$ on $\Psi_f(\mathcal{L})$ has been defined by the logarithm of the monodromy $\mathcal{N}$ in the abelian category of perverse sheaves (object not trivial to compute). In characteristic zero this has been successfully related to the theory of differential modules [29]. In this work we study the construction of the weight filtration given in the note [14] with new general proofs of the purity and decomposition statements there. The key result that enables us to simplify most of the proofs is the following decomposition (see §2):

$$Gr_r^{W^M} L = \oplus_{\Sigma_{i\in M} m_i = r} Gr_{m_n}^{W^n} \cdots Gr_{m_1}^{W^1} L$$

and in fact the intersection complexes defined by the local systems on Y_M^* with fiber:

$$\oplus_{\Sigma_{i\in M} m_i = r-|M|, m_i\geq 0} Gr_{m_n}^{W^n} \cdots Gr_{m_1}^{W^1} (L/(N_1L + \cdots + N_nL))$$

will be the Hodge components of the decomposition of the graded part of the weight filtrations (of the primitive parts in the case of nearby cocycles). This main result in the open case, its extension to the case of nearby cocycles and its comparison with Kashiwara and Saito's results form the contents of the article in the second and third sections. We hope that this comparison will be helpful to the reader who doesn't want to go immediately through the whole subject of differential modules.

One important improvement in the new purity theory with respect to the theory of mixed Hodge complexes in [8], [9] is the fact that the objects can be defined locally: these objects are the intermediate extensions of VHS on Y_M^* for some $M \subset I$ since such extension is defined on Y_M even if it is not proper which makes such objects very pleasant to use especially in the Ψ_f case. We adopt here the convention to use the terminology of perverse sheaves up to a shift in degrees, although it is very important to give explicitly this shift when needed in a proof. Let us review the contents: the definition of the weight is in $(\S 1, II, 2)$ formula (14); the main results are in §2, namely the key lemma for the decomposition at the level of the weights $(I,1)$, the local purity $(I.3)$, the local decomposition (II) and the global decomposition (III). In §3, the weight of the nilpotent action on Ψ_f is in (I), the local decomposition in $(I,1)$, the global decomposition in $(I,2)$, the comparison in (II) and in an example we apply this theory to remove the base change in Steenbrink's work. In the last part we just state the results for good $VMHS$ as there is no new difficulties in the proofs. Finally we suggest strongly to the reader to follow the proofs on an example, sometimes on the surface case as in $(II.1.5)$, or for X a line and $Y = 0$ a point, then the fiber at 0 of $\mathbf{j}_\star\mathcal{L}$ is a complex $L \xrightarrow{N} L$ where (L,N) is the nilpotent orbit of weight m defined at y and the weight on the complex is $\mathcal{W}[m]$ defined by $\mathcal{W}[m]_{r+m} = \mathcal{W}_r = (W_{r+1}L \xrightarrow{N} W_{r-1}L)$. This example will be again useful for $\Psi_f\mathcal{L}$ in §3.

I. Preliminaries on perverse extensions and nilpotent orbits

In the neighbourhood of a point y in Y, we can suppose $X \simeq D^{n+k}$ and $X^* \simeq (D^*)^n \times D^k$ where D is a complex disc, denoted with a star when the origin is deleted. The fundamental group $\Pi_1(X^*)$ is a free abelian group generated by n elements representing classes of closed paths around the origin, one for each D^* in the various axis with one dimensional coordinate z_i (the hypersurface Y_i is defined by the equation $z_i = 0$) . Then the local system $\mathcal{L}$ corresponds to a representation of $\Pi_1(X^*)$ in a vector space L, i.e the action of commuting automorphisms T_i for $i \in [1,n]$ indexed by the local components Y_i of Y and called monodromy action around Y_i. The automorphisms T_i decomposes as a product of commuting automorphisms, semi-simple and unipotent $T_i = T_i^s T_i^u$. When L is a $\mathbb{C}$ - vector space, T_i^s can be represented by the diagonal matrix of its eigenvalues. If we consider sequences of eigenvalues a_i for each T_i we have the spectral decomposition of L

$$(1) \qquad L = \oplus_{a.} L^{a.} \qquad , \qquad L^{a.} = \cap_{i\in[1,n]} (\cup_{j>0} \ker\, (T_i - a_i I)^j)$$

where the direct sum is over all families $(a.) \in \mathbb{C}^n$. The logarithm of T_i is defined as the sum

$$(2) \qquad Log T_i = -2i\pi(D_i + N_i) = Log T_i^s + Log T_i^u$$

where $-2i\pi D_i = Log T_i^s$ is the diagonal matrix formed by $Log\, a_i$ for all eigenvalues a_i of T_i^s and for a fixed determination of Log on $\mathbb{C}^*$, while $-2i\pi N_i = Log T_i^u$ is defined by the polynomial function $-2i\pi N_i = \Sigma_{k\geq 1}(1/k)(I - T_i^u)^k$ in the nilpotent $(I - T_i^u)$ so that the sum is finite.
Now we describe various local extensions of $\mathcal{L}$.

1.1 The (higher) direct image $\mathbf{j}_\star\mathcal{L}$, local and global description The complex $\mathbf{j}_*\mathcal{L}$ is perverse and its fiber at the origin in D^{n+k} is quasi-isomorphic to a Koszul complex as follows. We associate to $(L, (D_i + N_i), i \in [1,n])$ a strict simplicial vector space such that for all sequence $(i.) = (i_1 < \cdots < i_p)$

$$L(i.) = L \qquad , \qquad D_{i_j} + N_{i_j} \colon L(i. - i_j) \to L(i.).$$

The associated simple complex is the Koszul complex or the exterior algebra defined by $(L, D_i + N_i)$ denoted by $\Omega(L, D. + N.) := s(L(J), D. + N.)_{J\subset[1,n]}$ where J is identified with the strictly increasing sequence of its elements and where $L(J) = L$. It can be checked that its cohomology is the same as $\Omega(L, Id - T.)$, the Koszul complex defined by $(L, Id - T_i), i \in [1,n]$.

When we fix a family $\alpha_j \in [0,1[$ for $j \in [1,n]$ such that $e(\alpha_j) = e^{-2i\pi\alpha_j} = a_j$ is an eigenvalue for D_j, we have

$$(3) \qquad \Omega(L, D. + N.) := s(L(J), D. + N.)_{J\subset[1,n]} = \oplus_{\alpha.} \Omega(L^{e(\alpha.)}, \alpha.Id + N.).$$

In particular each sub-complex is acyclic when α_j is not zero since then $\alpha_j Id + N_j$ is an isomorphism. This local setting compares to the global case via Grothendieck and Deligne DeRham cohomology results. Let $y \in Y_M^\star$, then

$$(4) \qquad (\mathbf{j}_*\mathcal{L})_y \simeq (\Omega_X^*(Log Y) \otimes \mathcal{L}_X)_y \simeq \Omega(L, D_j + N_j, j \in M)$$

When we start with a $\mathbb{Q}$ local system, $\mathcal{L}_X$ is equal to the canonical extension with residue $\alpha_j \in [0,1[\cap\mathbb{Q}$ such that $a_j = e^{-2i\pi\alpha_j}$. An element v of $L_J^{e(\alpha.)} = L^{e(\alpha.)}, J \subset M$, corresponds to the section $\tilde{v} \otimes \frac{dz_J}{z_J}$ of $(\mathcal{L}_X \otimes \frac{dz_J}{z_J})_y$, where $\frac{dz_J}{z_J} = \Lambda_{j\in J}\frac{dz_j}{z_j}$ by the formula

$$(5) \qquad \tilde{v}(z) = (exp(\Sigma_{j\in J} log z_j(\alpha_j + N_j))).v = \Pi_{j\in J} {z_j}^{\alpha_j} \exp(\Sigma_{j\in J} log z_j N_j).v$$

a basis of L is sent on a basis of $(\mathcal{L}_X)_y$, the endomorphisms $D_j + N_j$ defines corresponding endomorphisms denoted by the same symbols on the image sections ṽ and we have

$$\nabla\tilde{v} = \Sigma_{j\in J}(\alpha_j + N_j).\tilde{v} \otimes \frac{dz_j}{z_j} \tag{6}$$

This description of $(\mathbf{j}_*\mathcal{L})_y$ is the model for the description of the next various perverse sheaves.

1.2 The intermediate extension $\mathbf{j}_{!\star}\mathcal{L}$ Let $(D+N)_J = \Pi_{j\in J}(D_j+N_j)$ denotes a composition of endomorphisms of L, we consider the strict simplicial sub-complex of the DeRham logarithmic complex (2.4) defined by $Im(D+N)_J$ in $L(J) = L$. The associated simple complex will be denoted by

$$IC(L) := s((D+N)_J L, (D. + N.))_{J\subset[1,n]} \tag{7}$$

The intermediate extension $\mathbf{j}_{!*}\mathcal{L}$ of $\mathcal{L}$ is defined by an explicit formula in terms of the stratification [24, §3]. Locally its fiber at a point $y \in Y_M^*$ is given in terms of the above complex

$$\mathbf{j}_{!*}(\mathcal{L})_y \simeq IC(L) \simeq s((D+N)_J L, (D. + N.))_{J\subset M} \tag{8}$$

The corresponding global DeRham description is given as a sub-complex $IC(X,\mathcal{L})$ of $\Omega_X^*(LogY)\otimes \mathcal{L}_X$. The residue of the connection ∇ along each Y_j defines an endomorphism $(\mathcal{D}_j + \mathcal{N}_j)$ on the restriction $\mathcal{L}_{Y_j}$ of $\mathcal{L}_X$, then in terms of a set of n coordinates $y_i, i \in [1,n]$ defining Y_M on an open set U_y containing $y \in Y_M^*$ where we identify M with $[1,n] = M$ ($n =\mid M \mid$) and a section

$$f = \Sigma_{J\subset M, J'\cap M=\varnothing} f_{J,J'}(\tfrac{dy_J}{y_J}) \wedge dy_{J'}$$

where we define $y_J = \Pi_{j\in J} y_j$ and $dy_J = \wedge_{j\in J} dy_j$, we have

$$f \in IC(U_y,\mathcal{L}) \Leftrightarrow \forall J \subset M,\ f_{J,J'}/Y_J \in (\mathcal{D}_J + \mathcal{N}_J)(\mathcal{L}_{Y_J}) \tag{9}$$

A global definition of $IC(X,\mathcal{L})$, using notations of [26], is given as follows. Consider $M \subset I$ and for all $J \subset M$ the families of sub-bundles $(\mathcal{D}_J + \mathcal{N}_J)(\mathcal{L}_{Y_J})$ of $\mathcal{L}_{Y_J}$, then define for each M the sub-module $\mathcal{I}(M)$ of $\mathcal{L}_X$ consisting of sections v satisfying:

$$v \in \mathcal{I}(M) \Leftrightarrow \forall J \subset I, v/Y_J \in (\mathcal{D}_{M\cap J} + \mathcal{N}_{M\cap J})(\mathcal{L}_{Y_J})$$

This submodule is well defined since we have, for $J \subset K$, the inclusion $(\mathcal{D}_J + \mathcal{N}_J)(\mathcal{L}_{Y_J})_{Y_K} \supset (\mathcal{D}_K + \mathcal{N}_K)(\mathcal{L}_{/Y_K})$. Let $\Omega(M)$ be the $\mathcal{O}_X$ sub-exterior algebra of $\Omega_X^*(LogY)$ generated by $\Omega_X^1(LogY_i)$ for i in M. Then we can define

$$IC(X,\mathcal{L}) = \Sigma_{M\subset I}\Omega(M) \otimes \mathcal{I}(M) \subset \Omega_X^*(LogY) \otimes \mathcal{L}_X. \tag{10}$$

In terms of the decomposition (3), since the endomorphism $(\alpha_j Id + \mathcal{N}_j)$ is an isomorphism on $\mathcal{L}_{Y_j}^{e(\alpha.)}$ whenever $\alpha_j \neq 0$, we introduce for each set $\alpha.$ the subset $I(\alpha.) \subset [1,n]$ such that $j \in I(\alpha.)$ iff $\alpha_j = 0$, then for each $J \subset [1,n]$, $(D+N)_J L^{e(\alpha.)} = N_{J\cap I(\alpha.)} L^{e(\alpha.)}$ where $N_{J\cap I(\alpha.)} = \Pi_{j\in J\cap I(\alpha.)} N_j$ (the identity if $J \cap I(\alpha.)$ is empty). Then locally the fiber of $\mathbf{j}_{!*}\mathcal{L}$ at a point $y \in Y_M^*$ is

$$\mathbf{j}_{!*}(\mathcal{L})_y \cong IC(L) \simeq \oplus_{\alpha.} s(N_{J\cap I(\alpha.)} L^{e(\alpha.)})_{J\subset M} \tag{11}$$

Remark: The fiber of the complex $(\Omega_X^*(LogY) \otimes \mathcal{L}_X^{e(\alpha.)})_y$ is acyclic if there exists an index $j \in M$ such that $\alpha_j \neq 0$.

1.3 Hodge filtration, Nilpotent orbits and Purity *Hypothesis 2: Variation of Hodge structures (VHS).* Consider the flat bundle $(\mathcal{L}_X, \nabla)$ in the previous hypothesis and suppose now that $\mathcal{L}_{X^*}$ underlies a VHS that is a polarised filtration by subbundles F of weight m satisfying Griffith's conditions [19].

The *nilpotent orbit* theorem [19], [4], [24], [25], states that F extends to a filtration by subbundles F of $\mathcal{L}_X$ such that the restrictions to open intersections Y_M^* of components of Y underly a variation of mixed Hodge structures $VMHS$ where the weight filtration is defined by the nilpotent endomorphism $\mathcal{N}_M$ defined by the connection.

Local version.

Near a point $y \in Y_M^*$ with $|M| = n$ a neighbourhood of y in the fiber of the normal bundle looks like a disc D^n and the above hypothesis reduces to

Local Hypothesis 2: Nilpotent orbits [4]. Let

$$(L, N_i, F, P, m, i \in M = [1, n]) \tag{12}$$

be defined by the above hypothesis, that is a $\mathbb{Q}$ vector space L with endomorphisms N_i viewed as defined by the horizontal (zero) sections of the connection on $(D^*)^n$, a Hodge structure F on $L^{\mathbb{C}} = L \otimes_{\mathbb{Q}} \mathbb{C}$ viewed as the fiber of the vector bundle $\mathcal{L}_X$ at y (here $y = 0$), a natural integer m the weight and the polarisation P.

The main theorem [4] states that for all $N = \Sigma_{i \in M} \lambda_i N_i$ with $\lambda_i > 0$ in $\mathbf{R}$ the filtration $W(N)$ (with center 0) is independant of N when λ_i vary and $W(N)[m]$ is the weight filtration of a polarised MHS called the limit MHS of weight m ($L, F, W(N)[m]$).

Remark: $W(N)[m]$ is $W(N)$ with indices shifted by m to the right: $(W(N)[m])_r := W_{r-m}(N)$, the convention being a shift to left for a decreasing filtration and to right for an increasing filtration.

We say that $W(N)$ defines a MHS of weight m on L. It is very important to notice that the same orbit underlies other different orbits depending on the intersection of components of Y (here the intersection of the axis of D^n) where the point z near y is considered, in particular $F_z \neq F_y$. In this case when we restrict the orbit to $J \subset M$, we should write

$$(L, N_i, F(J), P, m, i \in J \subset M)$$

Finally we will need the following result [4 p 505]: Let $I, J \subset M$ and let $N_J = \Sigma_{i \in J} N_i$ then $W(N_{I \cup J})$ is the weight filtration of N_J relative to $W(N_I)$

$$\forall j, i \geq 0, N_J^i : Gr_{i+j}^{W(N_{I \cup J})} Gr_j^{W(N_I)} \xrightarrow{\sim} Gr_{j-i}^{W(N_{I \cup J})} Gr_j^{W(N_I)}.$$

II. The weight filtration on the logarithmic complex

Now we want to give the construction of the weight filtration given in [14] and based on a general formula of the intersection complex given by Kashiwara and Kawai[26]. Earlier work in the surface case using ad hoc methods showed that the purity and the decomposition theorems could be obtained out of similar considerations.

To this purpose we introduce a category $S(I) = S$ attached to a set I. We start with a local study, that is to say with the hypothesis of a polarised nilpotent orbit and we describe the weight filtration W on the DeRham complex $\Omega(L, N.)$. In fact the filtration W is defined on a quasi-isomorphic complex and may appear unrealistic at first sight, however the features of the purity theory will appear relatively quickly. First we ask the reader to take some time to get acquainted with the new category $S(I)$ serving as indices for the new complex. The lowest weight is given by the intermediate extension of $\mathcal{L}$ or $IC(L)$, then for the higher weights we need to introduce the complexes $C_r^{KM}L$ for $K \subset M \subset I$ which describe the geometry of the decomposition theorem (§2, II) and the purity theory (§2, I.3) where the proof reflects deep relations between the weifgt filtrations of the various N_i.

1.4 Complexes with indices in the category $S(I)$**.** We introduce a category $S(I) = S$ attached to a set I, whose objects consist of sequences of increasing subsets of I of the following form:

$$(s_.) = (I = s_1 \supsetneqq s_2 \ldots \supsetneqq s_p \neq \emptyset), \ (p > 0)$$

Substracting a subset s_i from a sequence $s_.$ defines a morphism $\delta_i(s.) : (s. - s_i) \to s.$ and more generally $Hom(s'., s.)$ is equal to one element iff $(s.') \leq (s.)$. We write $s. \in S$ and define its degree $\mid s. \mid$ as the number of subsets s_i in $s_.$ or length of the sequence.

Correspondence with an open simplex. If $I = \{1, \ldots, n\}$ is finite, $S(I)$ can be realised as a barycentric subdivision of a simplex of dimension $n - 1$, a subset K corresponding to the barycenter of the vertices in K and a sequence of subsets to the simplex defined by the vertices associated to the subsets. Since all sequences contain I, all corresponding simplices must have the barycenter as vertex, that is: $S(I)$ define a simplicial object computing the n^{th} homology group with closed support of the open simplex. This remark leads us to the next definition.

Simplicial complex defined by complexes with indices in S(I). An algebraic variety over a fixed variety X with indices in S is a covariant functor $\Pi : X_{s.} \to X$ for $s. \in S$. An abelian sheaf (resp. complex of abelian sheaves)$\mathcal{F}$ is a family of abelian sheaves (resp. complex of abelian sheaves) $\mathcal{F}_{s.}$ over $X_{s.}$ and functorial morphisms $\mathcal{F}_{s.} \to \mathcal{F}_{s.'}$ for $(s.') \leq (s.)$.

The direct image of an abelian sheaf (resp. complex of sheaves) denoted $\Pi_* \mathcal{F}$ or $s(\mathcal{F}_{s.})_{s. \in S}$ is the simple complex(resp.simple complex associated to a double complex) on X:

$$\Pi_* \mathcal{F} := \oplus_{s. \in S} (\Pi_* \mathcal{F}_{s.})[\mid s. \mid - \mid I \mid], \ d = \Sigma_{i \in [1, |s.|]} (-1)^i \mathcal{F}(\delta_i(s.)).$$

The variety X defines the constant variety $X_{s.} = X$. The constant sheaf $\mathbb{Z}$ lifts to a sheaf on $X_{s.}$ such that the diagonal morphism : $\mathbb{Z}_X \to \oplus_{|(s.)|=|I|} \mathbb{Z}_{X_{s.}}$ (that is : $n \in \mathbb{Z} \to (\ldots, n_{s.}, \ldots) \in \oplus_{|(s.)|=|I|} \mathbb{Z}$ defines a quasi-isomorphism $\mathbb{Z}_X \cong \Pi_* \Pi^* (\mathbb{Z}_X)$. This is true since $S(I)$ is isomorphic to the category defined by the barycentric subdivision of an open simplex of dimension $\mid I \mid -1$.

