The First Eigenvalue of the Dirac Operator on Compact Spin Symmetric Spaces

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Abstract. We give a formula for the first eigenvalue of the Dirac operator acting on spinor fields of a spin compact irreducible symmetric space G/K.

1. Introduction

It is well-known that symmetric spaces provide examples where detailed information on the spectrum of Laplace or Dirac operators can be obtained. Indeed, for those manifolds, the computation of the spectrum can be (theoretically) done using group theoretical methods. However the explicit computation is far from being simple in general and only few examples are known. On the other hand, many results require some information about the first (nonzero) eigenvalue, so it seems interesting to get this eigenvalue without computing all the spectrum. In that direction, the aim of this paper is to prove the following formula for the first eigenvalue of the Dirac operator:

Theorem 1.1. *Let G/K be a compact, simply-connected, n-dimensional irreducible symmetric space with G compact and simply-connected, endowed with the metric induced by the Killing form of G sign-changed. Assume that G and K have same rank and that G/K has a spin structure. Let β_k, $k = 1, \ldots, p$, be the K-dominant weights occurring in the decomposition into irreducible components of the spin representation under the action of K. Then the square of the first eigenvalue of the Dirac operator is*

$$2 \min_{1 \leq k \leq p} \|\beta_k\|^2 + n/8\,, \tag{1}$$

where $\|\cdot\|$ is the norm associated to the scalar product $<\,,\,>$ induced by the Killing form of G sign-changed.

Remark 1.2. *The proof uses a lemma of R. Parthasarathy in* [Par71]*, which allows to express (1) in the following way. Let T be a fixed common maximal torus of G and K. Let Φ be the set of non-zero roots of G with respect to T. Let δ_G, (resp. δ_K) be the half-sum of the positive roots of G, (resp. K), with respect to a fixed lexicographic ordering in Φ. Then the square of the first eigenvalue of the Dirac operator is given by*

$$2\,\|\delta_G\|^2 + 2\,\|\delta_K\|^2 - 4 \max_{w \in W} < w \cdot \delta_G, \delta_K > + n/8\,, \tag{2}$$

where W is a certain (well-defined) subset of the Weyl group of G.

2. The Dirac Operator on a Spin Compact Symmetric Space

We first review some results about the Dirac operator on a spin symmetric space, cf. for instance [CFG89] or [Bär91]. A detailed survey on the subject may be found, among other topics, in the reference [BHMM]. Let G/K be a spin compact symmetric space. We assume that G/K is simply connected, so G may be chosen to be compact and simply connected and K is the connected subgroup formed by the fixed elements of an involution σ of G, cf. [Hel78]. This involution induces the Cartan decomposition of the Lie algebra $\mathfrak{G}$ of G into

$$\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}\,,$$

where $\mathfrak{K}$ is the Lie algebra of K and $\mathfrak{P}$ is the vector space $\{X \in \mathfrak{G}\,;\, \sigma_* \cdot X = -X\}$. This space $\mathfrak{P}$ is canonically identified with the tangent space to G/K at the point o, o being the class of the neutral element of G. We also assume that the symmetric space G/K is irreducible, so all the G-invariant scalar products on $\mathfrak{P}$, hence all the G-invariant Riemannian metrics on G/K are proportional. We consider the metric induced by the Killing form of G sign-changed. With this metric, G/K is an Einstein space with scalar curvature $\mathrm{Scal} = n/2$. The spin condition implies that the homomorphism $\alpha : K \to \mathrm{SO}(\mathfrak{P}) \simeq \mathrm{SO}_n$, $k \mapsto \mathrm{Ad}_G(k)_{|\mathfrak{P}}$ lifts to a homomorphism $\widetilde{\alpha} : K \to \mathrm{Spin}_n$, cf. [CG88]. Let $\rho : \mathrm{Spin}_n \to \mathrm{Hom}_{\mathbb{C}}(\Sigma, \Sigma)$ be the spin representation. The composition $\rho \circ \widetilde{\alpha}$ defines a "spin" representation of K which is denoted ρ_K. The spinor bundle is then isomorphic to the vector bundle

$$\mathbf{\Sigma} := G \times_{\rho_K} \Sigma\,.$$

Spinor fields on G/K are then viewed as K-equivariant functions $G \to \Sigma$, i.e. functions:

$$\Psi : G \to \Sigma \quad \text{s.t.} \quad \forall g \in G\,,\ \forall k \in K\,,\ \Psi(gk) = \rho_K(k^{-1}) \cdot \Psi(g)\,.$$

