An inverse scattering problem for short-range systems in a time-periodic electric field

François Nicoleau

▶ **To cite this version:**

An inverse scattering problem for short-range systems in a time-periodic electric field.

François Nicoleau

Laboratoire Jean Leray
UMR CNRS-UN 6629
Département de Mathématiques
2, rue de la Houssinière BP 92208
F-44322 Nantes cedex 03

e-mail : nicoleau@math.univ-nantes.fr

Abstract

We consider the time-dependent Hamiltonian $H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t,x)$ on $L^2(I\!\!R^n)$, where the external electric field $E(t)$ and the short-range electric potential $V(t,x)$ are time-periodic with the same period. It is well-known that the short-range notion depends on the mean value E_0 of the external electric field. When $E_0 = 0$, we show that the high energy limit of the scattering operators determines uniquely $V(t,x)$. When $E_0 \neq 0$, the same result holds in dimension $n \geq 3$ for generic short-range potentials. In dimension $n = 2$, one has to assume a stronger decay on the electric potential.

1   Introduction.

In this note, we study an inverse scattering problem for a two-body short-range system in the presence of an external time-periodic electric field $E(t)$ and a time-periodic short-range potential $V(t,x)$ (with the same period T). For the sake of simplicity, we assume that the period $T = 1$.

The corresponding Hamiltonian is given on $L^2(I\!\!R^n)$ by :

$$(1.1) \qquad H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t,x),$$

where $p = -i\partial_x$. When $E(t) = 0$, the Hamiltonian $H(t)$ describes the dynamics of the hydrogen atom placed in a linearly polarized monochromatic electric field, or a light particle in the restricted three-body problem in which two other heavy particles are set on prescribed periodic orbits. When $E(t) = \cos(2\pi t) E$ with $E \in I\!\!R^n$, the Hamiltonian describes the well-known AC-Stark effect in the E-direction [7].

In this paper, we assume that the external electric field $E(t)$ satisfies :

(A_1) $\qquad\qquad\qquad\qquad t \to E(t) \in L^1_{loc}(I\!\!R; I\!\!R^n)$, $\quad E(t+1) = E(t)$ a.e .

Moreover, we assume that the potential $V \in C^\infty(I\!\!R \times I\!\!R^n)$, is time-periodic with period 1, and satisfies the following estimations :

(A_2) $\qquad\qquad \forall \alpha \in I\!\!N^n, \; \forall k \in I\!\!N, \; | \partial_t^k \partial_x^\alpha V(t,x) | \leq C_{k,\alpha} <x>^{-\delta-|\alpha|}$, with $\delta > 0,$

where $<x> = (1+x^2)^{\frac{1}{2}}$. Actually, we can accommodate more singular potentials (see [10], [11], [12] for example) and we need (A_2) for only k, α with finite order . It is well-known that under assumptions $(A_1) - (A_2)$, $H(t)$ is essentially self-adjoint on $\mathcal{S}(I\!\!R^n)$ the Schwartz space, [16]. We denote $H(t)$ the self-adjoint realization with domain $D(H(t))$.

Now, let us recall some well-known results in scattering theory for time-periodic electric fields. We denote $H_0(t)$ the free Hamiltonian :

$$(1.2) \qquad\qquad\qquad\qquad H_0(t) = \frac{1}{2}p^2 - E(t) \cdot x \; ,$$

and let $U_0(t,s)$, (resp. $U(t,s)$) be the unitary propagator associated with $H_0(t)$, (resp. $H(t)$) (see section 2 for details).

For short-range potentials, the wave operators are defined for $s \in I\!\!R$ and $\Phi \in L^2(I\!\!R^n)$ by :

$$(1.3) \qquad\qquad\qquad W^\pm(s) \; \Phi = \lim_{t \to \pm\infty} \; U(s,t) \; U_0(t,s) \; \Phi.$$

We emphasize that the short-range condition depends on the value of the mean of the external electric field :

$$(1.4) \qquad\qquad\qquad\qquad E_0 \; = \; \int_0^1 \; E(t) \; dt \; .$$

• The case $E_0 = 0$.

By virtue of the Avron-Herbst formula (see section 2), this case falls under the category of two-body systems with time-periodic potentials and this case was studied by Kitada and Yajima ([10], [11]), Yokoyama [22].