1.5 Local definition of the weight filtration.

Our hypothesis here consists again of the nilpotent orbit $(L, (N_i)_{i \in M}, F, m, P)$ of weight m and polarisation P and the corresponding filtrations $(W^J)_{J \subset M}$.

We will use the category $S(M)$ attached to M whose objects consist of sequences of decreasing subsets of M of the form $(s_.) = (M = s_1 \supsetneqq s_2 \ldots \supsetneqq s_p \neq \emptyset)$, $p > 0$.

In this construction we will need double complexes, more precisely complexes of the previously defined exterior complexes. They correspond to objects with indices in the category $M_.^+ \times S(M)$ where the objects of $M_.^+$ are the subsets $J \subset M$ including the empty set. Geometrically M corresponds to a normal section to Y_M^* in X and J to $\wedge_{i \in J} dz_i$ in the exterior DeRham complex written as $s(L_J)_{J\subset M}$ on the normal section to Y_M^* and the decomposition $M_{\cdot}^+ \simeq (M-K)_{\cdot}^+ \times K_{\cdot}^+$ corresponds to the isomorphism $\mathbb{C}^M \simeq \mathbb{C}^{(M-K)} \times \mathbb{C}^K$.

Notations. For each $s. \in S(M)$, $J \subset M$ and integer r we define $a_{s_\lambda}(J,r) = \mid s_\lambda \mid -2 \mid s_\lambda \cap J \mid +r$, and for all $(J,s.) \in M_{\cdot}^+ \times S(M)$ the vector spaces

$$W_r(J,s.) := \bigcap_{s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r)} L, \quad F^r(J,s.) := F^{r-|J|}L, \ \ (a_{s_\lambda}(J,r) = \mid s_\lambda \mid -2 \mid s_\lambda \cap J \mid +r)$$

where W^{s_λ} is centered at 0, then we define on the DeRham complex $\Omega(L,N.)$, the filtrations by sub-complexes $W(s.)$ (weight) and $F(s.)$ (Hodge) as

$$(13) \qquad W_r(s.) = s(W_r(J,s.))_{J\subset M}, \quad F^r(s.) := s(F^r(J,s.))_{J\subset M}$$

and finally

Definition: The weight (centered at zero) and Hodge filtrations on the combinatorial DeRham complex $\Omega^\star L = s(\Omega(L,N.))_{s.\in S(M)}$ are defined by "summing" over s. as:

(14)

$$F^r(\Omega^\star L) := s(F^r(s.))_{s.\in S(M)} \subset s(L(s.))_{s.\in S(M)}, \ \mathcal{W}_r(\Omega^\star L) := s(W_r(s.))_{s.\in S(M)} \subset s(L(s.))_{s.\in S(M)}$$

Ex in dimension 2.

Let $W^{1,2} = W(N_1+N_2), W^1 = W(N_1)$ and $W^2 = W(N_2)$, the weight $\mathcal{W}_r$ is a double complex where the first line is the direct sum for $\{1,2\} \supsetneq 1$ and $\{1,2\} \supsetneq 2$ of:

$$W^{1,2}_{r+2} \cap W^1_{r+2} \overset{(N_1,\, N_2)}{\longrightarrow} W^{1,2}_r \cap W^1_r \oplus W^{1,2}_r \cap W^1_{r+2} \overset{(-N_2,\, N_1)}{\longrightarrow} W^{1,2}_{r-2} \cap W^1_r$$

and

$$W^{1,2}_{r+2} \cap W^2_{r+2} \overset{(N_1,\, N_2)}{\longrightarrow} W^{1,2}_r \cap W^2_{r+2} \oplus W^{1,2}_r \cap W^2_r \overset{(-N_2,\, N_1)}{\longrightarrow} W^{1,2}_{r-2} \cap W^2_r$$

The second line for $\{1,2\}$ is

$$W^{1,2}_{r+2} \overset{(N_1,\, N_2)}{\longrightarrow} W^{1,2}_r \oplus W^{1,2}_r \overset{(-N_2,\, N_1)}{\longrightarrow} W^{1,2}_{r-2}.$$

which reduces to the formula in [26] for $r = m$.

1.6 The Complexes $C_r^{KM}L$ and $C_r^K L$.

To study the graded part of the weight, we need to introduce the following subcategories: For each subset $K \subset M$, let $S_K(M) = \{s. \in S(M) : K \in s.\}$ and consider the isomorphism of categories:

$$S(K) \times S(M-K) \tilde{\to} S_K(M), \ (s., s'_{\cdot}) \to (K \cup s'_{\cdot}, s.)$$

We consider the vector space with indices $(J,s.) \in M.^+ \times S_K M$, and its associated complex

$$(15) \qquad C_r(J,s.)L := \bigcap_{K \neq s_\lambda \in s.} W^{s_\lambda}_{a_\lambda(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L, \quad C_r(s.)L := s(C_r(J,s.)L)_{J\in M_{\cdot}^+}$$

we define as well $C_r(J,s.)N_J L$ and $C_r(J,s.)(L/N_J L)$ by replacing L with $N_J L$ and $L/N_J L$, then $C_r(s.)ICL := s(C_r(J,s.)N_J L)_{J\in M_{\cdot}^+}$ and $C_r(s.)QL := s(C_r(J,s.)L/N_J L)_{J\in M_{\cdot}^+}$

Definition: For $K \subset M$ the complex $C_r^{KM}L$ is defined as

(16)

$$C_r^{KM}L := s(C_r(s.)L)_{s.\in S_K(M)}, \ C_r^{KM}ICL := s(C_r(s.)ICL)_{s.\in S_K(M)}, \ C_r^{KM}QL := s(C_r(s.)QL)_{s.\in S_K(M)}$$

In the case $K = M$ we write $C_r^K L$ (resp. $C_r^K ICL$, $C_r^K QL$ for $C_r^{KK}L$ (resp. ICL, QL instead of L

$$(17) \qquad C_r^K L = s(C_r(J,s.))_{(J,s.)\in K_{\cdot}^+ \times S(K)} = s((\cap_{K \supsetneq s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L)_{(J,s.)\in K_{\cdot}^+ \times S(K)}$$

2§. Main theorems on the properties of the weight filtration

In this section we aim to prove that the filtration is actually the weight of what would be in the proper case a mixed Hodge complex in Deligne's terminology, that is the induced filtration by F on the graded parts $Gr^{\mathcal{W}}$ of $\mathcal{W}$ is a Hodge filtration. For this we need to decompose the complex as a direct sum of intermediate extensions of variations of Hodge structures (which has a meaning locally) whose cohomologies are pure Hodge structures [5] and [24] in the proper case. This is done in the following three subsections. In the first we prove a key lemma that apply to prove the purity of the complex $C_r^K L$. Once this purity result is established, we can easily prove in the second subsection the decomposition theorem after a careful study of the category of indices $S(I)$. In the third subsection we give the global statements for a filtered combinatorial logarithmic complex. For this we use the above local decomposition to obtain a global decomposition of the graded weight into intermediate extensions of polarised VHS on the various intersections of components of Y. This last statement uses the formula announced by Kashiwara and Kawai [26] that we prove since we have no reference for its proof.

I. Purity of the cohomology of the complex $C_r^K L$

In this subsection we introduce the fiber of the variations of Hodge structures needed in the decomposition of $Gr^{\mathcal{W}}$. The result here is the fundamental step in the general proof. The plan of this subsection is as follows. First we start with a key lemma relating the various relative monodromy weight fltrations (centered at zero) associated to a nilpotent orbit L; N_i is compatible with $W(N_j)$ for $i \neq j$ but shift Hodge filtration by -1, hence it is not clear whether it is strict, however we need technical results to establish the purity and decomposition properties and this key lemma provides what seems to be the elementary property at the level of a nilpotent orbit that leads to the decomposition. Second we present a set of elementary complexes. Finally we state the purity result on the complexes $C_r^K L$ which behave as a direct sum of elementary complexes.

2.1 Properties of the relative weight filtrations

Given the nilpotent orbit we may consider various filtrations $W^J = W(\Sigma_{i \in J} N_i)$ for various $J \subset M$. They are centered at 0, preserved by N_i for $i \in M$ and shifted by -2 for $i \in J$: $N_i W_r^J \subset N_i W_{r-2}^J$. We will need more subtle relations between these filtrations that we discuss in this subsection.

Key lemma (Decomposition of the relative weight filtrations) : Let $(L, N_i, i \in [1,n], F)$ be a polarised nilpotent orbit and let $W^i := W(N_i)$ (all weights centered at 0), then :
i) For each subset $A = \{i_1, \dots, i_j\} \subset [1,n]$, of length $\mid A \mid = j$

$$Gr_r^{W^A} L \simeq \oplus_{m. \in X_r^A} Gr_{m_{i_j}}^{W^{i_j}} \cdots Gr_{m_{i_1}}^{W^{i_1}} L \text{ where } X_r^A = \{m. \in \mathbb{Z}^j : \Sigma_{i_l \in A} m_{i_l} = r\}$$

and if $A = B \cup C$

$$Gr_{b+c}^{W^A} Gr_c^{W^C} L \simeq Gr_b^{W^B} Gr_c^{W^C} L \simeq Gr_{b+c}^{W^A} Gr_b^{W^B} L$$

ii) Let N_i' denotes the restriction of N_i to $Gr_c^{W^C}$ and $N_B' = \Sigma_{i \in B} N_i'$, then W_b^B induces $W_b(N_B')$ on $Gr_c^{W^C}$.
iii) In particular the repeated graded objects in (i) do not depend on the order of the elements in A.

Remark: This result give relations between various weight filtrations in terms of the elementary ones $W^i := W(N_i)$ and will be extremely useful in the study of the properties of the weight filtration on the higher direct image. For example for $N = N_1 + \cdots + N_n$, there exists a canonical decomoposition

$$Gr_r^{W(N)} L = \oplus_{m.\in X_r} Gr_{m_n}^{W^n} \cdots Gr_{m_1}^{W^1} L \text{ where } X_r = \{m. \in \mathbb{Z}^n : \Sigma_{i\in[1,n]} m_i = r\}.$$

The proof by induction on n is based on the following important result of Kashiwara [25, thm 3.2.9, p 1002]:

Let (L, N, W) consists of a vector space endowed with an increasing filtration W preserved by a nilpotent endomorphism N on L and suppose that the relative filtration $M = M(N, W)$ exists, then there exists a canonical decomposition:

$$Gr_l^M L = \oplus_k Gr_l^M Gr_k^W L.$$

Proof of the key lemma. To stress the properties of commutativity of the graded operation for the filtrations, we prove first

Sublemma: For all subsets $[1, n] \supset A \supset \{B, C\}$, the isomorphism of Zassenhaus $Gr_b^{W^B} Gr_c^{W^C} L \simeq Gr_c^{W^C} Gr_b^{W^B} L$ is an isomorphism of MHS with weight filtration (up to a shift) $W = W^A$ and Hodge filtration $F = F_A$, hence compatible with the third filtration W^A or F_A.

Proof: Recall that both spaces $Gr_b^{W^B} Gr_c^{W^C}$ and $Gr_c^{W^C} Gr_b^{W^B}$ are isomorphic to $W_b^B \cap W_c^C$ modulo $W_c^C \cap W_{b-1}^B + W_b^B \cap W_{c-1}^C$. In this isomorphism a third filtration like F_A (resp. W^A) is induced on one side by $F'_k = (F_A^k \cap W_c^C) + W_{c-1}^C$ (resp. $W'_k = W_k^A \cap W_c^C) + W_{c-1}^C$) and on the second side by $F''_k = (F_A^k \cap W_b^B) + W_{b-1}^B$ (resp. $W''_k = W_k^A \cap W_b^B) + W_{b-1}^B$). We introduce the third filtration $F'''_k = F_A^k \cap W_b^B \cap W_c^C$ (resp. $W'''_k = W_k^A \cap W_b^B \cap W_c^C$) and we notice that all these spaces are in the category of MHS, hence the isomorphism of Zassenhaus which must be strict, is compatible with the third filtrations induced by F_A (resp. W^A).

Proof. i) Let $A \subset [1, n]$ and $i \in A$, then W^A exists and induces the relative weight filtration for N_i with respect to $W^{(A-i)}$. Then we have by Kashiwara's result $Gr_l^{W^A} L = \oplus_k Gr_l^{W^A} Gr_k^{W(A-i)} L$. Let us attach to each point (k, l) in the plane the space $Gr_l^{W^A} Gr_k^{W(A-i)} L$ and let $M_j = \oplus_l Gr_l^{W^A} Gr_{l-j}^{W(A-i)} L$ be the direct sum along indices on a parallel to the diagonal (shifted by j) in the plane (k, l). Then we have for $j > 0$

$$(N_i)^j : Gr_{k+j}^{W^A} Gr_k^{W(A-i)} L \simeq Gr_{k-j}^{W^A} Gr_k^{W(A-i)} L, \ (N_i)^j : M_j \simeq M_{-j}.$$

This property leads us to introduce the space $V = \oplus_l Gr_l^{W^A} L \simeq \oplus_{l,k} Gr_l^{W^A} Gr_k^{W(A-i)} L$, then N_i on L extends to a nilpotent endomorphism on V, $N_i : V \to V$ inducing $N_i : Gr_l^{W^A} L \to Gr_{l-2}^{W^A} L$ on each $l-$component of V. We consider on V two increasing filtrations $W'_s := \oplus_{l-k\leq s} Gr_l^{W^A} Gr_k^{W(A-i)} L$ and $W''_s := \oplus_l W_s^i Gr_l^{W^A} L$. Then N_i shift these filtrations by -2. In fact $N_i : W'_s \to W'_{s-2}$ sends $Gr_l^{W^A} Gr_k^{W(A-i)} L$ to $Gr_{l-2}^{W^A} Gr_k^{W(A-i)} L$ and $(N_i)^j$ induces an isomorphism $Gr_j^{W'} V \simeq Gr_{-j}^{W'} V$. As well we have an isomorphism $Gr_j^{W''} V \simeq Gr_{-j}^{W''} V$, since $(N_i)^j : (Gr_j^{W^i} L, W^A, F_A) \simeq (Gr_{-j}^{W^i} L, W^A, F_A)$ is an isomorphism of MHS up to a shift in indices, hence strict on W^A and F_A and induces an isomorphism $Gr_j^{W^i} Gr_l^{W^A} \simeq Gr_{-j}^{W^i} Gr_{l-2j}^{W^A}$. Then these two filtrations W'_s and W''_s are equal by uniqueness of the weight filtration of N_i on V, that is

$$W''_s = \oplus_{k,l} W_s^i Gr_l^{W^A} Gr_k^{W(A-i)} L = W'_s = \oplus_{l-k\leq s} Gr_l^{W^A} Gr_k^{W(A-i)} L$$

that is $W^i_s Gr^{W^A}_l Gr^{W^{(A-i)}}_k L = Gr^{W^A}_l Gr^{W^{(A-i)}}_k L$ if $l-k \leq s$ and $W^i_s Gr^{W^A}_l Gr^{W^{(A-i)}}_k L = 0$ otherwise, or

$$Gr^{W^A}_l Gr^{W^i}_j L \simeq Gr^{W^A}_l Gr^{W^i}_j Gr^{W^{A-i}}_{l-j} L, \text{and for all } l \neq k+k',\ Gr^{W^A}_l Gr^{W^i}_k Gr^{W^{(A-i)}}_{k'} L \simeq 0.$$

In other words: W^A induces a trivial filtration on $Gr^{W^i}_k Gr^{W^{(A-i)}}_{k'} L$ of weight $k+k'$ that is

$$Gr^{W^A}_l L \simeq \oplus_k Gr^{W^A}_l Gr^{W^{A-i}}_k L \simeq \oplus_k Gr^{W^A}_l Gr^{W^{A-i}}_k Gr^{W^i}_{l-k} L \simeq \oplus_k Gr^{W^{(A-i)}}_k Gr^{W^i}_{l-k} L.$$

Now if we suppose by induction on length of A, the decomposition true for $A-i$, we deduce easily the decomposition for A from the above result.
ii) We restate here the property of the relative monodromy for W^A with respect to W^C.
iii)In the proof above we can start with any i in A, hence the decomposition is symmetric in elements in A. It follows that the graded objects of the filtrations $W^i, W^r, W^{\{i,r,j\}}$ commute and since W^j can be expressed using these filtrations, we deduce that W^i, W^r, W^j also commute, for example: $Gr^{W^{\{i,r,j\}}}_{a+b+c} Gr^{W^{\{i,j\}}}_{a+b} Gr^{W^i}_a \simeq Gr^{W^r}_c Gr^{W^j}_b Gr^{W^i}_a$ is symmetric in i, j, r.
Corollary: The morphism N_i induces for all j, exact sequences for all integers r :
$0 \to W^j_r \cap ker N_i \to W^j_r L \to W^j_r \cap N_i L \to 0$ *and* $0 \to W^j_r \cap N_i L \to W^j_r L \to W^j_r (L/N_i L) \to 0.$

Proof: N_i is strict for W^i and $W^{\{i,j\}}$ hence we have the above exact sequences for $Gr^{W^{\{i,j\}}}_{a+b} Gr^{W^i}_b = Gr^{W^j}_a Gr^{W^i}_b$.

2.2 Elementary complexes The proof of the purity uses the following simplicial vector spaces. For each $J \subset [1,n]$, let

$$K((m_1,\cdots,m_n),J)L = Gr^{W^n}_{m_n-2|n\cap J|} \cdots Gr^{W^r}_{m_r-2|r\cap J|} \cdots Gr^{W^1}_{m_1-2|1\cap J|} L$$

(resp. for $L/N_J L$ and $N_J L$). For all $i \notin J$, the endomorphism N_i induces a morphism denoted also $N_i : K((m_1,\cdots,m_n),J)L \to K((m_1,\cdots,m_n),J\cup i)L$, (resp. for $L/N_J L$ and $N_J L$ instead of L), then we consider the following elementary complexes defined as simple associated complexes:

$$K(m_1,\cdots,m_n)L := s(K((m_1,\cdots,m_n),J)L, N_i)_{J\subset[1,n]} \tag{18}$$

(resp. $K(m_1,\cdots,m_n)QL := s(K((m_1,\cdots,m_n),J)L/N_J L, N_i)_{J\subset[1,n]}$
and $K(m_1,\cdots,m_n)ICL := s(K((m_1,\cdots,m_n),J)N_J L, N_i)_{J\subset[1,n]}$).
*Proposition: i) For any $((m_1,\cdots,m_n) \in \mathbb{Z}^n$ let $J(m.) = \{i \in [1,n] : m_i \geq 1\}$, then the cohomology of an elementary complex $K(m_1,\cdots,m_n)L$ is a subquotient of $K((m_1,\cdots,m_n),J(m.))L$, hence concentrated in degree $\mid J(m.) \mid$. Moreover it vanishes iff there exists at least one $m_i = 1$. More precisely, if no $m_i = 1$, the cohomology is isomorphic to $K((m_1,\cdots,m_n),J(m.))[(\cap_{i\notin J(m.)}(ker N_i : L/(\Sigma_{j\in J(m.)} N_j L) \to L/(\Sigma_{j\in J(m.)} N_j L))]$,
moreover this object is symmetric in the operations kernel and cokernel and is isomorphic to $K((m_1,\cdots,m_n),J(m.))[(\cap_{\{i:m_i<1\}} ker N_i)/(\Sigma_{\{j:m_j>1\}} N_j(\cap_{\{i:m_i<1\}} ker N_i))]$,
that is at each process of taking $Gr^{W^i}_{m_i}$ we apply the functor ker if $m_i \notin J(m.)$ and coker if $m_i \in J(m.)$.
ii) If there exists $m_i > 0$, then $K(m_1,\cdots,m_n)ICL \simeq 0$, hence
$K(m_1,\cdots,m_n)L \simeq K(m_1,\cdots,m_n)QL$.
iii) If all $m_i \leq 0$, then*

$K(m_1,\cdots,m_n)ICL \simeq K(m_1,\cdots,m_n)L \simeq Gr^{W^n}_{m_n}\cdots Gr^{W^1}_{m_1}\cap_{i\in[1,n]} ker(N_i : L\to L)$.