Let $L^2_K(G, \Sigma)$ be the Hilbert space of L^2 K-equivariant functions $G \to \Sigma$. The Dirac operator $\mathcal{D}$ extends to a self-adjoint operator on $L^2_K(G, \Sigma)$. Since it is an elliptic operator, it has a (real) discrete spectrum. Now if the spinor field Ψ is an eigenvector of $\mathcal{D}$ for the eigenvalue λ, then the spinor field $\sigma^* \cdot \Psi$ is an eigenvector for the eigenvalue $-\lambda$, hence the spectrum of the Dirac operator is symmetric with respect to the origin. Thus the spectrum of $\mathcal{D}$ may be deduced from the spectrum of its square $\mathcal{D}^2$. By the Peter-Weyl theorem, the natural unitary representation of G on the Hilbert space $L^2_K(G, \Sigma)$ decomposes into the Hilbert sum

$$\bigoplus_{\gamma \in \widehat{G}} V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, \Sigma)\,,$$

where $\widehat{G}$ is the set of equivalence classes of irreducible unitary complex representations of G, (ρ_γ, V_γ) represents an element $\gamma \in \widehat{G}$ and $\mathrm{Hom}_K(V_\gamma, \Sigma)$ is the vector space of K-equivariant homomorphisms $V_\gamma \to \Sigma$, i.e.

$$\mathrm{Hom}_K(V_\gamma, \Sigma) = \{A \in \mathrm{Hom}(V_\gamma, \Sigma) \text{ s.t. } \forall k \in K\,, A \circ \rho_\gamma(k) = \rho_K(k) \circ A\}\,.$$

The injection $V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, \Sigma) \hookrightarrow L^2_K(G, \Sigma)$ is given by

$$v \otimes A \mapsto \Big(g \mapsto (A \circ \rho_\gamma(g^{-1})\,) \cdot v\Big)\,.$$

Note that $V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, \Sigma)$ consists of $\mathcal{C}^\infty$ spinor fields to which the Dirac operator can be applied. The restriction of $\mathcal{D}^2$ to the space $V_\gamma \otimes \mathrm{Hom}_K(V_\gamma, \Sigma)$ is given by the Parthasaraty formula, [Par71]:

$$\mathcal{D}^2(v \otimes A) = v \otimes (A \circ \mathcal{C}_\gamma) + \frac{\mathrm{Scal}}{8}\, v \otimes A\,, \tag{3}$$

where $\mathcal{C}_\gamma$ is the Casimir operator of the representation (ρ_γ, V_γ). Now since the representation is irreducible, the Casimir operator is a scalar multiple of identity, $\mathcal{C}_\gamma = c_\gamma\,\mathrm{id}$, where the eigenvalue c_γ only depends of $\gamma \in \widehat{G}$. Hence if $\mathrm{Hom}_K(V_\gamma, \Sigma) \neq \{0\}$, $c_\gamma + n/16$ belongs to the spectrum of $\mathcal{D}^2$. Let $\rho_K = \oplus \rho_{K,k}$ be the decomposition of the spin representation $K \to \Sigma$ into irreducible components. Denote by $\mathrm{m}(\rho_{\gamma|K}, \rho_{K,k})$ the multiplicity of the irreducible K-representation $\rho_{K,k}$ in the representation ρ_γ restricted to K. Then

$$\dim\, \mathrm{Hom}_K(V_\gamma, \Sigma) = \sum_k \mathrm{m}(\rho_{\gamma|K}, \rho_{K,k})\,.$$

So the spectrum of the square of the Dirac operator is

$$\mathrm{Spec}(\mathcal{D}^2) = \{c_\gamma + n/16\ ;\ \gamma \in \widehat{G}\ \text{s.t.}\ \exists k\ \text{s.t.}\ \mathrm{m}(\rho_{\gamma|K}, \rho_{K,k}) \neq 0\}\,. \tag{4}$$