We recall that for a unitary or self-adjoint operator U, $\mathcal{H}_c(U)$, $\mathcal{H}_{ac}(U)$, $\mathcal{H}_{sc}(U)$ and $\mathcal{H}_p(U)$ are, respectively, continuous, absolutely continuous, singular continuous and point spectral subspace of U.

We have the following result ([10], [11], [21]) :

Theorem 1
Assume that hypotheses (A_1), (A_2) are satisfied with $\delta > 1$ and with $E_0 = 0$.
 Then : (i) the wave operators $W^{\pm}(s)$ exist for all $s \in I\!R$.
        (ii) $W^{\pm}(s+1) = W^{\pm}(s)$ and $U(s+1,s)\, W^{\pm}(s) = W^{\pm}(s)\, U_0(s+1,s)$.
        (iii) $Ran\ (W^{\pm}(s)) = \mathcal{H}_{ac}\,(U(s+1,s))$ and $\mathcal{H}_{sc}\,(U(s+1,s)) = \emptyset$.
        (iv) the purely point spectrum $\sigma_p(U(s+1,s))$ is discrete outside $\{1\}$.

Comments.

1 - The unitary operators $U(s+1,s)$ are called the Floquet operators and they are mutually equivalent. The Floquet operators play a central role in the analysis of time periodic systems. The eigenvalues of these operators are called Floquet multipliers. In [5], Galtbayar, Jensen and Yajima improve assertion (iv) : for $n = 3$ and $\delta > 2$, $\mathcal{H}_p\,(U(s+1,s))$ is finite dimensional.

2 - For general $\delta > 0$, $W^{\pm}(s)$ do not exist and we have to define other wave operators. In ([10], [11]), Kitada and Yajima have constructed modified wave operators $W_{HJ}^{\pm}$ by solving an Hamilton-Jacobi equation.

● **The case $E_0 \neq 0$.**

This case was studied by Moller [12] : using the Avron-Herbst formula, it suffices to examine Hamiltonians with a constant external electric field, (Stark Hamiltonians) : the spectral and the scattering theory for Stark Hamiltonians are well established [2]. In particular, a Stark Hamiltonian with a potential V satisfying (A_2) has no eigenvalues [2]. The following theorem, due to Moller, is a time-periodic version of these results.

Theorem 2
Assume that hypotheses (A_1), (A_2) are satisfied with $\delta > \frac{1}{2}$ and with $E_0 \neq 0$.
 Then : (i) the Floquet operators $U(s+1,s)$ have purely absolutely continuous spectrum.
        (ii) the wave operators $W^{\pm}(s)$ exist for all $s \in I\!R$ and are unitary.
        (iii) $W^{\pm}(s+1) = W^{\pm}(s)$ and $U(s+1,s)\, W^{\pm}(s) = W^{\pm}(s)\, U_0(s+1,s)$.

The inverse scattering problem.

For $s \in I\!R$, we define the scattering operators $S(s) = W^{+*}(s)\, W^{-}(s)$. It is clear that the scattering operators $S(s)$ are periodic with period 1.

The inverse scattering problem consists to reconstruct the perturbation $V(s,x)$ from the scattering operators $S(s)$, $s \in [0,1]$.

In this paper, we prove the following result :

3

Theorem 3

Assume that $E(t)$ satisfies (A_1) and let V_j, $j = 1, 2$ be potentials satisfying (A_2). We assume that $\delta > 1$ (if $E_0 = 0$), $\delta > \frac{1}{2}$ (if $E_0 \neq 0$ and $n \geq 3$), $\delta > \frac{3}{4}$ (if $E_0 \neq 0$ and $n = 2$). Let $S_j(s)$ be the corresponding scattering operators.
Then :

$$\forall s \in [0, 1], \ S_1(s) = S_2(s) \iff V_1 = V_2 \ .$$

We prove Theorem 3 by studying the high energy limit of $[S(s), p]$, (Enss-Weder's approach [4]). We need $n \geq 3$ in the case $E_0 \neq 0$ in order to use the inversion of the Radon transform [6] on the orthogonal hyperplane to E_0. See also [15] for a similar problem with a Stark Hamiltonian.

We can also remark that if we know the free propagator $U_0(t, s)$, $s, t \in I\!\!R$, then by virtue of the following relation :

(1.5) $$S(t) = U_0(t, s) \ S(s) \ U_0(s, t) \ ,$$

the potential $V(t, x)$ is uniquely reconstructed from the scattering operator $S(s)$ at only one initial time.