Proof. i) It is enough to notice that N_i is injective if $m_i > 0$, surjective if $m_i < 2$ and bijective if $m_i = 1$. In fact, given N_i we can view $K(m_1,\cdots,m_n)L$ as the cone over
$N_i : K(m_1,\cdots,\widehat{m_i},\cdots,m_n)(Gr^{W^i}_{m_i}L) \xrightarrow{N_i} K(m_1,\cdots,\widehat{m_i},\cdots,m_n)(Gr^{W^i}_{m_i-2}L)$.
Hence if $m_i > 0$ (resp. $m_i < 2$), that is N_i is injective on $Gr^{W^i}_{m_i}$ (resp. surjective onto $Gr^{W^i}_{m_i-2}$),
$K(m_1,\cdots,m_i,\cdots,m_n)L \cong K(m_1,\cdots,\widehat{m_i},\cdots,m_n)(Gr^{W^i}_{m_i-2}(L/N_iL))[-1]$
(resp. $K(m_1,\cdots,\widehat{m_i},\cdots,m_n)(Gr^{W^i}_{m_i}(kerN_i : L\to L))$
where $K(m_1,\cdots,\widehat{m_i},\cdots,m_n)$ is viewed for the nilpotent orbit $Gr^{W^i}_{m_i-2}L/N_iL$
(resp.$Gr^{W^i}_{m_i}(kerN_i : L\to L)$) with the nilpotent endomorphisms N'_j induced by N_j for $j\neq i$.
ii) $K(m_1,\cdots,m_n)ICL$ can be viewed as a cone over

$N_i : K(m_1,\cdots,\widehat{m_i},\cdots,m_n)IC(Gr^{W^i}_{m_i}L) \to K(m_1,\cdots,\widehat{m_i},\cdots,m_n)IC(Gr^{W^i}_{m_i-2}N_iL)$

where $K(m_1,\cdots,\widehat{m_i},\cdots,m_n)(Gr^{W^i}_{m_i}L$ is viewed for the $(n-1)-dim$ nilpotent orbit $Gr^{W^i}_{m_i}L$. If $m_i > 0$, then N_i is an isomorphism.
iii) If $m_i \leq 0$, then the above N_i is just surjective and $K(m_1,\cdots,m_n)ICL$ is isomorphic to $K(m_1,\cdots,\widehat{m_i},\cdots,m_n)IC(Gr^{W^i}_{m_i}ker(N_i : L\to N_iL)$, then (iii) follows by induction on i.

2.3 Main result *Theorem (Purity). Let L be a polarised nilpotent orbit (local hypothesis 2 (§1, I.3)), then the complexes C^K_rL in (§1, II.3) where we suppose $K = M$ of length $\mid K\mid = n$, satisfy the following properties*

i) Let $r > 0$ and $T(r) = \{(m_1,\cdots,m_n)\in \mathbf{N}^n : \forall i\in K, m_i\geq 2$ and $\Sigma_{j\in K}m_j = \mid K\mid + r\}$ then the cohomology of the complex C^K_rL is isomorphic to that of the following complex

$$C^K_rL \cong C(T(r)) \simeq \oplus_{(m_1,\cdots,m_n)\in T(r)}K(m_1,\ldots,m_n)L$$

In particular its cohomology, concentrated in degree $\mid K\mid$, is isomorphic to

$Gr^{W^K}_{r-|K|}[L/(\Sigma_{i\in K}N_iL)] \simeq \oplus_{(m_1,\cdots,m_n)\in T(r)}Gr^{W^n}_{m_n-2}\cdots Gr^{W^i}_{m_i-2}\cdots Gr^{W^1}_{m_1-2}[L/(\Sigma_{i\in K}N_iL)]$

it is a polarised Hodge structure of weight $r+m-\mid K\mid$ with the induced filtrations W^K (shifted by m) and F^K. Moreover, if $r = 0$, the complex C^K_rL is acyclic.

ii) Dually, for $r < 0$, let $T'(r) = \{(m_1,\cdots,m_n)\in\mathbb{Z}^n : \forall i\in K, m_i\leq 0, \Sigma_{j\in K}m_j = \mid K\mid + r\}$, then the complex C^K_rL is isomorphic to the following complex

$$C^K_rL \cong C(T'(r))[1-\mid K\mid] \simeq \oplus_{(m_1,\cdots,m_n)\in T'(r)}K(m_1,\ldots,m_n)L[1-\mid K\mid]$$

In particular its cohomology, concentrated in degree $\mid K\mid -1$, is isomorphic to
$Gr^{W^K}_{r+|K|}[(\cap_{i\in K}(kerN_i : L\to L] \simeq \oplus_{(m_1,\cdots,m_n)\in T'(r)}Gr^{W^n}_{m_n}\cdots Gr^{W^i}_{m_i}\cdots Gr^{W^1}_{m_1}[(\cap_{i\in K}(kerN_i : L\to L]$
it is a polarised Hodge structure of weight $r+m+\mid K\mid$ with the induced filtrations W^K (shifted by m) and F^K .

iii) The complex C^K_rL is quasi-isomorphic to C^K_rQ (16) for $r\geq 0$ and to the complex C^K_rICL (16) for $r\leq 0$.

Remark: If $r\in[1,\mid K\mid -1]$, $T(r)$ is empty and C^K_rL is acyclic. If $r\in[-\mid K\mid +1, 0]$, $T'(r)$ is empty and C^K_rL is acyclic. In all cases C^K_rL appears in $Gr^{\mathcal{W}}\Omega^*(\mathcal{L})$.

Proof: The important fact used here is the particular decomposition for a nilpotent orbit of the relative filtrations, that is the isomorphism, functorial for the differentials of C^K_rL

$$Gr^{W^K}_{a_{K(J,r)}}(\cap_{K\supsetneq s_\lambda\in s.}W^{s_\lambda}_{a_\lambda(J,r-1)})L \simeq \oplus_{m.\in X(J,s.,r)}Gr^{W^n}_{m_n}\cdots Gr^{W^i}_{m_i}\cdots Gr^{W^1}_{m_1}L,$$

where for all $(J, s.) \in K^+_. \times S(K)$,

$$X(J, s., r) = \{m. \in \mathbb{Z}^n : \Sigma_{i\in K} m_i = a_K(J, r) \text{ and } \forall s_\lambda \in s., \Sigma_{i\in s_\lambda} m_i \leq a_\lambda(J, r-1)\}$$

In particular, if we define for $J = \varnothing, X(s., r) = X(\varnothing, s., r)$,
$X(s., r) = \{m. \in \mathbb{Z}^n : \Sigma_{i\in K} m_i = \mid K \mid +r \text{ and } \forall s_\lambda \in s., \Sigma_{i\in s_\lambda} m_i \leq \mid s_\lambda \mid +r-1\}$,
the complex $C_r^K(s.)$ splits as a direct sum of elementary complexes

$$C_r^K(s.) \simeq \oplus_{X(s.,r)} K(m_1, \dots, m_n).$$

On the otherside for a fixed $J \subset K$ we consider the complex defined by the column of vector spaces

$$C_r^K(J) = s[C_r^K(J, s.)]_{s.\in S(K)} \cong s[\oplus_{m.\in X(J,s.,r)} K((m_1, \dots, m_n), J)]_{s.\in S(K)}.$$

We want to show that each column is an acyclic complex if $((m_1, \cdots, m_n) \notin T(r)$ and a resolution of $K((m_1, \dots, m_n), J)$ otherwise (if $((m_1, \cdots, m_n) \in T(r))$.
This is just a combinatorial study, which helped to formulate the statement after an explicit study of the theorem in case $n = 2$ and $n = 3$. We give a proof based on the following facts:

Lemma: i) Let $r \geq 0$*, then for each* $i \in K$ *the sub-complexes*

$$C_r^K(W_1^i L) \simeq s(\oplus_{m.\in X(s.,r) \text{ and } m_i<2} K(m_1, \cdots, m_n)L)_{s.\in S(K)}$$

as well $C_r^K(IC(W_1^i L))$ *are acyclic. More precisely for* $r > 0$*, each column* $C_r^K(J)(W_1^i L) = s(C_r^K(J, s.)(W_1^i L)_{s.\in S(K)}$ *is acyclic.*
ii) Dually, for $r < 0$ *and for each* $i \in K$ *the quotient complexes*

$$C_r^K(L/W_1^i L) \simeq s(\oplus_{m.\in X(s.,r) \text{ and } m_i\geq 2} K(m_1, \cdots, m_n)L)_{s.\in S(K)}$$

are acyclic column by column.
Proof: We distinguish in $S(K)$ the subcategory S_i'' whose objects $s.$ contain K and $K - i$. The complement $S(K) - S_i''$ is a full subcategory of $S(K)$ since if we delete a subset in $s. \in S(K) - S_i''$ we still have an object in this subcategory. Hence the sum $C_i'' := s[C_r^K(s.)]_{s.\in S(K)-S_i''}$ is a subcomplex of $C_r^K L$ whose quotient complex is $C(S_i'') := s[C_r^K(s.)]_{s.\in S_i''}$.

Dually, we consider $S_i''' \subset S_i'$ whose objects $s.$ contain $\{i\}$. The complement $S(K) - S_i'''$ is a full subcategory of $S(K)$ since if we delete a subset in $s. \in S(K) - S_i'''$ we still have an object in this subcategory. Hence the sum $C_i''' := s[C_r^K(s.)]_{s.\in S(K)-S_i'''}$ is a subcomplex of $C_r^K L$ whose quotient complex is $C(S_i''') := s[C_r^K(s.)]_{s.\in S_i'''}$.

Sublemma: In the exact sequence

$$0 \to C_i''(W_1^i L) \to C_r^K(W_1^i L)[1] \to C(S_i'')(W_1^i L)[1] \to 0$$

the complexes at each side are acyclic (column by column if $r > 0$*), so is the middle complex. Dually, in the exact sequence*

$$0 \to C_i'''(L/W_0^i L) \to C_r^K(L/W_0^i L)[1] \to C(S_i''')(L/W_0^i L)[1] \to 0$$

the complexes at each side are acyclic (column by column if $r < 0$*), so is the middle complex.*

Proof. We write $s.$ as $(s.' \supset s_v \cup i \supset s_{v-1} \supset s.'')$ where$i \notin s_{v-1}$ and distinguish in the objects of $S(K)$ two families : S_i whose objects are defined by the $s.$ satisfying $s_v \supsetneqq s_{v-1}$ (including the case $s_{v-1} = \varnothing$) and S'_i whose objects $s.$ satisfy $s_{v-1} = s_v$ (including the case $s_{v-1} = \varnothing$, that is $s_v \cup i = i$).

We form the complexes $C(S'_i - S''_i) := s[C^K_r(s.)]_{s.\in S'_i - S''_i}$ (resp. $C(S_i) := s[C^K_r(s.)]_{s.\in S_i}$. If we delete $s_v \cup i$ in $s. \in S'_i - S''_i$ we get an element in S_i . then removing $s_v \cup i$ can be viewed as a morphism $d_{s_v \cup i} : C(S'_i - S''_i) \to C(S_i)$ and the cone over this $d_{s_v\cup i}$ is equal to $C''_i[1]$.

Now, if we reduce the construction to $W^i_1 L$ and if $s.$ is an object of S'_i, the condition $m. \in X(s., r)$ associated to $s.$ when $s_{v-1} = s_v \neq \varnothing$ (resp. $s_v \cup i$) is $\Sigma_{j\in s_v} m_j \leq\mid s_v \mid +r-1$ (resp. $m_i + \Sigma_{j\in s_v} m_j \leq\mid s_v \cup i \mid +r-1$), but precisely when $m_i < 2$ (that is in $W^i_1 L$ the condition for s_v is equivalent to the union of the conditions for s_v and $s_v \cup i$, that is $d_{s_v\cup i}$ induces an isomorphism for such object in $S'_i - S''_i$.

When $s_{v-1} = \varnothing$ and if $r > 0$, the condition $m_i \leq r$ is irrelevant since already $m_i \leq 1$ and $r \geq 1$, so that $d_{s_v\cup i}$ induces an isomorphism for all objects in $S'_i - S''_i$ (if $r = 0$ the difference are complexes $K(m_1, \ldots, m_n)$ with some $m_i = 1$, hence acyclic).

Dually, S'''_i whose objects $s.$ satisfy $s_{v-1} = s_v = \varnothing$, that is $i \in s.$ is contained in S'_i , then the cone over the morphism $d_{s_{v-1}} : C(S'_i - S'''_i)(L) \to C(S_i)(L)$ is isomorphic to $C'''_i[1]$. When $m_i > 0$ the condition for $s_v \cup i$ is equivalent to the union of the conditions for $s_v = s_{v-1} \neq \varnothing$ and $s_v \cup i$, that is $d_{s_{v-1}}$ induces an isomorphism for such object in $S'_i - S'''_i$.

Finally we prove $C(S''_i)(W^i_1 L) = 0$. In fact the conditions for K and $K - i$ in any $s. \in S''_i$ are $\Sigma_{j\in K} m_j =\mid K \mid +r$ (resp. $\Sigma_{j\in (K-i)} m_j \leq\mid K - i \mid +r-1$), hence we get by difference $m_i \geq 2$.

Dually,$C(S'''_i)(L/W^i_0 L) = 0$. In fact the condition $m_i \leq r$ corresponding to $i \in s. \in S'''_i$ is not compatible with $m_i > 0$ when $r < 0$ which ends the proof of the sublemma.

On the otherside, it is easy to check that

Lemma: The complex $C(T(r))$ is contained in each $C^K_r(s.)$ that is $T(r) \subset X(s., r)$. Dually, the complex $C(T'(r))$ is contained only in $C^K_r(s.)$ for $s. = K$.

We check the condition $\forall s_\lambda \in s., \Sigma_{i\in s_\lambda} m_i \leq\mid s_\lambda \mid +r-1\}$ for all $m. \in T(r)$ by induction: $\Sigma_{j\in K} m_j =\mid K \mid +r \Rightarrow \forall A = K - k \subset K, \Sigma_{j\in A} m_j \leq\mid A \mid +r-1$ by substracting $m_k > 1$. If this is true for all $A :\mid A \mid= a$ then $\forall B = A - k \subset A, \Sigma_{j\in B} m_j \leq\mid B \mid +r-2$ as well.

Dually, the condition for $K, \Sigma_{j\in K} m_j =\mid K \mid +r \Rightarrow \forall A = K-k \subset K, \Sigma_{j\in A} m_j >\mid A \mid +r-1$ by substracting $m_k < 1$. If this is true for all $A :\mid A \mid= a$ then $\forall B = A-k \subset A, \Sigma_{j\in B} m_j >\mid B \mid +r-1$ as well.

Finally, we form the complex $C(r) = s(C(T(r))_{s.\in S(K)} \subset C^K_r L$ and we deduce

Lemma: i) The quotient $(C^K_r L/C(r)) \cong 0$ is acyclic.

ii) For any maximal index $s. \in S(K)$, the embeddding $C(T(r)) \subset C^K_r(s.)$ induces a quasi-isomorphism $C(T(r)) \cong C^K_r L$.

iii) Dually, the quotient $[C^K_r L/C(T'(r))(s. = K)] \cong 0$ is acyclic.

Proof. i) Since any element of the quotient can be represented by an element in a subcomplex $K(m_1, \ldots, m_n)$ with some $m_i < 2$ we can apply the lemma above for some $W^i_1 L$.

ii) $d_{s_v\cup i}$ induces an isomorphism on the complexes obtained as sum of $C(T(r))$ over $S'_i - S''_i$ and S_i, hence the cohomology of $C(r)$ comes from $C(S''_i)$. All elements $s.$ in S''_i contain $K \supset K-i$, so we can repeat the same arguments in the category $S(K - i)$ but for $j \neq i$, so (ii) follows by induction.

iii) The assertions for $C^K_r ICL$ and $C^K_r QL$ (20) follow from the the corresponding isomorphisms $C(T(r))L \simeq (C(T(r))ICL$ and $(C(T'(r))L \simeq (C(T'(r))ICL$.

Remark (duality). Given a polarised nilpotent orbit $(L, N_i(i \in M), P)$, the local duality induces an isomorphism:

$$d_r^K : C_r^K L[2 \mid K \mid -1] \simeq Hom_{\mathbb{Q}}(C_{-r}^K L, \mathbb{Q})$$

hence $H^i(C_r^K L)[2 \mid K \mid -1]) = H^{i+2|K|-1} \simeq Hom_{\mathbb{Q}}(H^{-i}(C_{-r}^K L), \mathbb{Q})$ and for $i = 1- \mid K \mid$:

$$d_r^K : H^{|K|}(C_r^K L) \simeq Hom_{\mathbb{Q}}(H^{|K|-1}(C_{-r}^K L), \mathbb{Q})$$

The duality is constructed as follows: For each $s.$ we define $C(s.) = \{s'. \in S(K) :\mid s'. \mid + \mid s. \mid = \mid K \mid +1$ and $s. \cup s'.$ is maximal $\}$, that is to say $s.$ is complementary to $s'.$ except that both must contain K, then we define: $P^* : C_r(J, s.) \otimes C_{-r}(J', s'.)) \to \mathbb{Q}$ as $P^*(a,b) = P(a,b)$ for $J' = K - J$ and $s'. \in C(s.)$ and zero otherwise (P^* is non zero on $C_r(J, s.) \otimes (\oplus_{s'. \in C(s.)} C_{-r}(K - J, s'.))$.

It can be checked that the induced morphism $P^* : C_r^K L[2 \mid K \mid -1] \to Hom_{\mathbb{Q}}(C_{-r}^K L, \mathbb{Q})$ commutes with differentials.

The relation between $C_r^K L$ and $C_r^{KM} L$

The following result will be important in the general proof of the decomposition of $Gr_r^{\mathcal{W}} \Omega^* L$ as direct sum of intersection complexes.