3. Proof of the result

We assume that G and K have same rank. Let T be a fixed common maximal torus. Let Φ be the set of non-zero roots of the group G with respect to T. According to a classical terminology, a root θ is called compact if the corresponding root space is contained in $\mathfrak{K}_{\mathbb{C}}$ (that is, θ is a root of K with respect to T) and noncompact if the root space is contained in $\mathfrak{P}_{\mathbb{C}}$. Let Φ_G^+ be the set of positive roots of G, Φ_K^+ be the set of positive roots of K, and Φ_n^+ be the set of positive noncompact roots with respect to a fixed lexicographic ordering in Φ. The half-sums of the positive roots of G and K are respectively denoted δ_G and δ_K and the half-sum of noncompact positive roots is denoted by δ_n. The Weyl group of G is denoted W_G. The space of weights is endowed with the W_G-invariant scalar product $<\,,\,>$ induced by the Killing form of G sign-changed.

Let

$$W := \{w \in W_G\ ;\ w \cdot \Phi_G^+ \supset \Phi_K^+\}\,. \tag{5}$$

By a result of R. Parthasaraty, cf. lemma 2.2 in [Par71], the spin representation ρ_K of K decomposes into the irreducible sum

$$\rho_K = \bigoplus_{w \in W} \rho_{K,w}\ , \tag{6}$$

where $\rho_{K,w}$ has for dominant weight

$$\beta_w := w \cdot \delta_G - \delta_K\,. \tag{7}$$

Now define $w_0 \in W$ such that

$$\|\beta_{w_0}\|^2 = \min_{w \in W} \|\beta_w\|^2\,, \tag{8}$$

and

(9)

$$\text{if there exists a } w_1 \neq w_0 \in W \text{ such that } \|\beta_{w_1}\|^2 = \min_{w\in W} \|\beta_w\|^2\,,\ \text{then } \beta_{w_1} \prec \beta_{w_0}\,,$$

where $\prec$ is the usual ordering on weights.

Lemma 3.1. *The weight*

$$\beta^G_{w_0} := w_0^{-1} \cdot \beta_{w_0} = \delta_G - w_0^{-1} \cdot \delta_K \,,$$

is G-dominant.

Proof. Let $\Pi_G = \{\theta_1, \dots, \theta_r\} \subset \Phi^+_G$ be the set of simple roots. It is sufficient to prove that $2\,\frac{<\beta^G_{w_0},\theta_i>}{<\theta_i,\theta_i>}$ is a non-negative integer for any simple root θ_i. Since T is a maximal common torus of G and K, β_{w_0}, which is an integral weight for K is also an integral weight for G. Now since the Weyl group W_G permutes the weights, $\beta^G_{w_0} = w_0^{-1} \cdot \beta_{w_0}$ is also a integral weight for G, hence $2\,\frac{<\beta^G_{w_0},\theta_i>}{<\theta_i,\theta_i>}$ is an integer for any simple root θ_i. So we only have to prove that this integer is non-negative.
Let θ_i be a simple root. Since $2\,\frac{<\delta_G,\theta_i>}{<\theta_i,\theta_i>} = 1$, (see for instance § 10.2 in [Hum72]) and since the scalar product $< \cdot, \cdot >$ is W_G-invariant, one gets

$$2\,\frac{< \beta^G_{w_0}, \theta_i >}{< \theta_i, \theta_i >} = 1 - 2\,\frac{< \delta_K, w_0 \cdot \theta_i >}{< \theta_i, \theta_i >} \,. \tag{10}$$