In [21], Yajima proves uniqueness for the case of time-periodic potential with the condition $\delta > \frac{n}{2} + 1$ and with $E(t) = 0$ by studying the scattering matrices in a high energy regime.
In [20], for a time-periodic potential that decays exponentially at infinity, Weder proves uniqueness at a fixed quasi-energy.

Note also that inverse scattering for long-range time-dependent potentials without external electric fields was studied by Weder [18] with the Enss-Weder time-dependent method, and by Ito for time-dependent electromagnetic potentials for Dirac equations [8].

2   Proof of Theorem 3.

2.1   The Avron-Herbst formula.

First, let us recall some basic definitions for time-dependent Hamiltonians. Let $\{H(t)\}_{t \in I\!\!R}$ be a family of selfadjoint operators on $L^2(I\!\!R^n)$ such that $\mathcal{S}(I\!\!R^n) \subset D(H(t))$ for all $t \in I\!\!R$.

Definition.
We call *propagator* a family of unitary operators on $L^2(I\!\!R^n)$, $U(t, s)$, $t, s \in I\!\!R$ such that :

    1 - $U(t, s)$ is a strongly continuous fonction of $(t, s) \in I\!\!R^2$.
    2 - $U(t, s) \, U(s, r) = \, U(t, r)$ for all $t, s, r \in I\!\!R$.
    3 - $U(t, s) \, (\mathcal{S}(I\!\!R^n)) \subset \mathcal{S}(I\!\!R^n)$ for all $t, s \in I\!\!R$.
    4 - If $\Phi \in \mathcal{S}(I\!\!R^n)$, $U(t, s)\Phi$ is continuously differentiable in t and s and satisfies :

$$i \, \frac{\partial}{\partial t} U(t, s) \ \Phi = \ H(t) \, U(t, s) \ \Phi \ , \quad i \, \frac{\partial}{\partial s} U(t, s) \ \Phi = \ -U(t, s) \, H(s) \ \Phi \ .$$

To prove the existence and the uniqueness of the propagator for our Hamiltonians $H(t)$, we use a generalization of the Avron-Herbst formula close to the one given in [3].

In [12], the author gives, for $E_0 \neq 0$, a different formula which has the advantage to be time-periodic. To study our inverse scattering problem, we use here a different one, which is defined for all E_0. We emphasize that with our choice, $c(t)$ (see below for the definition of $c(t)$) is also periodic with period 1; in particular $c(t) = O(1)$.

The basic idea is to generalize the well-known Avron-Herbst formula for a Stark Hamiltonian with a constant electric field E_0, [2]; if we consider the Hamiltonian B_0 on $L^2(\mathbb{R}^n)$,

$$(2.1) \qquad\qquad B_0 = \frac{1}{2}p^2 - E_0 \cdot x \ ,$$

we have the following formula :

$$(2.2) \qquad\qquad e^{-itB_0} = e^{-i\frac{E_0^2}{6}t^3} \ e^{itE_0 \cdot x} \ e^{-i\frac{t^2}{2}E_0 \cdot p} \ e^{-it\frac{p^2}{2}} \ .$$

In the next definition, we give a similar formula for time-dependent electric fields.

Definition.

We consider the family of unitary operators $T(t)$, for $t \in \mathbb{R}$:

$$T(t) = \ e^{-ia(t)} \ e^{-ib(t) \cdot x} \ e^{-ic(t) \cdot p} \ ,$$

where :

$$(2.3) \qquad b(t) = \ - \int_0^t (E(s) - E_0) \ ds - \int_0^1 \int_0^t (E(s) - E_0) \ ds \ dt \ .$$

$$(2.4) \qquad\qquad c(t) = \ - \int_0^t b(s) \ ds \ .$$

$$(2.5) \qquad\qquad a(t) = \int_0^t \left(\frac{1}{2} \ b^2(s) - E_0 \cdot c(s) \right) \ ds \ .$$

Lemma 4

The family $\{H_0(t)\}_{t \in \mathbb{R}}$ has an unique propagator $U_0(t, s)$ defined by :

$$(2.6) \qquad\qquad U_0(t, s) = \ T(t) \ e^{-i(t-s)B_0} \ T^*(s) \ .$$

Proof.