Proposition: Let $H = H^*(C_r^K L)$, considered as a nilpotent orbit with indices $i \in M - K$, then
i) We have : $C_r^{KM} L \simeq \mathcal{W}_{-1} \Omega^*(H)$
ii) For $r \geq 0$, $C_r^{KM} L \simeq C_r^{KM} QL$

Proof. i) We can write $(K_.^+ \times S(K) \times (M-K)_.^+ \times S(M-K)) \simeq M_.^+ \times S_K(M)$, with the correspondence $(J, s., J', s.') \to ((J, J'), (s_.' \cup K, s.))$ then using the relations:
1)$a_{s_\lambda}((J, J'), r-1) = a_{s_\lambda}(J, r-1)$ when $s_\lambda \subset K$
2)$a_{s'_\lambda}((J, J'), r-1) = a_{s'_\lambda}(J', -1) + a_K(J, r)$ and
$W^{s'_\lambda \cup K}_{a_{s'_\lambda \cup K}((J,J'),r-1)}(Gr^{W^K}_{a_K(J,r)}) = W^{s'_\lambda}_{a_{s'_\lambda}(J',-1)}(Gr^{W^K}_{a_K(J,r)})$ since $W^{s'_\lambda \cup K}$ is relative to W^K
we find
$C_r^{KM} L := s(\cap_{K \neq s_\lambda \in (s.' \cup K, s.)} W^{s_\lambda}_{a_{s_\lambda}((J,J'),r-1)} Gr^{W^K}_{a_K((J,J'),r)} L)_{((J,J'),(s_.' \cup K, s.)) \in M_.^+ \times S_K M} \simeq$
$s[(\cap_{s'_\lambda \in s.'} W^{K \cup s'_\lambda}_{a_{s'_\lambda}(J', a_K(J,r-1)}(s[Gr^{W^K}_{a_K(J,r)}(\cap_{K \supsetneq s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L]_{(J,s.) \in K_.^+ \times S(K)})]_{(J',s.') \in (M-K)_.^+ \times S(M-K)}$
$\simeq s[\cap_{s'_\lambda \in s.'} W^{s'_\lambda}_{a_{s'_\lambda}(J',-1)}(C_r^K L)]_{(J',s.') \in (M-K)_.^+ \times S(M-K)}$
where $\cap_{s'_\lambda \in s.'} W^{s'_\lambda}_{a_{s'_\lambda}(J',-1)}(C_r^K L) = C_r^K(\cap_{s'_\lambda \in s.'} W^{K \cup s'_\lambda}_{a_{s'_\lambda}(J',-1)} L)$ is defined as above for each subset of L.
This formula shows that $C_r^{KM} L$ is constructed in two times, once as C^K over $K_.^+ \times S(K)$ (that is a space normal to Y_K) and once as a weight filtration over $(M-K)_.^+ \times S(M-K)$ (that is the space Y_K).

ii) Let $H' = H^*(C_r^K IC(L))$ then the above proof apply word for word to show (notation 18): $C_r^{KM} IC(L) \simeq \mathcal{W}_{-1} \Omega^*(H')$. For $r \geq 0, H' = 0$, hence $C_r^{KM} IC(L) \cong 0$ and the isomorphism in (ii) follows since $C_r^{KM} Q(L) \simeq C_r^{KM} L / C_r^{KM} IC(L)$.

II. Local decomposition.

Theorem (decomposition): For a nilpotent otbit L of dimension n, there exist canonical injections of $C_r^{KM} L$ in $Gr_r^{\mathcal{W}}(\Omega^\star L)$ which decomposes in the category of perverse sheaves (up to a shift in *degrees) as a direct sum*

$$Gr_r^{\mathcal{W}}(\Omega^\star L) \simeq \oplus_{K\subset M} C_r^{KM} L.$$

Moreover $Gr_0^{\mathcal{W}}(\Omega^\star L) \simeq 0$ *is acyclic.*

To carry out the proof by induction on n, we use only the property $Gr_0^{\mathcal{W}}(\Omega^\star L) \simeq 0$ in dimension $n-1$ to get the decomposition in *dim* n, then we use the fact that $C_0^K L$ for all K is acyclic to get again $Gr_0^{\mathcal{W}}(\Omega^\star L) \simeq 0$ in dimension n so to complete the induction step. For $n = 1$, K and M reduces to one element 1 and the theorem reduces to

$$Gr_r^W \simeq C_r^1 L := Gr_{r+1}^{W^1} L \xrightarrow{N_1} Gr_{r-1}^{W^1} L$$

By the elementary properties of the weight filtration of N_1, it is quasi-isomorphic to $Gr_{r-1}^{W^1}(L/N_1 L)[-1]$ if $r > 0$, $Gr_{r+1}^{W^1}(ker N_1 : L \to L)$ if $r < 0$ and $Gr_0^W L \simeq 0$.

For higher dimensions, the proof is carried in various steps.

The complexes $A_r^{KM}L$, $B_r^{KM}L$ *and* $D_r^{KM}L$

Fixing the *dim.* n, the proof is by induction on the length $\mid K \mid$ of K in M. For $r \in \mathbb{Z}$, K fixed and $(J, s.) \in M^+_{\cdot} \times S_K M$ we consider the space $V(J, s.) = \bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L$, and the filtrations of V: $W_t^1(J, s.) = W^K_{a_K(J,r+t)}(V(J, s.))$, $W_t^2(J, s.) = \bigcap_{K \subsetneq s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)}(V(J, s.))$ so that $W_t^1(J, s.) \cap W_t^2(J, s.) = \bigcap_{K \subset s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)} L$.

By summing over $(J, s.)$, we define the complexes

$$A_r^{KM} = s[W^K_{a_K(J,r)} \cap W_0^2(V(J, s.))/W^K_{a_K(J,r-1)} \cap W^2_{-1}(V(J, s.))]_{(J,s.)\in M.^+ \times S_K(M)}$$

$$B_r^{KM} = s[Gr_0^{W^2} W^K_{a_K(J,r-1)}(\textstyle\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)})L]_{(J,s.)\in M.^+ \times S_K(M)}$$

$$D_r^{KM} = s[Gr_0^{W^2(J,s.)} Gr^{W^K}_{a_K(J,r)}(\textstyle\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)]_{(J,s.)\in M.^+ \times S_K(M)}$$

Lemma: For all $K \subset M$, *there exists a natural quasi-isomorphism*

$$B_r^{KM} L \oplus C_r^{KM} L \cong A_r^{KM} L$$

The proof of this lemma reduces to two sublemmas.

Sublemma: For all $K \subset M$, *there exists an exact sequence of complexes*

$$0 \to B_r^{KM} \oplus C_r^{KM} \to A_r^{KM} \to D_r^{KM} \to 0$$

The proof is based on the following elementary remark:

Let W^i *for* $i = 1, 2$ *be two increasing filtrations on an object* V *of an abelian category and* a_i *two integers, then we have an exact sequence:*

$$0 \to W^2_{a_2-1} Gr^{W^1}_{a_1} \oplus W^1_{a_1-1} Gr^{W^2}_{a_2} \to W^1_{a_1} \cap W^2_{a_2} / W^1_{a_1-1} \cap W^2_{a_2-1} \to Gr^{W^1}_{a_1} Gr^{W^2}_{a_2} \to 0$$

We apply this remark to the space $V(J, s.) = \bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L$, and the filtrations W^1 and W^2 of V for $a_1 = 0$ and $a_2 = 0$, then we deduce from the above sequence an exact sequence of vector spaces

$$0 \to Gr_0^{W^2} W^K_{a_K(J,r-1)}(V) \oplus W^2_{-1}(J, s.) Gr^{W^K}_{a_K(J,r)}(V)$$

$$\to W^K_{a_K(J,r)} \cap W_0^2(V)/W^K_{a_K(J,r-1)} \cap W^2_{-1}(V) \to Gr_0^{W^2} Gr^{W^K}_{a_K(J,r)}(V) \to 0$$

The sublemma follows by summing over $(J, s.)$.

Next we prove by induction on n for the general theorem, (not only the lemma)

Sublemma: $D_r^{KM} \cong 0$.

Proof. The idea of the proof is to write D_r^{KM} as $Gr_0^{\mathcal{W}}(\Omega^\star(C_r^K L)) \simeq 0$ where $C_r^K L$ is viewed as a nilpotent orbit on $M-K$ (that is the fiber of a local system on Y^*_{M-K}) and use the induction to prove it is zero. We can either use that $C_r^K L$ is reduced to its unique non zero cohomology or as well prove the acyclicity for each term in $C_r^K L$, what we do as follows.

We simplify the notation from W_t^2 above to
$W_t^0(K, J, s.) := \bigcap_{K \subsetneq s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)} Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)$ so to write the complex as $D_r^{KM} L = s(Gr_0^{W^0(K,J,s.)} L)_{(J,s.)\in M.^+ \times S_K(M)}$, then we use the decomposition of $S_K(M)$ to rewrite W_t^0 as
$W_t^0(K, (J, J'), (K \cup s.', s.)) = \bigcap_{s'_\alpha \in s.'} W^{K \cup s'_\alpha}_{a_{s'_\alpha}(J', a_K(J,r)+t)} Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}((J,J'),r-1)} L)$
$= \bigcap_{s'_\alpha \in s.'} W^{s'_\alpha}_{a_{s'_\alpha}(J',t)} Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_\lambda(J,r-1)} L)$
and the complex
$D_r^{KM} L = s[s[Gr_0^{W^0}(K, (J, J')(K \cup s.', s.))L]_{(J,s.)\in K.^+ \times S(K)}]_{(J',s.')\in (M-K).^+ \times S(M-K)}$
as a sum in two times over $(J, s.) \in K.^+ \times S(K)$ and $(J', s.') \in (M-K).^+ \times S(M-K))$. For a fixed $(J, s.)$ we consider $L(r, J, s.) := Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)$ and the filtration by subspaces $W_t'^0(L(r, J, s.)) := \bigcap_{s'_\lambda \in s.'} W^{s'_\lambda}_{a_{s'_\lambda}(J',t)}(L(r, J, s.)$ and finally the complex
$D(M-K)(L(r, J, s.) := s[Gr_0^{W'^0}(L(r, J, s.))]_{(J',s.')\in (M-K).^+ \times S(M-K)}$.
By construction

$$D_r^{KM} L = s[D(M-K)(L(r, J, s.)]_{(J,s.)\in K.^+ \times S(K)}.$$

We prove by induction on n: $D(M-K)(L(r, J, s.) \simeq 0$. First we embed $D(M-K)(L(r, J, s.))$ in the complex

$$D'(M-K)(L(r, J)) := s[Gr_0^{W'^0}(Gr^{W^K}_{a_K(J,r)} L)]_{(J',s.')\in (M-K).^+ \times S(M-K)}$$

using the embedding $L(r, J, s.) \subset Gr^{W^K}_{a_K(J,r)} L$. Now we introduce the weight filtration $\mathcal{W}$ on the combinatorial DeRham complex $\Omega^\star(Gr^{W^K}_{a_K(J,r)} L)$ for the nilpotent orbit $Gr^{W^K}_{a_K(J,r)} L$ of dimension strictly less then n and weight $a_K(J, r)$ and we notice that $D'(M-K)(L(r, J)) \simeq Gr_0^{\mathcal{W}}(\Omega^\star(Gr^{W^K}_{a_K(J,r)} L)) \simeq 0$ which is zero by induction in dimension $n-1$. Now $D'(M-K)(L(r, J))$ is a complex of MHS and $W^{s_\lambda}_{a_\lambda(J,r-1)}$ (up to a shift) is a filtration by subcomplexes of MHS, so we deduce by strictness that for each $r, J, s.$ the complex $D(M-K)(L(r, J, s.) \simeq 0$ is zero. This ends the proof of the sublemma and hence the lemma.

Proof of the decomposition theorem

For each $i \in \mathbf{N}$ we define a map $\varphi_i : S(M) \to \mathcal{P}(M)$ to the subsets of M such that $M \supset \varphi_i(s.) = Sup\{s_\lambda : \mid s_\lambda \mid \leq i\}$ and for each $(J, s.) \in M.^+ \times S(M)$, the filtration with index t of L,
$W_t^2(\varphi_i(s.), J, s.) := \bigcap_{\varphi_i(s.) \subsetneq s_\lambda \in s.} (W^{s_\lambda}_{a_{s_\lambda}(J,r+t)})(\bigcap_{s_\lambda \subset \varphi_i(s.), s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)$,
We define $G_i(J, s.)(L) = Gr_0^{W^2(\varphi_i(s.),J,s.)} L$ then we consider the complex

$$G_i(s.) = s(G_i(J, s.)L)_{J\subset M} = s(Gr_0^{W^2(\varphi_i(s.),J,s.)} L)_{J\subset M},\ G_i^{SM} L = s(G_i(s.))_{s.\in S(M)}$$

In particular, when $\varphi_i(s.) = \varnothing$, $G_i(s.) = Gr_r^{W(s.)} L^{s.}$ so that for $i \leq 0$, $G_i^{SM} L = Gr_r^{W^{SM}} L^{SM}$ and when $i = \mid M \mid -1$, $G_i^{SM} L = C_r^M L$. Hence the proof of the decomposition theorem will follows from the

Lemma.

$$G_i^{SM} \cong G_{i+1}^{SM} \oplus C_r^{KM}$$

Now in order to compare $G_i^{SM}L$ and $G_{i+1}^{SM}L$, we consider the category $S^i(M) = \{s. \in S(M) : \mid s. \mid = i\}$ which is not a subcategory of $S(M)$ but $S(M) - S^{i+1}(M)$ is a subcategory such that the restrictions of G_i and G_{i+1} define two subcomplexes: $G'_i = s(G_i(s.))_{s.\in S(M)-S^{i+1}(M)} \subset G_i^{SM}L$ and $G'_{i+1} = s(G_{i+1}(s.))_{s.\in S(M)-S^{i+1}(M)} \subset G_{i+1}^{SM}L$. We have $G'_{i+1} = G'_i$ since $\varphi_i = \varphi_{i+1}$ on $S(M) - S^{i+1}(M)$.
The next step is to compute the quotient complexes, for this we remark: $S^{i+1}(M) \simeq \oplus_{|K|=i+1} S_K(M)$ and $\varphi_{i+1}(s.) = K$ for $s. \in S_K M$, then :

$$G_{i+1}^{S^{i+1}(M)} = G_{i+1}^{SM}L/G'_{i+1} \simeq s(G_{i+1}(s.))_{s.\in S^{i+1}(M)} \simeq \oplus_{|K|=i+1} s(G_{i+1}(s.))_{s.\in S_K(M)} \simeq \oplus_{|K|=i+1} B_r^K M$$

On the other side, when $s. \in S_K M$ where $\mid K \mid = i+1$, $G_i(J, s.) = Gr_0^{W^1(\varphi_i(s.),J,s.)}L$, and $G_i^{S^{i+1}(M)} = s(G_i(s.))_{s.\in S_K M} = A_r^K M$, so that

$$G_i^{S^{i+1}(M)} = G_i^{SM}L/G'_i \simeq s(G_i(s.))_{s.\in S^{i+1}(M)} \simeq \oplus_{|K|=i+1} s(G_i(s.))_{s.\in S_K(M)} \simeq \oplus_{|K|=i+1} A_r^K M.$$

Now we deduce from the quasi-isomorphism $B_r^K M \oplus C_r^K M \cong A_r^K M$ that $G_i^{S^{i+1}(M)} = A_r^K M \cong G_{i+1}^{S^{i+1}(M)} \oplus C_r^{KM}$ hence $G_i^{SM}/G'_i \cong (G_{i+1}^{SM}/G'_{i+1}) \oplus C_r^{KM}$, which proves the lemma since $G'_i \cong G'_{i+1}$.

III. Global construction of the weight filtration.

For each subset $s_\lambda \subset I$ such that $Y_{s_\lambda} \neq \varnothing$ we write $\mathcal{W}^{s_\lambda} = \mathcal{W}(\Sigma_{i\in s_\lambda}\mathcal{N}_i)$ for the filtration by subbundles defined by the nilpotent endomorphisms of the restriction $\mathcal{L}_{Y_{s_\lambda}}$ of $\mathcal{L}_X$ to $Y^*_{s_\lambda}$, inducing also by restriction and for each subset $K \supset s_\lambda$ a filtration on $\mathcal{L}_{Y_K}$. Since $(\Omega^*_X(LogY) \otimes \mathcal{L}_X^{e(\alpha.)})_y$ is acyclic if there exists an index $j \in M$ such that $\alpha_j \neq 0$ (see the formula (11) and the remark below), we can suppose from now on the local system unipotent.

Definition (the weight and Hodge filtrations). The weight filtration is defined for $\mathcal{L}$ unipotent, on the following combinatorial logarithmic complex

$$\Omega^*(\mathcal{L}) = s(\Omega^*_{X_{s.}}(LogY) \otimes \mathcal{L}_{X_{s.}})_{s.\in S}$$

as follows: let $M \subset I, \mid M \mid = p$ and $y \in Y^*_M$, then in terms of a set of n coordinates $y_i, i \in [1, n]$ where we identify M with $[1, p]$ on an open set $U_y \simeq D^{|M|} \times D^{n-p}$ containing y and a section $f = (f^{s.})_{s.\in S}$

$$f^{s.} = \Sigma_{J\subset M, J'\cap M=\varnothing} f^{s.}_{J,J'} \frac{dy_J}{y_J} \wedge dy_{J'}$$

$$f = (f^{s.})_{s.\in S} \in \mathcal{W}_r(\Omega^*(\mathcal{L}))_{/U_y} \Leftrightarrow \forall J, N \subset M,\ f^{s.}_{J,J'}/Y_N \cap U_y \in \bigcap_{s_\lambda \in s., s_\lambda \subset N} \mathcal{W}^{s_\lambda}_{a_\lambda(J,r)}(\mathcal{L}_{/Y_N\cap U_y})$$

By convention we let for all integers r, $\mathcal{W}_r/X - Y = \Omega^*(\mathcal{L})/X - Y$, so that $\mathcal{W}_r$ is the direct image for r big enough and the extension by zero for r small enough. It is a filtration by subcomplexes *of analytic subsheaves globally defined on X. The Hodge filtration F is constant in $(s.)$ and deduced from Schmid's extension to $\mathcal{L}_X$*

$$F^p(s.) = 0 \to F^p\mathcal{L}_X \ldots \to \Omega^i_{X_{s.}}(LogY) \otimes F^{p-i}\mathcal{L}_{X_{s.}} \to \ldots, \; F^p = s(F^p(s.))_{s.\in S}$$

Theorem: Consider a unipotent local system $\mathcal{L}$ underlying a variation of polarised Hodge structures of weight m; then the complex

$$(\Omega^*(\mathcal{L}), \mathcal{W}[m], F) \tag{19}$$

with the filtrations $\mathcal{W}$ and F defined above satisfy the decomposition and purity properties. More precisely, for all subset $K \subset I$ and all integers $r > 0$ (resp. $r < 0$), let $(\mathcal{L}^K_r, W, F)$ denotes the bifiltered local system underlying a polarised VHS on Y^*_K of weight $r - |K| + m$ (resp.$r + |K| + m$) and of general fiber $(H^{|K|}(C^K_r L), W, F)$ of weight induced by $W^{s_K}_{r-|K|+m}$ and F defined by L (resp. $(H^{|K|-1}(C^K_r L), W, F)$ for $r < 0$ of weight induced by $W^{s_K}_{r+|K|+m}$). Then we have the following decomposition into intermediate extensions (up to shift in degrees) of (VHS) $\mathcal{L}^K_r$ compatible with the local decomposition

$$(Gr^{\mathcal{W}[m]}_{r+m}\Omega^*\mathcal{L}, F) \simeq \oplus_{K\subset I} j^K_{!*}(\mathcal{L}^K_r[- \mid K \mid], W[2 \mid K \mid]], F[- \mid K \mid], \text{ for } r > 0 \text{ and } j^K : Y^*_K \to Y_K$$
$$(Gr^{\mathcal{W}[m]}_{r+m}\Omega^*\mathcal{L}, F) \simeq \oplus_{K\subset I} j^K_{!*}\mathcal{L}^K_r[1- \mid K \mid], W[-1], F), \text{ for } r < 0$$
$$(Gr^{\mathcal{W}[m]}_{m}\Omega^*\mathcal{L}, F) \simeq 0$$

that is for $r \geq 0$ the weight is coincides with the weight for Hodge structures but for $r < 0$ the true weight for Hodge structures is $r + m + 1$.

iii) The projection on the quotient complex $(\Omega^*(\mathcal{L})/j_{!*}\mathcal{L}, \mathcal{W}[m], F)$ with the induced filtrations, induces a filtered quasi- isomorphism on $(Gr^{\mathcal{W}[m]}_{r+m}, F)$ for $r > 0$.

Proof. The decomposition of $(Gr^{\mathcal{W}[m]}_{r+m}\Omega^*(\mathcal{L}), F)$ reduces near a point $y \in Y^*_M$ to the local decomposition of $Gr^{\mathcal{W}}_{r+m}\Omega^* L$ for the nilpotent orbit L defined at the point y by the local system since $C^{KM}_r L$ is precisely the fiber of $j^K_{!*}\mathcal{L}^K_r[- \mid K \mid]$ for $r > 0$ (resp. $j^K_{!*}\mathcal{L}^K_r[1- \mid K \mid]$ for $r < 0$. The count of weight takes into account for $r > 0$ the residue in the isomorphism with L that shifts W and F but also the shift in degrees, while for $r < 0$ there is no residue but only a shift in degrees, the rule being as follows:

Let (K, W, F) be a mixed Hodge complex then for all $m, h \in \mathbb{Z}$, $(K', W', F') = (K[m], W[m - 2h], F[h])$ is also a mixed Hodge complex.