Suppose first that $w_0 \cdot \theta_i \in \Phi_K$. If $w_0 \cdot \theta_i$ is positive then $w_0 \cdot \theta_i$ is necessarily a K-simple root. Indeed let $\Pi_K = \{\theta'_1, \dots, \theta'_l\} \subset \Phi^+_K$ be the set of K-simple roots. One has $w_0 \cdot \theta_i = \sum_{j=1}^l b_{ij}\,\theta'_j$, where the b_{ij} are non-negative integers. But since $w_0 \in W$, there are l positive roots $\alpha_1, \dots, \alpha_l$ in Φ^+_G such that $w_0 \cdot \alpha_j = \theta'_j$, $j = 1, \dots, l$. So $\theta_i = \sum_{j=1}^l b_{ij}\,\alpha_j$. Now each α_j is a sum of simple roots $\sum_{k=1}^r a_{jk}\,\theta_k$, where the a_{jk} are non-negative integers. So $\theta_i = \sum_{j,k} b_{ij}\,a_{jk}\,\theta_k$. By the linear independence of simple roots, one gets $\sum_j b_{ij}\,a_{jk} = 0$ if $k \neq i$, and $\sum_j b_{ij}\,a_{ji} = 1$. Hence there exists a j_0 such that $b_{ij_0} = a_{j_0 i} = 1$, the other coefficients being zero. So $w_0 \cdot \theta_i = \theta'_{j_0}$ is a K-simple root. Now since $2\,\frac{<\delta_K,w_0\cdot\theta_i>}{<\theta_i,\theta_i>} = 2\,\frac{<\delta_K,w_0\cdot\theta_i>}{<w_0\cdot\theta_i,w_0\cdot\theta_i>} = 1$, one gets $2\,\frac{<\beta^G_{w_0},\theta_i>}{<\theta_i,\theta_i>} = 0$, hence $2\,\frac{<\beta^G_{w_0},\theta_i>}{<\theta_i,\theta_i>} \geq 0$. Now, the same conclusion holds if $w_0 \cdot \theta_i$ is a negative root of K, since $2\,\frac{<\delta_K,w_0\cdot\theta_i>}{<\theta_i,\theta_i>} = -2\,\frac{<\delta_K,-w_0\cdot\theta_i>}{<\theta_i,\theta_i>} = -1$, hence $2\,\frac{<\beta^G_{w_0},\theta_i>}{<\theta_i,\theta_i>} = 2$.
Suppose now that $w_0 \cdot \theta_i \notin \Phi_K$, that is $w_0 \cdot \theta_i$ is a noncompact root. This implies that $w_0\sigma_i$, where σ_i is the reflection across the hyperplane $\theta_i^\perp$, is an element of W. Let $\alpha_1, \dots, \alpha_m$ be the positive roots in Φ^+_G such that $w_0 \cdot \alpha_j = \alpha'_j$, where the α'_j, $j = 1, \dots, m$ are the positive roots of K. Since σ_i permutes the positive roots other than θ_i, (cf. for instance Lemma B, § 10.2 in [Hum72]), and since θ_i can not be one of the roots $\alpha_1, \dots, \alpha_m$ (otherwise $w_0 \cdot \theta_i \in \Phi^+_K$), each root $\sigma_i \cdot \alpha_j$ is positive. So $w_0\sigma_i \in W$ since $w_0\sigma_i \cdot (\sigma_i \cdot \alpha_j) = \alpha'_j$, $j = 1, \dots, m$.

We now claim that $2\,\frac{<\beta^G_{w_0},\theta_i>}{<\theta_i,\theta_i>} < 0$, which is equivalent to $2\,\frac{<\delta_K,w_0\cdot\theta_i>}{<\theta_i,\theta_i>} > 1$, is impossible.
Suppose that

$$2\,\frac{< \delta_K, w_0 \cdot \theta_i >}{< \theta_i, \theta_i >} > 1 \,. \tag{11}$$

Since δ_K can be expressed as $\delta_K = \sum_{i=1}^l c_i\,\theta'_i$, where the c_i are nonnegative, there exists a K-simple root θ'_j such that $< \theta'_j, w_0 \cdot \theta_i > 0$, and since $2\,\frac{<\theta'_j,w_0\cdot\theta_i>}{<\theta'_j,\theta'_j>}$ is an integer, this implies that

$$2\,\frac{<\theta'_j, w_0\cdot\theta_i>}{<\theta'_j,\theta'_j>} \geq 1\,. \tag{12}$$

So $\theta'_j - w_0\cdot\theta_i$ is a root (cf. for instance § 9.4 in [Hum72]). Moreover, from the bracket relation $[\mathfrak{K},\mathfrak{P}]\subset\mathfrak{P}$, it is a noncompact root. Now $\pm(\theta'_j - w_0\cdot\theta_i)$ is a positive noncompact root, so by the description of the weights of the spin representation ρ_K, (they are of the form: δ_n-(a sum of distinct positive noncompact roots), cf. §2 in [Par71]),

$$(w_0\cdot\delta_G-\delta_K)\pm(\theta'_j - w_0\cdot\theta_i) \text{ is a weight of } \rho_K\,.$$