We can always assume $s = \ 0$ and we make the following ansatz :

$$(2.7) \qquad\qquad U_0(t, 0) = \ e^{-ia(t)} \ e^{-ib(t) \cdot x} \ e^{-ic(t) \cdot p} \ e^{-itB_0} \ .$$

Since on the Schwartz space, $U_0(t, 0)$ must satisfy :

$$(2.8) \qquad\qquad i \, \frac{\partial}{\partial t} \, U_0(t, 0) = H_0(t) \, U_0(t, 0) \, ,$$

the functions $a(t), \ b(t), c(t)$ solve :

$$(2.9) \qquad\qquad \dot{b}(t) = -E(t) + E_0, \ \dot{c}(t) = -b(t), \ \dot{a}(t) = \frac{1}{2} \, b^2(t) - E_0 \cdot c(t).$$

We refer to [3] for details and [12] for a different formula. □

In the same way, in order to define the propagator corresponding to the family $\{H(t)\}$, we consider a Stark Hamiltonian with a time-periodic potential $V_1(t, x)$, (we recall that $c(t)$ a is C^1-periodic function) :

$$(2.10) \qquad B(t) = \ B_0 + V_1(t, x) \ \text{ where } \ V_1(t, x) = \ e^{ic(t) \cdot p} \, V(t, x) \, e^{-ic(t) \cdot p} = \ V(t, x + c(t)).$$

Then, $B(t)$ has an unique propagator $R(t, s)$, (see [16] for the case $E_0 = 0$ and [12] for the case $E_0 \neq 0$). It is easy to see that the propagator $U(t, s)$ for the family $\{H(t)\}$ is defined by :

$$(2.11) \qquad\qquad U(t, s) = \ T(t) \, R(t, s) \, T^*(s).$$

Comments.

Since the Hamiltonians $H_0(t)$ and $H(t)$ are time-periodic with period 1, one has for all $t, s \in I\!R$:

$$(2.12) \qquad\qquad U_0(t + 1, s + 1) = U_0(t, s) \ , \ \ U(t + 1, s + 1) = U(t, s) \, .$$

Thus, the wave operators satisfy $W^{\pm}(s + 1) = W^{\pm}(s)$.

2.2   The high energy limit of the scattering operators.

In this section, we study the high energy limit of the scattering operators by using the well-known Enss-Weder's time-dependent method [4]. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(I\!R^n)$ [1], the Dirac equation [9], the N-body case [4], the Stark effect ([15], [17]), the Aharonov-Bohm effect [18].

In [13], [14] a stationary approach, based on the same ideas, is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect.

Before giving the main result of this section, we need some notation.

- Φ, Ψ are the Fourier transforms of functions in $C_0^\infty(\mathbb{R}^n)$.
- $\omega \in S^{n-1} \cap \Pi_{E_0}$ is fixed, where Π_{E_0} is the orthogonal hyperplane to E_0.
- $\Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Phi, \ \Psi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Psi$.

We have the following high energy asymptotics where $<,>$ is the usual scalar product in $L^2(\mathbb{R}^n)$:

Proposition 5

Under the assumptions of Theorem 3, we have for all $s \in [0, 1]$,

$$< [S(s), p] \ \Phi_{\lambda,\omega} \ , \ \Psi_{\lambda,\omega} > = \ \lambda^{-\frac{1}{2}} \ < \left(\int_{-\infty}^{+\infty} \partial_x V(s, x + t\omega) \ dt \right) \ \Phi \ , \ \Psi > + o \left(\lambda^{-\frac{1}{2}} \right) .$$

Comments.

Actually, for the case $n = 2$, $E_0 \neq 0$ and $\delta > \frac{3}{4}$, Proposition 5 is also valid for $\omega \in S^{n-1}$ satisfying $| \omega \cdot E_0 | < | E_0 |$, (see ([18], [15]).

Then, Theorem 3 follows from Proposition 5 and the inversion of Radon transform ([6] and [15], Section 2.3).

Proof of Proposition 5.

For example, let us show Proposition 5 for the case $E_0 \neq 0$ and $n \geq 3$, the other cases are similar. More precisely, see [18] for the case $E_0 = 0$, and for the case $n = 2$, $E_0 \neq 0$, see ([17], Theorem 2.4) and ([15], Theorem 4).

Step 1.