The same proof apply for $r = 0$, hence $\mathcal{W}_{-1} \simeq \mathcal{W}_0$ is isomorphic to the intermediate extension of $\mathcal{L}$ by Kashiwara and Kawai's formula, that we prove below. The assertion (iii) follows from the assertion (iii) in the purity theorem corresponding to a result on $C^{KM}_r QL$.

Proof of Kashiwara and Kawai's formula: $j_{!*}\mathcal{L}[2n] \simeq \mathcal{W}_0\Omega^*(\mathcal{L}[2n])$.

In this subsection we give a proof of the formula of the intermediate extension of $\mathcal{L}[2n]$, announced in [26], which is in fact the subcomplex $\mathcal{W}_0\Omega^*(\mathcal{L}[2n])$. It follows easily from the local decomposition of the graded parts of the weight filtration, by induction on the dimension n.

Theorem. The subcomplex $\mathcal{W}_0\Omega^*(\mathcal{L}[2n])$ is quasi-isomorphic to the intermediate extension of $\mathcal{L}[2n]$.

The proof of this theorem is by induction on the dimension n. It is true in dimension 1 and if we suppose the result true in dimension strictly less than n, we can apply the result for local systems defined on open subsets of the closed sets Y_K, namely the local system $\mathcal{L}_r^K[- \mid K \mid])$ for $r > 0$ (resp.$\mathcal{L}_r^K[1- \mid K \mid])$ for $r < 0$) whose fiber at at each point $y \in Y_K^*$ is quasi-isomorphic to $C_r^K L$. Let $j^K : Y_K^* \to Y_K$ be the open embedding in Y_K and consider the associated DeRham complex $\Omega^*(\mathcal{L}_r^K)$ on Y_K whose weight filtration will be denoted locally near a point in Y_M^* by $\mathcal{W}^{M-K}$ for $K \subset M$; then by the induction hypothesis we have at the point y: $\mathcal{W}_{-1}\Omega^* L \simeq \mathcal{W}_0 \Omega^* L$ is also quasi-isomorphic to the fiber of the intermediate extension of $\mathcal{L}$, that is

$$\forall r > 0, C_r^{KM}(L) \simeq (j_{!*}^K \mathcal{L}_r^K[- \mid K \mid])_y \simeq \mathcal{W}_{-1}^{M-K} C_r^K L$$

and similarly for $r < 0$.

We use the following criteria caracterising intermediate extension [17]:
Consider the stratification defined by Y on X and the middle perversity $p(2k) = k-1$ associated to the closed subset $Y^{2k} = \cup_{|K|=k} Y_K$ of real codimension $2k$. We let $Y^{2k-1} = Y^{2k}$ and $p(2k-1) = k-1$. For any complex of sheaves $S.$ on X which is constructible with respect to the stratification , let $S.^{2k} = S.^{2k-1} = S. \mid X - Y^{2k}$ and consider the four properties:

a) Normalisation: $S. \mid X - Y^2 \cong \mathcal{L}[2n]$

b) Lower bound: $\mathbf{H}^i(S.) = 0$ for all $i < -2n$

c) vanishing condition: $\mathbf{H}^m(S.^{2(k+1)}) = \mathbf{H}^m(S.^{2k+1}) = 0$ for all $m > k - 2n$

d) dual condition: $\mathbf{H}^m(j_{2k}^! S.^{2(k+1)}) = 0$ for all $k \geq 1$ and all $m > k - 2n$ where $j_{2k} : Y^{2k} - Y^{2(k+1)} \to X - Y^{2(k+1)}$ is the closed embedding,

then $S.$ is the intermediate extension of $\mathcal{L}[2n]$.
In order to prove the result for n we check the above four properties for $\mathcal{W}_0\Omega^*(\mathcal{L}[2n])$. The first two are clear and we use the exact sequences

$$0 \to \mathcal{W}_{r-1} \to \mathcal{W}_r \to Gr_r^{\mathcal{W}} \to 0$$

to prove d)(resp. c)) by descending (resp. ascending)indices from $\mathcal{W}_r$ to $\mathcal{W}_{r-1}$ for $r \geq 0$ (resp. $r-1$ to r for $r < 0$) applying at each step the inductive hypothesis to $Gr_r^{\mathcal{W}}$.

Proof of d). The dual condition is true for r big enough since then $\mathcal{W}_r$ coincides with the whole complex, that is the higher direct image of $\mathcal{L}[2n]$ on $X - Y$. Now we apply d) on $Y_{K'}$ with $\mid K' \mid = k'$ for $j'_{2k} : Y^{2k} \cap Y_{K'} - Y^{2(k+1)} \cap Y_{K'} \to Y_{K'} - Y^{2(k+1)} \cap Y_{K'}$ where we suppose $k > k'$ (notice that $Y^{2k} \cap Y_{K'} = (Y \cap Y_{K'})^{2(k-k')}$, then for $S.'$ equal to the intermediate extension of $\mathcal{L}_r^{K'}[2n - 2k']$ on $Y_{K'}$ we have the property $\mathbf{H}^m({j'}_{2k}^! S.'^{2(k-k')+1)}) = 0$ for all $(k - k') \geq 1$ and all $m > k - k' - 2(n - k') = k + k' - 2n$ which gives for $S.'[k']$ on X: $\mathbf{H}^m(j_{2k}^! S.'^{2(k+1)}[k']) = 0$ for all $k > k'$ and all $m > k - 2n$, hence d) is true.

If $k = k'$, then $Y^{2k} \cap Y_{K'} = Y_{K'}$ and we have a local system in degree $k' - 2n$ on $Y_{K'} - Y^{2(k+1)} \cap Y_{K'}$ hence d) is still true and for $k < k'$, the support $Y_{K'} - Y^{2(k+1)} \cap Y_{K'}$ of $S.'$ is empty. From the decomposition theorem and the induction, this argument apply to $Gr_r^{\mathcal{W}}$ and hence apply by induction on $r \geq 0$ to $\mathcal{W}_0$ and also to $\mathcal{W}_{-1}$.

Proof of c). Dually, the vanishing condition is true for r small enough since then $\mathcal{W}_r$ coincides with the extension by zero of $\mathcal{L}[2n]$ on $X - Y$.
Now we use the filtration for $r < 0$, for $S.'$ equal to the intermediate extension of $\mathcal{L}_r^{K'}[2n - 2k']$ on $Y^{K'}$ we have for $k > k'$: $\mathbf{H}^m(S.'^{2(k-k')+1}) = 0$ for all $m > k + k' - 2n$, which gives for $S.'[k'+1]$, $(r < 0)$ on X: $\mathbf{H}^m(S.^{2(k+1)}) = \mathbf{H}^m(S.^{2k+1}) = 0$ for all $m > k - 1 - 2n$. If $k = k'$, then $S.'[k'+1]$ is a local system in degree $-2n + k - 1$ on $Y_{K'} - Y^{k+1}$ and for $k < k'$, $Y_{K'} - Y^{k+1}$ is empty.

Corollary: If we suppose X proper and we replace the filtration $\mathcal{W}$ by $\mathcal{W}''$ with $\mathcal{W}''_i = \mathcal{W}_i$ for $i \geq 0$ and $\mathcal{W}''_{-1} = 0$, then the bifiltered complex

$$(\Omega^*(\mathcal{L}), \mathcal{W}''[m], F)$$

is a mixed Hodge complex .

3§. The complex of nearby cycles $\Psi_f(\mathcal{L})$.

Let $f: X \to D$ and suppose $Y = f^{-1}(0)$; the definition of the complex of sheaves of nearby cocycles on Y is given in [11]; its cohomology fiber at a point y equals the cohomology of the Milnor fiber F_y at y in Y. The monodromy $\mathcal{T}$ induces an action on the cohomology $H^i(\Psi_f(\mathcal{L})_y) \simeq H^i(F_y, \mathcal{L})$ and on the complex itself viewed in the abelian category of perverse sheaves. It is important to point out that the action on the complex is related to the action on cohomology through a spectral sequence and precisely in our subject we need to use the weight filtration on the complex itself and not on its cohomology.
The aim of this section is to describe the weight filtration on $\Psi_f(\mathcal{L})$. This problem is closely related to the weight filtration in the open case since there exists a close *relation between $\Psi^u_f(\mathcal{L})$, the direct image $\mathbf{j}_*\mathcal{L}$ and $\mathbf{j}_{!*}\mathcal{L}$* as explained in [2] (and previously in a private letter by Deligne and Gabber)

Proposition [2]: Let $\mathcal{N} = Log\mathcal{T}^u$ denotes the logarithm of the unipotent part of the monodromy, then we have the following isomorphism in the abelian category of perverse sheaves

$$\mathbf{j}_*\mathcal{L}/\mathbf{j}_{!*}\mathcal{L} \simeq Coker(\mathcal{N}: \Psi^u_f(\mathcal{L}) \to \Psi^u_f(\mathcal{L}))[-1] \tag{20}$$

The filtration $W(\mathcal{N})$ on $\Psi^u_f(\mathcal{L})$ induces a filtration $\mathcal{W}$ on $Coker\mathcal{N}/\Psi^u_f(\mathcal{L})$, hence on $\mathbf{j}_*\mathcal{L}/\mathbf{j}_{!*}\mathcal{L}$.

The induced filtration on $\mathbf{j}_*\mathcal{L}/\mathbf{j}_{!*}\mathcal{L}$ is independant of the choice of f. For a rigorous proof one should use the result of Verdier [34]. To prove the independance of f we can use a path in the space of functions between two local equations f and f' of Y and defines by parallel transport an isomorphism between $\Psi_f(\mathcal{L})$ and $\Psi_{f'}(\mathcal{L})$; modulo *coker*$\mathcal{N}$, this isomorphism is independant of the path.

I. The weight filtration on the nearby cycles $\Psi_f(\mathcal{L})$

The method to compute Ψ_f as explained in [11] uses the restriction $i^*_Y\mathbf{j}_*\mathcal{L}$ of the higher direct image of $\mathcal{L}$ to Y and the cup-product $H^i(X^*, \mathcal{L}) \otimes H^1(X^*, \mathbb{Q}) \xrightarrow{\smile\eta} H^{i+1}(X^*, \mathcal{L})$ by the inverse image $\eta = f^*c \in H^1(X^*, \mathbb{Q})$ of a generator c of the cohomology $H^1(D^*, \mathbb{Q})$. Thus one defines a morphism (of degree 1), η: $i^*_Y\mathbf{j}_*\mathcal{L} \to i^*_Y\mathbf{j}_*\mathcal{L}[1]$ such that $\eta^2 = 0$ so to get a double complex whose simple associated complex is quasi-isomorphic to $\Psi^u_f(\mathcal{L})$, the unipotent part of $\Psi_f(\mathcal{L})$ under the monodromy action $\mathcal{T}$

$$\Psi^u_f(\mathcal{L}) \simeq s(i^*_Y\mathbf{j}_*\mathcal{L}[p], \eta)_{p\leq 0} \tag{21}$$

In order to get the full $\Psi_f(\mathcal{L})$ (not only the unipotent part under the action of $\mathcal{T}$) Deligne introduced local systems of rank one $\mathcal{V}_\beta$ on the disc with monodromy $e(\beta) = exp(-2i\pi\beta)$ and proved the following isomorphism

$$\Psi_f(\mathcal{L}) \simeq \oplus_{\beta\in\mathbb{C}} \Psi^u_f(\mathcal{L} \otimes f^{-1}\mathcal{V}_\beta) \tag{22}$$

When $\mathcal{L}$ is quasi-unipotent we need only to consider $\beta \in \mathbb{Q} \cap [0,1[$. Moreover, near a point $y \in Y$ such that $f = \Pi_{j\in M} z_j^{n_j}$, the tensor product $\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_\beta$ is unipotent near y if and only if $\forall j \in M, \alpha_j + n_j\beta \in \mathbf{N}$, then

$$(23) \qquad \Psi_f(\mathcal{L}^{e(\alpha.)}) \simeq \oplus_{\beta\in S} \Psi_f^u(\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_\beta), \quad S = \{\beta \in \mathbb{C} : \forall j \in M, \alpha_j + n_j\beta \in \mathbf{N}\}.$$

Due to this formula, the problem can be reduced later in the article to study the unipotent part Ψ_f^u. We recall that in order to construct $\mathcal{L}_X$ we need to choose a section as follows
Definition: We define $\tau : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ *as the section of* $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ *such that* $Re(\tau) \in [0,1[$.
Local description . Near a point $y \in Y$, where $f = \prod_{i=1}^{n} z_i^{n_i}$ for non zero n_i, in DeRham cohomology $\eta = f^*(\frac{dt}{t}) = \sum_{i=1}^{n} n_i \frac{dz_i}{z_i}$. The morphism η on $\Omega(L, D.+N.)$ (3) is defined by $\varepsilon_{i_j} n_{i_j} Id : L(i.-i_j) = L \to L = L(i.)$ where ε_{ij} is the signature of the permutation which order strictly $(i.-i_j, i_j)$ for various i_j. For each complex number β, we consider the following complexes where $L^{e(\alpha.)} = \cap_{i\in[1,n]} L^{e(\alpha_i)}$ is the intersection of the eigenspaces for $T_i^s, i \in [1,n]$ with eigenvalues $e(\alpha_i)$.

$$(24) \qquad \Psi_p^\beta(L) = \oplus_{\alpha.} \Omega(L^{e(\alpha.)}, \tau(\alpha_i + \beta n_i)Id + N_i)_{i\in[1,n]}) [p], \ p \le 0$$

where η: $\Psi_p^\beta(L) \to \Psi_{p+1}^\beta(L)$ is a complex morphism satisfying $\eta^2 = 0$, the (Ψ_p^β, η) form a double complex for $p \le 0$. Let $\Psi^\beta(L)$ denotes the associated simple complex. In order to take into account the action of $N = -1/2i\pi Log T^u$ we write after Kashiwara, $L[N^p]$ for $L[p]$ and $L[N^{-1}]$ for the direct sum over p, so that the action of N is just multiplication by N

$$(25) \qquad \Psi^\beta(L) = s(\Psi_p^\beta(L), \eta)_{p\le 0} \simeq \oplus_\alpha. \Omega(L^{e(\alpha.)}[N^{-1}], \tau(\alpha_i + \beta n_i)Id + N_i - n_i N)_{i\in[1,n]}).$$

It is isomorphic to the direct sum of Koszul complexes defined by $(L^{e(\alpha.)}[N^{-1}], \tau(\alpha_i+\beta n_i)Id + N_i - n_i N)_{i\in[1,n]}$. The complex $\Omega(L^{e(\alpha.)}, \tau(\beta n. + \alpha.)Id + N.)$ is acyclic unless $(n_i\beta + \alpha_i) \in \mathbf{N}$ for all $i \in M$, hence $\Psi^\beta(L)$ is acyclic but for a finite number of β such that $e(\beta)$ is an eigenvalue of the monodromy action. The proof of Deligne's result [11] reduces to:
The fiber at zero in D^{n+k} *of* $\Psi_f(\mathcal{L})$ *(resp.* $\Psi_f^u(\mathcal{L})$*) is quasi-isomorphic to a (finite) direct sum of* $\Psi^\beta(L)$*(25)(resp. to* $\Psi^0(L)$ *for* $\beta = 0$*)*

$$(26) \qquad \Psi_f(\mathcal{L})_0 \simeq \oplus_{\beta\in\mathbb{C}} \Psi^\beta L \quad , \quad \Psi_f^u(\mathcal{L})_0 \simeq \Psi^0 L$$

3.1 The weight and Hodge filtrations on $\Psi^0 L$ We consider again a nilpotent orbit L. To describe the weight in terms of the filtrations $(\Omega^* L, \mathcal{W}, F)$ associated to L, we need to use the constant complex with index $s. \in S(M)$, $\Psi^0 L(s.) = \Psi^0 L$ and introduce the complex

$$(27) \qquad (\Psi^0 L)_{S(M)} := s(\Psi^0 L(s.))_{s.\in S(M)}$$

which can be viewed also as $s(\Omega^* L[p], \eta)_{p\le 0}$, then we define on it the weight filtration

$$(28) \qquad \mathcal{W}_r(\Psi^0 L)_{S(M)} = s(\mathcal{W}_{r+2p-1}\Omega^* L[p], \eta)_{p\le 0}, \ \ F^r(\Psi^0 L)_{S(M)} = s(F^{r+p}\Omega^* L[p], \eta)_{p\le 0}.$$

Monodromy.

The logarithm $\mathcal{N}$ of the monodromy is defined by an endomorphism ν of the complex $\Psi^0(L)_{S(M)}$, given by the formula

$$\forall a. = \Sigma_{p\leq 0} a_p \in (\Psi^0 L)_{S(M)}, (\nu(a.))_p = a_{p-1} \ \forall p \leq 0$$

such that $\nu(\mathcal{W}_r) \subset \mathcal{W}_{r-2}$ and $\nu(F^r) \subset F^{r-1}$.

Decomposition of $Gr_r^{\mathcal{W}}$

The morphism η induces a morphism denoted also by $\eta : C_r^{KM} L \to C_{r+2}^{KM} L[1]$ so that we can define a double complex and the associated simple complex

$$\Psi_r^{KM} L = s(C_{r+2p-1}^{KM} L[p], \eta)_{p\leq 0}, \ \Psi_r^K L := \Psi_r^{KK} L$$

We will see soon that this complex decomposes into a direct sum.

Lemma: There exists natural injections of $\Psi_r^{KM} L$ *into* $Gr_r^{\mathcal{W}}(\Psi^0 L)_{S(M)}$ *and a decomposition*

$$Gr_r^{\mathcal{W}}(\Psi^0 L)_{S(M)} \simeq \oplus_{K\subset M} \Psi_r^{KM} L \tag{29}$$

Proof: By the spectral sequence of a double complex, it is enough to check the decomposition on the columns where the proof reduces to the decomposition in the open case.

Theorem: The weight filtration (28) coincides with $W(\mathcal{N})$ *defined by the logarithm of the monodromy in the abelian category of perverse sheaves.*

The proof in two steps reduces to the lamma and the proposition below.

Lemma: The following statements are equivalent
i) For all $i \geq 1, \ \nu^i : Gr_i^{\mathcal{W}}(\Psi^0 L)_{S(M)} \simeq Gr_{-i}^{\mathcal{W}}(\Psi^0 L)_{S(M)}$.
ii) For all $i \geq 1, \ Gr_i^{\mathcal{W}} ker \ \nu^i = s(Gr_{i+2p-1}^{\mathcal{W}} \Omega^* L[p], \eta)_{-i<p\leq 0} \cong 0$.

Proof: The morphism ν^i on $(\Psi^0 L)_{S(M)}$ is surjective and its kernel is sum of the columns of index $-i < p \leq 0$.

Remark: It may be interesting for the reader to check the statement on the example of a line with f equivalent at 0 to z^n on the fiber of $(\Psi^0 L)$ at the point 0 for $L = \mathbb{C}$ and $N = 0$, where the similarity and the differences with Steenbrink's construction appears already.

Proposition: For all $i \geq 1, \ Gr_i^{\mathcal{W}} ker \ \nu^i = s(Gr_{i+2p-1}^{\mathcal{W}} \Omega^* L[p], \eta)_{-i<p\leq 0} \cong 0$.