Now, $(w_0\cdot\delta_G-\delta_K)+(\theta'_j - w_0\cdot\theta_i)$ can not be a weight of ρ_K. Otherwise since $\sigma_i\cdot\delta_G=\delta_G-\theta_i$, $(w_0\sigma_i\cdot\delta_G-\delta_K)+\theta'_j$ is a weight of ρ_K. But since $w_0\sigma_i\in W$, $\mu:=w_0\sigma_i\cdot\delta_G-\delta_K$ is a dominant weight of ρ_K. So μ is a dominant weight but not the highest weight of an irreducible component of ρ_K. Hence there exists an irreducible representation of ρ_K with dominant weight $\lambda=w\cdot\delta_G-\delta_K$, $w\in W$, whose set of weights Π contains μ. Furthermore $\mu\prec\lambda$. Now since $\mu\in\Pi$, $\|\mu+\delta_K\|^2\leq\|\lambda+\delta_K\|^2$, with equality only if $\mu=\lambda$, (cf. for instance Lemma C, §13.4 in [Hum72]). But $\|\mu+\delta_K\|^2=\|\delta_G\|^2=\|\lambda+\delta_K\|^2$, so $\mu=\lambda$, contradicting the fact that $\mu\prec\lambda$. Thus only

$$\mu_0 := (w_0\cdot\delta_G-\delta_K)-(\theta'_j - w_0\cdot\theta_i)\,, \tag{13}$$

can be a weight of ρ_K. Now one has

$$\begin{aligned}\|\mu_0\|^2 = \;& \|w_0\cdot\delta_G-\delta_K+w_0\cdot\theta_i\|^2\\ & -2 <w_0\cdot\delta_G-\delta_K+w_0\cdot\theta_i,\theta'_j> +\|\theta'_j\|^2\,.\end{aligned}$$

Since $w_0\cdot\delta_G-\delta_K$ is a dominant weight, $<w_0\cdot\delta_G-\delta_K,\theta'_j>\geq 0$, and from (12), $2<w_0\cdot\theta_i,\theta'_j>-\|\theta'_j\|^2\geq 0$, so

$$\|\mu_0\|^2\leq\|(w_0\cdot\delta_G-\delta_K)+w_0\cdot\theta_i\|^2\,.$$

Now

$$\begin{aligned}\|(w_0\cdot\delta_G-\delta_K)+w_0\cdot\theta_i\|^2 = \;& \|w_0\cdot\delta_G-\delta_K\|^2\\ & +2<\delta_G-w_0^{-1}\cdot\delta_K,\theta_i> +\|\theta_i\|^2\,.\end{aligned}$$

But, as we supposed $2\,\frac{<\beta^G_{w_0},\theta_i>}{<\theta_i,\theta_i>}<0$, one has $\frac{2<\delta_G-w_0^{-1}\cdot\delta_K,\theta_i>}{\|\theta_i\|^2}\leq -1$, so $2<\delta_G-w_0^{-1}\cdot\delta_K,\theta_i>+\|\theta_i\|^2\leq 0$, hence

$$\|(w_0\cdot\delta_G-\delta_K)+w_0\cdot\theta_i\|^2\leq\|w_0\cdot\delta_G-\delta_K\|^2\,,$$

so

$$\|\mu_0\|^2\leq\|w_0\cdot\delta_G-\delta_K\|^2\,.$$

Now, being a weight of ρ_K, μ_0 is conjugate under the Weyl group of K to a dominant weight of ρ_K, say $w_1\cdot\delta_G-\delta_K$, with $w_1\in W$. Note that $w_1\neq w_0$, otherwise since $\mu_0\prec w_1\cdot\delta_G-\delta_K$, (cf. Lemma A, § 13.2 in [Hum72]), the noncompact root $\theta'_j-w_0\cdot\theta_i$ should be a linear combination with integral coefficients of compact simple roots. But, by the bracket relation $[\mathfrak{K},\mathfrak{K}]\subset\mathfrak{K}$, that is impossible. Thus, by the definition of w_0, cf. (8), $\|w_0\cdot\delta_G-\delta_K\|^2\leq\|w_1\cdot\delta_G-\delta_K\|^2=\|\mu_0\|^2$, so

$$\|\mu_0\|^2=\|w_1\cdot\delta_G-\delta_K\|^2=\|w_0\cdot\delta_G-\delta_K\|^2\,.$$