Since $c(t)$ is periodic, $c(t) = O(1)$. Then, $V_1(t, x)$ is a short-range perturbation of B_0, and we can define the usual wave operators for the pair of Hamiltonians $(B(t), B_0)$:

$$(2.13) \qquad\qquad \Omega^{\pm}(s) = \ s - \lim_{t \to \pm\infty} \ R(s, t) \ e^{-i(t-s)B_0} \ .$$

Consider also the scattering operators $S_1(s) = \ \Omega^{+*}(s) \ \Omega^-(s)$. By virtue of (2.6) and (2.11), it is clear that :

$$(2.14) \qquad\qquad S(s) = T(s) \ S_1(s) \ T^*(s) \ .$$

Using the fact that $e^{-ib(s)\cdot x} \ p \ e^{ib(s)\cdot x} = p + b(s)$, we have :

$$(2.15) \qquad\qquad [S(s), p] = [S(s), p + b(s)] = T(s) \ [S_1(s), p] \ T^*(s) \ .$$

Thus,

$$(2.16) \qquad < [S(s), p] \ \Phi_{\lambda,\omega} \ , \ \Psi_{\lambda,\omega} > = \ < [S_1(s), p] \ T^*(s) \ \Phi_{\lambda,\omega} \ , \ T^*(s) \ \Psi_{\lambda,\omega} > \ .$$

7

In other hand,

$$(2.17) \qquad T^*(s) \ \Phi_{\lambda,\omega} = \ e^{i\sqrt{\lambda}x.\omega} \ e^{ic(s)\cdot(p+\sqrt{\lambda}\omega)} \ e^{ib(s)\cdot x} \ e^{ia(s)} \ \Phi.$$

So, we obtain :

$$(2.18) \qquad < [S(s),p] \ \Phi_{\lambda,\omega} \ , \ \Psi_{\lambda,\omega} > \ = \ < [S_1(s),p] \ f_{\lambda,\omega}, \ g_{\lambda,\omega} >,$$

where

$$(2.19) \qquad f = \ e^{ic(s)\cdot p} \ e^{ib(s)\cdot x} \ \Phi \text{ and } g = \ e^{ic(s)\cdot p} \ e^{ib(s)\cdot x} \ \Psi .$$

Clearly, f, g are the Fourier transforms of functions in $C_0^\infty(I\!R^n)$.

- **Step 2 : Modified wave operators.**

Now, we follow a strategy close to [15] for time-dependent potentials. First, let us define a free-modified dynamic $U_D(t,s)$ by :

$$(2.20) \qquad U_D(t,s) = e^{-i(t-s)B_0} \ e^{-i\int_0^{t-s} \ V_1(u+s,up'+\frac{1}{2}u^2 E_0) \ du} \ ,$$

where p' is the projection of p on the orthogonal hyperplane to E_0.

We define the modified wave operators :

$$(2.21) \qquad \Omega_D^\pm(s) = \text{s} - \lim_{t\to\pm\infty} \ R(s,t) \ U_D(t,s) .$$

It is clear that :

$$(2.22) \qquad \Omega_D^\pm(s) = \Omega^\pm(s) \ e^{-ig^\pm(s,p')} \ ,$$

where

$$(2.23) \qquad g^\pm(s,p') \ = \ \int_0^{\pm\infty} \ V_1(u+s, up' + \frac{1}{2}u^2 E_0) \ du .$$

Thus, if we set $S_D(s) = \Omega_D^{+*}(s)\Omega_D^-(s)$, one has :

$$(2.24) \qquad S_1(s) = e^{-ig^+(s,p')} \ S_D(s) \ e^{ig^-(s,p')}$$

- **Step 3 : High energy estimates.**

Denote $\rho = min \ (1,\delta)$. We have the following estimations, (the proof is exactly the same as in ([15], Lemma 3) for time-independent potentials).