Proof: Let $\nu_i^{KM} := s[C_{i+2p-1}^{KM} L[p], \eta]_{-i<p\leq 0}$, then by the decomposition theorem we have: $Gr_i^{\mathcal{W}} ker \ \nu^i \cong \oplus_{K\subset M} \nu_i^{KM}$. Denotes ν_i^{KK} by ν_i^K, then we can easily check that ν_i^{KM} is the intermediate extension of ν_i^K and is quasi-isomorphic to zero if $\nu_i^K \cong 0$, so we reduce the proof to

Lemma: For all $i \geq 1, \ \nu_i^K := s[C_{i+2p-1}^K L[p], \eta]_{-i<p\leq 0} \cong 0$

Proof: In order to give a proof by induction for i assuming the result for $i-2$, we write ν_i^K as:

$s[C_{-(i-1)}^K L[-(i-1)], s[C_{i+2p-1}^K L[p], \eta]_{-(i-2)<p\leq 0}[-1], C_{i-1}^K L, \eta] \cong 0$

We know that $C_{-(i-1)}^K L[\mid K \mid -1] \cong Gr_{|K|-(i-1)}^{W^K}[(\cap_{i\in K} ker N_i : L \to L] \simeq$

$\oplus_{(m_1,\cdots,m_n)\in T'(r)} Gr_{m_n}^{W^n} \cdots Gr_{m_i}^{W^i} \cdots Gr_{m_1}^{W^1}[(\cap_{i\in K}(ker N_i : L \to L]$

and $C_{(i-1)}^K L[\mid K \mid] \cong Gr_{r-|K|}^{W^K}[L/(\Sigma_{i\in K} N_i L)]$

$\simeq \oplus_{(m_1,\cdots,m_n)\in T(r)} Gr_{m_n-2}^{W^n} \cdots Gr_{m_i-2}^{W^i} \cdots Gr_{m_1-2}^{W^1}[L/(\Sigma_{i\in K} N_i L)]$.

Given a nilpotent orbit (L, N_i), we denote in general the primitive part of $(Gr_r^{W(N)} L)$ by $(Gr_r^{W(N)} L)^0$, then we have the following isomorphisms :

$Gr_r^{W(N)}(L/NL) \simeq (Gr_r^{W(N)} L)^0 \xrightarrow{N^r} (Gr_{-r}^{W(N)} kerN)$,

so we can deduce in general:

$N_1^{m_1} \cdots N_n^{m_n} : Gr_{m_n}^{W^n} \cdots Gr_{m_i}^{W^i} \cdots Gr_{m_1}^{W^1} [L/(\Sigma_{i\in K} N_i L)] \simeq Gr_{m_n}^{W^n} \cdots Gr_{m_i}^{W^i} \cdots Gr_{m_1}^{W^1} \cap_{i\in K} kerN_i$

the sum over $\{(m_1 \geq 0, \cdots, m_n \geq 0)\} : (\Sigma_{i\in K} m_i = i - 1- \mid K \mid$ induces an isomorphism:

$\gamma : Gr_{i-1-|K|}^{W^K} [L/(\Sigma_{i\in K} N_i L)] \to Gr_{-(i-1)+|K|}^{W^K} [(\cap_{i\in K} (kerN_i : L \to L]$

since $Gr_{-(i-1)+|K|}^{W^K} [\cap_{i\in K} kerN_i : L \to L]$ is isomorphic to

$\oplus_{\{(m_1\leq 0,\cdots,m_n\leq 0)\}:(\sum_{i\in K} m_i = |K|-i+1} \cdots \oplus Gr_{m_n}^{W^n} \cdots Gr_{m_i}^{W^i} \cdots Gr_{m_1}^{W^1}$.

Then γ induces a quasi-isomorphism from

$C_{(i-1)}^K L[\mid K \mid] \cong Gr_{i-1-|K|}^{W^K} [L/(\Sigma_{i\in K} N_i L)] \cong Gr_{i-1-|K|}^{W^K} [(L/(N_1 L)/N_2(L/(N_1 L)]$

to $C_{(1-i)}^K L[\mid K \mid -1]$.

A diagram chasing shows that ν_i^K is in fact a cone over γ^{-1}, hence zero, which establishes the lemma and the proposition.

Corollary: The graded part of $(\Psi^0 L)_{S(M)}$ is non zero for only a finite number of indices for which it reduces to a double complex of finite terms. More precisely, let i_0 be an integer large enough to have $Gr_j^{\mathcal{W}} \Omega^* \mathcal{L} = 0$ for all $j > i_0$, then for a given $i \geq 1$:

$$Gr_i^{\mathcal{W}} (\Psi^0 L)_{S(M)} \cong s(Gr_{i+2p-1}^{\mathcal{W}} \Omega^* L[p], \eta)_{p\leq -i}$$

where only a finite number of p such that $-i_0 \leq i + 2p - 1 \leq -i - 1$ are non zero in the right term. For $i \leq -1$ we use the isomorphism ν^{-i}. In particular, $Gr_i^{\mathcal{W}} (\Psi^0 L)_{S(M)} \cong 0$ for all i such that $|i| \geq i_0$. We have a direct sum of intermediate extensions (up to shift in degrees) of VHS of weight $i + m + 1$.

Proof: Suppose $i > 0$, then $Gr_i^{\mathcal{W}} (\Psi^0 L)_{S(M)} = s(Gr_{i+2p-1}^{\mathcal{W}} \Omega^* L[p], \eta)_{p\leq 0}$ is the cone over

$\eta : s(Gr_{i+2p-1}^{\mathcal{W}} \Omega^* L[p], \eta)_{-i<p\leq 0}[-1] \to s(Gr_{i+2p-1}^{\mathcal{W}} \Omega^* L[p], \eta)_{p\leq -i}$

where the first complex is $Gr_i^{\mathcal{W}} ker\ \nu^i$ hence quasi-isomorphic to zero, then the corollary follows.

Remark: i) This corollary, shows that the weight filtration behaves like a finite one, so that we can apply in the proper case the results on mixed Hodge complex where the weight filtration is supposed to be finite.

ii) In Steenbrink's case that is $\mathcal{L} = \mathbb{C}$, $s(\mathcal{W}_{p-1} \Omega^* L[p], \eta)_{-i_0<p\leq 0}$ is a subcomplex quasi-isomorphic to $(\Psi^0 L)_{S(M)}$. For a general $\mathcal{L}$, it is not a complex, nevertheless the graded part behaves like if we restrict to such object.

iii) Dually, we could define $(\Psi^0 L)_{S(M)}$ as $i_Y^* s(\Omega^* L[p], \eta)_{p\geq 0}[1]$, with the filtrations

$$\mathcal{W}_r = s(\mathcal{W}_{r+2p+1} \Omega^* L[p], \eta)_{p\geq 0}[1],\ F^r = s(F^{r+p+1} \Omega^* L[p], \eta)_{p\geq 0}[1]$$

then the above results show that the two definitions give quasi-isomorphic complexes and the formula for $Gr^{\mathcal{W}}$ behaves like if we could use the quotient by $\mathcal{W}_p \Omega^* L$ in each column p.

We will see later that we can take the quotient with the subcomplex generated by ICL for $p = 0$ and then use the induced filtrations on the quotient.

Corollary (decomposition): Let $I(p) = \{p \leq 0, -i_0 \leq i + 2p - 1 \leq -i - 1\}$, then:

$$Gr_i^{\mathcal{W}} (\Psi^0 L)_{S(M)} = s(Gr_{i+2p-1}^{\mathcal{W}} \Omega^* L[p], \eta)_{p\in I(p)} = \oplus_{p\in I(p)} Gr_{i+2p-1}^{\mathcal{W}} \Omega^* L[p].$$

Proof : It follows from the remark that $Gr_i^{\mathcal{W}}(\Psi^0 L)_{S(M)}$ can be computed for a finite number of columns such that $r = i + 2p - 1 \leq p - 1 < 0$ is negative and where each term is a direct sum of $C_r^{KM} L$, intermediate extension of $C_r^K L$ whose cohomology is concentrated in degree $|K| - 1$, hence the map η is zero and we get a direct sum instead of a double complex.

3.2 The global weighted complex $(\Psi_f^u(\mathcal{L}), \mathcal{W}, F)$

Returning to the global situation, we need to define the Hodge filtration on $\Psi_f(\mathcal{L}_X)$. First F extends to the logarithmic complex by the formula: $F^p(\Omega_X^*(LogY) \otimes \mathcal{L}_X) = s(\Omega_X^q(LogY) \otimes F^{p-q}(\mathcal{L}_X), \nabla_X)_{p\leq 0})$, then F extends to $(\Psi_f(\mathcal{L}_X)$ via the formula

$$F^r(i_Y^\star s(\Omega_X^*(LogY) \otimes \mathcal{L}_X[i])_{i\leq 0}) = i_Y^\star s(F^{r+p+1}(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[i], \eta)_{i\leq 0}$$

The definition of the global weight filtration reduces to the local construction at a point $y \in Y_M^*$, using the quasi-isomorphism $(\Psi_f\ (\mathcal{L}^{e(\alpha.)}))_y \ \simeq\ \oplus_\beta\ \Psi^\beta(L^{e(\alpha.)})$.

We suppose again $\mathcal{L}$ unipotent and define as previously $(\Psi_f^u\mathcal{L})_{S(M)} := s(\Psi_f^u\mathcal{L}(s.))_{s.\in S(M)}$ which can be viewed also as $s(\Omega^*\mathcal{L}[p], \eta)_{p\leq 0}$, then we define on it the weight filtration

$$(30)\quad \mathcal{W}_r(\Psi_f^u\mathcal{L})_{S(M)} = i_Y^\star s(\mathcal{W}_{r+2p-1}\Omega^*\mathcal{L}[p], \eta)_{p\leq 0},\ F^r(\Psi_f^u\mathcal{L})_{S(M)} = i_Y^\star s(F^{r+p+1}\Omega^*\mathcal{L}[p], \eta)_{p\leq 0}$$

The logarithm of the monodromy $\mathcal{N}$ is defined on this complex as in the local case. The filtration $W(\mathcal{N})$ is defined on $(\Psi_f^u\mathcal{L})_{S(M)}$ in the abelian category of perverse sheaves.

Theorem: Suppose $\mathcal{L}$ underlies a unipotent variation of polarised Hodge structures of weight m, then the graded part of the weight filtration (30) of the complex

$$(\Psi_f^u(\mathcal{L}_X), \mathcal{W}[m], F)$$

decomposes into a direct sum of intermediate extension of VHS; moreover we have $W(\mathcal{N}) = \mathcal{W}$.

The proof of this theorem reduces by definition to show that $(Gr_r^{\mathcal{W}[m]}, F)$ decomposes which result can be reduced to the local case where it has been checked in the above corollaries.

Remark: We could as well define the complex $(\Psi^0 L)_{S(M)}$ by summing over $p \geq 0$:

$$(31)\qquad (\Psi^0\mathcal{L})_{S(M)} = i_Y^\star s[(s(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[p], \eta)]_{p\geq 0}[1]$$

By the above remarks the two definitions give quasi-isomorphic complexes.

3.3 The global weighted complex of $(\Psi_f(\mathcal{L}), \mathcal{W}, \mathcal{F})$

Let $y \in Y_M^*$, we deduce from the isomorphism

$$(32)\qquad (\Psi_f\ (\mathcal{L}^{e(\alpha.)}))_y \ \simeq\ \oplus_\beta\ \Psi^\beta(L^{e(\alpha.)})$$

the global weight filtration in the abelian category of perverse sheaves on Deligne's extension $(\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_\beta)_X$ and the associated combinatorial logarithmic complex $\Omega^*(\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_\beta)$ where we define the global weight filtration in the abelian category of perverse sheaves at points y such that $\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_\beta$ is unipotent since otherwise it is acyclic near y and doesn't contribute to cohomology.

Finally we can define the combinatorial logarithmic filtered complex as:

$$(\Psi_f(\mathcal{L}), \mathcal{W}) := \oplus_{(\alpha.,\beta)}(\Psi_f^u(\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_\beta), \mathcal{W}))$$

The Hodge filtration F extends to the logarithmic complex and to $\Psi_f(\mathcal{L}_X)$.

Theorem: Suppose $\mathcal{L}$ underlies a variation of polarised Hodge structures of weight m, then the complex

$$(\Psi_f(\mathcal{L}_X), \mathcal{W}[m], F)$$

decomposes into a direct sum of intermediate extension of VHS; moreover we have $W(\mathcal{N}) = \mathcal{W}$ where $\mathcal{N}$ is the logarithm of the unipotent part of the monodromy T^u.

II. The weight filtration after M. Kashiwara and M. Saito

Local situation. We give in this subsection Kashiwara and Saito's constructions and indications on the proofs of the decomposition and the purity results for $\Psi_f^u(\mathcal{L}$ [29] in order to compare the two constructions. In the reference this result is embedded in the language and theory of Hodge modules, a theory adapted for general pushforward results but not necessary at this stage.

Given $\alpha_i \in [0,1[$ for $i \in [1,n]$ (or equivalently a section τ with value in $]0,1]$), we consider the polynomial ring $\mathbb{C}[N]$ in one variable (resp. the field $\mathbb{C}[N,N^{-1}]$) and the module $L[N] = L\otimes_{\mathbb{C}}\mathbb{C}[N]$ (resp. $L[N,N^{-1}]$) endowed with commuting endomorphisms $(\alpha_i Id + N_i) \otimes Id$ where N_i is nilpotent, denoted also by $\alpha_i Id + N_i$, and multiplication by N denoted also by N. For each family of integers $n_i > 0$, for $i \in [1,n]$, we consider the endomorphisms $A_i = \alpha_i Id + N_i - n_i N$ on $L[N]$ (resp. $L[N,N^{-1}]$). When $\alpha_i \neq 0$, A_i is invertible on $L[N]$ and when $\alpha_i = 0$, the inverse of the endomorphisms A_i are defined on $L[N,N^{-1}]$ and equal to

$$A_i^{-1} = -\Sigma_{j\geq 0}(N_i)^j/(n_i N)^{j+1}$$

where the sum is finite since N_i is nilpotent for all i. In particular A_i and $A_J = \Pi_{i\in J}A_i$, $J \subset [1,n]$, are injective on $L[N]$ so that we can deduce

Lemma. Given $(L, \alpha_i Id + N_i)_{i\in[1,n]}$, we let $I(\alpha.) = \{i \in [1,n] : \alpha_i = 0\}$
i) The complex $\Omega(L[N,N^{-1}],\ A_i = \alpha_i Id + N_i - n_i N, i \in [1,n])$ is acyclic.
ii) The following complexes are isomorphic

$$\Omega(L[N^{-1}], A.) \xrightarrow{\sim} \Omega(L[N], A.)[1]$$

iii) The complex

$$IC(L[N], A.) = s(Im A_{J\cap I(\alpha.)})_{J\subset[1,n]} \simeq 0, \quad (Im A_{J\cap I(\alpha.)} = Im A_J) \tag{33}$$

is an acyclic sub-complex of the Koszul complex $\Omega(L[N], A.)$.
iv) Let $\Psi_J(L) := L[N]/Im A_{J\cap I(\alpha.)}$ and $\Psi^0 L = s(\Psi_J(L)_{J\subset[1,n]}, (A_i)_{i\in[1,n]})[1]$, be associated to the simplicial complex with differential induced by $A_i : \Psi_{J-i}(L) \to \Psi_J(L)$, then we have the following isomorphism

$$\Omega(L[N], A.) \overset{\Pi}{\xrightarrow{\sim}} \Psi^0 L = s(\Psi_J(L)_{J\subset[1,n]}, A.)[1] = s(L[N]/Im(A_{J\cap I(\alpha.)}), A.)_{J\subset[1,n]}[1] \tag{34}$$

We give the statement for $\Psi^\beta(L^{e(\alpha.)})$ in general
Proposition: Given $(\alpha.) = (\alpha_j)_{j\in[1,n]}$ and β, we consider on $L^{e(\alpha.)}$ the endomorphisms $A_i = (\tau(\alpha_i + \beta n_i)Id + N_i - n_i N)$ for $i \in [1,n]$ then we have the isomorphisms:

$$\Psi^\beta(L^{e(\alpha.)}) = \Omega(L^{e(\alpha.)}[N^{-1}], A.) \simeq \Omega(L^{e(\alpha.)}[N], A.)[1].$$

Let $Im A_J = Im(A_{J\cap I(\tau(\alpha.+\beta n.)}$ in $L^{e(\alpha.)}$ denotes the image of the composition $A_J = \Pi_{j\in J}A_j$ then this Koszul complex is isomorphic to

$$\Psi^\beta(L^{e(\alpha.)}) \ = \ s((L^{e(\alpha.)}[N]/Im A_{J\cap I(\tau(\alpha.+\beta n.))}), A.)_{J\subset[1,n]}[1]. \tag{35}$$

Let $S(\alpha.) = \{\gamma \in \mathbf{C} : \forall j \in M, \alpha_j + n_j\gamma \in \mathbf{N}\}$. Only for $\beta \in S(\alpha.)$ the complex is not acyclic.

Remark i) The importance of the introduction of $\Psi_J(L)$ is that they are canonically associated to the perverse sheaf $\Psi^0(L)$, so that the construction of the weight filtration reduce to its construction on these vector spaces. It is more precise to work on these vector spaces then on the cohomology of the perverse sheaf, the relation being a kind of spectral sequence.

ii) We can give now a proof of the isomorphism (20) of the proposition in this paragraph. Recall that $\Psi_J L = L[N]/Im A_J$ and we have $Coker N/\Psi_J L \simeq Coker N_J/L$ since $A_i = N_i - n_i N$ is equal to N_i modulo N, so that we have locally at $y \in Y^*_M$ the isomorphisms

$$(\mathbf{j}_*\mathcal{L}/\mathbf{j}_{!*}\mathcal{L})_y \simeq s(Coker N_J/L)_{J\subset[1,n]} \simeq Coker N/s(\Psi_J L)_{J\subset[1,n]} \simeq Coker \mathcal{N}/\Psi^u_f\mathcal{L}[-1]$$

which establishes (20).

iii) In general, the graded part of the cokernel is the primitive part $P_k(N)$ for all $k \geq 0$: $Gr_k^{W(N)}(Coker N/\Psi_J L) \simeq P_k(N)$.

The filtration $W(N)$ on $\Psi_J L$ defines a filtration by sub-complexes of $\Psi^0 L$ and corresponds to the filtration $W(\mathcal{N})$ of $\Psi^u_f(\mathcal{L})$.

Now in order to study the weight filtration we need to consider this complex as a perverse sheaf in the corresponding abelian category. That is why we recall here basic facts on this category needed to understand the construction.

The *category of perverse sheaves* $\mathcal{L}^\cdot$ on X with respect to the natural stratification $Y^\star_M$ defined by Y (i.e such that for each $M \subset I$, the cohomology of $\mathcal{L}^\cdot/Y^*_M$ is locally constant), are described locally at a point y considered as the center of a polydisc $(D^*)^M$, from a topological view point, by the following combinatorial construction in [25, p 996] (see also [16], [2]).

The category $\mathcal{P}$ of perverse sheaves $\mathcal{L}^\cdot$ on $(D^*)^M$, with respect to its NCD stratification is equivalent to the abelian category defined as follows :
i) A family of vector spaces L_A for $A \subset M$,
ii) A family of morphisms
$f_{AB}: L_B \to L_A$ and $h_{BA}: L_A \to L_B$ for $B \subset A \subset I$ such that :
$f_{AB} \circ f_{BC} = f_{AC}$, $h_{CB} \circ h_{BA} = h_{CA}$ for $C \subset B \subset A$
$f_{AA} = h_{AA} = id$, $h_{A,A\cup B} \circ f_{A\cup B,B} = f_{A,A\cap B} \circ h_{A\cap B,B}$ for all A, B
and if $A \supset B, \mid A \mid = \mid B \mid +1$, then $1 - h_{BA} f_{AB}$ is invertible.