But by the condition (9), the last equality is impossible, otherwise since $\mu_0\prec w_1\cdot\delta_G-\delta_K$ and $w_1\cdot\delta_G-\delta_K\prec w_0\cdot\delta_G-\delta_K$, the noncompact root $\theta'_j-w_0\cdot\theta_i$

should be a linear combination with integral coefficients of compact simple roots. Hence $2\,\frac{<\beta^G_{w_0},\theta_i>}{<\theta_i,\theta_i>} \geq 0$ also if $w_0\cdot\theta_i \notin \Phi_K$. $\square$

Now let (ρ_0, V_0) be an irreducible representation of G with dominant weight $\beta^G_{w_0}$. The fact that $\beta_{w_0} = w_0\cdot\beta^G_{w_0}$ is a weight of ρ_0 is an indication that $\rho_{0|K}$ may contain the irreducible representation ρ_{K,w_0}. This is actually true:

Lemma 3.2. *With the notations above,*

$$\mathrm{m}(\rho_{0|K}, \rho_{K,w_0}) \geq 1\,.$$

Proof. Let v_0 be the maximal vector in V_0, (it is unique up to a nonzero scalar multiple). Let $g_0 \in T$ be a representative of w_0. Then $g_0\cdot v_0$ is a weight vector for the weight β_{w_0}, since for any X in the Lie algebra $\mathfrak{T}$ of T:

$$\begin{aligned} X\cdot(g_0\cdot v_0) &= \tfrac{d}{dt}\Big((\exp(tX)\,g_0)\cdot v_0\Big)_{|t=0} &&= \tfrac{d}{dt}\Big(\Big(g_0\,g_0^{-1}\exp(tX)\,g_0\Big)\cdot v_0\Big)_{|t=0}\\ &= g_0\cdot\Big(\Big(\mathrm{Ad}(g_0^{-1})\cdot X\Big)\cdot v_0\Big) &&= \beta^G_{w_0}(w_0^{-1}\cdot X)\,(g_0\cdot v_0)\\ &= (w_0\cdot\beta^G_{w_0})(X)\,(g_0\cdot v_0) &&= \beta_{w_0}(X)\,(g_0\cdot v_0)\,. \end{aligned}$$

In order to prove the result, we only have to prove that $g_0\cdot v_0$ is a maximal vector (for the action K), hence is killed by root-vectors corresponding to simple roots of K. So let θ_i' be a simple root of K and E_i' be a root-vector corresponding to that simple root. Since $w_0 \in W$, there exists a positive root $\alpha_i \in \Phi_G^+$ such that $w_0\cdot\alpha_i = \theta_i'$. Then $E_i := \mathrm{Ad}(g_0^{-1})(E_i')$ is a root-vector corresponding to the root α_i since for any X in $\mathfrak{T}$

$$\begin{aligned} [X, E_i] &= [X, \mathrm{Ad}(g_0^{-1})(E_i')] &&= \mathrm{Ad}(g_0^{-1})\cdot[\mathrm{Ad}(g_0)(X), E_i']\\ &= \mathrm{Ad}(g_0^{-1})\cdot[w_0\cdot X, E_i'] &&= \Big((w_0^{-1}\cdot\theta_i')(X)\Big)\,\mathrm{Ad}(g_0^{-1})\cdot E_i'\\ &= \alpha_i(X)\,E_i\,. \end{aligned}$$

But since v_0 is killed by the action of the root-vectors corresponding to positive roots in Φ_G^+, one gets

$$\begin{aligned} E_i'\cdot(g_0\cdot v_0) &= \tfrac{d}{dt}\Big(\Big(g_0 g_0^{-1}\exp(t\,E_i')g_0\Big)\cdot v_0\Big)_{|t=0}\\ &= \tfrac{d}{dt}\Big(\Big(g_0\exp\Big(t\,\mathrm{Ad}(g_0^{-1})\cdot E_i'\Big)\Big)\cdot v_0\Big)_{|t=0}\\ &= g_0\cdot\Big(E_i\cdot v_0\Big)\\ &= 0\,. \end{aligned}$$

Hence the result. $\square$

From the result (4), we may then conclude:

Lemma 3.3.

$$2\,\|\beta_{w_0}\|^2 + n/8\,,$$

is an eigenvalue of the square of the Dirac operator.