Lemma 6
For $\lambda \gg 1$, we have :

$(i) \qquad \| \ \left(V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) \ U_D(t,s) \ e^{ig^\pm(s,p')} f_{\lambda,\omega} \ \|$

$$\leq \ C \ (1+ |\ (t-s)\sqrt{\lambda} \ |)^{-\frac{1}{2}-\rho} \ .$$

$(ii) \qquad \| \ \left(R(t,s)\Omega_D^\pm(s) - U_D(t,s) \right) e^{ig^\pm(s,p')} f_{\lambda,\omega} \ \| = \ O \ (\lambda^{-\frac{1}{2}}) \ , \text{ uniformly for } t, \ s \in I\!R .$

8

- **Step 4.**

We denote $F(s, \lambda, \omega) = < [S_1(s), p]\ f_{\lambda,\omega}\ ,\ g_{\lambda,\omega} >$. Using (2.24), we have :

$$
\begin{aligned}
F(s, \lambda, \omega) &= < [e^{-ig^+(s,p')}\ S_D(s)\ e^{ig^-(s,p')}, p]\ f_{\lambda,\omega}\ ,\ g_{\lambda,\omega} > \\
&= < [S_D(s), p]\ e^{ig^-(s,p')} f_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} g_{\lambda,\omega} > \\
&= < [S_D(s) - 1, p - \sqrt{\lambda}\omega]\ e^{ig^-(s,p')} f_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} g_{\lambda,\omega} > \\
&= < (S_D(s) - 1)\ e^{ig^-(s,p')} (pf)_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} g_{\lambda,\omega} > \\
&\quad - < (S_D(s) - 1)\ e^{ig^-(s,p')} f_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} (pg)_{\lambda,\omega} > \\
&:= F_1(s, \lambda, \omega) - F_2(s, \lambda, \omega).
\end{aligned}
$$

First, let us study $F_1(s, \lambda, \omega)$. Writing $S_D(s) - 1 = (\Omega_D^+(s) - \Omega_D^-(s))^*\ \Omega_D^-(s)$ and using

$$(2.25)\quad \Omega_D^+(s) - \Omega_D^-(s) = i \int_{-\infty}^{+\infty} R(s,t) \left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s)\ dt\ ,$$

we obtain :

$$(2.26)\qquad S_D(s) - 1 = -i \int_{-\infty}^{+\infty} U_D(t, s)^* \left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right)$$
$$R(t, s)\ \Omega_D^-(s)\ dt\ .$$

Thus,

$$
\begin{aligned}
F_1(s, \lambda, \omega) = -i \int_{-\infty}^{+\infty} &< R(t, s)\ \Omega_D^-(s)\ e^{ig^-(s,p')} (pf)_{\lambda,\omega}\ , \\
&\left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ dt \\
= -i \int_{-\infty}^{+\infty} &< U_D(t, s)\ e^{ig^-(s,p')} (pf)_{\lambda,\omega}\ , \\
&\left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ dt \\
&+ R_1(s, \lambda, \omega)\ ,
\end{aligned}
$$

where :

$$(2.27)\qquad R_1(s, \lambda, \omega) = -i \int_{-\infty}^{+\infty} < \left(R(t, s)\ \Omega_D^-(s) - U_D(t, s) \right)\ e^{ig^-(s,p')} (pf)_{\lambda,\omega}\ ,$$
$$\left(V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t, s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ dt\ .$$

By Lemma 6, it is clear that $R_1(s, \lambda, \omega) = O(\lambda^{-1})$. Thus, writing $t = \frac{\tau}{\sqrt{\lambda}} + s$, we obtain :

$$(2.28) \qquad F_1(s,\lambda,\omega) \;=\; -\frac{i}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} \; < U_D(\frac{\tau}{\sqrt{\lambda}}+s,s) \; e^{ig^-(s,p')} \; (pf)_{\lambda,\omega} \;,$$

$$\left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}p'+\frac{\tau^2}{2\lambda}E_0) \right)$$

$$U_D(\frac{\tau}{\sqrt{\lambda}}+s,s) \; e^{ig^+(s,p')} g_{\lambda,\omega} > \; d\tau + \; O\left(\lambda^{-1}\right) \;.$$

Denote by $f_1(\tau,s,\lambda,\omega)$ the integrand of the (R.H.S) of (2.28). By Lemma 6 (i),

$$(2.29) \qquad | \, f_1(\tau,s,\lambda,\omega) \, | \;\leq\; C \, (1+ | \, \tau \, |)^{-\frac{1}{2}-\rho} \;.$$

So, by Lebesgue's theorem, to obtain the asymptotics of $F_1(s,\lambda,\omega)$, it suffices to determine $\lim\limits_{\lambda \to +\infty} f_1(\tau,s,\lambda,\omega)$.