Minimal extensions

We will need the following description for $A \subset M$ of the caegory $\mathcal{M}_A$ of the minimal extensions of a locally constant sheaf $\mathcal{L}$ on $X^\star_A$: in terms of the family of vector spaces L_B for $B \subset M$; it is equivalent to $L_B = 0$ for $A \not\subseteq B$, and f_{BA} is surjective and g_{AB} is injective for $A \subset B$. We denote by $\mathcal{M}$ the objects isomorphic to a direct sum of objects in $\cup_A \mathcal{M}_A$.

The category $\mathcal{M}$ of sums of minimal extensions

A result of Kashiwara states [25, p 997]

A perverse sheaf $\mathcal{L}^\cdot \in \mathcal{P}$ is a direct sum of minimal extensions (in $\mathcal{M}$) if and only if

$$(36) \qquad \forall A, B \subset M,\ B \subset A, \quad L_A \simeq Im f_{AB} \oplus Ker g_{BA}$$

Moreover, it is enough to consider $|A| = |B| + 1$.
The above condition is equivalent to the isomorphism:

$$(37) \qquad \oplus_{B\subset A} f_{AB}(P_B(\mathcal{L}^\cdot)) \xrightarrow{\sim} L_A$$

where $P_B(\mathcal{L}^{\cdot}) = \cap_{C \subsetneq B} Ker\, g_{CB}$*, then moreover* $g_{BA} : f_{AB}(P_B(\mathcal{L}^{\cdot})) \to P_B(\mathcal{L}^{\cdot})$ *is injective for* $B \subset A$.

Description of the weight filtraion in the category of perverse sheaves.
The family $\Psi^{\beta}_J(L^{e(\alpha.)}) = L^{e(\alpha.)}[N]/ImA_{J\cap I(\tau(\alpha.+\beta n.)}$ for $J \subset M, J \neq \varnothing$ gives precisely the description of $\Psi^{\beta}(L^{e(\alpha.)})$ as a perverse sheaf, where for $i \in J$, the morphisms $f_{J(J-i)} = A_i : \Psi^{\beta}_{J-i}(L^{e(\alpha.)}) \to \Psi^{\beta}_J(L^{e(\alpha.)})$ and $g_{J(J-i)} = p_i : \Psi^{\beta}_J(L^{e(\alpha.)}) \to \Psi^{\beta}_{J-i}(L^{e(\alpha.)})$ is the canonical projection. The product by N induces on each $\Psi^{\beta}_J(L^{e(\alpha.)})$ a nilpotent endomorphism denoted also by N which commutes with A_i and p_i, hence these morphisms are compatible with $W(N)$; they send $W_{r-1}(N)$ into itself (it is enough to show that for $b \in \Psi_{J-i}(L^{e(\alpha.)})$, if $N^s(b) = 0$ for $s \geq r$, $N^s(A_i(b)) = A_i(N^s(b) = 0$ (resp.for p_i)).

For each integer r, let $Gr^{W(N)}_r \Psi^{\beta}_J(L^{e(\alpha.)}), p^r_i, A^r_i$ denotes the corresponding perverse graded objects, then we define

$$K^r_i = Ker\ p^r_i : Gr^{W(N)}_r \Psi^{\beta}_J(L^{e(\alpha.)}) \to Gr^{W(N)}_r \Psi^{\beta}_{J-i}(L^{e(\alpha.)}), \quad K^r_J :\ \cap_{i \in J} K^r_i \tag{38}$$

in particular $\forall i \in J, K^r_J \subset KerN_i \subset Gr^{W(N)}_r \Psi^{\beta}_J(L^{e(\alpha.)})$. The aim of the next part is to deduce the decomposition property (37) via the proof of (36) in presence of a polarised Hodge filtration.

First we give a global setting of the problem.

3.4 The global weighted complex of nearby cycles $(\Psi_f(\mathcal{L}), \mathcal{W}(\mathcal{N})$ In this subsection, we define the weight filtration abstractly without going back to an explicit formula as used on the combinatorial logarithmic complex. The filtration $W(N)$ on each $\Psi^{\beta}(L^{e(\alpha.)})$ defines a filtration by sub-complexes on $\oplus_{\alpha.}\Psi^{\beta}L^{e(\alpha.)}$ and corresponds via (32) to the filtration $W(\mathcal{N})$ of $\Psi_f(\mathcal{L})_y$ in the abelian category of perverse sheaves where $\mathcal{N} = -1/2i\pi LogT^u$.
Consider Deligne's extension $(\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_{\beta})_X$ and the associated logarithmic complex

$$i^{\star}_Y s(\Omega^*_X(LogY) \otimes (\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_{\beta})_X[p], \eta)_{p \geq 0}.$$

There exists a global acyclic sub-complex $IC((\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_{\beta})_X[N])$ inducing at each fiber at the point y the complex $IC(L^{e(\alpha.)}[N], A_i = (\tau(\beta n_i + \alpha_i) + N_i - n_i N))$. We define the weight filtration $W(\mathcal{N})$ on the quotient complex

$$\Psi^u_f((\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_{\beta})_X) := (s(\Omega^*_X(LogY) \otimes (\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_{\beta})_X[p], \eta)_{p \geq 0} / IC(\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_{\beta})_X[N]))[1]$$

as the filtration inducing $W(N)$ at each fiber $\Psi^{\beta}(L^{e(\alpha.)})$ at points of Y. Finally we can define the logarithmic filtered complex as:

$$(\Psi_f(\mathcal{L}_X), W(\mathcal{N})) := \oplus_{(\alpha.,\beta)}(\Psi^u_f((\mathcal{L}^{e(\alpha.)} \otimes f^{-1}\mathcal{V}_{\beta}))_X, W(\mathcal{N})) \tag{39}$$

The filtration F extends to $(\Psi_f(\mathcal{L}_X)$ via its extension to $\Omega^*_X(LogY) \otimes \mathcal{L}_X[N][1]$ by the formula

$$F^p(i^{\star}_Y s(\Omega^*_X(LogY) \otimes \mathcal{L}_X[i])_{i \geq 0})[1] = i^{\star}_Y s(F^{p+i+1}(\Omega^*_X(LogY) \otimes \mathcal{L}_X)[i], \eta)_{i \geq 0}[1]$$

$$= i^{\star}_Y s[(s(\Omega^q_X(LogY) \otimes F^{p+i+1-q}\mathcal{L}_X)_{q \geq 0})[i], \eta)]_{i \geq 0}[1].$$

Theorem: Suppose $\mathcal{L}$ *underlies a variation of polarised Hodge structures of weight* m*, then graded part of the weight on the complex (39) with the filtration* F *defined above*

$$(\Psi_f(\mathcal{L}_X), W(\mathcal{N})[m], F) \tag{40)}$$

decomposes into intermediate extensions of VHS.

The proof of this theorem reduces by definition to show that $(Gr_r^{W(\mathcal{N})[m]}, F)$ decomposes. This can be checked locally via (36) and (37). That is we need to use the following decomposition theorem based on results due to Kashiwara [25] in characteristic zero and proved in [2] in the language of purity in positive characteristic.

Theorem (decomposition)(Kashiwara - Saito): For each onteger a, $Gr_a^{W(\mathcal{N})}(\Psi_f(\mathcal{L}^{e(\alpha.)}))_y \simeq Gr_a^{W(N)}\Psi^\beta(L^{e(\alpha.)})$ *is isomorphic to a direct sum of fibers at* y *of various intermediate extension of variations of polarised Hodge structures. Precisely*

$$Gr_a^{W(N)}\Psi^\beta(L^{e(\alpha.)}) \simeq \oplus_{J\subset M} IC((K_J^a(L^{e(\alpha.)}), N_i, i \in M - J) \qquad (41))$$

where K_J^a*, defined by* (38)*, is a pure Hodge structure of weight* $a+m$ *with the induced Hogde filtration* F.

i) Elements of Kashiwara's proof [30, prop. 3.19, and Appendix]. We will write L for $L^{e(\alpha.)}$ and associate to $(L, F, P, N_i, i \in M = [1, n])$ the module $L[N]$ where N is a polynomial variable, endowed with two filtrations as follows. Consider $W(L) = W(\Sigma_{i\in M} N_i)[m]$ and F on L, then define

$$W_k(L[N]) = \Sigma_j W_{k+2j} L \otimes N^j\ ,\ F^p(L[N]) = \Sigma_j F^{p+j} L \otimes N^j \qquad (42)$$

Since the endomorphisms $A_i = N_i - n_i N$ shift W by -2 and F by -1, the two filtrations induce a MHS on the cokernel $\Psi_J = L[N]/ImA_J$ for $J \subset M$. We have an isomorphism compatible with the filtrations

$$(\oplus_{j\leq l-1} L \otimes N^j, W, F) \quad \simeq \quad (\Psi_J L, W, F) \qquad (43)$$

obtained via the composition of the natural embedding in $L[N]$ with the projection on $\Psi_J L$, where W anf F are defined on the left term as in the formula (42) above. In fact the relation : $N^l = \Sigma_{j\in[1,l]}(-1)^{j+1}\sigma_j(\frac{N_i}{n_i}, i \in J) \otimes N^{l-j}$ where $l =\mid J \mid$ and σ_j is the j^{th} elementary symmetric function of $\frac{N_i}{n_i}, i \in J$, on the quotient of the right term leads to the definition of the action of N on the left term by the formula :

$$N(a \otimes N^{l-1}) = \Sigma_{1\leq j\leq l}(-1)^{j+1}\sigma_j((N_i/n_i), i \in J)(a) \otimes N^{l-j}.$$

In order to define a polarisation we introduce a product P_J on $\Psi_J(L)$ as follows

$$P_J(aN^i, bN^j) = P(a, (-1)^i \quad Res\ (A_J^{-1}(b \otimes N^{i+j})))$$

where A_J^{-1} is defined on $L[N, N^{-1}]$, N is considered as a variable x and the residue Res is equal to the coefficient of $1/N$ in the fraction in N. This formula shows directly that the product is well defined on $Coker A_J$; in fact, $P_J(aN^i, A_J(c) = P(a, (-1)^i \quad \mathrm{Res}(c \otimes N^i) = 0$ since the residue is zero. Using an explicit expression of A_i^{-1}, we find $P_J(aN^p, bN^q) = (-1)^p P(a, \Sigma_{a_i}(\Pi_i(N_i^{a_i}(b)/n_i^{a_i+1}))$ where $a_i \geq 0$ and $\Sigma_i a_i = i + j - l + 1$. In particular

$$P_J(a, bN^r) = (1/\Pi_i n_i)P(a, b) \ \text{ if } r = l - 1, \text{ and zero otherwise}$$

$$P_J(aN^i, bN^j) = (-1)^i P_J(a, bN^{i+j}).$$

In [30], the following result is attributed to Kashiwara

Theorem : With the previous notations, namely W *and* F

$$\Psi_J(L) = (L[N]/ImA_J, N_1, \dots, N_n, N; W, F, P_J) \qquad (44)$$

underlies a polarised nilpotent orbit of weight $m+1-|J|$, *that is: the weight filtration* $W(N+\Sigma_{i\in M}N_i)[m+1-|J|]=W$ *underlies the weight of a* MHS *on* $\Psi_J(L)$ *with the Hodge filtration* F.

ii)(see M. Saito [29,5.2.15, 5.2.14], [30,3.20.4]). The induced morphisms N, N_i and A_i shift W by -2 and F by -1. Since $W(N+\Sigma_{i\in J}N_i)$ is the weight filtration of the endomorphism $\Sigma_{i\in J}N_i$ relative to $W(N)$ that is for all $\Psi_J(L)$:

$$Gr_{j+r}^{W(N+\Sigma_{i\in J}N_i)}Gr_r^{W(N)}\overset{(\Sigma_{i\in J}N_i)^j}{\longrightarrow}Gr_{r-j}^{W(N+\Sigma_{i\in J}N_i)}Gr_r^{W(N)}$$

we have:

$$Gr_{j+r}^{W(N+\Sigma_{i\in J}N_i)}Gr_r^{W(N)}\simeq Gr_j^{W(\Sigma_{i\in J}N_i)}Gr_r^{W(N)}\simeq Gr_j^{W(\Sigma_{i\in J}A_i)}Gr_r^{W(N)}$$

Now we may consider the orbit with only two endomorphisms $(\Psi_J(L), N_i, N, F=F(N_i))$ (F is the limit along the axis Y_i), then we deduce commutative diagrams for j varying in an interval of $\mathbf{Z}$ symmetric with center 0 with at left HS of weight $n+j$ where $n=m+a-|J|$ and to the right $n+j-1$

$$\begin{array}{ccc} Gr_j^{W(N_i)}Gr_a^{W(N)}\Psi_{J-i}L & \overset{A_i}{\longrightarrow} & Gr_{j-1}^{W(N_i)}Gr_a^{W(N)}\Psi_J L(-1) \\ \downarrow N_i & \swarrow p_i & \downarrow N_i \\ Gr_{j-2}^{W(N_i)}Gr_a^{W(N)}\Psi_{J-i}L(-1) & \overset{A_i}{\longrightarrow} & Gr_{j-3}^{W(N_i)}Gr_a^{W(N)}\Psi_J L(-2) \end{array} \tag{45}$$

moreover we have: $P_J(A_iu,v)=P_{J-i}(u,p_iv)$ for all $u\in\Psi_{J-i}L$ and $v\in\Psi_JL$
In this situation, a result of M. Saito [MI,5.2.15] applies and shows

Proposition : For all $J\subset I$ *and* $i\in J$, *consider the morphisms*

$$Gr_a^{W(N)}\Psi_{J-i}L\overset{A_i}{\longrightarrow}Gr_a^{W(N)}\Psi_JL(-1)\overset{p_i}{\longrightarrow}Gr_a^{W(N)}\Psi_{J-i}L(-1)$$

then we have a decomposition

$$Gr_a^{W(N)}\Psi_JL \quad\simeq\quad ImA_i\oplus ker\,p_i \tag{46}$$

compatible with the primitive decomposition . In particular, p_i *induces an isomorphism of* ImA_i *in* $Gr_a^{W(N)}\Psi_JL$ *onto* ImA_i *in* $Gr_a^{W(N)}\Psi_{J-i}L$.

The result is deduced from the sequence in the proposition by taking its graded version $Gr_{j-1}^{W(N_i)}$ for various j as in (45) and using the polarisation of HS induced on, to prove for each j,

$$ImA_i\oplus Kerp_i\simeq Gr_{j-1}^{W(N_i)}Gr_a^{W(N)}\Psi_JL(-1)$$

hence, since A_i and p_i are compatible with the MHS of weight $W(N_i)$, we get:
$ImA_i\oplus Kerp_i\simeq Gr_a^{W(N)}\Psi_JL(-1)$.

Now, to finish the proof of the decomposition theorem, it remains to show that K_J^a is pure and polarised in two steps:

Lemma: i)$K_J^a\subset W_0(\Sigma_{i\in J}N_i)Gr_a^{W(N)}\Psi_JL$.
ii) $K_J^a\cap(W_{-1}(\Sigma_{i\in J}N_i)Gr_a^{W(N)}\Psi_JL)=0$.

Proof. i) the assertion (i) follows from the relation: $Ker(\Sigma_{i\in J}N_i)\subset W_0(\Sigma_{i\in J}N_i)Gr_a^{W(N)}\Psi_JL$.
ii) Suppose $x\in W_{-s}(\Sigma_{i\in J}N_i)Gr_a^{W(N)}\Psi_JL\cap K_J^a$ where $-s\leq -1$, then there exists $y\in W_s(\Sigma_{i\in J}N_i)Gr_a^{W(N)}\Psi_JL = W_s(\Sigma_{i\in J}A_i)Gr_a^{W(N)}\Psi_JL$ such that $x=(\Sigma_{i\in J}A_i)^s(y)$ (by surjectivity of $\Sigma_{i\in J}A_i$ on negative weights) then for each i, we have $(N_i)^s(y)\in ImA_i\ mod\ W_{-s-1}$,

hence $x = \sum_i N_i^s(y)$ is in $(\cap_i ker p_i) \cap \sum_i Im A_i = 0 \; mod \; W_{s-1}$, that is $x \in W_{-s-1}(\Sigma_{i\in J} N_i) Gr_a^{W(N)} \Psi_J L$. We deduce (ii) by a descending inductive argument on $-s$.

We deduce from the lemma that K_J^a is pure of weight a which ends the proof of the theorem.

III. Example : Rank one local system $\mathcal{L}$ on $X - Y$.

We apply the above theory to remove the base change in Steenbrink's work. In this case the monodromy of $\mathcal{L}$ around components of Y is of the following form: $\forall i \in I, T_i = \alpha_i Id, N_i = 0$. For $\beta \in \mathbb{Q} \cap [0,1[$, let $\mathcal{V}_\beta$ denotes the rank one local system on the punctured disc with monodromy $e^{-2i\pi\beta}$, $\mathcal{S} := \mathcal{L} \otimes f^{-1}\mathcal{V}_\beta$ and $\mathcal{S}_X$ its Deligne's extension; then we can define the weight filtration $W(\mathcal{N})$ explicitly on the complex

$$\Psi_f^u((\mathcal{S}_X) := i_Y^\star s(\Omega_X^*(LogY) \otimes \mathcal{S}_X)[p], \eta)_{p\geq 0}[1]$$

First we define W on $\Omega_X^*(LogY) \otimes \mathcal{S}_X$. Let $I(\beta) : \{i \in I : \alpha_i + \beta n_i \in \mathbb{Z}\}$, $Y(\beta) = \cup_{i\in I(\beta)} Y_i$, $C(\beta) = \cup_{i\in I-I(\beta)} Y_i$, notice that $\mathcal{S}_X$ is locally trivial along $Y(\beta) - C(\beta)$ and its logarithmic complex is acyclic along $C(\beta)$, that is for $j : X - Y \to X$, we have: $(\mathbf{j}_*\mathcal{S})_{/X-Y(\beta)}) = (\mathbf{j}_!\mathcal{S})_{/X-Y(\beta)}$.
We write $\Omega_X^*(LogY)$ as $\Omega_X^*(LogY(\beta) \otimes \Omega_X^*(Log\, C(\beta))$ and extend the logarithmic weight filtration $W^{Y(\beta)}$ along $Y(\beta)$ to the whole complex by
$W := [W^{Y(\beta)}(\Omega_X^*(LogY(\beta))] \otimes \Omega_X^*(LogC(\beta)) \otimes \mathcal{S}_X$. Then we have
$Gr_i^W(\Omega_X^*(LogY) \otimes \mathcal{S}_X) \simeq (Gr_i^{W^{Y(\beta)}}(\Omega_X^*(LogY(\beta))) \otimes \Omega_X^*(LogC(\beta)) \otimes \mathcal{S}_X \simeq$
$\oplus_{J\subset I(\beta), |J|=i} \Omega_{Y_J}^*(Log(C(\beta) \cap Y_J))[-i] \otimes \mathcal{S}_{Y_J}$
with the differential of the induced connection on $\mathcal{S}_{Y_J}$.
Locally, let L denotes the general fiber of $\mathcal{S}$, then the fiber of the logarithmic complex at a point $y \in Y_M^*$ is isomorphic to the Koszul complex $(\Omega(L, \tau(\alpha_i + \beta n_i)Id, i \in M)$ where L_J corresponds to $L \otimes \wedge_{i\in J} \frac{dz_i}{z_i}$. For each $i \in M$, let $(i,\beta) = \{j \in I - I(\beta) : Y_j \cap Y_i \neq \varnothing\}$, then the fiber of Gr_i^W is:

$$Gr_i^W \simeq \oplus_{i\in I(\beta)} (\Omega(L, \tau(\alpha_j + \beta n_j)Id, j \in (i,\beta))$$