Proof. By the Freudenthal's formula, the Casimir eigenvalue c_{γ_0} of the representation (ρ_0, V_0) is given by

$$\|\beta^G_{w_0} + \delta_G\|^2 - \|\delta_G\|^2 = 3\,\|\delta_G\|^2 + \|\delta_K\|^2 - 4\,<w_0\cdot\delta_G, \delta_K>\,.$$

On the other hand

$$\|\beta_{w_0}\|^2 = \|\delta_G\|^2 + \|\delta_K\|^2 - 2\,<w_0\cdot\delta_G, \delta_K>\,.$$

Hence

$$c_{\gamma_0} = 2\,\|\beta_{w_0}\|^2 + \|\delta_G\|^2 - \|\delta_K\|^2\,.$$

Now, the Casimir operator of $\mathfrak{K}$ acts on the spin representation ρ_K as scalar multiplication by $\|\delta_G\|^2 - \|\delta_K\|^2$, (cf. lemma 2.2 in [Par71]). Indeed, each dominant weight of ρ_K being of the form $w \cdot \delta_G - \delta_K$, $w \in W$, the eigenvalue of the Casimir operator on each irreducible component is given by:

$$\|(w \cdot \delta_G - \delta_K) + \delta_K\|^2 - \|\delta_K\|^2 = \|w \cdot \delta_G\|^2 - \|\delta_K\|^2 = \|\delta_G\|^2 - \|\delta_K\|^2\,.$$

On the other hand, the proof of the formula (3) shows that the Casimir operator of $\mathfrak{K}$ acts on the spin representation ρ_K as scalar multiplication by $\frac{\mathrm{Scal}}{8} = n/16$ (cf. [Sul79]), hence

$$\|\delta_G\|^2 - \|\delta_K\|^2 = n/16\,. \tag{14}$$

So

$$c_{\gamma_0} + n/16 = 2\,\|\beta_{w_0}\|^2 + n/8\,.$$

□

In order to conclude, we have to prove that

Lemma 3.4.

$$2\,\|\beta_{w_0}\|^2 + n/8\,,$$

is the lowest eigenvalue of the square of the Dirac operator.

Proof. Let $\gamma \in \widehat{G}$ such that there exists $w \in W$ such that $\mathrm{m}(\rho_{\gamma_{|K}}, \rho_{K,w}) \geq 1$. Let β_γ be the dominant weight of ρ_γ. First, since the Weyl group permutes the weights of ρ_γ, $w^{-1} \cdot \beta_w = \delta_G - w^{-1} \cdot \delta_K$ is a weight of ρ_γ. Hence

$$\|\beta_\gamma + \delta_G\|^2 \geq \|w^{-1} \cdot \beta_w + \delta_G\|^2\,,$$

(cf. for instance Lemma C, §13.4 in [Hum72]). So, from the Freudenthal formula,

$$c_\gamma = \|\beta_\gamma + \delta_G\|^2 - \|\delta_G\|^2 \geq \|w^{-1} \cdot \beta_w + \delta_G\|^2 - \|\delta_G\|^2\,.$$

But, using (14)

$$\|w^{-1} \cdot \beta_w + \delta_G\|^2 - \|\delta_G\|^2 = 2\,\|\beta_w\|^2 + \|\delta_G\|^2 - \|\delta_K\|^2 = 2\,\|\beta_w\|^2 + n/16\,.$$

Hence by the definition of β_{w_0},

$$c_\gamma \geq 2\,\|\beta_w\|^2 + n/16 \geq 2\,\|\beta_{w_0}\|^2 + n/16\,.$$

Hence the result. □

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