Let us denote :

$$(2.30) \qquad U^\pm(t,s,p') = e^{i \int_t^{\pm\infty} V_1(u+s,up'+\frac{1}{2}u^2 E_0) \, du} \;.$$

We have :

$$(2.31) \qquad f_1(\tau,s,\lambda,\omega) \;=\; < e^{-i\frac{\tau}{\sqrt{\lambda}}B_0} \; U^-(\frac{\tau}{\sqrt{\lambda}},s,p') \; (pf)_{\lambda,\omega} \;,$$

$$\left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}p'+\frac{\tau^2}{2\lambda}E_0) \right) e^{-i\frac{\tau}{\sqrt{\lambda}}B_0} \; U^+(\frac{\tau}{\sqrt{\lambda}},s,p') \, g_{\lambda,\omega} > \;.$$

Using the Avron-Herbst formula (2.2), we deduce that :

$$(2.32) \qquad f_1(\tau,s,\lambda,\omega) \;=\; < e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \; U^-(\frac{\tau}{\sqrt{\lambda}},s,p') \; (pf)_{\lambda,\omega} \;,$$

$$\left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x+\frac{\tau^2}{2\lambda}E_0) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}p'+\frac{\tau^2}{2\lambda}E_0) \right) e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \; U^+(\frac{\tau}{\sqrt{\lambda}},s,p') \, g_{\lambda,\omega} > \;.$$

Then, we obtain :

$$(2.33) \qquad f_1(\tau,s,\lambda,\omega) \;=\; < e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \; U^-(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega) \; pf \;,$$

$$\left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x+\frac{\tau^2}{2\lambda}E_0) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}(p'+\sqrt{\lambda}\omega)+\frac{\tau^2}{2\lambda}E_0) \right)$$

$$e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \; U^+(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega) \, g > \;.$$

Since

$$(2.34) \qquad e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \;=\; e^{-i\frac{\tau\sqrt{\lambda}}{2}} \; e^{-i\tau\omega.p} \; e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \;,$$

we have

(2.35) $\quad f_1(\tau, s, \lambda, \omega) \; = \; < e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \, U^-(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) \;\; pf \; ,$

$$\left(V_1(\frac{\tau}{\sqrt{\lambda}} + s, x + \tau\omega + \frac{\tau^2}{2\lambda} \, E_0) - V_1(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}}(p' + \sqrt{\lambda}\omega) + \frac{\tau^2}{2\lambda} \, E_0) \right)$$

$$e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2} \, U^+(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) \; g > \; .$$

Since $\mid V_1(u + s, u(p' + \sqrt{\lambda}\omega) + \frac{1}{2}u^2 E_0)) \mid \; \leq \; C \, (u^2 + 1)^{-\delta} \in L^1(I\!R^+, \; du)$, it is easy to show (using Lebesgue's theorem again) that :

(2.38) $$s - \lim_{\lambda \to +\infty} U^\pm(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega) \; = 1 \; .$$

Then,

(2.39) $$\lim_{\lambda \to +\infty} f_1(\tau, s, \lambda, \omega) \; = \; < pf \; , \; (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) \; g > \; .$$

So, we have obtained :

(2.40) $\quad F_1(s, \lambda, \omega) \; = \; -\frac{i}{\sqrt{\lambda}} \; < pf, \; \left(\int_{-\infty}^{+\infty} (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) \, d\tau \right) \; g > \; + o(\frac{1}{\sqrt{\lambda}}).$

In the same way, we obtain

(2.41) $\quad F_2(s, \lambda, \omega) \; = \; -\frac{i}{\sqrt{\lambda}} \; < f, \; \left(\int_{-\infty}^{+\infty} (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) \, d\tau \right) \; pg > \; + o(\frac{1}{\sqrt{\lambda}}) \; ,$

so

(2.42) $$F(s, \lambda, \omega) \; = \; F_1(s, \lambda, \omega) - F_2(s, \lambda, \omega)$$

(2.43) $$= \; \frac{1}{\sqrt{\lambda}} \; < f, \; \left(\int_{-\infty}^{+\infty} \partial_x V_1(s, x + \tau\omega) \, d\tau \right) \; g > \; + o(\frac{1}{\sqrt{\lambda}}) \; .$$

Using (2.19) and $\partial_x V(s, x + \tau\omega) = e^{-ic(s)\cdot p} \, \partial_x V_1(s, x + \tau\omega) \, e^{ic(s)\cdot p}$, we obtain :

(2.44) $\quad F(s, \lambda, \omega) \; = \; \frac{1}{\sqrt{\lambda}} \; < \Phi, \; \left(\int_{-\infty}^{+\infty} \partial_x V(s, x + \tau\omega) \, d\tau \right) \; \Psi > \; + o(\frac{1}{\sqrt{\lambda}}) \; . \; \Box$

References

[1]   S. Arians : "Geometric approach to inverse scattering for the Schrödinger equation with magnetic and electric potentials", J. Math. Phys. 38 (6), 2761-2773, (1997).