This weight filtration and the Hodge filtration extend to $s(\Omega_X^*(LogY) \otimes \mathcal{S}_X[p], \eta)_{p\geq}[1]$ by the formula

$$W_i = \oplus_{p\geq 0} W_{i+2p+1}, \;\; F^i = \oplus_{p\geq 0} F^{i+p+1}.$$

Here the fiber of the double complex at y is isomorphic to the Koszul complex $(\Omega(L[N], A_i = \tau(\alpha_i + \beta n_i)Id - n_i N)$ where A_i is an isomorphism whenever $\tau(\alpha_i + \beta n_i) \neq 0$, that is $i \in I - I(\beta)$.
Now we introduce the acyclic subcomplex $\mathcal{K} \simeq \oplus_{p\geq 0} W_p[p+1]$, whose fiber at y in $(\Omega(L[N], A_i = \tau(\alpha_i + \beta n_i)Id - n_i N, i \in M)$ is given as $s(K_J)_{J\subset M}$ where $K_J = Im A_J = N^r L[N]$ whenever $\mid J \cap I(\beta) = J(\beta) \mid = r$ $(A_J = \Pi_{i\in J} A_i)$.
The quotient complex with induced filtration:

$$\Psi_f^\beta(\mathcal{L}) \cong (i_Y^\star[s(\Omega_X^*(LogY) \otimes \mathcal{S}_X[p], \eta)_{p\geq 0}[1]]/\mathcal{K}, W, F)$$

is the bifiltered complex computing $\Psi_f^\beta(\mathcal{L})$. The quotient complex has fiber at y:

$$s(\Psi_J(L), A_i, i \in M)_{J\subset M}, \Psi_J(L) = (L[N]/Im A_J) \simeq \oplus_{p\in[0,r-1]} LN^p.$$

Let $\mathcal{K}_p$ denotes the pth column of $\mathcal{K}$, then the graded object with respect to W is:

$$Gr_i^W(\Psi_f^\beta(\mathcal{L}) \simeq i_Y^\star s(Gr_{i+2p+1}^W[(\Omega_X^*(LogY) \otimes \mathcal{S}_X)[p]]/\mathcal{K}_p)_{p\geq 0}[1] \simeq$$

$$s(\oplus_{J\subset I(\beta),|J|=i+2p+1}\Omega^*_{Y^J}(Log(C(\beta)\cap Y^J))\otimes \mathcal{S}_{Y^J}[-i-p])_{p\geq 0,p\geq -i,p\leq n=dim\, X}$$

Proposition: ${}_W E_1^{p,q}((\Psi_f^\beta(\mathcal{L})) = \oplus_{i\geq 0, i\geq p}\oplus_{J\subset\beta} H_c^{2p+q-2i}(Y_J - C(\beta))^{-p+2i+1}, \mathcal{S}_/)$
where $\mathcal{S}_/$ *is a local system on* Y_J *deduced from* $\mathcal{S}$ *as restriction of Deligne's extension* $\mathcal{S}_X$ *and has zero restriction to* $C(\beta)$*.*

The weight spectral sequence is: ${}_W E_1^{p,q}((\Psi_f^\beta(\mathcal{L})) :=$
$H^{p+q}(X, Gr^W_{-p}((\Psi_f^\beta(\mathcal{L})) = \oplus_{i\geq 0,i\geq p} H^{p+q-i}(X, Gr^W_{-p+2i+1}(i_Y^\star[(\Omega_X^*(LogY)\otimes\mathcal{S}_X)[i]]/\mathcal{K}_i)[1]) =$
$\oplus_{i\geq 0,i\geq p} H^{p+q+1}(\tilde{Y}, \Omega^*_{\tilde{Y}(\beta)^{-p+2i+1}}(Log(C(\beta)\cap\tilde{Y}(\beta)^{-p+2i+1})\otimes\mathcal{S}_{\tilde{Y}(\beta)^{-p+2i+1}}[p-2i-1]) =$
$\oplus_{i\geq 0,i\geq p} H_c^{2p+q-2i}(\tilde{Y}(\beta)-C(\beta))^{-p+2i+1}, \mathcal{S}_/)$ where $\mathcal{S}_/\simeq\mathcal{S}_{\tilde{Y}(\beta)^{-p+2i+1}}$ is induced by $\mathcal{S}_X$ and has zero restriction to $C(\beta)$.

4§. Variation of Mixed Hodge structures

Let (L, W^0) be a filtered object in an abelian category and N a nilpotent endomorphism of (L, W^0). Deligne [10, (6.1.13)] introduced the notion of relative weight filtration W of N with respect to W^0 on L and showed that if it exists, it is the unique filtration satisfying for all $a\in\mathbb{Z}, b\in\mathbf{N}$

$$NW_a\subset W_{a-2} \text{ and } N^b: Gr^W_{a+b}Gr^{W^0}_a L\simeq Gr^W_{a-b}Gr^{W^0}_a L$$

A variation of mixed Hodge structures $(VMHS)$: (L, W^0, F) is called good [10,(1.8.15)] if there exists a relative weight filtration W for the action of the logarithm of the monodromy N. We showed in three notes developed in [13], the existence of a limit relative weight filtration W for geometric $VMHS$ inducing on $Gr^{W^0}L$ the limit of the VHS on $(Gr^{W^0}L, F)$. Steenbrink and Zucker called an axiomatic $VMHS$ admissible if it is good and satisfy a set of properties all satisfied by the geometric case [25], [34]. In this section we show that the definition of the weight filtration extends to this case without major difficulties using only the ingredients of proofs already introduced in the previous cases.

4.1 Good $VMHS$.

Let $V_X = (\mathcal{L}, (\mathcal{L}^{\mathbb{Q}}, W^0), (\mathcal{L}^{\mathbb{C}}, W^0\otimes\mathbb{C}, F))$ be a unipotent $VMHS$ on $X-Y$ and $\mathcal{L}_X$ its canonical extension, then W^0 (finite) extends to a filtration by subbundles. We say that V_X is good if the following properties are satisfied :
i) the filtration F extends to $\mathcal{L}_X$ as a filtration by sub-bundles
ii) for all $J\subset I$, the relative filtration $M_J := W(N_J, W^0_{Y_J^*})$ exists (it is a filtration by sub-local systems).
iii) The limit filtrations (M_J, F) define a $VMHS$: $V_{Y_J^*}$ on Y_J^*; moreover $W^0_{Y_J^*}$ is a filtration by sub-$VMHS$ such that the induced $VMHS$ on $Gr^{W^0}\mathcal{L}^{\mathbb{Q}}_{Y_J^*}$ coincides with the limit of the VHS on $Gr^{W^0}\mathcal{L}^{\mathbb{Q}}_{X^*}$ ($V_{Y_J^*}$ is called the limit of V_{X^*} on Y_J^*).
iv) Compatibility: let $K, J\subset I$, then the $VMHS$ $V_{Y_J^*}$ (iii) satisfy (ii) on Y_J^* and its limit $VMHS$ on $Y^*_{J\cup K}$ coincides with the $VMHS$: $V_{Y^*_{J\cup K}}$ limit of the $VMHS$: V_{X^*}, that is to say

$$W(N_{J\cup K}, W^0_{Y^*_{J\cup K}}) = W(N_K, W(N_J, W^0_{Y_J^*})).$$

The last property is to be understood at each point $y\in Y^*_{J\cup K}$ where $W(N_J, W^0_{Y_J^*})$ extends, moreover the above properties are not independant. In a study [25], Kashiwara deduce the properties (ii) to (iv) from the existence of M_i for $i\in I$.
The $VMHS$ is said to be graded polarised if for all r, $Gr_r^{W^0}\mathcal{L}$ is polarised.

4.2 Local definition of the weight filtration.

For each subset $s_\lambda \subset I$ such that $Y_{s_\lambda} \neq \varnothing$ we write $\mathcal{W}^{s_\lambda} = W(\Sigma_{i\in s_\lambda}\mathcal{N}_i, W^0)$ for the filtration by subbundles defined by the nilpotent endomorphisms of the restriction $\mathcal{L}_{Y_{s_\lambda}}$ of $\mathcal{L}_X$ to $Y^*_{s_\lambda}$ (which exists by hypothesis), inducing also by restriction and for each subset $K \supset s_\lambda$ a filtration on $\mathcal{L}_{Y_K}$. We define the weight filtration $\mathcal{W}'(s.)$ for $s. \in S(I)$, locally near each point $y \in Y^*_M$, as in the case of VHS, in terms of a set of n coordinates $y_i, i \in [1,n]$ where we identify M with $[1,p]$ on an open set $U_y \simeq D^{|M|} \times D^{n-p}$ containing y and a section $f^{s.} = \Sigma_{J\subset M, J'\cap M=\varnothing} f^{s.}_{J,J'}\frac{dy_J}{y_J} \wedge dy_{J'}$ of $\Omega^*_{X_{s.}} \otimes \mathcal{L}_{X_{s.}}$

$$f^{s.} \in \mathcal{W}'_r(\Omega^*_{X_{s.}} \otimes \mathcal{L}_{X_{s.}})_{/U_y} \Leftrightarrow \forall J, N \subset M,\ f^{s.}_{J,J'}/Y_N \cap U_y \in \bigcap_{s_\lambda \in s.,\, s_\lambda \subset N} \mathcal{W}^{s_\lambda}_{a_\lambda(J,r)}\mathcal{L}_{(Y_N\cap U_y)}$$

It is a filtration by subcomplexes of analytic subsheaves globally defined on X. We deduce two filtrations $\mathcal{W}'$ and $\mathcal{W}$ of $\Omega^*\mathcal{L}$ as follows:

$$\mathcal{W}' = s(\mathcal{W}'(s.))_{s.\in S(I)},\ \ \mathcal{W}_r(s.) = \mathcal{W}'_r(s.) \cap W^0_r,\ \ \mathcal{W} = s(\mathcal{W}(s.))_{s.\in S(I)}$$

moreover the filtration W^0 extends as a constant filtration for all $s. \in S(I)$ and the previous definition of the Hodge filtration F remains unchanged. The filtrations $\mathcal{W}'$ and $\mathcal{W}$ will have different applications, the first leads to the filtration $W(\mathcal{N})$ on the nearby-cocycles and the second defines a MHS on $X - Y$ when X is proper.

Let $b \in \mathbb{Z}$ and $K \subset M$; when the point y is in $Y^*_K \subset Y^*_M$, the previous study apply to the nilpotent orbit $Gr^{W^0}_b L$ defined at the point y so that we can conclude that for $a > b$ (resp. $a < b$) the complex $C^K_a(Gr^{W^0}_b L)$ is concentrated in degree $\mid K \mid$ (resp. $\mid K \mid -1$ and acyclic for $a = b$. When the point y vary in Y^*_K, the cohomology $H^{|K|}(C^K_a Gr^{W^0}_b L)$ is of weight $a- \mid K \mid$ induced by $W^K_{a-|K|}(N, W^0 L)$ and F (resp. $H^{|K|-1}(C^K_a(Gr^{W^0}_b L)))$ of weight $a+ \mid K \mid$), defines a local system $\mathcal{L}^K_{a,b}$ on Y^*_K which underlies a VHS induced by F. We define as well the local system $\mathcal{L}^K_a$ of general fiber the cohomology of $C^K_a(W^0_{a-1}L)$ equal to $H^{|K|}(C^K_a W^0_{a-1}L)$ is of weight $a- \mid K \mid$ induced by $W^K_{a-|K|}(N, W^0 L)$ and F, underlies a polarised VHS induced by F graded by W^0.

Theorem. Let $(\mathcal{L}, \mathcal{W}', \mathcal{F})$ be a unipotent graded polarised good variation of mixed Hodge structures on X-Y; then the complex

$$(\Omega^*\mathcal{L}, \mathcal{W}', \mathcal{W}, W^0, F)$$

with the filtrations $\mathcal{W}'$, $\mathcal{W}$, W^0 and F defined above satisfy the following decomposition and purity properties

i) Purity: For all subset $K \subset I$ and all integers $a > b$ (resp. $a < b$), the VHS : $(\mathcal{L}^K_{a,b}, W, F)$ of general fiber $H^{|K|}(C^K_a(Gr^{W^0}_b L))$ (resp. $H^{|K|-1}(C^K_a(Gr^{W^0}_b L)))$ on Y^*_K, as well the VHS : $(\mathcal{L}^K_a, W, F)$ of general fiber $C^K_a(W^0_{a-1}L)$, graded polarised by W^0.

ii) The complex $Gr^{\mathcal{W}'}_a Gr^{W^0}_a \Omega^*\mathcal{L}$ is acyclic, hence

$$Gr^{\mathcal{W}}_a \Omega^*\mathcal{L} \simeq Gr^{\mathcal{W}'}_a W^0_{a-1}\Omega^*\mathcal{L} \oplus Gr^{W^0}_a \mathcal{W}'_{a-1}\Omega^*\mathcal{L}$$

iii)Decomposition : We have the following decomposition into intermediate extensions on Y_K of VHS $\mathcal{L}^K_{a,b}$ (resp. $\mathcal{L}^K_a$)

$$(Gr^{\mathcal{W}'}_a Gr^{W^0}_b \Omega^*\mathcal{L}, F) \simeq \oplus_{K\neq\varnothing, K\subset I} j^K_{!*}(\mathcal{L}^K_{a,b}[- \mid K \mid], W[2 \mid K \mid], F[- \mid K \mid])\ \ \textit{if } a > b,\ \ (j^K : Y^*_K \to Y_K),$$

$$(Gr^{\mathcal{W}'}_a Gr^{W^0}_b \Omega^*\mathcal{L}, F) \simeq \oplus_{K\neq\varnothing, K\subset I} j^K_{!*}(\mathcal{L}^K_{a,b}[1- \mid K \mid], W[-1], F)\ \ \textit{ifa} < b,$$

$$Gr^{\mathcal{W}}_a \Omega^*\mathcal{L} \simeq \oplus_{K\subset I} j^K_{!*}(\mathcal{L}^K_a[- \mid K \mid], W[2 \mid K \mid], F[- \mid K \mid]).$$

iv) When X is proper, the filtrations $\mathcal{W}$ anf F define a MHS on the cohomology of $X - Y$ with value in $\mathcal{L}$ and the filtration W^0 induces a filtration by $sub - MHS$.

Proof. i) Locally we reduce the problem to the study for $K \neq \emptyset, K \subset M \subset I$ of $C_a^K(Gr_b^{W^0} L)$ (resp. $C_a^{KM}(Gr_b^{W^0} L)$) previously seen for the polarised nilpotent orbit $Gr_b^{W^0} L$, while the fiber of $j_{!*}^K \mathcal{L}_a^K$ at y is the complex $C_a^{KM}(W_{a-1}^0 L)$. The complex $C_a^K(W_{a-1}^0 L)$ for a nilpotent orbit L defined at y in Y_K^* is a VHS by successive extensions of $Gr_i^{W^0} L$ for $i < a$.

ii) The acyclicity reduces to the case of the VHS : $Gr_a^{\mathcal{W}'} \Omega^* Gr_a^{W^0} \mathcal{L} \simeq 0$, while the direct sum is similar to the case studied in (§2.II). Notice that: $Gr_a^{\mathcal{W}'} W_{a-1}^0 \Omega^* \mathcal{L} = Gr_a^{\mathcal{W}'} \Omega^* W_{a-1}^0 \mathcal{L}$.

iii) The assertions for $Gr_b^{W^0} L$ follow from the case of VHS, while the assertion for $\mathcal{W}$ follows from (ii) whose right term appears in the case of $K = \emptyset$ and $Gr_a^{\mathcal{W}'} {W^0}_{a-1} L$ appears for $K \neq \emptyset$ and can be checked as successive extensions of $Gr_i^{W^0} L$ for $i < a$ using the count of weight as in the case of VHS and following remark

Remark: We use in the proof the fact that a bifiltered complex (K, W^0, F) with (K, W^0) defined over $\mathbb{R}$ and a finite increasing filtration W^0 such that $(Gr_r^{W^0} K, F)$ is a Hodge complex of weight a for all r, then (K, F) is a Hodge complex of weight a and W^0 induces on cohomolgy a filtration by $sub - HS$.

iv) When X is proper, the decomposition and purity results prove that the complex $(\Omega^* \mathcal{L}, \mathcal{W}', W^0, F)$ is a filtered mixed Hodge complex according to the terminology of [13].

4.3 Nearby-cocycles $\Psi_f^u \mathcal{L}$

The constructions for $\Psi_f^u \mathcal{L}$ are similar to the case of VHS. We define the filtrations

$$\mathcal{W}'_r(\Psi_f^u \mathcal{L})_{S(M)} = i_Y^\star s(\mathcal{W}'_{r+2p-1} \Omega^* \mathcal{L}[p], \eta)_{p \leq 0},$$
$$W_r^0(\Psi_f^u \mathcal{L})_{S(M)} = i_Y^\star s(\Omega^*(W_r^0 \mathcal{L})[p], \eta)_{p \leq 0},$$
$$F^r(\Psi_f^u \mathcal{L})_{S(M)} = i_Y^\star s(F^{r+p} \Omega^* \mathcal{L}[p], \eta)_{p \leq 0}$$

the action of the logarithm of the monodromy ν is defined similarly, then the relation for $a \geq 1$ and $b \in \mathbb{Z}$

$$\nu^a : Gr_{b+a}^{\mathcal{W}'} Gr_b^{W^0} \Psi_f^u \mathcal{L} \simeq Gr_{b-a}^{\mathcal{W}'} Gr_b^{W^0} \Psi_f^u \mathcal{L}$$

follows from the corresponding relation for $Gr_b^{W^0} L$ as a VHS and this concludes that: $\mathcal{W}'$ is the relative monodromy filtration with respect to W^0 on $\Psi_f^u \mathcal{L}$. This isomorphism is equivalent to

Corollary: i) For all $a \geq 1$, $Gr_{b+a}^{\mathcal{W}'} Gr_b^{W^0} ker \nu^a = s(Gr_{b+a+2p-1}^{\mathcal{W}'} Gr_b^{W^0} \Omega^* L[p], \eta)_{-a < p \leq 0} \cong 0$

ii) for $a > 0$, we have; $Gr_{b+a}^{\mathcal{W}'} Gr_b^{W^0} \Psi_f^u \mathcal{L} \simeq s(Gr_{b+a+2p-1}^{\mathcal{W}'} Gr_b^{W^0} \Omega^* L[p], \eta))_{p \leq -a} \simeq$ $\oplus_{a+2p-1 \leq -a-1} Gr_{b+a+2p-1}^{\mathcal{W}'} Gr_b^{W^0} \Omega^* L[p]$ where only a finite number of p give non zero terms.

iii) for $t > v \in \mathbb{Z}$, $Gr_r^{\mathcal{W}'}(W_t^0 / W_v^0) \Psi_f^u \mathcal{L}$ decomposes into a direct sum of intermediate extensions of VHS of weight r graded polarised with respect to an induced filtration by W^0.

iv)for a proper morphism f, the complex $(\Psi_f^u \mathcal{L}, \mathcal{W}', W^0, F)$ is a filtered mixed Hodge complex in the sense that for all $b < a$, $(W_a^0 / W_b^0) \Psi_f^u \mathcal{L}, \mathcal{W}', F)$ is a mixed Hodge complex.

The proof is similar to the case of VHS and uses the remark above for successive extensions of $Gr_i^{W^0} L$. Notice the adjustment of the weight to r in (iii) since we take the sum for $r = a + b$ with $a + 2p - 1 \leq 0$ that is the case of shift by 1 of the weight $s + 1$ of $(Gr_s^{\mathcal{W}'} \Omega^* Gr_u^{W^0} L$ for $s \leq u$ which is compensated by -1 in the formula $r + 2p - 1$, while $2p$ is compensated by p in the formula for F.

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