[2]   J. E. Avron - I. W. Herbst, "Spectral and scattering theory for Schrödinger operators related to the Stark effect, Comm. Math. Phys. 52, 239-254, (1977).

[3]   H. L. Cycon - R. G. Froese - W. Kirsch -B. Simon, "Schrödinger operators with application to quntum mechanics and global geometry", Texts and Monographs in Physics, Springer Study Edition (Springer Verlag, BNerlin) (1987).

[4]   V. Enss - R. Weder, "The geometrical approach to multidimensional inverse scattering", J. Math. Phys, Vol. 36 (8), 3902-3921, (1995).

[5]   A. Galtbayar - A. Jensen - K. Yajima : "Local time decay of solutions to Schrödinger equations with time-periodic potentials", J. Stat. Phys. 116, n^0 1-4, 231-282, (2004).

[6]   S. Helgason, "The Radon Transform", Progress in Mathematics 5, Birkhäuser, (1980).

[7]   J. S. Howland, "Two problems with time-dependent Hamiltonians", Mathematical Methods and Application of Scattering Theory, ed. J. A. DeSanto, A. W. Saenz and W.W. Zachary, Lecture Notes in Physics, 130, Springer, (1980).

[8]   H. T. Ito, "An inverse scattering problem for Dirac equations with time-dependent electromagnetic potential", Publi. Res. Inst. Math. Sci. 34, n4, p. 355-381, (1998).

[9]   W. Jung, "Geometric approach to inverse scattering for Dirac equation", J. Math. Phys. 36 (8), 3902-3921, (1995).

[10]   H. Kitada - K. Yajima, "A scattering theory for time-dependent long-range potentials", Duke Math. Journal, Vol. 49, number 2, 341-376, (1982).

[11]   H. Kitada - K. Yajima, "Remarks on our paper : A scattering theory for time-dependent long-range potentials", Duke Math. Journal, Vol. 50, number 4, 1005-1016, (1983).

[12]   J. S. Moller, "Two-body short-range systems in a time-periodic electric field", Duke Math. Journal, Vol. 105, number 1, 135-166, (2000).

[13]   F. Nicoleau, "A stationary approach to inverse scattering for Schrödinger operators with first order perturbation", Communication in P.D.E, Vol 22 (3-4), 527-553, (1997).

[14]   F. Nicoleau, "An inverse scattering problem with the Aharonov-Bohm effect", Journal of Mathematical Physics, Issue 8, pp. 5223-5237, (2000).

[15]   F. Nicoleau, "Inverse scattering for Stark Hamiltonians with short-range potentials", Asymptotic Analysis, 35 (3-4), 349-359, (2003),

[16]   M. Reed - R. Simon, "Methods of mathematical physics", Vol. 2, Academic Press, (1978).

[17]   R. Weder, "Multidimensional inverse scattering in an electric field", Journal of Functional Analysis, Vol. 139 (2), 441-465, (1996).

[18]   R. Weder, "Inverse scattering for N-body systems with time dependent potentials", Inverse Problems of Wave Propagation and Diffraction, Eds. G. Chavent, P. C. Sabatier, Lecture Notes in Physics 486, Springer Verlag, (1997).

[19]   R. Weder, "The Aharonov-Bohm effect and time-dependent inverse scattering theory, Inverse Problems, Vol. 18 (4), 1041-1056, (2002).

[20]   R. Weder, " Inverse scattering at a fixed quasi-energy for potentials periodic in time", Inverse Problems 20, no. 3, 893–917, (2004).

[21]   K. Yajima, "Time periodic Schrödinger equations, Topics in the theory of Schrödinger equations, H. Araki - H. Ezawa editors, World Scientific Publishing Co. Pte. Ltd, 9-70, (2004).

[22]   K. Yokoyama, "Mourre theory for time-periodic systems", Nagoya Math. J. 149, 193-210, (1998).